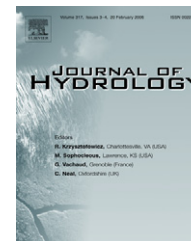




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# Analytical solution for groundwater flow in an anisotropic sloping aquifer with arbitrarily located multiwells

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## KEYWORDS

Analytical solution;  
Unconfined aquifer;  
Anisotropic;  
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**Summary** This paper presents a new analytical solution to describe the hydraulic head distribution and flow system in an anisotropic unconfined aquifer with a sloping bed and arbitrarily located multiwells under transient recharge. Unlike the existing analytical solutions for delineating the capture zones, the solution presented in this study is easy and convenient to calculate flow field in some complicated systems, and it can provide useful information for the pump-and-treat design. In addition, the presented solution may have the possible application in evaluating the sensitivity of hydrogeological parameters, or estimating the hydraulic parameters when coupled with an optimization approach to analyze the aquifer data.

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## Introduction

The pump-and-treat method is a commonly applied technology for the remediation of contaminated groundwater plumes since it can be used to exert hydraulic control over predictable aquifer extents and can capture and treat the dissolved contaminants. Optimal design of a contaminated groundwater capture system is crucial to the success of the pump-and-treat effort. Therefore, analytical solutions for determining capture zones and stagnation points have been suggested by many researchers (Javandel and Tsang,

1986; Lerner, 1992; Grubb, 1993; Faybishenko et al., 1995; Schafer, 1996; Shan, 1999; Zlotnik, 1997). These analytical solutions provide useful tools for delineating the capture zone easily; yet, Yeo and Lee (2003) indicated that these analytical solutions have some limitations. For example, they are applicable only to extraction wells situated in homogeneous aquifers that are isotropic in the horizontal plane. Additionally, they cannot be applied to the situations in which water is reinjected into the aquifer through injection wells after treatment.

Daly and Morel-Seytoux (1981) developed analytical solutions to calculate the head distribution in a heterogeneous rectangular aquifer with source and sink. Amadei and Illangasekare (1992) adopted their solutions to model fluid flow in a fractured rock. Their solutions can deal with complexities

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### Nomenclature

$W$	length of the aquifer [L]	$p$	number of injection/extraction wells
$L$	width of the aquifer [L]	$x, y$	coordinate axes
$D$	mean depth of saturation [L]	$x_i, y_i$	coordinates of the injection/extraction of the $i$ th well
$S_y$	specific yields	$x_2 - x_1$	length of the recharge basin [L]
$h$	variable groundwater head [L]	$y_2 - y_1$	width of the recharge basin [L]
$K_x, K_y$	hydraulic conductivity in $x$ and $y$ direction (L/T)	$Q_i$	injection/extraction rate of the $i$ th well [L <sup>3</sup> /T]
$R(t)$	transient recharge rate (L/T)	$\delta(x)$	Dirac delta function [1/L]
$q$	slope of the aquifer base in percentage	$a$	$q/2D$ [1/L]
$r$	decay constant (1/T)	$\kappa$	$D/S_y$ [L]
$t$	time of observation (T)		

such as heterogeneity and a large number of arbitrarily located wells at different rates. However, the existing solutions for delineating capture zones are very complicated to use in calculating the flow field if compared with the simple analytical or numerical approaches. Thus, this fact limits the use of those solutions in pump-and-treat design. Yeo and Lee (2003) provided a simple steady state solution for a confined aquifer which is homogeneous and anisotropic with arbitrarily located multiwells. Their solution is easy and convenient to use for head calculation in complicated multiwells aquifer systems. However, an analytical solution for head distribution in unconfined aquifers with arbitrarily located extraction and/or injection wells is still not available.

In unconfined aquifers, groundwater may be replenished by natural precipitation, irrigation, or artificial recharge. Most of the analytical solutions available in literature attempt to describe the water table fluctuations between equally spaced drains located on flat land in response to a uniform recharge or a recharge rate varying exponentially or linearly with time. However, there are problem areas of sloping lands in many parts of the world. Ram and Chauhan (1987) derived an analytical solution for unsteady state groundwater flow in a homogeneous, isotropic, unconfined aquifer lying over a sloping impermeable bed and receiving time-varying recharge. Singh et al. (1991) extended their work to more general transient recharge functions using the method of eigenvalue–eigenfunction expansion. Ramana et al. (1995) investigated the problem of water table fluctuation in a 2-D finite aquifer system with inclined base.

The objective of this paper is to develop a new analytical solution that is applicable for delineating the flow field in anisotropic and sloping unconfined aquifers with exponentially varying recharge and arbitrarily located multiwells. This solution contains double infinite summations of trigonometric functions that are difficult to accurately evaluate due to their oscillatory nature and slow convergence. The Levin transform method (Levin, 1973), is proposed to accelerate the convergence of evaluation of the solution. The presented analytical solution can be used to deal with the complex flow field generated by large number of multiinjections and/or extraction wells operating at different rates in a sloping unconfined aquifer that receives an exponentially varying recharge. Unlike existing solutions, this solution is applicable to a wide range of field situations. This can simplify the design of pump-and-treat system in unconfined sloping aquifers. Two examples are given to illustrate the use of this new solution.

### Mathematical model

#### Governing equation and related boundary and initial conditions

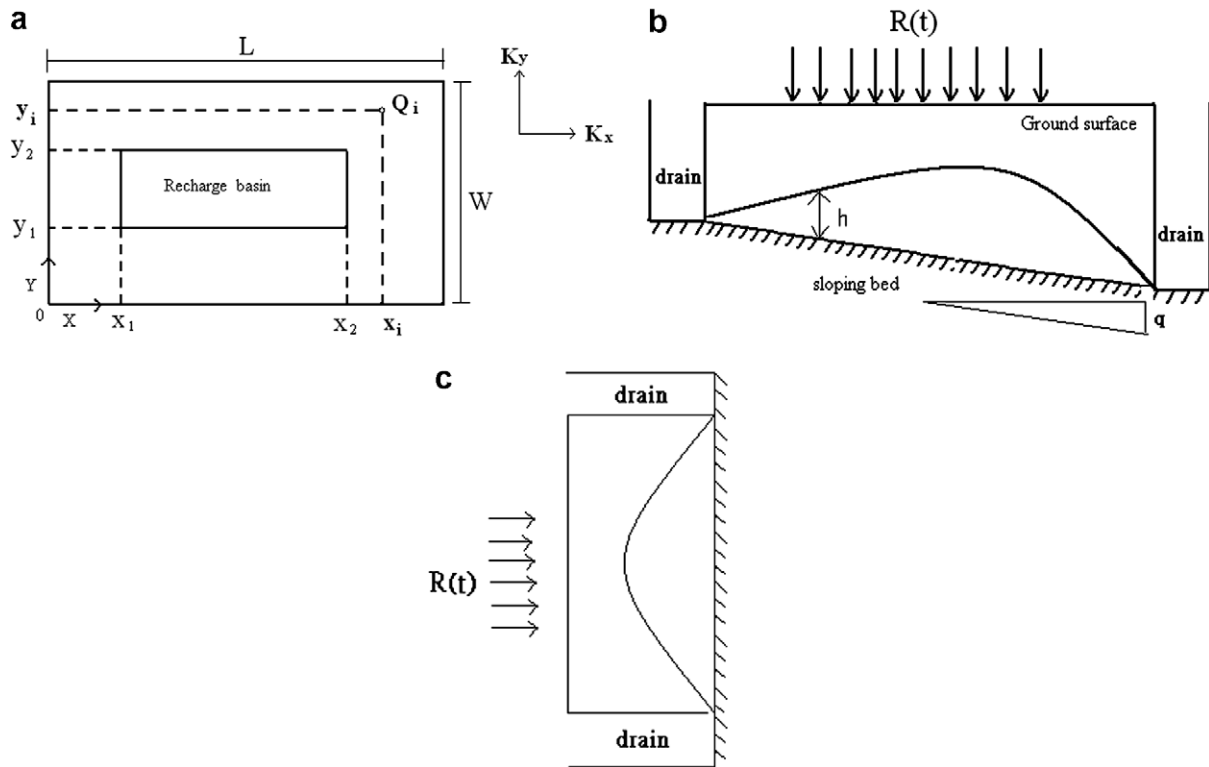
Surface and cross-sectional views along the  $x$ - and  $y$ -axes illustrate the zone of recharge and sloping aquifer for the case considered (Fig. 1a–c). The wells can be injected and pumped at multiple arbitrary points. The aquifer is receiving localized time varying recharge from a rectangular basin, and the impervious base of the aquifer is inclined along the  $x$ -axis. Assume that the aquifer is homogeneous and anisotropic in the horizontal plane and the rate of recharge compared to the hydraulic conductivity being so small that the vertically added water to the water table flows almost horizontally. Furthermore, the Dupuit's assumptions are applicable in this aquifer that the vertical hydraulic gradient is small and negligible (Schwartz and Zhang, 2003). The equation governing the groundwater flows with multiple injection–extraction wells is formulated as

$$K_x \frac{\partial^2 H}{\partial x^2} + K_y \frac{\partial^2 H}{\partial y^2} - 2aK_x \frac{\partial H}{\partial x} + 2R(t) = \frac{1}{\kappa} \frac{\partial H}{\partial t} + \sum_{i=1}^p Q_i \delta(x - x_i) \delta(y - y_i) \quad (1)$$

where  $H = h^2$ ,  $h$  is the variable groundwater head,  $W$  and  $L$  are the width and length of the aquifer,  $K_x$  and  $K_y$  are the hydraulic conductivity in the  $x$  and  $y$  direction, respectively,  $x_i$  and  $y_i$  are the coordinates of the injection or extraction wells,  $p$  is the number of wells,  $\delta$  is the Dirac delta function,  $Q_i$  is the injection or extraction rate of the  $i$ th well,  $a = q/2D$  and  $q$  is the slope of the impervious barrier overlain by aquifer,  $D$  is the mean depth of saturation,  $\kappa = D/S_y$ ,  $S_y$  is the specific yield,  $t$  is the time of observation, and  $R(t)$  is the time varying rate of recharge. Since the rate of recharge depends on many hydrological conditions. For simplicity, the rate of recharge is defined as (Ramana et al., 1995)

$$R(t) = \begin{cases} R_1 + R_0 \exp(-rt), & x_1 \leq x \leq x_2; \quad y_1 \leq y \leq y_2 \\ 0, & 0 \leq x \leq x_1; \quad 0 \leq y \leq y_1 \\ 0, & x_2 \leq x \leq W; \quad y_2 \leq y \leq L \end{cases} \quad (2)$$

where  $r$  is the decay constant. Eq. (2) implies that the recharge rate exponentially decreases from an initial value



**Figure 1** (a) A top view for an unconfined aquifer with the recharge scheme and the injection or extraction wells at arbitrary location. (b) The vertical section in  $x$ -direction of the aquifer system. (c) The vertical section in  $y$ -direction of the aquifer system.

$R_1 + R_0$  to a lower value  $R_1$  and thereafter remains constant. The initial and boundary conditions are respectively described as

$$H(x, y, 0) = 0 \quad (3)$$

and

$$H(0, y, t) = H(L, y, t) = H(x, 0, t) = H(x, W, t) = 0 \quad (4)$$

Since the governing equation, the initial condition and boundary conditions are specified, the solution of hydraulic head in a homogeneous, anisotropic, and sloping unconfined aquifer with transient recharge and arbitrarily located multiwells can be obtained via the finite sine transform and eigenvalue–eigenfunction method for Eqs. (1)–(4). Detailed derivations for the solution are given in Appendix and the result is

$$H(x, y, t) = \frac{16}{K_x L \pi} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{1}{m} \sin\left(\frac{m\pi(y_1 + y_2)}{2W}\right) \times \sin\left(\frac{m\pi(y_2 - y_1)}{2W}\right) \alpha_1 \Omega_1 - \frac{4}{K_x W L} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \alpha_2 \Omega_2 \quad (5)$$

where

$$\alpha_1 = \frac{1}{a^2 + \lambda_n^2} \{ \exp(-a x_2) [-a \sin(\lambda_n x_2) - \lambda_n \cos(\lambda_n x_2)] - \exp(-a x_1) [-a \sin(\lambda_n x_1) - \lambda_n \cos(\lambda_n x_1)] \} \quad (6)$$

$$\alpha_2 = \sum_{i=1}^p Q_i \sin\left(\frac{m\pi y_i}{W}\right) \sin\left(\frac{n\pi x_i}{L}\right) \exp(-a x_i) \quad (7)$$

$$\Omega_1 = e^{ax} \left\{ R_0 \frac{e^{-rt} - e^{-\kappa K_x (a^2 + \beta_m^2 + \lambda_n^2)t}}{(a^2 + \beta_m^2 + \lambda_n^2) - \frac{r}{\kappa K_x}} + R_1 \frac{1 - e^{-\kappa K_x (a^2 + \beta_m^2 + \lambda_n^2)t}}{(a^2 + \beta_m^2 + \lambda_n^2)} \right\} \times \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{W}\right) \quad (8)$$

$$\Omega_2 = e^{ax} \left\{ \frac{1 - e^{-\kappa K_x (a^2 + \beta_m^2 + \lambda_n^2)t}}{(a^2 + \beta_m^2 + \lambda_n^2)} \right\} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{W}\right) \quad (9)$$

$$\beta_m = \sqrt{\frac{K_y m\pi}{K_x W}} \quad (10)$$

and

$$\lambda_n = \frac{n\pi}{L} \quad (11)$$

### Special cases

#### Ramana et al.'s solution (Ramana et al., 1995)

By substituting  $K_x = K_y = K$  and  $Q_i = 0$  into Eq. (5), the solution for isotropic aquifer without extraction and/or injection wells is obtained as

$$H(x, y, t) = \frac{16}{KL\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m} \sin\left(\frac{m\pi(y_1 + y_2)}{2W}\right) \times \sin\left(\frac{m\pi(y_2 - y_1)}{2W}\right) \alpha_1 \Omega'_1 \quad (12)$$

where

$$\Omega'_1 = e^{ax} \left\{ R_0 \frac{e^{-rt} - e^{-\kappa K (a^2 + \beta_m^2 + \lambda_n^2)t}}{(a^2 + \beta_m^2 + \lambda_n^2) - \frac{r}{\kappa K}} + R_1 \frac{1 - e^{-\kappa K (a^2 + \beta_m^2 + \lambda_n^2)t}}{(a^2 + \beta_m^2 + \lambda_n^2)} \right\} \times \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{W}\right) \quad (13)$$

with

$$\beta'_m = \frac{m\pi}{W} \tag{14}$$

Note that Eq. (12) is given in Ramana et al. (1995); thus, their solution can be considered as a special case of our solution, Eq. (5).

**Rao and Sarma’s solution (Rao and Sarma, 1981)**

By substituting  $K_x = K_y = K$ ,  $Q_d = 0$ ,  $R_0 = 0$  and  $a = 0$  into Eq. (5), the solution for isotropic horizontal aquifer with constant recharge can be obtained as following:

$$H(x, y, t) = \frac{16}{KL\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m} \times \sin\left(\frac{m\pi(y_1 + y_2)}{2W}\right) \sin\left(\frac{m\pi(y_2 - y_1)}{2W}\right) \alpha_1 \Omega'_1 \tag{15}$$

where

$$\Omega'_1 = \left\{ R_1 \frac{1 - e^{-\kappa K(\beta_m'^2 + \lambda_n^2)t}}{(\beta_m'^2 + \lambda_n^2)} \right\} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{W}\right) \tag{16}$$

Eq. (13) is the same as the solution of Rao and Sarma (1981), thus, Rao and Sarma’s solution can be also considered as a special case of our solution.

**Numerical evaluation**

The solution includes double infinite series and is laborious to directly evaluate due to the nature of alternate oscillation and slow convergence. The value of  $D$  was calculated as  $h_{max}/2$ , where  $h_{max}/2$  is the maximum height of water table at midpoint between the drains for a particular experiment (Ram and Chauhan, 1987). Fig. 2 demonstrates the plots of the summation of Eq. (5) versus  $n'$  for  $x = 150$  m,  $y = 350$  m,  $K_x = K_y = 50$  m/day,  $S_y = 0.05$ ,  $q = 4\%$ , and constant recharge rate  $R_1 = 0.005$  m/day in the whole aquifer

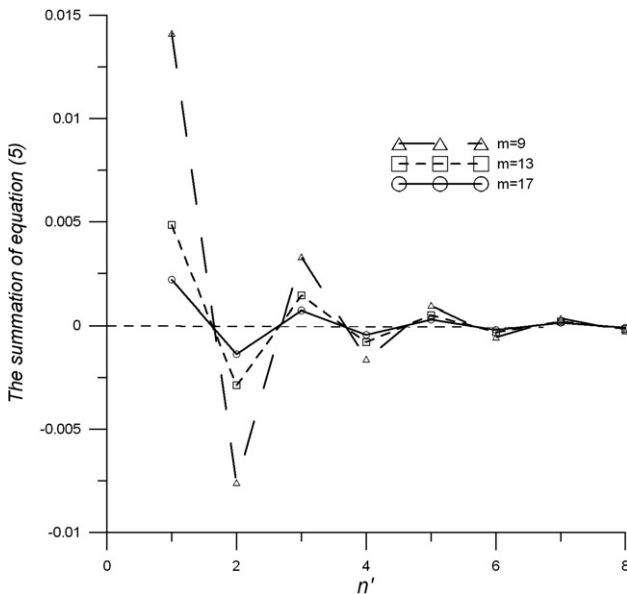


Figure 2 The summation of Eq. (5) versus the argument  $n'$ .

without injection or extraction wells, where  $n'$  is the new term for summation of continuous positive or negative values for  $n$ . This figure shows the oscillatory nature of the summation and also displays that the summation will converge slowly for large  $m$ . In addition, the value of the summation of Eq. (5) approaches zero when  $m$  is large. This curve demonstrates persistent oscillations over many cycles, and the amplitude of oscillation decreases with the horizontal axis. In addition, these formulas are essentially composed of infinite series and may also converge slowly. Thus, the Levin transform (Levin, 1973) is employed to accelerate the convergence when evaluating the infinite series.

**The Levin transforms**

Levin transform is a nonlinear acceleration method. Let  $\{S_n, n = 1, 2, \dots\}$  be an infinite sequence of real numbers tending to a limit  $S$ . The corresponding approximation  $T_{k,n}$  for  $S$  can be expressed by the sequence transform

$$T_{k,n} = S_n + g_{k,n} \Delta S_n \tag{17}$$

where  $T_{k,n}$  is the approximation of order  $k$  for the  $n$ th term of the transformed sequence,  $g_{k,n}$  is approximation of order  $k$  for the  $n$ th term of an associated sequence and  $\Delta S_n = S_{n+1} - S_n$ . Bhowmick et al. (1989) presented a form for the Levin transform, which is more convenient for numerical computations. The expression for the Levin transform provided by Bhowmick et al. (1989) can be expressed as

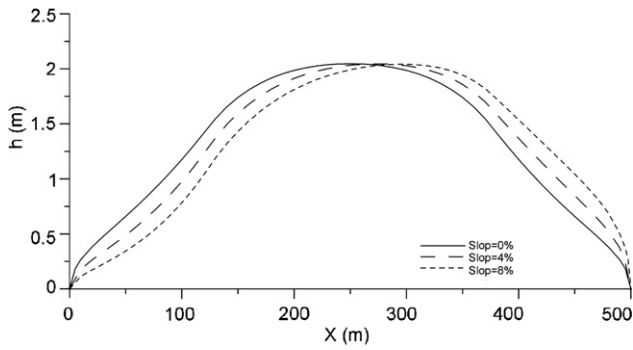
$$T_{k,n}(\{S_n\}) = \frac{\sum_{j=0}^k (-1)^j \binom{k}{j} (n+j)^{k-2} S_{n+j} (\Delta S_{n+j})^{-1}}{\sum_{j=0}^k (-1)^j \binom{k}{j} (n+j)^{k-2} (\Delta S_{n+j})^{-1}} \tag{18}$$

It is necessary to set a certain convergence criterion when applying the Levin transform to evaluate a given series. Accordingly, one may define a convergence factor, ERR, as

$$|T_{k+1,n}(\{S_n\}) - T_{k,n}(\{S_n\})| \leq \text{ERR} \tag{19}$$

**Results and discussion**

To test the performance of the analytical solution for a hypothetical problem, a groundwater flow system for sloping bed without injection–extraction wells was considered with the length and width of the aquifer are both 500 m, and the hydraulic conductivities in  $x$  and  $y$  direction are both 50 m/day. The specific yield is 0.05, and an exponentially decreasing recharge pattern represented by  $R = 0.0371e^{-0.571t}$  (Ramana et al., 1995) was assumed, where  $R$  is the recharge rate in meters per day and  $t$  is the time in days. The aquifer is receiving localized time varying recharge in the range of  $125 \text{ m} < x, y < 375 \text{ m}$ . Fig. 3 shows the water table profiles at  $y = 250$  m with the above exponential decreasing recharge at  $t = 1$  day in an aquifer with slopes of 0%, 4%, and 8%. This figure demonstrates that the water surface tends to shift toward the downslope as the slope of the impermeable bed increases. Fig. 4 illustrates the water table profiles at  $y = 250$  m in



**Figure 3** Water table profiles for different slope of the aquifer with localized recharge ( $y = 250$  m,  $K_x = K_y = 50$  m/day,  $t = 1$  day and  $R = 0.0371e^{-0.571t}$ ).

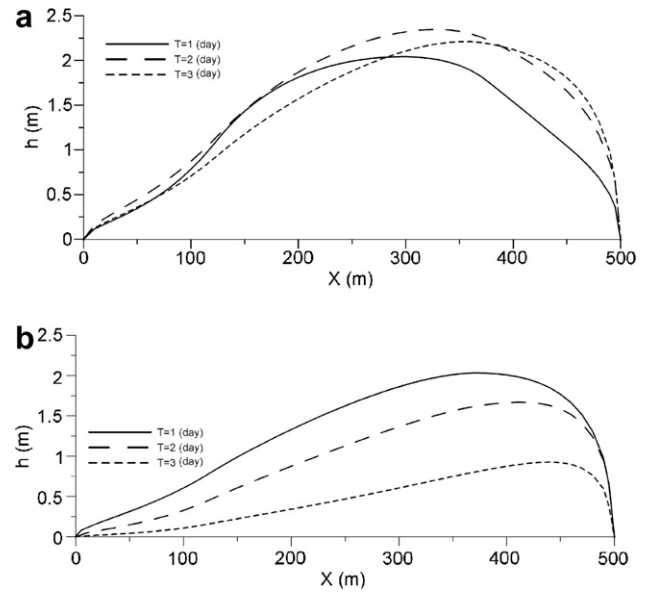
the aquifer with slope of 8% for different periods of water table rise. As shown in Fig. 4a, the water table rises at faster rate in the recharge region at the beginning. Yet, on the third day it starts decaying towards the upslope but still rises in the region away from the upslope. Thereafter, as shown in Fig. 4b, it starts decaying with time homogeneously in the entire region. It is clear from the figure that the point maximum growth is shifted with time towards downslope. Fig. 5 plots the hydraulic head distribution with the slope of 8% for different ratio of hydraulic conductivity in  $x$  direction to  $y$  direction for uniform recharge with the rate  $R = 0.0371e^{-0.571t}$  for 1 day within the whole domain of the aquifer. These figures demonstrate that the water table is steeper in  $y$  direction than in  $x$  direction as the ratio of hydraulic conductivity in  $x$ -direction to  $y$ -direction increases.

The following examples are presented to describe the hydraulic head distribution and flow system in an anisotropic unconfined aquifer with a sloping or horizontal bed and arbitrarily located multiwells under transient recharge.

**Example 1.** Fig. 6 shows that an aquifer with an area of  $500\text{ m} \times 500\text{ m}$ , two real extraction wells are located at  $(400\text{ m}, 200\text{ m})$  and  $(400\text{ m}, 300\text{ m})$ , and one real injection well is placed at  $(150\text{ m}, 250\text{ m})$  for example 1-1. The hydraulic conductivities  $K_x$  and  $K_y$  are 0.5 and 0.25 m/day, respectively. The boundary heads at  $x = 0$  and  $x = L$  are respectively kept at  $h_1 = 10$  m and  $h_2 = 5$  m, and no-flow boundaries are imposed on the sides of  $y = 0$  and  $y = W$  in the first example. Since the hydraulic head in Eq. (5) is derived from the condition that  $H = 0$  at  $x = 0$  and  $x = L$ , the new head distribution can be expressed as

$$h'(x, y, t) = \sqrt{H(x, y, t) + \frac{h_2^2 - h_1^2}{L}x + h_1^2} \quad (20)$$

where  $h_1$  and  $h_2$  are the boundary heads at  $x = 0$  and  $x = L$ , respectively. The hydraulic gradient at  $(x, y)$  can be determined by taking partial derivatives of Eq. (20) with respect to  $x$  and  $y$ , respectively. The field is extended for image wells, and the shaded area shown in Fig. 6a represents the concerned field with real wells and other dotted squares are for image wells. The number of image wells is infinite in theory, but it is known that pairs of image wells closest to real well yield an acceptable head change because others

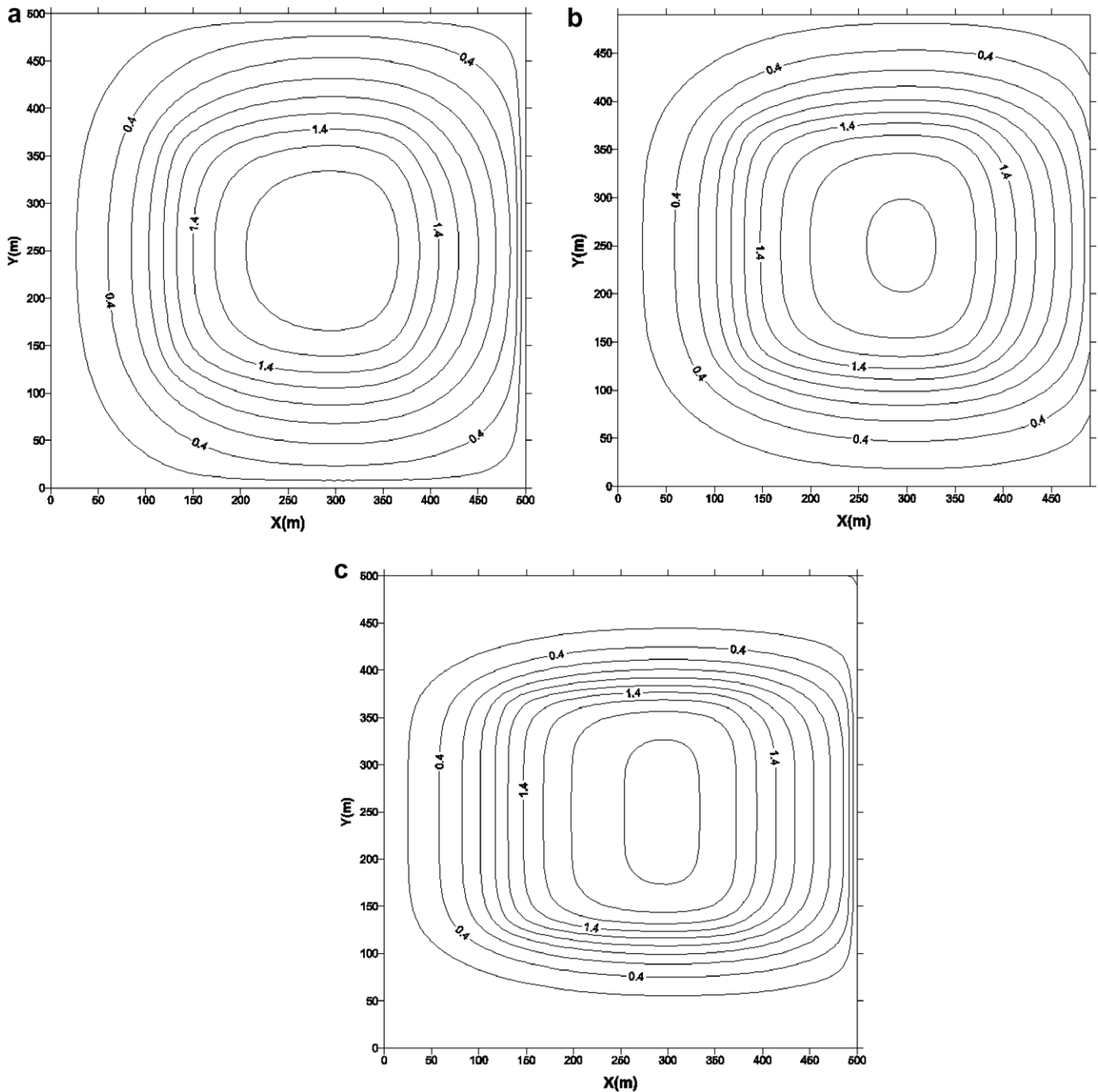


**Figure 4** Water table profiles at different times for sloping bed with localized recharge (a) rising, (b) decaying ( $y = 250$  m,  $K_x = K_y = 50$  m/day, slope = 8% and  $R = 0.0371e^{-0.571t}$ ).

have a negligible influence on the head change (Yeo and Lee, 2003). The exponentially recharge rate is  $0.0371e^{-0.571t}$  for 100 days, the slope of aquifer is 8% and the specific yield is 0.05. The image extraction and injection wells have the same rates as real wells.

These two real extraction wells with an extraction rate of  $8.64\text{ m}^3/\text{day}$  and the injection well with a rate of  $4.32\text{ m}^3/\text{day}$  are imposed for example 1-2 as shown in Fig. 6b. Fig. 6c illustrates the head distribution and gradient for the same scenario in example 1-1, but with a extraction rate of  $25.92\text{ m}^3/\text{day}$  and closer extraction wells. The real extraction wells with a rate of  $25.92\text{ m}^3/\text{day}$  in example 1-2 shown in Fig. 6c are located at  $(400\text{ m}, 225\text{ m})$  and  $(400\text{ m}, 275\text{ m})$ . The extraction rates captured in Fig. 6b is not achieved; however, the contaminated groundwater, if happened, can be restrained in the central area indicated in Fig. 6c by the new arrangement of wells and higher extraction rates.

**Example 2.** The analytical solution derived in this study can be applied to the situation that the no-flow boundaries are at  $y$  faces; however, for the no-flow boundaries at  $x$  faces, this solution gives only an approximate evaluation for the flow field due to the dipping direction of the image aquifer. Thus, Example 2 simulates the head distribution and gradients with the no-flow boundaries at  $x$  and  $y$  faces in horizontal base (no sloping) of the aquifer. Fig. 7 illustrates the geometry of real and images wells for the aquifer of  $500\text{ m} \times 500\text{ m}$  with two no-flow boundaries intercepted at a right angle for Example 2. The conductivities  $K_x$  and  $K_y$  are 0.5 m/day, and one real extraction well with the rate of  $8.64\text{ m}^3/\text{day}$  are located at  $(400\text{ m}, 400\text{ m})$ , and three real injection wells with the same rate are placed at  $(200\text{ m}, 250\text{ m})$ ,  $(100\text{ m}, 400\text{ m})$ , and  $(300\text{ m}, 150\text{ m})$ . Similarly, the exponentially recharge rate is  $0.0371e^{-0.571t}$  m/day for 100 days, and the specific yield is 0.05. Fig. 7 shows



**Figure 5** Water table profiles for different ratio of hydraulic conductivity: (a)  $K_x/K_y = 1.0$ , (b)  $K_x/K_y = 2.0$  and (c)  $K_x/K_y = 5.0$  (slope = 8%,  $t = 1$  day and  $R = 0.0371e^{-0.571t}$ ).

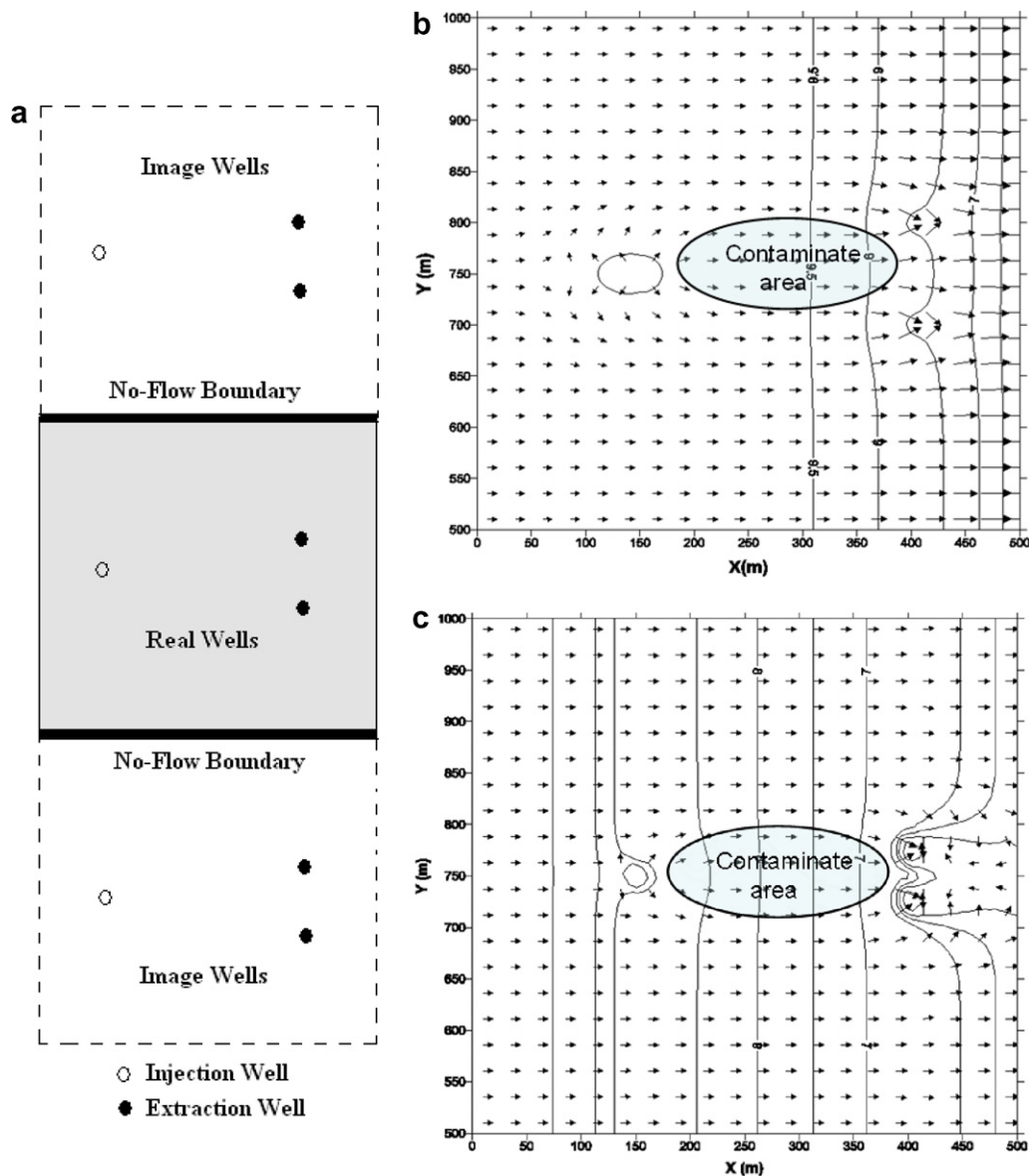
head distributions and gradients for horizontal base of the hypothetical aquifer with time varying recharge, which illustrates a good containment of contaminated groundwater by the multiwells.

These two examples demonstrate that this new analytical solution is useful for the easy calculation of the hydraulic distribution and gradient in a complex flow field by multiwells and time varying recharge in a sloping unconfined aquifer. Consequently, this solution can be a useful tool to quickly delineate the capture zone and aid in the design the pump-and-treat system for groundwater remediation.

## Conclusions

A mathematical model for representing the groundwater flow system in a homogeneous, anisotropic, and sloping unconfined aquifer with transient recharge and multiple injection and/or extraction wells is presented. The solution of the model derived via the Fourier finite sine transform and separation of variables consists of an infinite series which has poor convergence. Thus, the Levin method is employed for efficiently evaluating the analytical solutions. The results demonstrate that the water table profiles are significantly influenced by the transient recharge, the





**Figure 6** (a) Geometry of real and image wells, (b) head distributions and gradients of the aquifer for example 1-1, and (c) head distributions and gradients of the aquifer for example 1-2.

anisotropy, and the slope of the aquifer. Furthermore, the results also illustrate the hydraulic head distributions and gradients for the sloping unconfined aquifer with transient recharge and multiple wells.

This new solution can be used to describe the hydraulic head distribution and flow system in an anisotropic unconfined aquifer with a sloping bed and arbitrarily located multiwells under transient recharge. Since the existing analytical solutions for delineating the capture zones were complicated to calculate, the solution presented in this study is easy and convenient for head calculation in complicated systems, and it can provide useful information for the pump-and-treat design. In addition, the presented solution may have the possible application in evaluating the sensitivity of the hydrogeological parameters on the predicted capture zones, and estimating the hydraulic parameters when coupled with an optimization approach to analyze the aquifer data.

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## Appendix. Derivation of (5)

Taking the finite Fourier sine transform to  $H(x, y, t)$  in Eq. (1), it can be obtained as (Andrews and Shivamoggi, 1999)

$$F_s[H(x, y, t); y \rightarrow m] = \int_0^W H(x, y, t) \sin\left(\frac{m\pi y}{W}\right) dy = \bar{H}(x, m, t) \quad (\text{A.1})$$

The finite Fourier sine transform of the second partial derivatives of  $H(x, y, t)$  in Eq. (1) can be expressed as

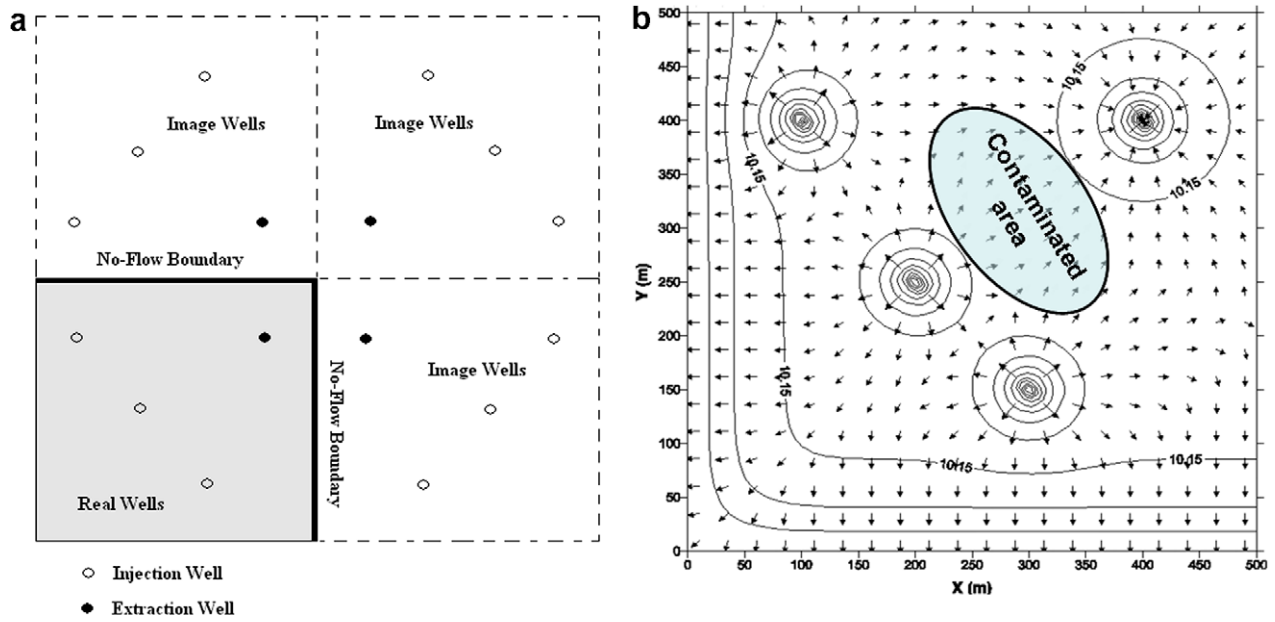


Figure 7 (a) Geometry of real and image wells and (b) head distributions and gradients of the aquifer for Example 2.

$$\int_0^W \frac{\partial^2 H}{\partial y^2} \sin\left(\frac{m\pi y}{W}\right) dy = -\left(\frac{m\pi}{W}\right)^2 \bar{H}(x, m, t) \quad (\text{A.2})$$

Following the properties of the Dirac Delta function, the finite Fourier sine transform of the second terms on the right hand side of Eq. (1) are evaluated as

$$\begin{aligned} & \int_0^W \sum_{i=1}^p Q_i \delta(x - x_i) \delta(y - y_i) \sin\left(\frac{m\pi y}{W}\right) dy \\ &= \sum_{i=1}^p Q_i \delta(x - x_i) \sin\left(\frac{m\pi y_i}{W}\right) \end{aligned} \quad (\text{A.3})$$

Replacing Eq. (1) with Eqs. (A.1)–(A.3) results the following finite Fourier sine-transformed equation:

$$\begin{aligned} & K_x \frac{\partial^2 \bar{H}}{\partial x^2} - K_y \left(\frac{m\pi}{W}\right)^2 \bar{H} - 2aK_x \frac{\partial \bar{H}}{\partial x} + 2\bar{R}(t) \\ &= \frac{1}{\kappa} \frac{\partial \bar{H}}{\partial t} + \sum_{i=1}^p Q_i \delta(x - x_i) \sin\left(\frac{m\pi y_i}{W}\right) \end{aligned} \quad (\text{A.4})$$

where  $\bar{R}(t)$ , the transformed form of  $R(t)$ , is

$$\bar{R}(t) = R(t) \frac{W}{m\pi} \left[ \cos\left(\frac{m\pi y_1}{W}\right) - \cos\left(\frac{m\pi y_2}{W}\right) \right] \quad (\text{A.5})$$

and the associated Fourier-transformed initial and boundary conditions are

$$\bar{H}(x, m, 0) = 0 \quad (\text{A.6})$$

and

$$\bar{H}(0, m, t) = \bar{H}(L, m, t) = 0 \quad (\text{A.7})$$

For the solution of Eq. (A.4), the following transformation is devised (Ozisk, 1980):

$$\bar{H}(x, m, t) = G(x, t) \exp(ax) \exp[-\kappa K_x t(a^2 + \beta_m^2)] \quad (\text{A.8})$$

with

$$\beta_m = \sqrt{\frac{K_y m\pi}{K_x W}} \quad (\text{A.9})$$

which transforms Eqs. (A.4)–(A.6) as follows:

$$\begin{aligned} & \frac{\partial^2 G}{\partial x^2} + \frac{2\bar{R}(t)}{K_x} e^{-ax} e^{\kappa K_x (a^2 + \beta_m^2) t} \\ &= \frac{1}{\kappa K_x} \frac{\partial G}{\partial t} + \sum_{i=1}^p Q_i \delta(x - x_i) \sin\left(\frac{m\pi y_i}{W}\right) \frac{1}{K_x} e^{-ax} e^{\kappa K_x (a^2 + \beta_m^2) t} \end{aligned} \quad (\text{A.10})$$

$$G(x, 0) = 0 \quad (\text{A.11})$$

$$G(0, t) = 0 \quad (\text{A.12})$$

and

$$G(L, t) = 0 \quad (\text{A.13})$$

The solution of Eq. (A.10) with the initial and boundary conditions may then be obtained by the separation of variables. It is convenient to first solve the homogeneous form of Eq. (A.10), i.e.,

$$\frac{\partial^2 G}{\partial x^2} = \frac{1}{\kappa K_x} \frac{\partial G}{\partial t} \quad (\text{A.14})$$

by substituting

$$G(x, t) = \phi(x) \varphi(t) \quad (\text{A.15})$$

into Eq. (A.10). Two separate equations may be obtained, one for space coordinate,  $x$ , and other for time,  $t$ , i.e.,

$$\frac{\partial^2 \phi}{\partial x^2} + \lambda^2 \phi = 0 \quad (\text{A.16})$$

and

$$\frac{\partial \varphi}{\partial t} + \lambda^2 \kappa K_x \varphi = 0 \quad (\text{A.17})$$

The corresponding boundary conditions are given as

$$\phi(0) = 0 \quad (\text{A.18})$$

and

$$\phi(L) = 0 \quad (\text{A.19})$$



Thus, the solution of Eq. (A.16) with the corresponding boundary conditions, Eqs. (A.18) and (A.19), is

$$\phi_n(x) = C_n \sin(\lambda_n x) \quad (\text{A.20})$$

where  $\phi_n(x)$ 's are eigenfunctions and  $\lambda_n$ 's are eigenvalues, given by

$$\lambda_n = \frac{n\pi}{L}, \quad n = 0, 1, 2, \dots \quad (\text{A.21})$$

After applying the orthogonal property of  $\phi_n(x)$  on Eq. (A.20),  $C_n$  is obtained as

$$C_n = \sqrt{\frac{2}{L}} \quad (\text{A.22})$$

Thus, Eq. (A.20) becomes

$$\phi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}x\right) \quad (\text{A.23})$$

For these values of  $\phi_n(x)$ , Eq. (A.15) becomes

$$G(x, t) = \sum_{n=0}^{\infty} \phi_n(x) \varphi_n(t) \quad (\text{A.24})$$

Substituting Eq. (A.24) into Eq. (A.10) and multiplying Eq. (A.10) throughout by  $C_n \sin(\lambda_n x)$  and integrating over the length of the aquifer obtains

$$\frac{d\varphi_n}{dt} + \lambda_n^2 \kappa K_x \varphi_n(t) = \sqrt{\frac{2}{L}} \kappa \exp[\kappa K_x (a^2 + \beta_m^2) t] \cdot [2\bar{R}(t)\alpha_1 - \alpha_2] \quad (\text{A.25})$$

where

$$\alpha_1 = \frac{1}{a^2 + \lambda_n^2} \{ \exp(-ax_2) [-a \sin(\lambda_n x_2) - \lambda_n \cos(\lambda_n x_2)] - \exp(-ax_1) [-a \sin(\lambda_n x_1) - \lambda_n \cos(\lambda_n x_1)] \} \quad (\text{A.26})$$

and

$$\alpha_2 = \sum_{i=1}^p Q_i \sin\left(\frac{m\pi y_i}{W}\right) \sin\left(\frac{n\pi x_i}{L}\right) \exp(-ax_i) \quad (\text{A.27})$$

The corresponding initial condition is given by

$$\varphi_n(t) = 0 \quad (\text{A.28})$$

The solution of Eq. (A.25) subject to the initial condition, Eq. (A.28), is

$$\varphi_n(t) = \frac{C_n}{K_x} \left\{ \frac{2R_0 \alpha_1 \alpha_3}{(a^2 + \beta_m^2 + \lambda_n^2 - \frac{r}{\kappa K_x})} \left[ \exp \kappa K_x \left( a^2 + \beta_m^2 - \frac{r}{\kappa K_x} \right) t - \exp(-\kappa K_x \lambda_n^2 t) \right] - \frac{2R_1 \alpha_1 \alpha_3 - \alpha_2}{(a^2 + \beta_m^2 + \lambda_n^2)} \left[ \exp \kappa K_x (a^2 + \beta_m^2) t - \exp(-\kappa K_x \lambda_n^2 t) \right] \right\} \quad (\text{A.29})$$

where

$$\alpha_3 = \frac{W}{m\pi} \left[ \cos\left(\frac{m\pi y_1}{W}\right) - \cos\left(\frac{m\pi y_2}{W}\right) \right]$$

From Eqs. (A.8) and (A.24), the transformed solution may be written as

$$\bar{H}(x, m, t) = \sum_{n=0}^{\infty} \phi_n(x) \varphi_n(t) \exp(ax) \exp[-\kappa K_x (a^2 + \beta_m^2) t] \quad (\text{A.30})$$

By taking the inverse finite Fourier sine transform of Eq. (A.30) and substituting the values of  $\phi_n(x)$  and  $\varphi_n(t)$  into Eq. (A.30), the solution of Eq. (1) can then be written as

$$H(x, y, t) = \frac{16}{K_x L \pi} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{1}{m} \sin\left(\frac{m\pi(y_1 + y_2)}{2W}\right) \times \sin\left(\frac{m\pi(y_2 - y_1)}{2W}\right) \alpha_1 \Omega_1 - \frac{4}{K_x W L} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \alpha_2 \Omega_2 \quad (\text{A.31})$$

where

$$\Omega_1 = e^{ax} \left\{ R_0 \frac{e^{-rt} - e^{-\kappa K_x (a^2 + \beta_m^2 + \lambda_n^2) t}}{(a^2 + \beta_m^2 + \lambda_n^2) - \frac{r}{\kappa K_x}} + R_1 \frac{1 - e^{-\kappa K_x (a^2 + \beta_m^2 + \lambda_n^2) t}}{(a^2 + \beta_m^2 + \lambda_n^2)} \right\} \times \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{W}\right) \quad (\text{A.32})$$

and

$$\Omega_2 = e^{ax} \left\{ \frac{1 - e^{-\kappa K_x (a^2 + \beta_m^2 + \lambda_n^2) t}}{(a^2 + \beta_m^2 + \lambda_n^2)} \right\} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{W}\right) \quad (\text{A.33})$$

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