行政院國家科學委員會專題研究計畫 成果報告

Stopping 及 non-stopping time 關聯性之探討
研究成果報告(精簡版)

計 畫 類 別 ： 個別型
計 畫 編 號 ： NSC 99-2115-M-009-003-
執 行 期 間 ： 99 年 08 月 01 日至 100 年 10 月 31 日
執 行 單 位 ： 國立交通大學應用數學系（所）

計 畫 主 持 人 ： 吳慶堂

計畫參與人員： 碩士班研究生-兼任助理人員：葉淑娟
碩士班研究生-兼任助理人員：許佩蓉

公 開 資 訊 ： 本計畫可公開查詢

中 華 民 國 101 年 01 月 31 日

中 文 摘 要 ： 在本計畫中我們想研究的是 stopping time 及 non-stopping time 之間的差別與差距。利用 conditional probability，local time 以及 enlargement of filtrations 等等的技巧，我們想試著造出幾種不同檢視 stopping time 及 non-stopping time 之間的差距的 measurements， 並利用幾個例子來看看各種不同的檢驗法來找出一個較合適、較好用的檢測法。

中文關鍵詞： Azema 上鞅, 最後通過時, 非停點時

英 文 摘 要 ： We investigate some measurements of distance between the ' non-stopping time' and the ' stopping time', in particular the non-stopping time of ends of some previsible sets. We aim to give some possible measurements and study several explicit examples using the technique of conditional probability, local time and the enlargement of filtration.

英文關鍵詞： Azema supermartingale, last passage times, non-stopping time

行政院國家科學委員會補助專題研究計畫 ■ 成 果 報 告
□期中進度報告

Stopping 與 non-stopping time 關聯性之探討
(Relationship between Stopping and Non-stopping Time)

計畫類別：■ 個別型計畫　　□ 整合型計畫
計畫編號：NSC　99－2115－M－009－003－
執行期間：　99 年　8 月　1 日至　100 年　10 月 31 日

執行機構及系所：　國立交通大學應用數學系

計畫主持人：吳慶堂(國立交通大學應用數學系）
計畫參與人員：
（1）葉淑娟：國立交通大學數學建模與科學計算研究所碩士班
（2）許佩蓉：國立交通大學數學建模與科學計算研究所碩士班

成果報告類型(依經費核定清單規定繳交)：■精簡報告　□完整報告

本成果報告包括以下應繳交之附件：
■赴國外出差或研習心得報告一份
□赴大陸地區出差或研習心得報告一份
□出席國際學術會議心得報告及發表之論文各一份
□國際合作研究計畫國外研究報告書一份

處理方式：除列管計畫及下列情形者外，得立即公開查詢
□涉及專利或其他智慧財產權，□一年□二年後可公開查詢

中　華　民　國　一百零一　年　一　月　三十一　日

附件二

國科會補助專題研究計畫成果報告自評表

請就研究內容與原計畫相符程度、達成預期目標情況、研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）、是否適合在學術期刊發表或申請專利、主要發現或其他有關價值等，作一綜合評估。

1. 請就研究內容與原計畫相符程度、達成預期目標情況作一綜合評估

■ 達成目標

□ 未達成目標（請說明，以 100 字為限）

□ 實驗失敗

□ 因故實驗中斷

□ 其他原因

說明：

2. 研究成果在學術期刊發表或申請專利等情形：

論文：□已發表 ■未發表之文稿 □撰寫中 □無

專利：□已獲得 □申請中 ■無

技轉：□已技轉 □洽談中 ■無

其他：（以 100 字為限）

3. 請依學術成就、技術創新、社會影響等方面，評估研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）（以 500 字為限）

此研究是針對定義在 previsible set 的 non-stopping time 與其最接近的 stopping time 的探討. 主要的研究成果是提供一些測量與檢驗方法, 並看看一些例子. 此理論目前看來僅是理論上有意思, 我們一直想看看 non-stopping time 與 stopping time 的差別到底有多大. 我門在這裡提供了一些可能的測量法. 至於應用價值或進一步的可能因時間關係尚未開始考慮.

本論文以已投稿台灣數學雜誌, 並獲得接受.

國科會補助專題研究計畫項下赴國外(或大陸地區)出差或研習心得報告

日期：101 年 1 月 31 日

計畫編號	NSC 99－2115－M－009－003－		
計畫名稱	Stopping 與 non-stopping time 關聯性之探討		
出國人員姓名		服務機構及職稱	
出國時間	年 月 日 至 年 月 日	出國地點	

一、國外(大陸)研究過程

二、研究成果

三、建議

四、其他

原本因為暑假開設中央研究院數學研究所暑期研習課程與交通大學之暑修課程，所以將本計畫延長三個月。原擬利用暑修課程與一百學年度開學間之空檔前往德國柏林及法國巴黎討論此研究計畫之結尾與後續發展，所有行程也以規劃好；但適拙荊家中有喪，必須予以延期。然因延期後學校已開學，前往歐洲又無法只利用三四天的時間研究訪問，因此只能將放棄此次出赴國外研究訪問的機會。

Measuring the "non-stopping timeness" of ends of previsible sets

Ching-Tang Wu[1]
Department of Applied Mathematics
National Chiao Tung University
No. 1001, University Road
30050 Hsinchu, Taiwan
email: ctwu@math.nctu.edu.tw

Ju-Yi Yen
Vanderbilt University, Nashville
Tennessee 37240, USA
email: ju-yi.yen@vanderbilt.edu

Marc Yor
Laboratoire de Probabilités et Modèles Aléatoires
Université Pierre et Marie Curie
Case Courrier 188, 4, Place Jussieu
75252 Paris, Cedex 05, France
and
Institut Universitaire de France

Abstract

In this paper, we propose some "measurements" of the "non-stopping timeness" of ends $\mathcal{G}$ of previsible sets, such that $\mathcal{G}$ avoids stopping times, in an ambiant filtration. We then study several explicit examples, involving last passage times of some remarkable martingales.

2000 Mathematics Subject Classification: 60G35, 60G40, 60G44.

Keywords: Azéma supermartingale, last passage times, non-stopping time.

[1]Corresponding author

1. Introduction: About ends of previsible sets

In this paper, we are interested in random times $\mathcal{G}$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ as ends of $(\mathcal{F}_t)$-previsible sets Γ, that is,

$$\mathcal{G} \equiv \mathcal{G}_\Gamma = \sup\{t : \ (t, \omega) \in \Gamma\}. \tag{1}$$

For simplicity, we shall make the following assumptions:

(C) All $((\mathcal{F}_t), P)$-martingales are continuous;
(A) For any $(\mathcal{F}_t)$-stopping time T, $P(\mathcal{G} = T) = 0$.

To such a random time, one associates the Azéma supermartingale

$$Z_t^{\mathcal{G}} = P(\mathcal{G} > t | \mathcal{F}_t),$$

which, under (C) and (A), admits a continuous version.

Theorem 1.1. *Under (C) and (A), there exists a unique positive local martingale $(N_t, t \geq 0)$, with $N_0 = 1$, such that*

$$Z_t^{\mathcal{G}} = P(\mathcal{G} > t | \mathcal{F}_t) = \frac{N_t}{S_t},$$

where $S_t := \sup_{s \leq t} N_s$ for $t \geq 0$.

Proof. See [8]: page 16, Proposition 1.3. □

Note that since $\mathcal{G} < \infty$ a.s., $N_t \underset{t \to \infty}{\longrightarrow} 0$ a.s. We note further that $\log(S_\infty)$ is distributed exponentially, since by Doob's maximal identity

$$\log(S_\infty) \overset{\text{(law)}}{=} \log\left(\frac{1}{U}\right),$$

where U is uniform on $[0, 1]$. Then, the additive decomposition of the supermartingale N_t/S_t is given by

$$\frac{N_t}{S_t} = 1 + \int_0^t \frac{dN_u}{S_u} - \log(S_t) = E[\log(S_\infty) | \mathcal{F}_t] - \log(S_t). \tag{2}$$

Note that the martingale $E[\log(S_\infty) | \mathcal{F}_t]$ belongs to BMO since from (2),

$$E[\log(S_\infty) - \log S_t | \mathcal{F}_t] \leq 1.$$

In a number of questions, it is very interesting to consider the smallest filtration $(\mathcal{F}'_t)_{t \geq 0}$, which contains $(\mathcal{F}_t)$, and makes $\mathcal{G}$ a stopping time; this filtration is usually denoted $(\mathcal{F}_t^{\mathcal{G}})_{t \geq 0}$. One of the interests of $(Z_t^{\mathcal{G}})$ is that it allows to write any $(\mathcal{F}_t)$-martingale as a semimartingale in $(\mathcal{F}_t^{\mathcal{G}})_{t \geq 0}$; see e.g. [2, 3, 8, 9], for both general formulae and many examples.

Recently, it has been understood that Black-Scholes like formulae are closely related with certain such $\mathcal{G}$'s, thus throwing a new light on a cornerstone of mathematical finance, see, e.g. [6, 7]. In the present paper, with (A) as our essential hypothesis, we would like to measure "how much $\mathcal{G}$ differs from an $(\mathcal{F}_t)$ stopping time". The remainder of this paper consists in two sections. In Section 2, we propose several criterions to measure the NST ($\equiv$ Non Stopping Timeness) of $\mathcal{G}$'s which satisfy (C) and (A). In Section 3, we compute explicitly this function $m_{\mathcal{G}}$ for various examples, where $\mathcal{G}$ is the last passage time at a level of a martingale which converges to 0, as $t \to \infty$.

2. Several possible "NST" criterions

Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, an $(\mathcal{F}_t)$-previsible set Γ and a random time $\mathcal{G}$ given by (1). Our aim is to discuss the difference between $\mathcal{G}$ and an $(\mathcal{F}_t)$-stopping time. A natural question is to consider the function

$$m_{\mathcal{G}}(t) = E\left[\left(1_{(\mathcal{G}\geq t)} - P(\mathcal{G} > t|\mathcal{F}_t)\right)^2\right].$$

If $\mathcal{G}$ is an $(\mathcal{F}_t)$-stopping time, the Azéma supermartingale $Z_t^{\mathcal{G}} \equiv P(\mathcal{G} \geq t|\mathcal{F}_t)$ is identically equal to $1_{(\mathcal{G}\geq t)}$. Thus, $m_{\mathcal{G}}(t) = 0$ for all t. If $\mathcal{G}$ is not an $(\mathcal{F}_t)$-stopping time, a simple but useful remark is

$$m_{\mathcal{G}}(t) = E\left[Z_t^{\mathcal{G}}\left(1 - Z_t^{\mathcal{G}}\right)\right]. \tag{3}$$

Instead of considering the "full"function $(m_{\mathcal{G}}(t),\ t \geq 0)$, we may consider only:

$$m_{\mathcal{G}}^* = \sup_{t\geq 0} m_{\mathcal{G}}(t) \tag{4}$$

as a "global" measurement of the NST of $\mathcal{G}$.

Here are two other, a priori natural, measurements of the NST of $\mathcal{G}$:

$$m_{\mathcal{G}}^{**} = E\left[\sup_{t\geq 0}\left(Z_t^{\mathcal{G}}\left(1 - Z_t^{\mathcal{G}}\right)\right)\right] \tag{5}$$

and

$$\widetilde{m}_{\mathcal{G}} = \sup_{T\geq 0} E\left[Z_T^{\mathcal{G}}\left(1 - Z_T^{\mathcal{G}}\right)\right] \tag{6}$$

where T runs over all $(\mathcal{F}_t)$ stopping times.

However, we cannot expect to learn very much from $m_{\mathcal{G}}^{**}$ and $\widetilde{m}_{\mathcal{G}}$, since it is easily shown the following result

Lemma 2.1.

$$m_{\mathcal{G}}^{**} = \widetilde{m}_{\mathcal{G}} = \frac{1}{4}. \tag{7}$$

Proof. (i) The fact that $m^{**}_{\mathcal{G}} = 1/4$ follows immediately from

$$\sup_{x\in[0,1]} x(1-x) = \frac{1}{4},$$

and the fact that, a.s., the range of the process $(Z^{\mathcal{G}}_t, t \geq 0)$ is $[0,1]$ since $Z^{\mathcal{G}}_0 = 1$, $Z^{\mathcal{G}}_\infty = 0$, and $(Z^{\mathcal{G}}_t, t \geq 0)$ is continuous.

(ii) Let us consider $T_a = \inf\{t : Z^{\mathcal{G}}_t = a\}$, for $0 < a < 1$. Then:

$$Z^{\mathcal{G}}_t(1 - Z^{\mathcal{G}}_t)\big|_{t=T_a} = a(1-a).$$

Hence,

$$\sup_{a\in]0,1[} \left[Z^{\mathcal{G}}_{T_a}\left(1 - Z^{\mathcal{G}}_{T_a}\right)\right] = \sup_{a\in]0,1[} (a(1-a)) = \frac{1}{4}.$$

□

An immediate result is that 1/4 is an upper bound of $m_{\mathcal{G}}$ due to the definition. Moreover, there are some other measurements which have been investigated in a number of literatures.

Remark 2.2. (1) (The optional stopping time discrepancy $\mu_{\mathcal{G}}$) It has been shown in [4], of stopping times, among random times, as the times τ such that for every bounded martingale $(M_t)_{t\geq 0}$ one has

$$M_\tau = E\left[M_\infty | \mathcal{F}_\tau\right],$$

where, under our hypothesis (C), we may define $\mathcal{F}_\tau = \sigma\{H_\tau;\ H \text{ previsible}\}$. Thus, another measurement of the NST of $\mathcal{G}$ is

$$\mu_{\mathcal{G}} = \sup_{\substack{M_\infty \in L^2(\mathcal{F}_\infty) \\ E(M^2_\infty)\leq 1}} E\left[\left(M_{\mathcal{G}} - E\left[M_\infty|\mathcal{F}_{\mathcal{G}}\right]\right)^2\right].$$

(2) (Distance from stopping times) We introduce

$$\nu_{\mathcal{G}} = \inf_{T\geq 0} E|\mathcal{G} - T|,$$

where T runs over all $(\mathcal{F}_t)$ stopping times. However, this quantity may be infinite as $\mathcal{G}$ may have infinite expectation. We note that this distance was precisely computed by du Toit-Peskir-Shiryaev in the example of [1]. A more adequate distance may be:

$$\nu'_{\mathcal{G}} = \inf_{T\geq 0}\left(E\left[\frac{|\mathcal{G} - T|}{1 + |\mathcal{G} - T|}\right]\right)$$

In this paper we concentrate uniquely on the study of $(m_{\mathcal{G}}(t),\ t \geq 0)$ using the technique of Azéma supermartingale and enlargement of filtration.

3. A study of several interesting examples of functions $m_{\mathcal{G}}(t)$

3.1. **Some general formulae.** We shall compute $(m_{\mathcal{G}}(t),\ t \geq 0)$ in some particular cases where

$$\mathcal{G} = \mathcal{G}_K = \sup\{t \geq 0 : M_t = K\}, \qquad K \leq 1,$$

with $M_0 = 1$, $M_t \geq 0$, a continuous local martingale such that $M_t \underset{t\to\infty}{\longrightarrow} 0$. We recall that (see, e.g. [2, 8]):

$$Z_t = P(\mathcal{G}_K \geq t|\mathcal{F}_t) = 1 \wedge \left(\frac{M_t}{K}\right).$$

Thus

$$m_K(t) = E\left[Z_t\left(1 - Z_t\right)\right] = \frac{1}{K^2}\, E\left[M_t\left(K - M_t\right)^+\right]. \tag{8}$$

Consider the particular case $M_t = \mathcal{E}_t = \exp(B_t - t/2)$, with (B_t) a standard Brownian motion, and $\mathcal{G}_K = \sup\{t : \mathcal{E}_t = K\}$ for $K \leq 1$. From formula (8), we deduce:

$$\begin{aligned}
m_K(t) &= \frac{1}{K^2}\, E\left[\mathcal{E}_t\left(K - \mathcal{E}_t\right)^+\right] \\
&= \frac{1}{K^2}\, E\left[\left(K - \exp\left(B_t + \frac{t}{2}\right)\right)^+\right] \quad \text{(by Cameron-Martin)} \\
&= \frac{1}{K^2}\, \left\{KP\left(\exp\left(B_t + \frac{t}{2}\right) < K\right) - E\left[1_{\left(\exp\left(B_t+\frac{t}{2}\right)<K\right)} \exp\left(B_t + \frac{t}{2}\right)\right]\right\}.
\end{aligned}$$

Set $K = e^l$, we have

$$\begin{aligned}
m_K(t) &= e^{-l}P\left(B_t + \frac{t}{2} < l\right) - e^t e^{-2l}P\left(B_t + \frac{3t}{2} < l\right) \\
&= \left(e^{-l} - e^{t-2l}\right) P\left(B_1 < -\frac{3\sqrt{t}}{2} + \frac{l}{\sqrt{t}}\right) + e^{-l}P\left(-\frac{3\sqrt{t}}{2} + \frac{l}{\sqrt{t}} < B_1 < -\frac{\sqrt{t}}{2} + \frac{l}{\sqrt{t}}\right).
\end{aligned}$$

In particular,

$$m_1(t) = \left(1 - e^t\right) P\left(B_1 < -\frac{3\sqrt{t}}{2}\right) + P\left(-\frac{3\sqrt{t}}{2} < B_1 < -\frac{\sqrt{t}}{2}\right).$$

Figure 1 presents the graphs of $m_K(t)$ for some K's.

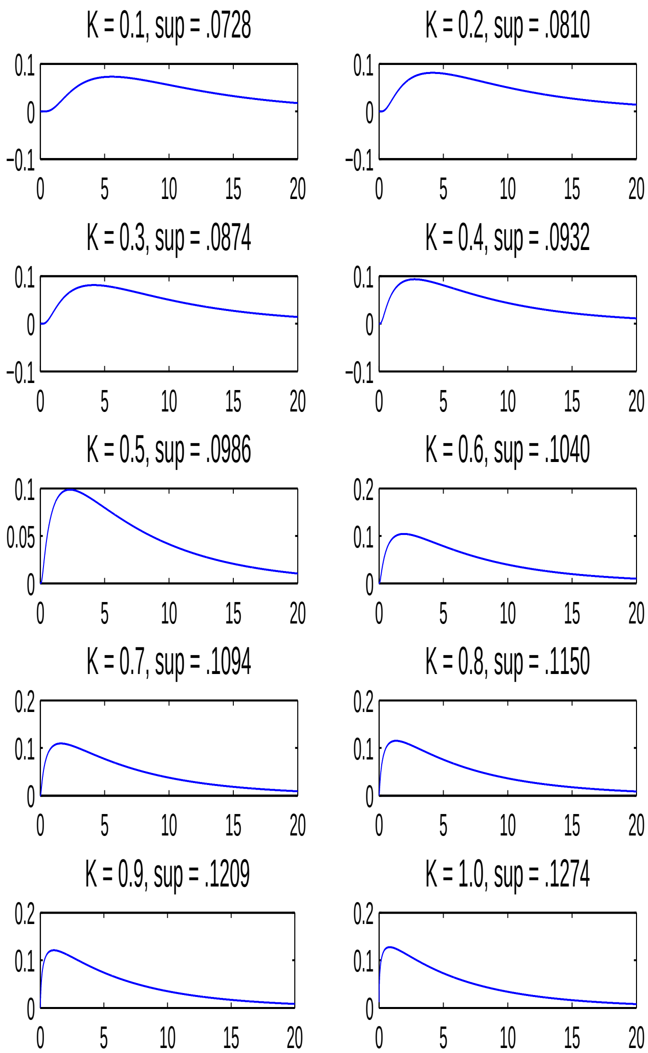

FIGURE 1. Graphs of $m_K(t)$, for $K = 0.1, 0.2, ...1$.

3.2. **The case $\mathcal{G} = \mathcal{G}_{\gamma_T^a} = \sup\{u \leq T : B_u = a\}$.** For fixed time T and $a \in \mathbb{R}$, the associated Azéma supermartingale is of the form

$$Z_t = \Phi\left(\frac{|B_t - a|}{\sqrt{T-t}}\right) 1_{\{t<T\}}$$

D	0	0.1	0.2	0.3	0.4
$m_{\mathcal{G}}^{a,T}$	0.17548	0.175531	0.173103	0.220612	0.244867
D	0.5	0.6	0.7	1	1.1
$m_{\mathcal{G}}^{a,T}$	0.249704	0.24059	0.218382	0.132556	0.105833
D	1.2	1.5	2	3	5
$m_{\mathcal{G}}^{a,T}$	0.0840563	0.0416004	0.0122678	0.000653202	1.30174×10^{-7}

Table 1. The values of $m_{\mathcal{G}}^{a,T}$ for some D

(see, e.g., Table (1α) of Progressive Enlargements, p.32 of [8]), where $\Phi(x) = \sqrt{\frac{2}{\pi}} \int_x^\infty e^{-u^2/2}\, du$. Then for $t < T$, using change of variables we have

$$m_{\mathcal{G}}^{a,T}(t) = E\left[\Phi\left(\frac{|\sqrt{t}\, B_1 - a|}{\sqrt{T-t}}\right)\left(1 - \Phi\left(\frac{|\sqrt{t}\, B_1 - a|}{\sqrt{T-t}}\right)\right)\right] = m^{a/\sqrt{T}}\left(\sqrt{\frac{T-t}{t}}\right),$$

where

$$m^D(c) := \frac{c}{\sqrt{2\pi}} \int_0^\infty \Phi(y)(1-\Phi(y)) \left(\exp\left(-\frac{(cy + D\sqrt{c^2+1})^2}{2}\right) + \exp\left(-\frac{(cy - D\sqrt{c^2+1})^2}{2}\right)\right) dy.$$

Hence

$$m_{\mathcal{G}}^{a,T} := \sup_{0 \le t \le T} m_{\mathcal{G}}^{a,T}(t) = \sup_{c \ge 0} m^{a/\sqrt{T}}(c).$$

Remark 3.1. (1) For $a \in \mathbb{R}$, $m_{\mathcal{G}}^{a,T} = m_{\mathcal{G}}^{-a,T}$, since $m^D(c) = m^{-D}(c)$.

(2) $m_{\mathcal{G}}^{0,T}$ is independent of T, since

$$m_{\mathcal{G}}^{0,T} = \sup_{c \ge 0} \frac{2c}{\sqrt{2\pi}} \int_0^\infty \Phi(y)(1 - \Phi(y)) \exp\left(-\frac{c^2 y^2}{2}\right) dy$$

is independent of T.

(3) the value of $m_{\mathcal{G}}^{a,T}$ depends only on $D := a/\sqrt{T}$, e.g., $(a,T) = (1,1)$ and $(a,T) = (1/2, 1/4)$ have the same $m_{\mathcal{G}}^{a,T}$ value, since $D = 1$ in both cases.

Remark 3.2. Table 1 gives the values of $m_{\mathcal{G}}^{a,T}$ for some D.

In fact, if D satisfies $\Phi(D) = \frac{1}{2}$ (i.e., D around 0.47693627), then $m_{\mathcal{G}}^{a,T} = \frac{1}{4}$ and the maximum occurs at $t = 0$. The same as $m_{\mathcal{G}}^{**}$ and $\tilde{m}_{\mathcal{G}}$.

Figure 2 - Figure 4 present the graphs $m^D(c)$ for some D. The horizontal axis is the value of $c = \sqrt{\dfrac{T-t}{t}}$ and the vertical axis is the value of $m^D(c)$, and its maximum is exactly $m_{\mathcal{G}}^{a,T}$.

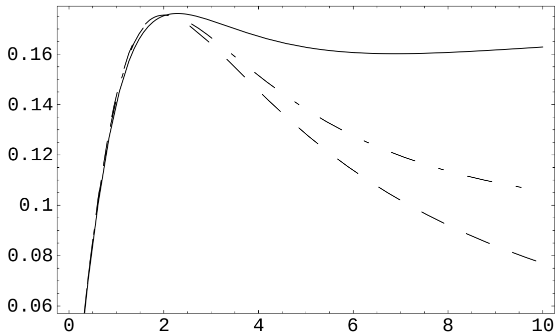

FIGURE 2. $D = 0$: — — — ; $D = 0.1$: — - — - ; $D = 0.2$: ————

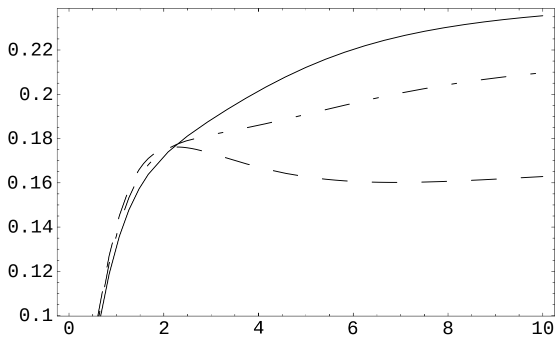

FIGURE 3. $D = 0.2$: — — — ; $D = 0.3$: — - — - ; $D = 0.4$: ————

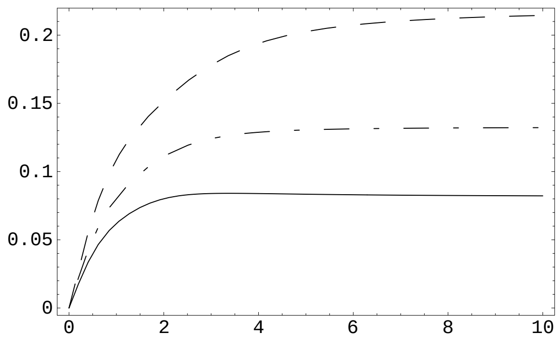

FIGURE 4. $D = 0.7$: — — — ; $D = 1$: — - — - ; $D = 1.2$: ————

3.3. **The case $\mathcal{G} = \mathcal{G}_{T_a} = \sup\{t < T_a : B_t = 0\}$.** , Here, we denote $T_a = \inf\{u : B_u = a\}$, for $a > 0$; and $S_t = \sup_{0\le u\le t} B_u$. The corresponding Azéma supermartingale is given by

$$Z_t = 1 - \frac{1}{a}\; B^+_{t\wedge T_a},$$

see, e.g., Table (1α) of Progressive Enlargements, p.32 of [8]. Thus, we obtain:

$$\begin{aligned} m_{\mathcal{G}}(t) &= E\left[\left(\frac{1}{a}\; B^+_{t\wedge T_a}\right)\left(1 - \frac{1}{a}\; B^+_{t\wedge T_a}\right)\right] \\ &= \frac{1}{a^2}\; E\left[1_{(t<T_a)}1_{(B_t>0)}B_t(a - B_t)\right] \\ &= \frac{1}{a^2}\; E\left[1_{(S_t<a)}1_{(B_t>0)}B_t(a - B_t)\right]. \end{aligned}$$

Let

$$\varphi(x) = E\left[1_{(S_1<x)}1_{(B_1>0)}B_1(x - B_1)\right],$$

then

$$m_{\mathcal{G}}(t) = \frac{t}{a^2}\;\varphi\left(\frac{a}{\sqrt{t}}\right).$$

Now, it remains to compute the function φ. We note that

$$\varphi(x) = E\left[B_1^+(x - B_1)^+\right] - E\left[1_{(S_1>x)}B_1^+(x - B_1)^+\right].$$

We shall take advantage of the very useful formula:

$$P(S_1 > x | B_1 = a) = \exp(-2x(x - a)), \qquad x \ge a > 0,$$

see, e.g., [5], p.425. Thus, we find

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}\int_0^x dy\; y(x - y)\left(\exp\left(-\frac{y^2}{2}\right) - \exp\left(-\frac{1}{2}(2x - y)^2\right)\right)$$

Thus,

$$\frac{\varphi(x)}{x^2} = \frac{x}{\sqrt{2\pi}}\int_0^1 du\; u(1 - u)\left(\exp\left(-\frac{x^2u^2}{2}\right) - \exp\left(-\frac{x^2}{2}(2 - u)^2\right)\right).$$

Note that the value of $\sup_{t>0} m_{\mathcal{G}}(t) = \sup_{x>0}\frac{\varphi(x)}{x^2}$ is independent of the value of a, since it depends only on the value of $x := a/\sqrt{t}$.

3.4. **The case $\mathcal{G} = L_a = \sup\{u : R_u = a\}$.** We have

$$Z_t = 1 \wedge \left(\frac{a}{R_t}\right)^{2\mu},$$

see, e.g., Table (1α) of Progressive Enlargements, p.32 of [8]. Here, (R_u) is $BES_0(d)$, and $\mu > 0$. Thus,

$$\begin{aligned} m_{\mathcal{G}}(t) &= E\left[\left(1 \wedge \left(\frac{a}{R_t}\right)^{2\mu}\right)\left(1 - 1 \wedge \left(\frac{a}{R_t}\right)^{2\mu}\right)\right] \\ &= E\left[1\left(\frac{a}{\sqrt{t}R_1} < 1\right)\left(\frac{a}{\sqrt{t}R_1}\right)^{2\mu}\left(1 - \left(\frac{a}{\sqrt{t}R_1}\right)^{2\mu}\right)\right]. \end{aligned}$$

Using the fact that $R_1^2 \overset{(\text{law})}{=} 2\gamma_{d/2}$ for $d = 2(\mu+1)$, we get

$$m_{\mathcal{G}}(t) = \varphi_\mu\left(\frac{a^2}{2t}\right),$$

where

$$\varphi_\mu(z) = \frac{1}{\Gamma(\mu+1)}\left\{z^\mu e^{-z} - z^{2\mu}\int_z^\infty \frac{du}{u^\mu} e^{-u}\right\}.$$

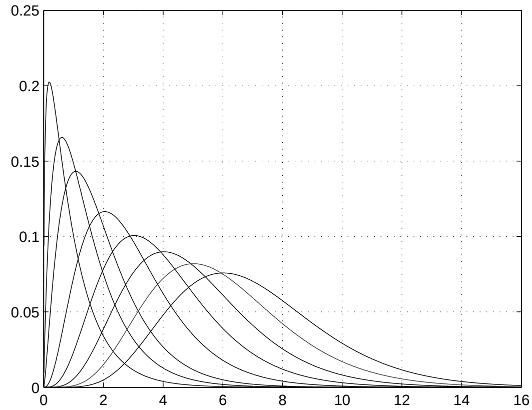

FIGURE 5. Graphs of $\varphi_\mu(z)$, for $\mu = 1/2,\ 1,\ 3/2,\ 5/2,\ 7/2,\ 9/2,\ 11/2,\ 13/2$, and that $z_{1/2} = 0.19$, $z_1 = 0.61$, $z_{3/2} = 1.08$, $z_{5/2} = 2.05$, $z_{7/2} = 3.04$, $z_{9/2} = 4.03$, $z_{11/2} = 5.02$, $z_{13/2} = 6.02$.

Figure 5 presents the graphs of φ_μ for $\mu = 1/2, 1, 3/2, 5/2, 7/2, 9/2, 11/2$ and $13/2$. We also approximate z_μ, the unique > 0 real which achieves the max of φ_μ. This will give us the value $m_\mu \overset{\text{def}}{=} m^*_{\mathcal{G}}$, for these $\mathcal{G} \equiv \mathcal{L}_a$ (note that, for a given μ, the value does not depend on a; this is because of the scaling property).

It is not difficult to show that: z_μ is the unique solution of

$$(E_\mu): \quad \frac{1}{2z} = \int_0^\infty \frac{dh}{(1+h)^\mu}\, e^{-hz}$$

and also

$$m_\mu = \frac{1}{\Gamma(\mu+1)}\, e^{-z_\mu}\, \frac{(z_\mu)^\mu}{2}.$$

Note that

$$m_\mu \leq m'_\mu \stackrel{\text{def}}{=} \frac{1}{\Gamma(\mu+1)} \sup_{z\geq 0} \left(e^{-z} \frac{z^\mu}{2} \right).$$

Figure 6 presents the graphs of m_μ and m'_μ.

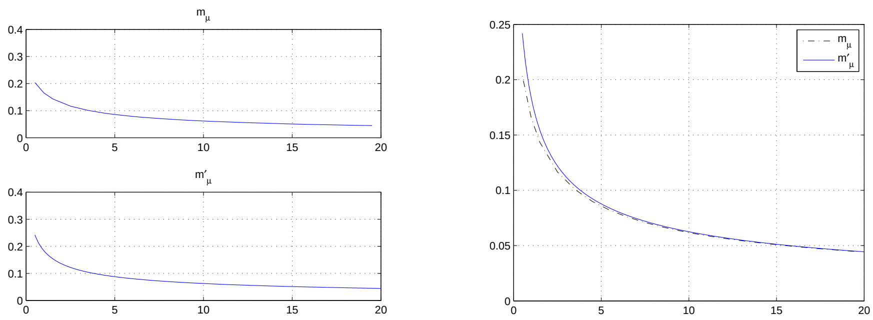

FIGURE 6. Graphs of m_μ and m'_μ

Acknowledgment: The first author's research is partially supported by the National Science Council under Grant #NSC 98-2115-M-3009-006-, National Center for Theoretical Sciences (NCTS) and Center of Mathematical Modeling and Scientific Computing (CMMSC). The second author was supported in part by NSF Grant DMS-0907513. The second author is grateful to the Academia Sinica Institute of Mathematics (Taipei, Taiwan) and the City University of Hong Kong for their hospitality and support during extended visits.

References

[1] J. du Toit, G. Peskir, G. and A. N. Shiryaev, Predicting the last zero of Brownian motion with drift, *Stochastics*, 80 (2008), 229-245.

[2] T. Jeulin, *Semi-martingales et grossissements d'une filtration*, LNM 833, Springer, Berlin, 1980.

[3] T. Jeulin and M. Yor, Eds. *Grossissements de filtrations: exemples et applications*, LNM 1118, Springer-Verlag, 1985.

[4] F. Knight and B. Maisonneuve, A characterization of stopping times, *The Annals of Probability*, Vol. 22, No. 3 (1994), 1600-1606.

[5] I. Karatzas and M. Shreve, *Brownian Motion and Stochastic Calculus, 2nd ed.* Springer, Berlin, 1991.

[6] D. Madan, R. Roynette and M. Yor, Option prices as probabilities, *Finance Research Letters* 5 (2008), 79-87. doi:10.1016/j.frl.2008.02.002

[7] D. Madan, R. Roynette and M. Yor, Unifying Black-Scholes type formulae which involve last passage times up to finite horizon. *Asia-Pacific Financial Markets*, 15 (2008), 97-115.

[8] R. Mansuy and M. Yor, *Random times and enlargements of filtrations in a Brownian setting*, LNM 1873, Springer, Berlin, 2006.

[9] A. Nikeghbali and M. Yor, Doob's maximal identity, multiplicative decomposition and enlargements of filtrations, *Ill. Journal of Maths.*, 50 (2006), 791-814.

國科會補助計畫衍生研發成果推廣資料表

日期:2012/01/31

<table>
<tr><td rowspan="3">國科會補助計畫</td><td>計畫名稱: Stopping 及 non-stopping time 關聯性之探討</td></tr>
<tr><td>計畫主持人: 吳慶堂</td></tr>
<tr><td>計畫編號: 99-2115-M-009-003- 學門領域: 機率</td></tr>
<tr><td colspan="2">無研發成果推廣資料</td></tr>
</table>

99 年度專題研究計畫研究成果彙整表

計畫主持人：吳慶堂			計畫編號：99-2115-M-009-003-				
計畫名稱：Stopping 及 non-stopping time 關聯性之探討							
成果項目			量化			單位	備註（質化說明：如數個計畫共同成果、成果列為該期刊之封面故事...等）
			實際已達成數（被接受或已發表）	預期總達成數(含實際已達成數)	本計畫實際貢獻百分比		
國內	論文著作	期刊論文	1	0	100%	篇	
		研究報告/技術報告	0	0	100%		
		研討會論文	0	0	100%		
		專書	0	0	100%		
	專利	申請中件數	0	0	100%	件	
		已獲得件數	0	0	100%		
	技術移轉	件數	0	0	100%	件	
		權利金	0	0	100%	千元	
	參與計畫人力（本國籍）	碩士生	0	0	100%	人次	
		博士生	0	0	100%		
		博士後研究員	0	0	100%		
		專任助理	0	0	100%		
國外	論文著作	期刊論文	0	0	100%	篇	
		研究報告/技術報告	0	0	100%		
		研討會論文	0	0	100%		
		專書	0	0	100%	章/本	
	專利	申請中件數	0	0	100%	件	
		已獲得件數	0	0	100%		
	技術移轉	件數	0	0	100%	件	
		權利金	0	0	100%	千元	
	參與計畫人力（外國籍）	碩士生	0	0	100%	人次	
		博士生	0	0	100%		
		博士後研究員	0	0	100%		
		專任助理	0	0	100%		

其他成果 (無法以量化表達之成果如辦理學術活動、獲得獎項、重要國際合作、研究成果國際影響力及其他協助產業技術發展之具體效益事項等，請以文字敘述填列。)	此篇論文為與法國巴黎第六大學 Marc Yor 教授與美國 Vanderbilt 大學顏如儀教授共同合作

	成果項目	量化	名稱或內容性質簡述
科教處計畫加填項目	測驗工具(含質性與量性)	0	
	課程/模組	0	
	電腦及網路系統或工具	0	
	教材	0	
	舉辦之活動/競賽	0	
	研討會/工作坊	0	
	電子報、網站	0	
	計畫成果推廣之參與（閱聽）人數	0	

國科會補助專題研究計畫成果報告自評表

請就研究內容與原計畫相符程度、達成預期目標情況、研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）、是否適合在學術期刊發表或申請專利、主要發現或其他有關價值等，作一綜合評估。

1. 請就研究內容與原計畫相符程度、達成預期目標情況作一綜合評估

■達成目標

□未達成目標（請說明，以 100 字為限）

□實驗失敗

□因故實驗中斷

□其他原因

說明：

2. 研究成果在學術期刊發表或申請專利等情形：

論文：□已發表 ■未發表之文稿 □撰寫中 □無

專利：□已獲得 □申請中 ■無

技轉：□已技轉 □洽談中 ■無

其他：（以 100 字為限）

3. 請依學術成就、技術創新、社會影響等方面，評估研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）（以 500 字為限）

此研究是針對定義在 previsible set 的 non-stopping time 與其最接近的 stopping time 的探討. 主要的研究成果是提供一些測量與檢驗方法, 並看看一些例子. 此理論目前看來僅是理論上有意思, 我們一直想看看 non-stopping time 與 stopping time 的差別到底有多大. 我門在這裡提供了一些可能的測量法. 至於應用價值或進一步的可能因時間關係尚未開始考慮.