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The globally Bi-3*-connected property of the honeycomb rectangular torus ☆

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Abstract

The honeycomb rectangular torus is an attractive alternative to existing networks such as mesh-connected networks in parallel and distributed applications because of its low network cost and well-structured connectivity. Assume that m and n are positive even integers with $n \geqslant 4$. It is known that every honeycomb rectangular torus $\mathrm{HReT}(m,n)$ is a 3-regular bipartite graph. We prove that in any $\mathrm{HReT}(m,n)$, there exist three internally-disjoint spanning paths joining x and y whenever x and y belong to different partite sets. Moreover, for any pair of vertices x and y in the same partite set, there exists a vertex z in the partite set not containing x and y, such that there exist three internally-disjoint spanning paths of $G - \{z\}$ joining x and y. Furthermore, for any three vertices x, y, and z of the same partite set there exist three internally-disjoint spanning paths of $G - \{z\}$ joining x and y if and only if $n \geqslant 6$ or $m = 2$.

Keywords: Hamiltonian; Honeycomb torus; Connectivity

1. Introduction

Network topology is a crucial factor for an interconnection network since it determines the performance of the network. One of the most popular network architectures is mesh-connected computers [7]. Each processor is placed into a square or rectangular grid and connected by a communication link to its neighbors in up to four directions. Some computer and communication networks have been built based on the mesh-connected structure. The honeycomb rectangular torus, introduced by Stojmenovic [11], is an alternative to existing networks such as mesh-connected networks in parallel and distributed computing because of its low network cost and well-structured connectivity. Network topology is usually represented by a graph where the vertices represent processors and the edges represent the links between processors. In this paper, for the graph definitions and notations we follow Harary [4]. Let $G = (V, E)$ be a *graph* if V is a finite set and E is a subset of

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$\{(u,v)|(u,v)$ is an unordered pair of $V\}$. We say that V is the *vertex set* and E is the *edge set of G*. Two vertices u and v are *adjacent* if $(u,v) \in E$. A *path P* of *length k* from x to y is a finite set of distinct vertices represented by $\langle v_0, v_1, v_2, \ldots, v_k \rangle$ where $x = v_0$, $y = v_k$, and (v_{i-1}, v_i) is an edge of E for all $1 \leqslant i \leqslant k$. We use P^{-1} to denote $\langle v_k, v_{k-1}, \ldots, v_1, v_0 \rangle$. A path is a *hamiltonian path* if its vertices are distinct and span V. A graph G is *hamiltonian connected* if there exists a hamiltonian path joining any two vertices of G. A *hamiltonian cycle* of G is a cycle that traverses every vertex of G exactly once. A graph G is *hamiltonian* if there exists a hamiltonian cycle in G. The hamiltonian properties are important aspects of designing an interconnection network. Many related works have appeared in the literature [3,6,8,12,13].

A *k-container* $C_k(x,y)$ in a graph G is a set of k internally vertex-disjoint paths between x and y. A *k^*-container* $C_{k^*}(x,y)$ in a graph G is a k-container such that every vertex of G is on some path in $C_k(x,y)$. Let G be a k-connected graph, it follows from Menger's theorem [9] that there exists a k-container between any two different vertices of G. A graph G is *k^*-connected* if there exists a k^*-container between any two distinct vertices in G. Obviously, a graph G is 1^*-connected if and only if it is hamiltonian connected. Moreover, a graph G is 2^*-connected if it is hamiltonian. The study of k^*-connected graph is motivated by the 3^*-connected graphs proposed by Albert et al. [1]. In [1], Albert et al. first studied those cubic 3-connected graphs such that there exists a 3^*-container between any pair of vertices. Such graphs are called *globally 3^*-connected graphs*.

Since every globally 3^*-connected graph is cubic, it contains an even number of vertices. Assume that $G = (V_1 \cup V_2, E)$ is a cubic 3-connected bipartite graphs with bipartition V_1 and V_2 such that $|V_1| \geqslant |V_2| \geqslant 2$. Let x and y be two distinct vertices in V_2. Assume that there exists a 3^*-container $C_{3^*}(x,y) = \{P_1, P_2, P_3\}$ in G. Suppose that there are a_i vertices of V_1 in P_i for $i = 1,2,3$. Obviously, there are $a_i + 1$ vertices of V_2 in P_i for $i = 1,2,3$. Hence, there are $a_1 + a_2 + a_3$ vertices of V_1 incidence with $P_1 \cup P_2 \cup P_3$ and there are $(a_1+1)+(a_2+1)+(a_3+1)-4 = a_1+a_2+a_3-1$ vertices of V_2 incidence with $P_1 \cup P_2 \cup P_3$. Therefore, any cubic 3-connected bipartite graph is not globally 3^*-connected.

For this reason, we say that a cubic bipartite graph $G = (V_1 \cup V_2, E)$ is *globally bi-3^*-connected* if there exists a 3^*-container between any pair of vertices of the different partite sets. Obviously, $|V_1| = |V_2|$ in any globally bi-3^*-connected with bipartition V_1 and V_2. Furthermore, a globally bi-3^*-connected graph is *hyper* if there exists a $C_{3^*}(x,y)$ in $G - \{z\}$ for any three vertices x,y, and z of the same partite set of G. A globally bi-3^*-connected graph is *strong* if for any x and y in the same partite set of G, there exists a vertex z of the same partite set as the one that contains x and y such that $G - \{z\}$ has a $C_{3^*}(x,y)$. Obviously, any globally bi-3^*-connected is strong if it is hyper. The concept of globally bi-3^*-connected, hyper globally bi-3^*-connected, and strong globally bi-3^*-connected was proposed by Kao et al. [5]. It is proved that $G - \{e\}$ is hamiltonian for any $e \in E(G)$ if G is globally bi-3^*-connected. Moreover, $G - \{x,y\}$ is hamiltonian for any $x \in V_1$ and $y \in V_2$ if G is hyper globally bi-3^*-connected.

Throughout this paper, we assume that m and n are positive even integers with $n \geqslant 4$. For any two positive integers r and s, we use $[r]_s$ to denote $r(\text{mod}\, s)$. We use the brick drawing, proposed in [11], to define the honeycomb rectangular torus. The honeycomb rectangular torus $\text{HReT}(m,n)$ is the graph with the vertex set $\{(i,j)|0 \leqslant i < m, 0 \leqslant j < n\}$ such that (i,j) and (k,l) are adjacent if they satisfy one of the following conditions:

1. $i = k$ and $j = [l \pm 1]_n$;
2. $j = l$ and $k = [i+1]_m$ if $i + j$ is odd; and
3. $j = l$ and $k = [i-1]_m$ if $i + j$ is even.

For example, the graph HReT(6,8) is shown in Fig. 1. It is easy to see that $\text{HReT}(m,n)$ is a bipartite graph with bipartition V_0 and V_1 where $V_0 = \{(i,j)|i+j \text{ is even}\}$ and $V_1 = \{(i,j)|i+j \text{ is odd}\}$. Moreover, $|V_0| = |V_1|$.

There are many studies on the properties of the honeycomb rectangular torus [3,8,11]. Stojmenovic [11] showed that the network cost of the honeycomb rectangular torus, which is defined as the product of degree and the diameter, is better than the other families based on mesh-connected computers and tori. Megson et al. [8] established the hamiltonian property of honeycomb torus. In particular, Cho and Hsu [3] proved that

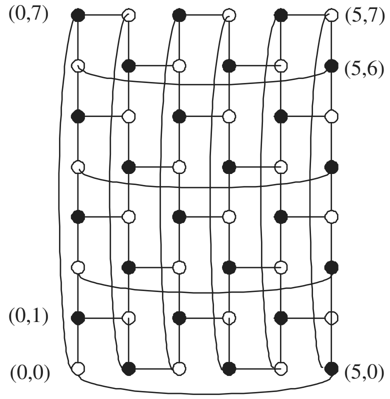

Fig. 1. The honeycomb rectangular torus HReT(6,8).

$\text{HReT}(m,n) - e$ is hamiltonian for any edge $e \in E(\text{HReT}(m,n))$. Furthermore, $\text{HReT}(m,n) - \{x,y\}$ is hamiltonian for any $x \in V_0$ and $y \in V_1$ if $n \geqslant 6$.

Based on Menger's Theorem [9], the 3-connected property of the honeycomb rectangular torus $\text{HReT}(m,n)$ can be derived. In this paper, we study the globally bi-3*-connected property of the honeycomb rectangular torus $\text{HReT}(m,n)$. We prove that any honeycomb rectangular torus $\text{HReT}(m,n)$ is strongly globally bi-3*-connected. Moreover, $\text{HReT}(m,n)$ is hyper globally bi-3*-connected if and only if $n \geqslant 6$ or $m = 2$.

2. A basic algorithm

In this section, we present an algorithm. The purpose of this algorithm is to extend a 3*-container $C_{3^*}(x,y) = \{P_1, P_2, P_3\}$ of $\text{HReT}(m,n)$ to a 3*-container of $\text{HReT}(m+2,n)$.

Algorithm 1. For $0 \leqslant i \leqslant m-1$, let $f_i : V(\text{HReT}(m,n)) \to V(\text{HReT}(m+2,n))$ be a function so assigned

$$f_i(k,l) = \begin{cases} (k,l) & \text{if } i \geqslant k \geqslant 0 \\ (k+2,l) & \text{otherwise.} \end{cases}$$

For $0 \leqslant i \leqslant m-1$ and $0 \leqslant j,k \leqslant n-1$, let $Q_i(j,[j+k]_n)$ denote the path $\langle (i,[j]_n), (i,[j+1]_n), (i,[j+2]_n), \ldots, (i,[j+k]_n) \rangle$ in $\text{HReT}(m,n)$. Suppose that $C_3(x,y)$ is a 3-container of $\text{HReT}(m,n)$ containing at least one edge joining vertices of column i to vertices of column $[i+1]_m$; i.e., $((i,j),([i+1]_m,j))$ in $E(C_3(x,y))$ for some $0 \leqslant j \leqslant n-1$. Let $0 \leqslant k_0 < k_1 < \cdots < k_t \leqslant n-1$ be the indices such that $((i,k_j),(i+1,k_j)) \in E(C_3(x,y))$. We construct $C'_{3,i}(x,y)$ as follows:

Let $\overline{C_{3,i}(x,y)}$ be the image of $C_3(x,y) - \{((i,k_j),(i+1,k_j)) | 0 \leqslant k_j \leqslant n-1\}$ under f_i. We set $j\prime = [j]_{(t+1)}$ and define A_j as

$$\langle (i,[k_j]_n), ([i+1]_{m+2},[k_j]_n), Q_{[i+1]_{m+2}}([k_j]_n,[k_{j'}-1]_n), ([i+1]_{m+2},[k_{j'}-1]_n),$$
$$([i+2]_{m+2},[k_{j'}-1]_n), Q^{-1}_{[i+2]_{m+2}}([k_j]_n,[k_{j'}-1]_n), ([i+2]_{m+2},[k_j]_n), ([i+3]_{m+2},[k_j]_n) \rangle.$$

Obviously, Aj is a path joining $(i,[k_j]_n)$ and $(i+3,[k_j]_n)$ for $0 \leqslant j < t$. It is easy to see that edges of $\overline{C_{3,i}(x,y)}$ together with edges of A_j, with $0 \leqslant j \leqslant t$ form a 3-container $C'_{3,i}(x,y)$ of $\text{HReT}(m+2,n)$. For example, a 3*-container $C_{3^*}((0,0),(2,2))$ of $\text{HReT}(4,12) - \{(1,7)\}$ is shown in Fig. 2a. The corresponding $C'_{3,1}((0,0),(2,2))$ is shown in Fig. 2b. We have the following lemma.

Lemma 1. *Suppose that $C_3(x,y)$ is a 3-container of HReT(m,n) containing at least one edge joining vertices of column i to vertices of column $[i+1]_m$. Then $C'_{3,i}(x,y)$ forms a 3-container of HReT$(m+2,n)$ containing at least one edge joining the vertices of column l to the vertices of column $[l+1]_m$ for any $l \in \{i,[i+1]_m,[i+2]_m\}$. Moreover, $C'_{3^*,i}(x,y)$ is a 3*-container of HReT$(m+2,n)$ if $C_{3^*}(x,y)$ is a 3*-container of HReT(m,n).*

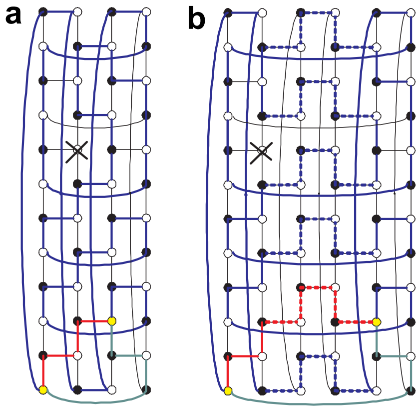

Fig. 2. Illustrations for Algorithm 1.

Furthermore, $C'_{3^*,i}(x,y)$ *is a* 3^**-container of* $HReT(m+2,n)-\{f_i(z)\}$ *if* $C_{3^*}(x,y)$ *is a* 3^**-container of* $HReT(m,n)-\{z\}$.

Lemma 2. *Suppose that* $C_3(x,y)$ *is a* 3*-container of* $HReT(2,n)$ *containing at least one edge in* $\{((0,j),(1,j))|j \text{ is odd}\}$ *and at least one edge in* $\{((0,j),(1,j))|j \text{ is even}\}$. *Then* $C'_{3,i}(x,y)$ *with* $i\in\{0,1\}$ *forms a* 3*-container of* $HReT(4,n)$ *containing at least one edge joining the vertices of column* l *to the vertices of column* $l+1$ *for any* $l\in\{0,1,2,3\}$. *Moreover,* $C'_{3^*,i}(x,y)$ *is a* 3^**-container of* $HReT(m+2,n)$ *if* $C_{3^*}(x,y)$ *is a* 3^**-container of* $HReT(m,n)$. *Furthermore,* $C'_{3^*,i}(x,y)$ *is a* 3^**-container of* $HReT(m+2,n)-\{f_i(z)\}$ *if* $C_{3^*}(x,y)$ *is a* 3^**-container of* $HReT(m,n)-\{z\}$.

With Lemmas 1 and 2, we say a 3-container $C_3(x,y)$ of $\mathrm{HReT}(2,n)$ is *regular* if $C_3(x,y)$ contains at least one edge in $\{((0,j),(1,j))|j \text{ is odd}\}$ and at least one edge in $\{((0,j),(1,j))|j \text{ is even}\}$. Assume that $m\geqslant 4$. We say a 3-container $C_3(x,y)$ of $\mathrm{HReT}(m,n)$ is *regular* if $C_3(x,y)$ contains at least one edge joining vertices in column i to vertices in column $[i+1]_m$ for $0\leqslant i\leqslant m-1$. We have the following lemma.

Lemma 3. *Suppose that* $C_{3^*}(x,y)$ *is a regular* 3^**-container for* $HReT(m,n)$. *Then* $C'_{3^*,i}(x,y)$ *is a regular* 3^**-container for* $HReT(m+2,n)$ *for every* $0\leqslant i<m$. *Moreover, suppose that* $C_{3^*}(x,y)$ *is a regular* 3^**-container for* $HReT(m,n)-\{z\}$. *Then* $C'_{3^*,i}(x,y)$ *is a regular* 3^**-container for* $HReT(m+2,n)-\{f_i(z)\}$ *for every* $0\leqslant i<m$.

3. The globally bi-3^*-connected properties of HReT(2,n)

For $h=\{0,1\}$ and $0\leqslant j,k\leqslant n-1$, let $R_h(j,[j+k]_n)$ denote the path $\langle(h,[j]_n),(h,[j+1]_n),([h+1]_m,[j+1]_n),([h+1]_m,[j+2]_n),(h,[j+2]_n),\ldots,([h+1]_m,[j+k-1]_n),(h,[j+k-1]_n),(h,[j+k]_n)\rangle$ in $\mathrm{HReT}(2,n)$.

Lemma 4. *Let* x *and* y *be any two vertices of* $HReT(2,n)=(V_0\cup V_1,E)$ *with* $x\in V_0$ *and* $y\in V_1$. *Then there exists a regular* 3^**-container* $C_{3^*}(x,y)$ *of* $HReT(2,n)$. *Hence* $HReT(2,n)$ *is globally bi-*3^**-connected.*

Proof. Without loss of generality, we may assume that $x=(0,0)$ and $y=(i,j)$. In order to prove this lemma, we will construct a regular 3^*-container $C_{3^*}(x,y)=\{P_1,P_2,P_3\}$ in $\mathrm{HReT}(2,n)$. We have the following cases:

Case 1: $i=0$ and j is odd. The corresponding paths are:

$P_1=\langle(0,0),Q_0(0,j),(0,j)\rangle;$

$P_2=\langle(0,j),R_0(j,0),(0,0)\rangle;$

$P_3=\langle(0,0),(1,0),Q_1(0,j),(1,j),(0,j)\rangle.$

Case 2: $i=1$ and j is even.
Case 2.1: $j=0$. The corresponding paths are:

$$P_1 = \langle (0,0), Q_0(0,n-2), (0,n-2), (1,n-2), Q_1^{-1}(0,n-2), (1,0) \rangle;$$

$$P_2 = \langle (0,0), (1,0) \rangle;$$

$$P_3 = \langle (0,0), (0,n-1), (1,n-1), (1,0) \rangle.$$

Case 2.2: $j>0$. The corresponding paths are:

$$P_1 = \langle (0,0), Q_0(0,j), (0,j), (1,j) \rangle;$$

$$P_2 = \langle (1,j), (1,j+1), (0,j+1), R_0(j+1,0), (0,0) \rangle;$$

$$P_3 = \langle (0,0), (1,0), Q_1(0,j), (1,j) \rangle.$$

Hence $\mathrm{HReT}(2,n)$ is globally bi-3^*-connected. See Fig. 3 for illustrations. □

Lemma 5. *Let x, y, and z be any three different vertices of $HReT(2,n) = (V_0 \cup V_1, E)$ in V_0. Then there exists a regular 3^*-container $C_{3^*}(x,y)$ of $HReT(2,n) - \{z\}$. Hence $HReT(2,n)$ is hyper globally bi-3^*-connected.*

Proof. Without loss of generality, we may assume that $x=(0,0)$, $y=(i,j)$, and $z=(k,l)$. In order to prove this lemma, we will construct a regular 3^*-container $C_{3^*}(x,y) = \{P_1,P_2,P_3\}$ in $\mathrm{HReT}(2,n) - \{z\}$. We have the following cases:

Case 1: $i=0$. Then j is even.
Case 1.1: $k=0$. Then l is even. By the symmetric property of $\mathrm{HReT}(2,n)$, we may assume that $l<j$. The corresponding paths are:

$$P_1 = \langle (0,j), Q_0(j,0), (0,0) \rangle;$$

$$P_2 = \langle (0,0), R_0(0,l-1), (0,l-1), (1,l-1), (1,l), (1,l+1), (0,l+1), R_0(l+1,j), (0,j) \rangle;$$

$$P_3 = \langle (0,j), (1,j), Q_1(j,0), (1,0), (0,0) \rangle.$$

Case 1.2: $k=1$. Then l is odd. By the symmetric property of $\mathrm{HReT}(2,n)$, we may assume that $l<j$. The corresponding paths are:

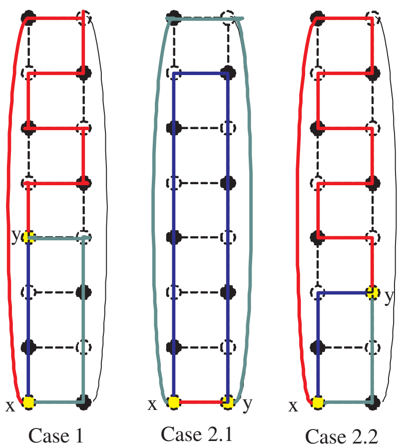

Fig. 3. Illustrations for Lemma 4.

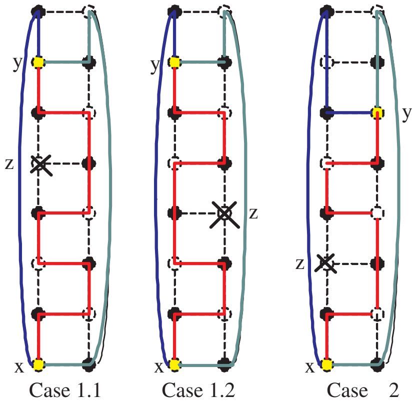

Fig. 4. Illustrations for Lemma 5.

$$P_1 = \langle (0,j), Q_0(j,0), (0,0)\rangle;$$
$$P_2 = \langle (0,0), R_0(0,l), (0,l), R_0(l,j), (0,j)\rangle;$$
$$P_3 = \langle (0,j), (1,j), Q_1(j,0), (1,0), (0,0)\rangle.$$

Case 2: $i=1$. Then j is odd. $k=0$. Then l is even. By the symmetric property of $\mathrm{HReT}(2,n)$, we may assume that $l<j$. The corresponding paths are:

$$P_1=\langle (1,j),(0,j),Q_0(j,0),(0,0)\rangle;$$
$$P_2=\langle (0,0),R_0(0,l-1),(0,l-1),(1,l-1),(1,l),(1,l+1),(0,l+1),R_0(l+1,j-1),(0,j-1),(1,j-1),(1,j)\rangle;$$
$$P_3=\langle (1,j),Q_1(j,0),(1,0),(0,0)\rangle.$$

Hence $\mathrm{HReT}(2,n)$ is hyper globally bi-3*-connected. See Fig. 4 for illustrations. □

4. The globally bi-3*-connected properties of HReT(4,n)

In this section, we need the following path patterns. For $0\leqslant i\leqslant m-1$ and $0\leqslant j,k\leqslant n-1$, we set

$$S_i^L(j) = \langle ([i]_m,[j]_n),([i-1]_m,[j]_n),([i-1]_m,[j+1]_n),([i-2]_m,[j+1]_n),([i-2]_m,[j+2]_n),([i-3]_m,[j+2]_n),$$
$$([i-3]_m,[j+3]_n),([i-4]_m,[j+3]_n),([i-4]_m,[j+2]_n)\rangle;$$
$$S_i^R(j) = \langle ([i]_m,[j]_n),([i+1]_m,[j]_n),([i+1]_m,[j+1]_n),([i+2]_m,[j+1]_n),([i+2]_m,[j+2]_n),$$
$$([i+3]_m,[j+2]_n),([i+3]_m,[j+3]_n),([i+4]_m,[j+3]_n),([i+4]_m,[j+2]_n)\rangle;$$
$$S_i^L(j,k) = \langle ([i]_m,[j]_n),S^L_{[i]_m}(j),([i-4]_m,[j+2]_n),S^L_{[i-4]_m}([j+2]_n),([i-8]_m,[j+4]_n),\ldots,$$
$$([i-2(k-j-2)]_m,[k-2]_n),S^L_{[i-2(k-j-2)]_m}([k-2]_n),([i-2(k-j)]_m,[k]_n)\rangle;$$

and

$$S_i^R(j,k) = \langle ([i]_m,[j]_n),S^R_{[i]_m}(j),([i+4]_m,[j+2]_n),S^R_{[i+4]_m}([j+2]_n),([i+8]_m,[j+4]_n),\ldots,$$
$$([i+2(k-j-2)]_m,[k-2]_n),S^R_{[i+2(k-j-2)]_m}([k-2]_n),([i+2(k-j)]_m,[k]_n)\rangle.$$

See Fig. 5 for illustrations.

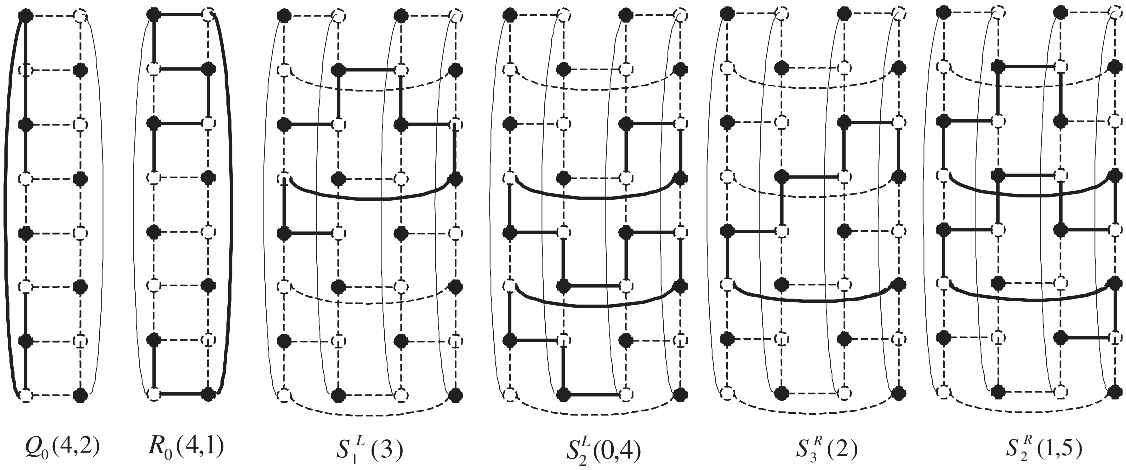

Fig. 5. The path patterns $Q_0(4,2)$, $R_0(4,1)$, $S_1^L(3)$, $S_2^L(0,4)$, $S_3^R(2)$, and $S_2^R(1,5)$.

Lemma 6. *Let x and y be any two vertices of* $HReT(4,n) = (V_0 \cup V_1, E)$ *with* $x \in V_0$ *and* $y \in V_1$*. Then there exists a regular* 3^**-container* $C_{3^*}(x,y)$ *of* $HReT(4,n)$*. Hence* $HReT(4,n)$ *is globally bi-*3^**-connected.*

Proof. Without loss of generality, we may assume that $x = (0,0)$ and $y = (i,j)$. In order to prove this lemma, we will construct a regular 3^*-container $C_{3^*}(x,y) = \{P_1, P_2, P_3\}$ in HReT$(4,n)$. By the symmetric property of HReT$(4,n)$, we may assume that $i \in \{0,1,2\}$. Hence we have the following cases:

Case 1: Suppose that $i \in \{0,1\}$. By Lemma 4, there exists a regular 3^*-container $C_{3^*}((0,0),(i,j))$ of HReT$(2,n)$. By Lemma 3, $C'_{3^*,1}((0,0),(i,j))$ forms a 3^*-container of HReT(4, n).

Case 2: $i = 2$. Then j is odd.

Case 2.1: Suppose that $j = 1$. The corresponding paths are:

$$P_1 = \langle (0,0),(0,n-1),(1,n-1),Q_1^{-1}(0,n-1),(1,0),(2,0),(2,1)\rangle;$$
$$P_2 = \langle (0,0),Q_0(0,n-2),(0,n-2),(3,n-2),Q_3^{-1}(1,n-2),(3,1),(2,1)\rangle;$$
$$P_3 = \langle (0,0),(3,0),(3,n-1),(2,n-1),Q_2^{-1}(1,n-1),(2,1)\rangle.$$

Case 2.2: Suppose that $j \neq 1$. The corresponding paths are:

$$P_1 = \langle (0,0),Q_0(0,j-1),(0,j-1),(3,j-1),(3,j),(2,j)\rangle;$$
$$P_2 = \langle (0,0),(3,0),Q_3(0,j-2),(3,j-2),(2,j-2),Q_2^{-1}(0,j-2),(2,0),(1,0),Q_1(0,j-1),$$
$$(1,j-1),(2,j-1),(2,j)\rangle;$$
$$P_3 = \langle (0,0),(0,n-1),S_L^{-1}(j+3,n-1),(0,j+3),(0,j+2),(1,j+2),(1,j+1),(1,j),(0,j),(0,j+1),$$
$$(3,j+1),(3,j+2),(2,j+2),(2,j+1),(2,j)\rangle.$$

Hence HReT$(4,n)$ is globally bi-3^*-connected. See Fig. 6 for illustrations. □

Lemma 7. *Let x, y, and z be any three different vertices of* $HReT(4,6) = (V_0 \cup V_1, E)$ *in* V_0*. Then there exists a regular* 3^**-container* $C_{3^*}(x,y)$ *of* $HReT(4,6) - \{z\}$*. Hence HReT(4,6) is hyper globally bi-*3^**-connected.*

Proof. Without loss of generality, we may assume that $x = (0,0)$, $y = (i,j)$, and $z = (k,l)$. The corresponding regular 3^*-container $C_{3^*}(x,y) = \{P_1, P_2, P_3\}$ in HReT$(4,6) - \{z\}$ are listed below.

Hence HReT(4,6) is hyper globally bi-3^*-connected. □

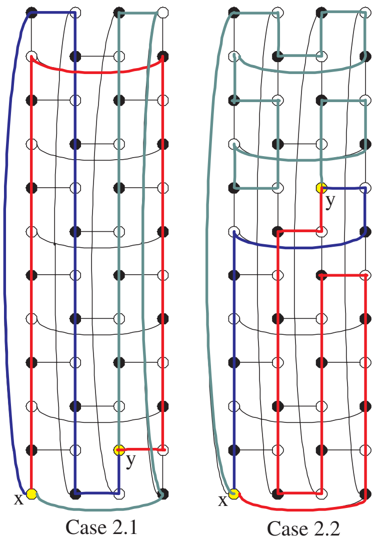

Fig. 6. Illustrations for Lemma 6.

y	z	$C_{3^*}(x,y)$
(0,2)	(2,2)	$\langle(0,0),(0,1),(0,2)\rangle$ $\langle(0,0),(0,5),(1,5),(1,0),Q_1(0,4),(1,4),(2,4),(2,3),(3,3),(3,2),(0,2)\rangle$ $\langle(0,0),(3,0),(3,1),(2,1),(2,0),(2,5),(3,5),(3,4),(0,4),(0,3),(0,2)\rangle$
(0,2)	(2,4)	$\langle(0,0),(0,1),(0,2)\rangle$ $\langle(0,2),(0,3),(0,4),(3,4),(3,5),(2,5),(2,0),(1,0),Q_1(0,5),(1,5),(0,5),(0,0)\rangle$ $\langle(0,0),(3,0),(3,1),(2,1),(2,2),(2,3),(3,3),(3,2),(0,2)\rangle$
(0,4)	(0,2)	$\langle(0,0),(0,5),(0,4)\rangle$ $\langle(0,0),(3,0),Q_3(0,3),(3,3),(2,3),(2,4),(2,5),(3,5),(3,4),(0,4)\rangle$ $\langle(0,0),(0,1),(1,1),(1,2),(2,2),(2,1),(2,0),(1,0),(1,5),(1,4),(1,3),(0,3),(0,4)\rangle$
(0,4)	(1,1)	$\langle(0,0),(0,5),(0,4)\rangle$ $\langle(0,0),(0,1),(0,2),(3,2),(3,3),(2,3),(2,2),(1,2),(1,3),(0,3),(0,4)\rangle$ $\langle(0,0),(3,0),(3,1),(2,1),(2,0),(1,0),(1,5),(1,4),(2,4),(2,5),(3,5),(3,4),(0,4)\rangle$
(1,3)	(0,2)	$\langle(0,0),(0,5),(0,4),(0,3),(1,3)\rangle$ $\langle(0,0),(0,1),(1,1),(1,2),(1,3)\rangle$ $\langle(0,0),(3,0),Q_3(0,5),(3,5),(2,5),Q_2^{-1}(0,5),(2,0),(1,0),(1,5),(1,4),(1,3)\rangle$
(1,5)	(0,2)	$\langle(0,0),(0,5),(1,5)\rangle$ $\langle(0,0),(0,1),(1,1),(1,2),(1,3),(0,3),(0,4),(3,4),(3,5),(2,5),(2,4),(1,4),(1,5)\rangle$ $\langle(0,0),(3,0),Q_3(0,3),(3,3),(2,3),Q_2^{-1}(0,3),(2,0),(1,0),(1,5)\rangle$
(1,1)	(2,0)	$\langle(0,0),(0,1),(1,1)\rangle$ $\langle(0,0),(3,0),(3,1),(2,1),(2,2),(1,2),(1,1)\rangle$ $\langle(0,0),(0,5),(0,4),(3,4),(3,5),(2,5),(2,4),(2,3),(3,3),(3,2),(0,2),(0,3),(1,3),(1,4),(1,5),(1,0),(1,1)\rangle$
(1,1)	(2,2)	$\langle(1,1),Q_1(1,4),(1,4),(2,4),(2,3),(3,3),(3,2),(0,2),(0,3),(0,4),(3,4),(3,5),(2,5),(2,0),(2,1),(3,1),(3,0),(0,0)\rangle$ $\langle(0,0),(0,1),(1,1)\rangle$ $\langle(0,0),(0,5),(1,5),(1,0),(1,1)\rangle$
(1,1)	(2,4)	$\langle(1,1),(1,0),(2,0),(2,5),(3,5),(3,4),(0,4),(0,3),(0,2),(3,2),(3,3),(2,3),(2,2),(2,1),(3,1),(3,0),(0,0)\rangle$ $\langle(0,0),(0,1),(1,1)\rangle$ $\langle(0,0),(0,5),(1,5),Q_1^{-1}(1,5),(1,1)\rangle$

y	z	$C_{3^*}(x,y)$
(1,3)	(2,0)	$\langle(0,0),(0,1),(1,1),(1,0),(1,5),(1,4),(1,3)\rangle$ $\langle(0,0),(3,0),(3,1),(2,1),(2,2),(1,2),(1,3)\rangle$ $\langle(0,0),(0,5),(0,4),(3,4),(3,5),(2,5),(2,4),(2,3),(3,3),(3,2),(0,2),(0,3),(1,3)\rangle$
(1,3)	(2,2)	$\langle(0,0),(0,5),(1,5),(1,4),(1,3)\rangle$ $\langle(0,0),(3,0),(3,5),(2,5),(2,4),(2,3),(3,3),(3,4),(0,4),(0,3),(1,3)\rangle$ $\langle(0,0),(0,1),(0,2),(3,2),(3,1),(2,1),(2,0),(1,0),Q_1(0,3),(1,3)\rangle$
(1,3)	(2,4)	$\langle(0,0),(0,5),(1,5),(1,4),(1,3)\rangle$ $\langle(0,0),(3,0),(3,5),(2,5),(2,0),(1,0),Q_1(0,3),(1,3)\rangle$ $\langle(0,0),(0,1),(0,2),(3,2),(3,1),(2,1),(2,2),(2,3),(3,3),(3,4),(0,4),(0,3),(1,3)\rangle$
(2,0)	(0,2)	$\langle(0,0),(3,0),Q_3(0,3),(3,3),(2,3),(2,4),(1,4),(1,3),(0,3),(0,4),(3,4),(3,5),(2,5),(2,0)\rangle$ $\langle(0,0),(0,5),(1,5),(1,0),(2,0)\rangle$ $\langle(0,0),(0,1),(1,1),(1,2),(2,2),(2,1),(2,0)\rangle$
(2,2)	(0,2)	$\langle(0,0),(0,1),(1,1),(1,0),(2,0),(2,1),(2,2)\rangle$ $\langle(0,0),(3,0),Q_3(0,3),(3,3),(2,3),(2,2)\rangle$ $\langle(0,0),(0,5),(1,5),(1,4),(2,4),(2,5),(3,5),(3,4),(0,4),(0,3),(1,3),(1,2),(2,2)\rangle$
(2,2)	(0,4)	$\langle(0,0),(0,5),(1,5),(1,0),(2,0),(2,5),(3,5),Q_3^{-1}(2,5),(3,2),(0,2),(0,3),(1,3),(1,4),(2,4),(2,3),(2,2)\rangle$ $\langle(0,0),(0,1),(1,1),(1,2),(2,2)\rangle$ $\langle(0,0),(3,0),(3,1),(2,1),(2,2)\rangle$
(2,2)	(1,1)	$\langle(0,0),(0,5),(1,5),(1,0),(2,0),(2,1),(2,2)\rangle$ $\langle(0,0),(3,0),Q_3(0,3),(3,3),(2,3),(2,2)\rangle$ $\langle(0,0),Q_0(0,4),(0,4),(3,4),(3,5),(2,5),(2,4),(1,4),(1,3),(1,2),(2,2)\rangle$

Lemma 8. *Assume that $n \geqslant 8$. Let x, y, and z be any three different vertices of $HReT(4,n) = (V_0 \cup V_1, E)$ in V_0. Then there exists a regular 3^*-container $C_{3^*}(x,y)$ of $HReT(4,n) - \{z\}$. Hence $HReT(4,n)$ is hyper globally bi-3^*-connected.*

Proof. Without loss of generality, we may assume that $x = (0,0)$, $y = (i,j)$, and $z = (k,l)$. In order to prove this lemma, we will construct a regular 3^*-container $C_{3^*}(x,y) = \{P_1, P_2, P_3\}$ in $\mathrm{HReT}(4,n) - \{z\}$. By the symmetric property of $\mathrm{HReT}(4,n)$, we may assume that $i \in \{0,1,2\}$. We have the following cases:

Case 1: Suppose that $i \in \{0,1\}$ and $z \in \{0,1\}$. By Lemma 5, there exists a regular 3^*-container $C_{3^*}((0,0),(i,j))$ of $\mathrm{HReT}(2,n) - \{(k,l)\}$. By Lemma 3, $C'_{3^*,1}((0,0),(i,j))$ forms a 3^*-container of $\mathrm{HReT}(4,n) - \{(k,l)\}$.

Case 2: $i = 0$ and $k = 2$. Then j and l are even. By the symmetric property, we have the following subcases.

Case 2.1: Suppose that $j = 4$ and $l = 2$. The corresponding paths are:

$$P_1 = \langle(0,0), Q_0(0,4), (0,4)\rangle;$$

$$P_2 = \langle(0,0),(0,n-1),(0,n-2),(3,n-2),Q_3^{-1}(4,n-2),(3,4),(0,4)\rangle;$$

$$P_3 = \langle(0,4),Q_0(4,n-3),(0,n-3),(1,n-3),Q_1^{-1}(0,n-3),(1,0),(1,n-1),(1,n-2),(2,n-2),$$
$$Q_2^{-1}(3,n-2),(2,3),(3,3),(3,2),(3,1),(2,1),(2,0),(2,n-1),(3,n-1),(3,0),(0,0)\rangle.$$

Case 2.2: Suppose that $n-4 > j \geqslant 2$ and $l = j+2$. The corresponding paths are:

$$P_1 = \langle(0,0), Q_0(0,j), (0,j)\rangle;$$

$$P_2 = \langle(0,0),(3,0),Q_3(0,j),(3,j),(0,j)\rangle;$$

$P_3 = \langle (0,j), Q_0(j,j+4), (0,j+4), (3,j+4), (3,j+5), (2,j+5), (2,j+4), (2,j+3), (3,j+3), (3,j+2), (3,j+1), (2,j+1), Q_2^{-1}(0,j+1), (2,0), (1,0), Q_1(0,j+5), (1,j+5), (0,j+5), (0,j+6), (3,j+6), (3,j+7), (2,j+7), (2,j+6), S_2^L(j+6,n-2), (2,n-2), (1,n-2), (1,n-1), (0,n-1), (0,0) \rangle$.

Case 2.3: Suppose that $n-6 > j \geqslant 2$ and $n-4 > l > j+2$. The corresponding paths are:

$P_1 = \langle (0,0), Q_0(0,j), (0,j) \rangle$;

$P_2 = \langle (0,0), (3,0), Q_3(0,j), (3,j), (0,j) \rangle$;

$P_3 = \langle (1,j), (1,j+1), (1,j+2), (3,j+2), (3,j+1), (2,j+1), Q_2^{-1}(0,j+1), (2,0), (1,0), Q_1(0,j+2), (1,j+2), (2,j+2), (2,j+3), (3,j+3), (3,j+4), (0,j+4), (0,j+3), S_0^R(j+3,l-3), (0,l-3), (1,l-3), (1,l-2), (2,l-2), (2,l-1), (3,l-1), (3,l), (3,l+1), (2,l+1), (2,l+2), (2,l+3), (3,l+3), (3,l+2), (0,l+2), Q_0^{-1}(l-1,l+2), (0,l-1), (1,l-1), Q_1(l-1,l+3), (1,l+3), (0,l+3), (0,l+4), (3,l+4), S_2^L(l+4,n-2), (2,n-2), (1,n-2), (1,n-1), (0,n-1), (0,0) \rangle$.

Case 2.4: Suppose that $n > 8$ and $j = l \geqslant 2$. The corresponding paths are:

$P_1 = \langle (0,0), Q_0(0,j), (0,j) \rangle$;

$P_2 = \langle (0,0), (3,0), Q_3(0,j-1), (3,j-1), (2,j-1), Q_2^{-1}(0,j-1), (2,0), (1,0), Q_1(0,j+1), (1,j+1), (0,j+1), (0,j) \rangle$;

$P_3 = \langle (0,j), (3,j), (3,j+1), (2,j+1), (2,j+2), (1,j+2), (1,j+3), (1,j+4), (2,j+4), (2,j+3), (3,j+3), (3,j+2), (0,j+2), (0,j+3), (0,j+4), (3,j+4), (3,j+5), (2,j+5), (2,j+6), (1,j+6), (1,j+5), S_1^L(j+5,n-5), (1,n-5), (0,n-5), (0,n-4), (3,n-4), (3,n-3), (2,n-3), (2,n-2), (2,n-1), (3,n-1), (3,n-2), (0,n-2), (0,n-3), (1,n-3), (1,n-2), (1,n-1), (0,n-1), (0,0) \rangle$.

Case 2.5: Suppose that $n = 8$, $j = 2$, and $l = 2$. The corresponding paths are:

$P_1 = \langle (0,0), (0,1), (0,2) \rangle$;

$P_2 = \langle (0,2), (0,3), (0,4), (3,4), Q_3(4,7), (3,7), (2,7), (2,0), (2,1), (3,1), (3,0), (0,0) \rangle$;

$P_3 = \langle (0,2), (3,2), (3,3), (2,3), Q_2(3,6), (2,6), (1,6), (1,7), (1,0), Q_1(0,5), (1,5), (0,5), (0,6), (0,7), (0,0) \rangle$.

Case 2.6: Suppose that $n = 8$, $j = 4$, and $l = 4$. The corresponding paths are:

$P_1 = \langle (0,0), Q_0(0,4), (0,4) \rangle$;

$P_2 = \langle (0,0), (0,7), (1,7), (1,0), Q_1(0,6), (1,6), (2,6), (2,5), (3,5), (3,4), (0,4) \rangle$;

$P_3 = \langle (0,0), (3,0), Q_3(0,3), (3,3), (2,3), Q_2^{-1}(0,3), (2,0), (2,7), (3,7), (3,6), (0,6), (0,5), (0,4) \rangle$.

Case 3: $i = 1$ and $k = 2$. Then j is odd and l is even. By the symmetric property, we have the following subcases.

Case 3.1: Suppose that $n-5>j\geqslant 1$ and $n-4>l>j+2$. The corresponding paths are:

$$P_1=\langle(0,0),Q_0(0,j),(0,j),(1,j)\rangle;$$

$$P_2=\langle(0,0),(3,0),Q_3(0,j),(3,j),(2,j),Q_2^{-1}(0,j),(2,0),(1,0),Q_1(0,j),(1,j)\rangle;$$

$$P_3=\langle(1,j),(1,j+1),(2,j+1),(2,j+2),(3,j+2),(3,j+1),S_3^L(j+1,l-2),(3,l-2),\\(0,l-2),(0,l-1),(1,l-1),(1,l),(1,l+1),(1,l+2),(2,l+2),(2,l+1),(3,l+1),\\(3,l),(0,l),(0,l+1),(0,l+2),(3,l+2),(3,l+3),(2,l+3),(2,l+4),(1,l+4),\\(1,l+3),S_1^L(l+3,n-5),(1,n-5),(0,n-5),(0,n-4),(3,n-4),(3,n-3),\\(2,n-3),(2,n-2),(2,n-1),(3,n-1),(3,n-2),(0,n-2),(0,n-3),(1,n-3),\\(1,n-2),(1,n-1),(0,n-1),(0,0)\rangle.$$

Case 3.2: Suppose that $n-5>j\geqslant 1$ and $l=j+1$. The corresponding paths are:

$$P_1=\langle(0,0),Q_0(0,j),(0,j),(1,j)\rangle;$$

$$P_2=\langle(0,0),(3,0),Q_3(0,j),(3,j),(2,j),Q_2^{-1}(0,j),(2,0),(1,0),Q_1(0,j),(1,j)\rangle;$$

$$P_3=\langle(1,j),Q_1(j,j+3),(1,j+3),(2,j+3),(2,j+2),(3,j+2),(3,j+1),(0,j+1),\\(0,j+2),(0,j+3),(3,j+3),(3,j+4),(2,j+4),(2,j+5),(1,j+5),(1,j+4),\\S_1^L(j+4,n-5),(1,n-5),(0,n-5),(0,n-4),(3,n-4),(3,n-3),(2,n-3),\\(2,n-2),(2,n-1),(3,n-1),(3,n-2),(0,n-2),(0,n-3),(1,n-3),(1,n-2),\\(1,n-1),(0,n-1),(0,0)\rangle.$$

Case 3.3: Suppose that $n-5>j\geqslant 1$ and $l=n-4$. The corresponding paths are:

$$P_1=\langle(0,0),Q_0(0,j),(0,j),(1,j)\rangle;$$

$$P_2=\langle(0,0),(0,n-1),(1,n-1),(1,0),Q_1(0,j),(1,j)\rangle;$$

$$P_3=\langle(1,j),(1,j+1),(2,j+1),(2,j+2),(3,j+2),(3,j+1),S_3^L(j+1,n-6),(0,n-6),\\(0,n-5),(1,n-5),Q_1(n-5,n-2),(1,n-2),(2,n-2),(2,n-3),(3,n-3),\\(3,n-4),(0,n-4),(0,n-3),(0,n-2),(3,n-2),(3,n-1),(2,n-1),(2,0),\\(2,1),(3,1),(3,0),(0,0)\rangle.$$

Case 3.4: Suppose that $j=n-5$ and $l=n-4$. The corresponding paths are:

$$P_1=\langle(0,0),Q_0(0,n-5),(0,n-5),(1,n-5)\rangle;$$

$$P_2=\langle(0,0),(0,n-1),(1,n-1),(1,0),Q_1(0,n-5),(1,n-5)\rangle;$$

$$P_3=\langle(1,n-5),Q_1(n-5,n-2),(1,n-2),(2,n-2),(2,n-3),(3,n-3),(3,n-4),\\(0,n-4),(0,n-3),(0,n-2),(3,n-2),(3,n-1),(2,n-1),(2,0),Q_2(0,n-5),\\(2,n-5),(3,n-5),Q_3^{-1}(0,n-5),(3,0),(0,0)\rangle.$$

Case 3.5: Suppose that $n-5>j\geqslant 1$ and $l=n-2$. The corresponding paths are:

$$P_1=\langle(0,0),Q_0(0,j),(0,j),(1,j)\rangle;$$

$$P_2=\langle(0,0),(3,0),(3,n-1),(2,n-1),(2,0),(1,0),Q_1(0,j),(1,j)\rangle;$$

$$P_3 = \langle (1,j),(1,j+1),(1,j+2),(0,j+2),(0,j+1),(3,j+1),Q_3^{-1}(1,j+1),(3,1),(2,1),$$
$$Q_2(1,j+2),(2,j+2),(3,j+2),(3,j+3),(0,j+3),(0,j+4),(1,j+4),(1,j+3),$$
$$S_1^R(j+3,n-6),(1,n-6),(2,n-6),(2,n-5),(3,n-5),(3,n-4),(0,n-4),$$
$$(0,n-3),(0,n-2),(3,n-2),(3,n-3),(2,n-3),(2,n-4),(1,n-4),$$
$$Q_1(n-4,n-1),(1,n-1),(0,n-1),(0,0)\rangle.$$

Case 4: $i=2$ and $k=0$. Then j and l are even. By the symmetric property, we have the following subcases.

Case 4.1: Suppose that $j=0$ are $l>0$. The corresponding paths are:

$$P_1 = \langle (0,0),(0,n-1),(1,n-1),(1,0),(2,0)\rangle;$$
$$P_2 = \langle (0,0),(0,1),(1,1),(1,2),(2,2),(2,1),(2,0)\rangle;$$
$$P_3 = \langle (0,0),(3,0),(3,1),(3,2),(0,2),(0,3),(1,3),(1,4),(2,4),(2,3),$$
$$S_2^R(3,j-1),(2,j-1),(3,j-1),(3,j),(3,j+1),(2,j+1),(2,j+2),(1,j+2),$$
$$(1,j+1),S_1^L(j+1,n-3),(1,n-3),(0,n-3),(0,n-2),(3,n-2),(3,n-1),$$
$$(2,n-1),(2,0)\rangle.$$

Case 4.2: Suppose that $l>j>0$. The corresponding paths are:

$$P_1 = \langle (0,0),(0,1),(1,1),Q_1(1,j),(1,j),(2,j)\rangle;$$
$$P_2 = \langle (0,0),(3,0),(3,1),(2,1),Q_2(1,j),(2,j)\rangle;$$
$$P_3 = \langle (2,j),(2,j+1),(2,j+2),(1,j+2),(1,j+1),(0,j+1),Q_0^{-1}(2,j+1),$$
$$(0,2),(3,2),Q_3(2,j+2),(3,j+2),(0,j+2),(0,j+3),(1,j+3),(1,j+4),$$
$$(2,j+4),(2,j+3),S_2^R(j+3,l-1),(2,l-1),(3,l-1),(3,l),(3,l+1),$$
$$(2,l+1),(2,l+2),(1,l+2),(1,l+1),S_1^L(l+1,n-1),(1,n-1),(0,n-1),(0,0)\rangle.$$

Case 4.3: Suppose that $j=l>0$. The corresponding paths are:

$$P_1 = \langle (0,0),Q_0(0,j-1),(0,j-1),(1,j-1),Q_1^{-1}(0,j-1),(1,0),(2,0),Q_2(0,j),(2,j)\rangle;$$
$$P_2 = \langle (0,0),(3,0),Q_3(0,j+1),(3,j+1),(2,j+1),(2,j)\rangle;$$
$$P_3 = \langle (2,j),S_2^L(j,n-1),(2,n-2),(1,n-2),(1,n-1),(0,n-1),(0,0)\rangle.$$

Case 5: $i=2$ and $k=1$. Then j is even and l is odd. By the symmetric property, we have the following subcases.

Case 5.1: Suppose that $j=0$ and $l=1$. The corresponding paths are:

$$P_1 = \langle (0,0),(0,n-1),(1,n-1),(1,0),(2,0)\rangle;$$
$$P_2 = \langle (0,0),(0,1),(0,2),(3,2),(3,1),(2,1),(2,0)\rangle;$$
$$P_3 = \langle (0,0),(3,0),(3,n-1),(3,n-2),(0,n-2),Q_0^{-1}(3,n-2),(0,3),(1,3),$$
$$(1,2),(2,2),(2,3),(3,3),Q_3(3,n-3),(3,n-3),(2,n-3),Q_2^{-1}(4,n-3),$$
$$(2,4),(1,4),Q_1(4,n-2),(1,n-2),(2,n-2),(2,n-1),(2,0)\rangle.$$

Case 5.2: Suppose that $j = 0$ and $n-1 > l > 1$. The corresponding paths are:

$P_1 = \langle (0,0),(0,n-1),(1,n-1),(1,0),(2,0)\rangle$;

$P_2 = \langle (0,0),(3,0),(3,1),(2,1),(2,0)\rangle$;

$P_3 = \langle (0,0),(0,1),(1,1),(1,2),(2,2),(2,3),(3,3),(3,2),S_3^L(2,j-3),(3,j-3),(0,j-3),$
$(0,j-2),(1,j-2),(1,j-1),(2,j-1),(2,j),(2,j+1),(1,j+1),(1,j+2),$
$(1,j+3),(2,j+3),(2,j+2),(3,j+2),Q_3^{-1}(j-1,j+2),(3,j-1),(0,j-1),$
$Q_0(j-1,j+3),(0,j+3),(3,j+3),(3,j+4),(2,j+4),(2,j+5),(1,j+5),$
$(1,j+4),S_1^L(j+4,n-3),(1,n-3),(0,n-3),(0,n-2),(3,n-2),(3,n-1),$
$(2,n-1),(2,0)\rangle$.

Case 5.3: Suppose that $n-1 > l > j+2$ and $j > 0$. The corresponding paths are:

$P_1 = \langle (0,0),(0,1),(1,1),Q_1(1,j),(1,j),(2,j)\rangle$;

$P_2 = \langle (0,0),(3,0),(3,1),(2,1),Q_2(1,j),(2,j)\rangle$;

$P_3 = \langle (2,j),(2,j+1),(3,j+1),(3,j),S_3^L(j,l-3),(3,l-3),(0,l-3),(0,l-2),(1,l-2),$
$(1,l-1),(2,l-1),(2,l),(2,l+1),(1,l+1),(1,l+2),(1,l+3),(2,l+3),(2,l+2),$
$(3,l+2),(3,l+1),(3,l),(3,l-1),(0,l-1),Q_0(l-1,l+3),(0,l+3),(3,l+3),$
$(3,l+4),(2,l+4),(2,l+5),(1,l+5),(1,l+4),S_1^L(l+4,n-1),(1,n-1),$
$(0,n-1),(0,0)\rangle$.

Case 5.4: Suppose that $n-2 > j$ and $l = n-1$. The corresponding paths are:

$P_1 = \langle (0,0),Q_0(0,j+1),(0,j+1),(1,j+1),(1,j+2),(2,j+2),(2,j+1),(2,j)\rangle$;

$P_2 = \langle (0,0),(3,0),(3,n-1),(2,n-1),(2,0),(1,0),Q_1(0,j),(1,j),(2,j)\rangle$;

$P_3 = \langle (2,j),Q_2^{-1}(1,j),(2,1),(3,1),Q_3(1,j+2),(3,j+2),(0,j+2),(0,j+3),(1,j+3),$
$(1,j+4),(2,j+4),(2,j+3),S_2^R(j+3,n-3),(2,n-3),(3,n-3),(3,n-2),$
$(0,n-2),(0,n-1),(0,0)\rangle$.

Case 5.5: Suppose that $j = n-2$ and $l = n-1$. The corresponding paths are:

$P_1 = \langle (0,0),(0,n-1),(0,n-2),(0,n-3),(1,n-3),(1,n-2),(2,n-2)\rangle$;

$P_2 = \langle (0,0),Q_0(0,n-4),(3,n-4),Q_3(n-4,n-1),(2,n-1),(2,n-2)\rangle$;

$P_3 = \langle (0,0),(3,0),Q_3(0,n-5),(3,n-5),(2,n-5),Q_2^{-1}(0,n-5),(2,0),Q_1(0,n-4),$
$(1,n-4),(2,n-4),(2,n-3),(2,n-2)\rangle$.

Hence HReT$(4,n)$ is hyper globally bi-3*-connected for $n \geqslant 8$. See Fig. 7 for illustrations. □

5. The globally bi-3*-connected properties of HReT(m,n)

Lemma 9. *Assume that m and n are positive even integers with $m,n \geqslant 4$. Let x and y be any two vertices of $HReT(m,n) = (V_0 \cup V_1, E)$ with $x \in V_0$ and $y \in V_1$. Then there exists a regular 3*-container $C_{3^*}(x,y)$ of $HReT(m,n)$.*

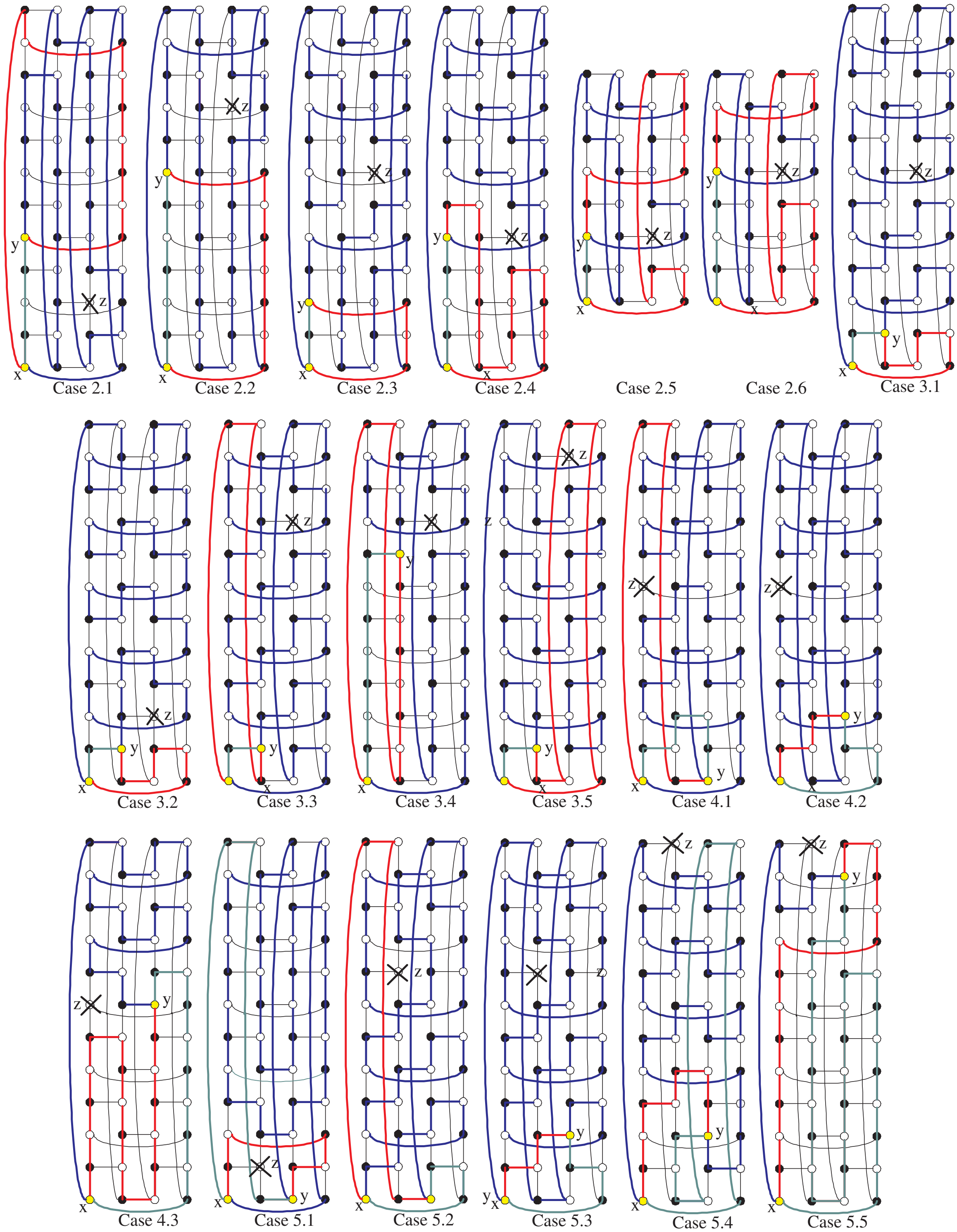

Fig. 7. Illustrations for Lemma 8.

Proof. Without loss of generality, we may assume that $x = (0,0)$ and $y = (i,j)$. In order to prove this lemma, we will construct a regular 3^*-container $C_{3^*}(x,y) = \{P_1, P_2, P_3\}$ in HReT(m,n). We prove the lemma by induction on m. With Lemma 6, our theorem holds for $m = 4$. Now, we consider the case that $m \geqslant 6$.

Suppose that $i < m-2$. By induction, there exists a regular 3^*-container $C_{3^*}(x,y) = \{P_1, P_2, P_3\}$ in $\mathrm{HReT}(m-2,n)$. By Lemma 3, $C'_{3^*,m-3}((0,0),(i,j))$ forms a 3^*-container of $\mathrm{HReT}(m,n)$. Suppose that $i \geqslant m-2$. By induction, there exists a regular $C_{3^*}(x,(i-2,j)) = \{P_1,P_2,P_3\}$ in $\mathrm{HReT}(m-2,n)$. By Lemma 3, $C'_{3^*,1}((0,0),(i,j))$ forms a 3^*-container of $\mathrm{HReT}(m,n)$. □

Lemma 10. *Assume that m and n are positive even integers with $m \geqslant 4$ and $n \geqslant 6$. Let x, y, and z be any three different vertices of $HReT(m,n) = (V_0 \cup V_1, E)$ in V_0. Then there exists a regular 3^*-container $C_{3^*}(x,y)$ of $HReT(m,n) - \{z\}$.*

Proof. Without loss of generality, we may assume that $x = (0,0)$, $y = (i,j)$, and $z = (k,l)$. In order to prove this lemma, we will construct a regular 3^*-container $C_{3^*}(x,y) = \{P_1,P_2,P_3\}$ in $\mathrm{HReT}(m,n) - \{z\}$. We prove the lemma by induction on m. With Lemmas 7 and 8, our theorem holds for $m = 4$. Now, we consider the case that $m \geqslant 6$.

Suppose that $i < m-2$ and $k < m-2$. By induction, there exists a regular 3^*-container $C_{3^*}(x,y) = \{P_1,P_2,P_3\}$ in $\mathrm{HReT}(m-2,n) - \{z\}$. By Lemma 3, $C'_{3^*,m-3}((0,0),(i,j))$ forms a 3^*-container of $\mathrm{HReT}(m,n) - \{z\}$. Suppose that $i < m-2$ and $k \geqslant m-2$. By induction, there exists a regular 3^*-container $C_{3^*}(x,y) = \{P_1,P_2,P_3\}$ in $\mathrm{HReT}(m-2,n) - (k-2,l)$. By Lemma 3, $C'_{3^*,i}((0,0),(i,j))$ forms a 3^*-container of $\mathrm{HReT}(m,n) - \{z\}$. Suppose that $i \geqslant m-2$ and $k < m-2$. By induction, there exists a regular 3^*-container $C_{3^*}(x,(i-2,j)) = \{P_1,P_2,P_3\}$ in $\mathrm{HReT}(m-2,n) - \{z\}$. By Lemma 3, $C'_{3^*,k}((0,0),(i,j))$ forms a 3^*-container of $\mathrm{HReT}(m,n) - \{z\}$. Suppose that $i \geqslant m-2$ and $k \geqslant m-2$. By induction, there exists a regular 3^*-container $C_{3^*}(x,(i-2,j)) = \{P_1,P_2,P_3\}$ in $\mathrm{HReT}(m-2,n) - (k-2,l)$. By Lemma 3, $C'_{3^*,1}((0,0),(i,j))$ forms a 3^*-container of $\mathrm{HReT}(m,n) - \{z\}$. □

Theorem 1. *Assume that m and n are positive even integers with $n \geqslant 4$. Then $HReT(m,n)$ is strongly globally bi-3^*-connected. Moreover, $HReT(m,n)$ is hyper globally bi-3^*-connected if and only if $n \geqslant 6$ or $m = 2$.*

Proof. With Lemmas 4 and 9, $\mathrm{HReT}(m,n)$ is globally bi-3^*-connected if m,n are even integers with $n \geqslant 4$.

By Lemmas 5 and 10, $\mathrm{HReT}(m,n)$ is hyper globally bi-3^*-connected if m,n are even integers with $n \geqslant 6$ or $m = 2$.

Now we consider the case $\mathrm{HReT}(m,4)$ with m is an even integer and $m \geqslant 4$. We first prove that such $\mathrm{HReT}(m,4)$ is not hyper globally bi-3^*-connected.

To prove this fact, let $x = (1,1)$, $y = (1,3)$ and $z = (0,2)$. Suppose that there exists a 3^*-container $C_{3^*}(x,y) = \{P_1,P_2,P_3\}$ of $\mathrm{HReT}(m,4) - \{z\}$. Since $deg_{HReT(m,4)-z}(v) = 2$ for $v \in \{(0,1),(0,3),(3,2)\}$, $\langle(1,1),(1,2),(1,3)\rangle$ and $\langle(1,1),(0,1),(0,0),(0,3),(1,3)\rangle$ are two paths in $C_{3^*}(x,y)$. Without loss of generality, we assmue that $P_1 = \langle(1,1),(1,2),(1,3)\rangle$ and $P_2 = \langle(1,1),(0,1),(0,0),(0,3),(1,3)\rangle$. Since $deg_{HReT(m,4)-z}((1,1)) = deg_{HReT(m,4)-z}((1,3)) = 3$, $((1,3),(1,0))$ and $((1,0),(1,1))$ are edges in P_3. Thus $P_3 = \langle(1,1),(1,0),(1,3)\rangle$. Obviously, $\{P_1 \cup P_2 \cup P_3\}$ does not span $\mathrm{HReT}(m,4) - \{z\}$. See Fig. 8 for an illustration. Hence $\mathrm{HReT}(m,4)$ is not hyper globally bi-3^*-connected.

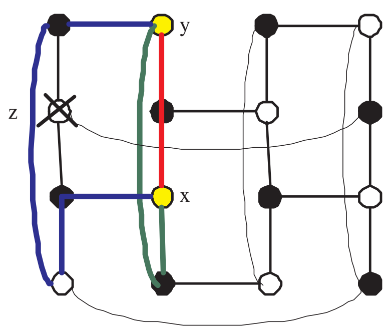

Fig. 8. Illustration for Theorem 1.

Although any HReT$(m,4)$ with m is an even integer and $m \geqslant 4$ is not hyper globally bi-3*-connected, we will prove that such HReT$(m,4)$ is strongly globally bi-3*-connected by induction.

We first prove that HReT(4,4) is strongly bi-3*-connected. Let x and y be any two different vertices in the same partite set of HReT(4,4). Without loss of generality, we may assume that x and y are vertices in V_0 and $x = (0,0)$. We need to find a vertex z in $V_0 - \{x,y\}$ such that there exists a 3*-container $C_{3^*}(x,y) = \{P_1,P_2,P_3\}$ of HReT(4,4)$-\{z\}$. The corresponding vertex z and 3*-container $C_{3^*}(x,y)$ are listed below.

y	z	$C_{3^*}(x,y)$
$(0,2)$	$(1,3)$	$\langle(0,0),(0,1),(0,2)\rangle$ $\langle(0,0),(0,3),(0,2)\rangle$ $\langle(0,0),(3,0),(3,1),(2,1),(2,0),(1,0),(1,1),(1,2),(2,2),(2,3),(3,3),(3,2),(0,2)\rangle$
$(1,1)$	$(1,3)$	$\langle(0,0),(0,1),(1,1)\rangle$ $\langle(0,0),(3,0),(3,1),(2,1),(2,0),(1,0),(1,1),\rangle$ $\langle(0,0),(0,3),(0,2),(3,2),(3,3),(2,3),(2,2),(1,2),(1,1)\rangle$
$(1,3)$	$(0,2)$	$\langle(0,0),(0,3),(1,3)\rangle$ $\langle(0,0),(0,1),(1,1),(1,2),(1,3)\rangle$ $\langle(0,0),(3,0),Q_3(0,3),(3,3),(2,3),Q_2^{-1}(0,3),(2,0),(1,0),(1,3)\rangle$
$(2,0)$	$(0,2)$	$\langle(0,0),(0,3),(1,3),(1,0),(2,0)\rangle$ $\langle(0,0),(3,0),(3,3),(3,2),(3,1),(2,1),(2,0)\rangle$ $\langle(0,0),(0,1),(1,1),(1,2),(2,2),(2,3),(2,0)\rangle$
$(2,2)$	$(0,2)$	$\langle(0,0),(3,0),Q_3(0,3),(3,3),(2,3),(2,2)\rangle$ $\langle(0,0),(0,3),(1,3),(1,0),(2,0),(2,1),(2,2)\rangle$ $\langle(0,0),(0,1),(1,1),(1,2),(2,2)\rangle$
$(3,1)$	$(0,2)$	$\langle(0,0),(3,0),(3,1)\rangle$ $\langle(0,0),(0,3),(1,3),(1,0),(2,0),(2,1),(3,1)\rangle$ $\langle(0,0),(0,1),(1,1),(1,2),(2,2),(2,3),(3,3),(3,2),(3,1)\rangle$
$(3,3)$	$(0,2)$	$\langle(0,0),(3,0),(3,3)\rangle$ $\langle(0,0),(0,3),(1,3),(1,0),(2,0),(2,1),(3,1),(3,2),(3,3)\rangle$ $\langle(0,0),(0,1),(1,1),(1,2),(2,2),(2,3),(3,3)\rangle$

Obviously, all these 3*-containers of HReT(4,4)$-\{z\}$ are regular.

Now we consider the case HReT$(m,4)$ with $m > 4$. Without loss of generality, we may assmue that $x = (0,0)$, $y = (i,j)$, and $z = (k,l)$. Suppose that $i < m-2$ and $k < m-2$. By induction, there exists a regular 3*-container $C_{3^*}(x,y) = \{P_1,P_2,P_3\}$ in HReT$(m-2,4) - \{z\}$. By Lemma 3, $C'_{3^*,m-3}((0,0),(i,j))$ forms a 3*-container of HReT$(m,4) - \{z\}$. Suppose that $i < m-2$ and $k \geqslant m-2$. By induction, there exists a regular 3*-container $C_{3^*}(x,y) = \{P_1,P_2,P_3\}$ in HReT$(m-2,4) - (k-2,l)$. By Lemma 3, $C'_{3^*,i}((0,0),(i,j))$ forms a 3*-container of HReT$(m,4) - \{z\}$. Suppose that $i \geqslant m-2$ and $k < m-2$. By induction, there exists a regular $C_{3^*}(x,(i-2,j)) = \{P_1,P_2,P_3\}$ in HReT$(m-2,4) - \{z\}$. By Lemma 3, $C'_{3^*,k}((0,0),(i,j))$ forms a 3*-container of HReT$(m,4) - \{z\}$. Suppose that $i \geqslant m-2$ and $k \geqslant m-2$. By induction, there exists a regular 3*-container $C_{3^*}(x,(i-2,j)) = \{P_1,P_2,P_3\}$ in HReT$(m-2,4) - (k-2,l)$. By Lemma 3, $C'_{3^*,1}((0,0),(i,j))$ forms a 3*-container of HReT$(m,4) - \{z\}$.

Thus the theorem is proved. □

6. Concluding remarks

The honeycomb networks have been proposed as attractive alternatives to mesh and torus interconnection networks for computer architectures, interconnection topologies, parallel processes and distributed systems.

Many investigations related to this family of the networks have been proposed in the literature [2,10–12]. In particular, the honeycomb rectangular torus $\text{HReT}(m,n)$ is a well-structured 3-connected cubic network. We study the globally bi-3*-connected property of the honeycomb rectangular torus $\text{HReT}(m,n)$ in this paper. We prove that any $\text{HReT}(m,n)$ is strongly globally bi-3*-connected. We also prove that $\text{HReT}(m,n)$ is hyper globally bi-3*-connected if and only if $n \geqslant 6$ or $m = 2$. Future work will try to find the globally 3*-connected property of other cubic interconnection networks such as honeycomb rhombic torus, which is another type of honeycomb network that is bipartite 3-connected cubic network introduced by Stojmenovic [11].

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