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Integral Equations and Operator Theory

Fredholmness of Linear Combinations of Two Idempotents

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Abstract. Let E and F be idempotent operators on a complex Hilbert space, and let a and b be nonzero scalars with $a + b \neq 0$. We prove that $aE + bF$ is Fredholm if and only if $E + F$ is, thus answering affirmatively a question asked by Koliha and Rakočević.

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In recent years, there has been some interest in the study of the invertibility and the Fredholmness of linear combinations of two idempotent operators on a complex Hilbert space. For example, if E and F are idempotents ($E^2 = E$ and $F^2 = F$) on a finite-dimensional Hilbert space and a and b are nonzero scalars with $a+b \neq 0$, then it was shown in [1] that the invertibility of $aE+bF$ and $E+F$ are equivalent. This is further strengthened in [5] to the equality of the nullities of $aE + bF$ and $E + F$. For E and F on a not necessarily finite-dimensional space, the equivalence of the invertibility is also true as proved in [3]. In connection with this, Koliha and Rakočević asked in [5] whether the Fredholmness of $aE + bF$ and $E + F$ are equivalent. The purpose of this note is to give an affirmative answer to this question.

Recall that an operator T on a Hilbert space is *Fredholm* if the nullities of T and T^* are finite and the range of T is closed. For a Fredholm T, its *index*, $\operatorname{ind} T$, is by definition nullity $T -$ nullity T^*. It is known that the Fredholmness of T is preserved under compact perturbations and is equivalent to the existence of an operator T' with $TT' - I$ and $T'T - I$ compact. An excellent reference for properties of Fredholm operators is [2, Chapter XI].

The main result of this note is the following theorem.

Theorem 1. *Let E and F be idempotents on a Hilbert space H, and a and b be nonzero scalars with $a + b \neq 0$. Then $aE + bF$ is Fredholm if and only if $E + F$ is. In this case*, $\operatorname{ind}(aE + bF) = \operatorname{ind}(E + F)$.

For its proof, we need the next lemma.

Lemma 2. *Let* $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *on* $H \oplus K$, *where* A *is Fredholm with* A' *on* H *satisfying* $AA' = I + K_1$ *and* $A'A = I + K_2$ *for some compact operators* K_1 *and* K_2*. Then* T *is Fredholm if and only if* $D - CA'B$ *is. In this case*, $\operatorname{ind} T = \operatorname{ind} A + \operatorname{ind}(D - CA'B)$.

This is due to Zhang [6, Theorem 1]. Here we give a much simplified proof. Note that this is an analogue of the invertible case: if $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with A invertible, then T is invertible if and only if $D - CA^{-1}B$, the Schur complement of A in T, is. In analogy to this, we may call, for the Fredholm case, $D - CA'B$ the *essential Schur complement of* A *in* T, which is, of course, determined only up to compact perturbations.

Proof of Lemma 2. Since

$$\begin{aligned} & \begin{bmatrix} I & 0 \\ CA' & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & D - CA'B \end{bmatrix} \begin{bmatrix} I & A'B \\ 0 & I \end{bmatrix} \\ = & \begin{bmatrix} A & AA'B \\ CA'A & CA'AA'B + D - CA'B \end{bmatrix} \\ = & \begin{bmatrix} A & (I + K_1)B \\ C(I + K_2) & C(I + K_2)A'B + D - CA'B \end{bmatrix} \\ = & \begin{bmatrix} A & B \\ C & D \end{bmatrix} + \begin{bmatrix} 0 & K_1B \\ CK_2 & CK_2A'B \end{bmatrix}, \end{aligned}$$

where

$$\begin{bmatrix} I & 0 \\ CA' & I \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} I & A'B \\ 0 & I \end{bmatrix}$$

are invertible and

$$\begin{bmatrix} 0 & K_1B \\ CK_2 & CK_2A'B \end{bmatrix}$$

is compact, we infer that T is Fredholm if and only if $A \oplus (D - CA'B)$ is. The latter is the case if and only if $D - CA'B$ is. In this case, we have

$$\operatorname{ind} T = \operatorname{ind}(A \oplus (D - CA'B)) = \operatorname{ind} A + \operatorname{ind}(D - CA'B),$$

completing the proof. □

Note that, in the preceding lemma, the Fredholmness of T does not imply that of A as the operator $T = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ on $l^2 \oplus l^2$ shows. However, if T is positive semidefinite, then we do have this implication.

Corollary 3. *Let* $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *on* $H \oplus K$ *be positive semidefinite. Then* T *is Fredholm if and only if* A *and* $D - CA'B$ *are, where* A' *is any operator on* H *with* $AA' - I$ *and* $A'A - I$ *compact.*

Proof. In light of Lemma 2, we need only prove that the Fredholmness of T implies that of A. Let $T = A_0 \oplus 0$ and $P = 0 \oplus I$ on $\operatorname{ran} T \oplus \ker T$. Since T is Fredholm, A_0 is invertible and hence so is $T+P = A_0 \oplus I$. Thus $\overline{W(T+P)} = \sigma(T+P)^\wedge \subseteq [\varepsilon, \infty)$ for some $\varepsilon > 0$, where $\overline{W(T+P)}$ and $\sigma(T+P)^\wedge$ denote the closure of the numerical range and the convex hull of the spectrum of $T+P$, respectively, and their equality is by [4, Problem 216]. Since

$$T+P = \begin{bmatrix} A & B \\ C & D \end{bmatrix} + \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix} = \begin{bmatrix} A+P_1 & B+P_2 \\ C+P_3 & D+P_4 \end{bmatrix} \text{ on } H \oplus K,$$

we infer that $\sigma(A+P_1) \subseteq \overline{W(A+P_1)} \subseteq \overline{W(T+P)} \subseteq [\varepsilon, \infty)$ (cf. [4, Problem 214]). This shows that $A + P_1$ is also invertible. Note that P is of finite rank and thus so is P_1. Therefore, A is Fredholm as asserted. □

We are now ready to prove Theorem 1. The proof models after the one for the invertible case as given in [3, Theorem 1].

Proof of Theorem 1. Since the idempotency and the Fredholmness are preserved under the similarity of operators, we may assume that one of E and F, say, F is an (orthogonal) projection ($F^2 = F = F^*$). We can express E and F on $H = \operatorname{ran} E \oplus \ker E^*$ as

$$E = \begin{bmatrix} I & E_1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} F_1 & F_1^{1/2} D F_2^{1/2} \\ F_2^{1/2} D^* F_1^{1/2} & F_2 \end{bmatrix},$$

where D is a contraction ($\|D\| \le 1$) from $\ker E^*$ to $\operatorname{ran} E$. We further decompose F_1 and F_2 as $F_1 = 0 \oplus I \oplus F_{11}$ and $F_2 = F_{22} \oplus I \oplus 0$ on $\operatorname{ran} E = \ker F_1 \oplus \ker (I - F_1) \oplus (\operatorname{ran} E \ominus (\ker F_1 \oplus \ker (I - F_1)))$ and $\ker E^* = (\ker E^* \ominus (\ker F_2 \oplus \ker (I - F_2))) \oplus \ker (I - F_2) \oplus \ker F_2$, respectively. Since F is positive semidefinite, we have

$$F = 0 \oplus I \oplus \begin{bmatrix} F_{11} & F_{11}^{1/2} D_1 F_{22}^{1/2} \\ F_{22}^{1/2} D_1^* F_{11}^{1/2} & F_{22} \end{bmatrix} \oplus I \oplus 0$$

for some contraction D_1 from $\ker E^* \ominus (\ker F_2 \oplus \ker (I - F_2))$ to $\operatorname{ran} E \ominus (\ker F_1 \oplus \ker (I - F_1))$. From

$$\begin{bmatrix} F_{11} & F_{11}^{1/2} D_1 F_{22}^{1/2} \\ F_{22}^{1/2} D_1^* F_{11}^{1/2} & F_{22} \end{bmatrix}^2 = \begin{bmatrix} F_{11} & F_{11}^{1/2} D_1 F_{22}^{1/2} \\ F_{22}^{1/2} D_1^* F_{11}^{1/2} & F_{22} \end{bmatrix},$$

we obtain

$$F_{11}^2 + F_{11}^{1/2} D_1 F_{22} D_1^* F_{11}^{1/2} = F_{11},$$
$$F_{11}^{3/2} D_1 F_{22}^{1/2} + F_{11}^{1/2} D_1 F_{22}^{3/2} = F_{11}^{1/2} D_1 F_{22}^{1/2}$$

and

$$F_{22}^{1/2} D_1^* F_{11} D_1 F_{22}^{1/2} + F_{22}^2 = F_{22}.$$

It can be derived using the injectivity of F_{jj} and $I - F_{jj}$, $j = 1, 2$, that

$$(*) \qquad D_1 D_1^* = I, \quad D_1^* D_1 = I \quad \text{and} \quad D_1^*(I - F_{11}) D_1 = F_{22}$$

(cf. [3, p. 1454]). Note that

$$(**)\quad aE+bF=\begin{bmatrix} aI & & & aE_{11} & aE_{12} & aE_{13} \\ & (a+b)I & & aE_{21} & aE_{22} & aE_{23} \\ & & aI+bF_{11} & aE_{31}+bF_{11}^{1/2}D_1F_{22}^{1/2} & aE_{32} & aE_{33} \\ & & bF_{22}^{1/2}D_1^*F_{11}^{1/2} & bF_{22} & & \\ & & & & bI & \\ & & & & & 0 \end{bmatrix}.$$

We claim that $aE+bF$ is Fredholm if and only if $I-F_{11}$ is invertible, $I-E_{31}D_1^*(I-F_{11})^{-1/2}F_{11}^{1/2}$ is Fredholm and $\dim\ker F_2<\infty$. Indeed, if $aE+bF$ is Fredholm, then, letting A be an operator on H such that $K\equiv(aE+bF)A-I$ is compact, we have, with

$$A=\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \quad\text{and}\quad K=\begin{bmatrix} K_1 & K_2 \\ K_3 & K_4 \end{bmatrix} \quad \text{on } H=\operatorname{ran}E\oplus\ker E^*,$$

$$\begin{bmatrix} aI+bF_1 & aE_1+bF_1^{1/2}DF_2^{1/2} \\ bF_2^{1/2}D^*F_1^{1/2} & bF_2 \end{bmatrix}\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}=\begin{bmatrix} I+K_1 & K_2 \\ K_3 & I+K_4 \end{bmatrix}.$$

Carrying out the multiplication here yields

$$bF_2^{1/2}D^*F_1^{1/2}A_2+bF_2A_4=I+K_4$$

or

$$bF_2^{1/2}(D^*F_1^{1/2}A_2+F_2^{1/2}A_4)=I+K_4.$$

This shows that $F_2^{1/2}$ is Fredholm and hence so is F_2. Therefore, F_{22} is invertible and thus so is $I-F_{11}$ by $(*)$. From Lemma 2, we derive that the Fredholmness of $aE+bF$ is equivalent to that of

$$\begin{bmatrix} aI+bF_{11} & aE_{31}+bF_{11}^{1/2}D_1F_{22}^{1/2} \\ bF_{22}^{1/2}D_1^*F_{11}^{1/2} & bF_{22} \end{bmatrix}$$

together with the finiteness of $\dim\ker F_2$. The Fredholmness of this latter operator is in turn equivalent to that of

$$aI+bF_{11}-(aE_{31}+bF_{11}^{1/2}D_1F_{22}^{1/2})F_{22}^{-1}(F_{22}^{1/2}D_1^*F_{11}^{1/2})$$

by Lemma 2. This is equal to

$$aI+bF_{11}-(aE_{31}+bF_{11}^{1/2}D_1D_1^*(I-F_{11})^{1/2}D_1)D_1^*(I-F_{11})^{-1/2}D_1D_1^*F_{11}^{1/2},$$

which can be further simplified to

$$a(I-E_{31}D_1^*(I-F_{11})^{-1/2}F_{11}^{1/2})$$

by $(*)$. This proves one direction. For the other, if $I-F_{11}$ is invertible, $I-E_{31}D_1^*(I-F_{11})^{-1/2}F_{11}^{1/2}$ is Fredholm and $\dim\ker F_2<\infty$, then we can reverse the above arguments to show that $aE+bF$ is Fredholm. The equivalence of the Fredholmness of $aE+bF$ and $E+F$ follows easily. Finally, we also have

$$\operatorname{ind}(aE+bF)=\operatorname{ind}(I-E_{31}D_1^*(I-F_{11})^{-1/2}F_{11}^{1/2})=\operatorname{ind}(E+F),$$

which completes the proof. □

The relation between the Fredholmness of $aE+bF$ with $a+b=0$ and $a+b \neq 0$ is given in the next final corollary.

Corollary 4. *Let E and F be idempotents on H.*

(a) *If $E-F$ is Fredholm, then so is $E+F$ and, in this case,* $\operatorname{ind}(E-F) = \operatorname{ind}(E+F) + \dim(\operatorname{ran} E \cap \operatorname{ran} F)$. *The converse is false.*
(b) *If* $\dim(\operatorname{ran} E \cap \operatorname{ran} F) < \infty$, *then $E-F$ is Fredholm if and only if $E+F$ is.*

Proof. As in the proof of Theorem 1, $E+F$ (resp., $E-F$) is Fredholm if and only if $I-F_{11}$ is invertible, $I - E_{31}D_1^*(I-F_{11})^{-1/2}F_{11}^{1/2}$ is Fredholm and $\dim\ker F_2 < \infty$ (and, in addition, $\dim\ker(I-F_1) < \infty$). Since $\ker(I-F_1) = \operatorname{ran} E \cap \operatorname{ran} F$, the assertions in (a) and (b) follow easily. □

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References

[1] J. K. Baksalary and O. M. Baksalary, Nonsingularity of linear combinations of idempotent matrices, *Linear Algebra Appl.* **388** (2004), 25–29.
[2] J. B. Conway, *A Course in Functional Analysis*, 2nd ed., Springer, New York, 1990.
[3] H. Du, X. Yao and C. Deng, Invertibility of linear combinations of two idempotents, *Proc. Amer. Math. Soc.* **134** (2006), 1451–1457.
[4] P. R. Halmos, *A Hilbert Space Problem Book*, 2nd ed., Springer, New York, 1982.
[5] J. J. Koliha and V. Rakočević, The nullity and rank of linear combinations of idempotent matrices, *Linear Algebra Appl.* **418** (2006), 11–14.
[6] G. Zhang, Wiener–Hopf operators and factorization of operators (Chinese), *Northeast. Math. J.* **4** (1988), 357–362.

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