Comment on "Evaluation of the Hantush's $M(\alpha, \beta)$ function using binomial coefficients" by B. A. Mamedov and A. S. Ekenoğlu

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[1] *Hantush* [1961a] presented a solution of the drawdown around a partially penetrating well with the $M(\alpha, \beta)$ function defined by the following definite integral [*Trefry*, 1998; *Mamedov and Ekenoğlu*, 2006]:

$$M(\alpha, \beta) = \frac{2}{\pi} \int_0^{\alpha} \frac{e^{-\beta(1+y^2)}}{1+y^2} dy \tag{1}$$

where y is a dummy variable and α and β are parameters related to the physical properties of unconfined aquifers. The values of the $M(\alpha, \beta)$ function was extensively tabulated by *Hantush* [1961b, 1964]. On the basis of binomial expansion theorem, *Mamedov and Ekenoğlu* [2006] developed an interesting algorithm for evaluating the Hantush's M function, which was expressed as

$$M(\alpha, \beta) = \frac{\alpha e^{-\beta}}{\pi} \int_0^1 \frac{e^{-\alpha^2 \beta x}}{\sqrt{x}(1+\alpha^2 x)} dx = \begin{cases} \dfrac{e^{-\beta}}{\pi} \lim\limits_{N\to\infty} \sum\limits_{i=0}^{N} F_i(-1)\beta^{-(i+1/2)} \gamma\left(i+1/2, \alpha^2\beta\right) & \text{for} \quad \alpha^2 \le 1 \quad (2a) \\ 1 + \dfrac{e^{-\beta}}{\pi} \lim\limits_{N'\to\infty} \sum\limits_{i=0}^{N'} F_i(-1)\beta^{(i+1/2)} \gamma\left(-i-1/2, \alpha^2\beta\right) & \text{for} \quad \alpha^2 \ge 1 \quad (2b) \end{cases}$$

with

$$F_i(-1) = \frac{(-1)(-2)\cdots(-i)}{i!} \tag{3}$$

and

$$\gamma\left(\pm(i+1/2), \alpha^2\beta\right) = \int_0^{\alpha^2\beta} e^{-t} t^{(\pm(i+1/2)-1)} dt \tag{4}$$

where $x = y^2/\alpha^2$. Equations (2)–(4) are evaluated by directly adding the infinite series; yet, the numerical evaluation is not straightforward and the accuracy of the results are not easy to evaluate because of the fact that the series involves the incomplete Gamma function and has a running sum from zero to infinity.

[2] In this comment, we provide a simple and efficient numerical approach as an alternative to evaluate the Hantush's M function. The Gaussian quadrature is employed to perform the numerical integration of equation (1) piecewise along the y axis from $(0, \alpha)$ to $(-1, 1)$ where a change of variable has been used. An n-point Gaussian quadrature formula may be written as [*Gerald and Wheatley*, 1989]

$$\int_{-1}^{1} f(\xi) d\xi = \sum_{i=1}^{n} W_i f(\xi_i) \tag{5}$$

where W_i is a weighting factor and ξ_i is an integration point. Both the six- and ten-term formulas of the Gaussian quadrature are used to carry out the integration for the same area under the integrand with the step size $\Delta y = \alpha$. The resulting values of the integration obtained from the six- and ten-term formulas are defined as A_6 and A_{10}, respectively. The absolute difference of these two results is defined by $\Delta A = |A_{10} - A_6|$. If $\Delta A > CTOL$, a half step size $(\alpha/2)$ will be used and the same integration procedure will be repeated until $\Delta A < CTOL$ which is a tolerance of accuracy. If $\Delta A < CTOL$, a double step size $(2\Delta y)$ is used for the next step. This procedure ensures that each numerical integration over a small step satisfies the required accuracy. Note that the last step size should be chosen such that the end of the step should be located right at a larger integrated range, i.e., α. In short, the integrand in equation (1) is obtained simply by adding all the resulting integration values. This approach has been successfully applied in some groundwater related problems [see, e.g., *Yang and Yeh*, 2002, 2006, 2007; *Yeh et al.*, 2003].

[3] For the case of $M(2, 3/2)$, the resulting numerical result for equation (2b) obtained by *Mamedov and Ekenoğlu* [2006] is 0.0832546934534 and the required upper limit of summations, L, is 20 as shown in Table 1 (for their equation (24b)), while the present approach takes only two steps to obtain the result. For the case of $M(3/5, 7/10)$, their approach requires L = 25 to converge the series of equation (2a) with the result of 0.1585691441918 shown in Table 2 (for their equation (24a)); however, the present approach also takes only two steps to obtain the result of

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0.1585691441919 which has an accuracy to 13 decimal places. In addition, we also examine the case of $M(13, 17)$ which is the extreme one given in Table 3 of *Mamedov and Ekenoğlu* [2006]. Their algorithm requires $L = 75$ to obtain the result of $0.551120725 \times 10^{-8}$ for equation (2b) (their equation (24b)), while the present approach needs just six steps to obtain the same result. The computation effort of the present approach in evaluating the Hantush's M function is significantly less if compared with that of *Mamedov and Ekenoğlu* [2006] approach using the transform function. Obviously, this present approach has the advantages of being easy, straightforward, and very efficient in evaluating the Hantush's M function.

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