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Chaos, Solitons and Fractals 34 (2007) 1649–1658
CHAOS SOLITONS & FRACTALS
www.elsevier.com/locate/chaos

Synchronization of complex chaotic systems in series expansion form

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Accepted 19 April 2006

Communicated by Prof. G. Iovane

Abstract

This paper studies the synchronization of complex chaotic systems in series expansion form by Lyapunov asymptotical stability theorem. A sufficient condition is given for the asymptotical stability of an error dynamics, and is applied to guiding the design of the secure communication. Finally, numerical results are studied for the Quantum-CNN oscillators synchronizing with unidirectional/bidirectional linear coupling to show the effectiveness of the proposed synchronization strategy.

1. Introduction

Since the discovery of chaos synchronization [1,2], it has been a focus of intensive research [3]. A main field of synchronization is the coupled identical chaotic system [4–10,13–20]. According to the condition of coupling signal, they can be classified into bidirectional [5–7] and unidirectional [8–10] coupling. Due to the simple configuration and easy implementation, the linearly error feedback coupling scheme can be adopted in many real fields. In practice, it is a key problem to determine the appropriate feedback gain or coupling parameters for realizing the synchronization. So far, there have been many specific results about determining the feedback gain or coupling parameters for particular systems [1–9]. A generic condition of global chaos synchronization for two coupled simple chaotic systems using the unidirectional linear error feedback has been studied [9,10]. In this paper, a generic criterion of the chaos synchronization is studied for complex chaotic systems in series expansion form with unidirectional/bidirectional coupling.

As the numerical example, recently developed Quantum Cellular Neural Network (Quantum-CNN) chaotic oscillator is used. Quantum-CNN oscillator equations are derived from a Schrödinger equation taking into account quantum dots cellular automata structures to which in the last decade a wide interest has been devoted with particular attention towards quantum computing [11,12].

This paper is organized as follows. In Section 2, by Lyapunov asymptotical stability theorem, a synchronization scheme is given. In Section 3 numerical simulations of the synchronization of two Quantum-CNN oscillator systems

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0960-0779/$ - see front matter
doi:10.1016/j.chaos.2006.04.072

by unidirectional and by bidirectional linear coupling are given, respectively. Finally, some concluding remarks are drawn in Section 4.

2. Synchronization scheme of complex chaotic systems in series expansion form

In this section, the synchronization by unidirectional/bidirectional linear coupling is studied. Based on Lyapunov asymptotical stabilization theory, we give a generic situation for judging chaos synchronization of two complex systems in series expansion form. The two chaos systems using bidirectional/unidirectional linear coupling can be written as [21–29]

$$\dot{x} = Ax + h(x) + k_1(y - x) \tag{1}$$

and

$$\dot{y} = Ay + h(y) + k_2(x - y), \tag{2}$$

where $x, y \in R^n$ represent the state vectors of the chaotic systems, $A \in R^{n\times n}$ is a constant matrix, $h \in R^{n\times n}$ is a continuous nonlinear function of x, y, k_1, and k_2 are constant gains which represent the coupled parameters.

Let

$$e = x - y, \tag{3}$$

then

$$\dot{e} = [A + M(x(t), y(t)) - (k_1 + k_2)]e + H(x(t), y(t), e), \tag{4}$$

where $M(x(t), y(t))e + H(x(t), y(t), e) = h(x) - h(y)$. The elements of $M(x(t), y(t))$ depend on state vectors x, y, and all of them are bounded convergent infinite series of x, y. $H(x(t), y(t), e)$ contains nonlinear terms of e.

In order to realize the chaos synchronization among these two chaotic systems, we should choose suitable coupled parameter matrices k_1 and k_2 so that

$$\lim_{t\to+\infty} e(t) = 0. \tag{5}$$

In the following, we give the generic criterion of local chaos synchronization of linear coupling systems using the bidirectional linear error feedback scheme.

Theorem. *If there exists a positive definite symmetric constant matrix* $\boldsymbol{P}$ *and a constant* $\varepsilon > 0$, *which satisfy*

$$B^{\mathrm{T}}P + PB \leqslant -\varepsilon I < 0 \tag{6}$$

uniformly for any x *and* y *in the phase space, where* $B = A + M(x(t), y(t)) - (k_1 + k_2)$ *then the error dynamics system* (4) *is locally asymptotically stable.*

Proof. For the nonautonomous error dynamic system (4), choose the following Lyapunov function:

$$V(t) = e^{\mathrm{T}}Pe, \tag{7}$$

then its derivative is

$$\begin{aligned}\dot{V}(t) &= \dot{e}^{\mathrm{T}}Pe + e^{\mathrm{T}}P\dot{e} = e^{\mathrm{T}}B^{\mathrm{T}}(t)Pe + e^{\mathrm{T}}PB(t)e + \text{higher order terms of } e = e^{\mathrm{T}}(B^{\mathrm{T}}(t)P + PB(t))e \\ &\quad + \cdots \leqslant -\varepsilon e^{\mathrm{T}}e < 0\end{aligned} \tag{8}$$

for all sufficient small $e \neq 0$. For sufficient small e, its higher order terms are indifferent to the sign of $\dot{V}(t)$. So, the theorem is proved [30–37]. □

3. Numerical results of the synchronization of two Quantum-CNN oscillator systems by unidirectional and by bidirectional linear coupling

Case 1 (*The synchronization by unidirectional linear coupling*). For a two-cell Quantum-CNN, the following differential equations are obtained [11,12]:

$$\begin{cases} \frac{d}{dt}x_1 = -2a_1\sqrt{1-x_1^2}\sin x_2, \\ \frac{d}{dt}x_2 = -\omega_1(x_1-x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2, \\ \frac{d}{dt}x_3 = -2a_2\sqrt{1-x_3^2}\sin x_4, \\ \frac{d}{dt}x_4 = -\omega_2(x_3-x_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4, \end{cases} \tag{9}$$

where x_1 and x_3 are polarizations, x_2 and x_4 are quantum phase displacements, a_1 and a_2 are proportional to the inter-dot energy inside each cell and ω_1 and ω_2 are parameters that weigh effects on the cell of the difference of the polarization of neighboring cells, like the cloning templates in traditional CNNs. Let $a_1 = a_2 = 13.4$, $\omega_1 = 11.9$, $\omega_2 = 6.04$. The initial values of linear coupled Quantum-CNN systems are taken as $x_1(0) = 0.8$, $x_2(0) = -0.77$, $x_3(0) = -0.72$, $x_4(0) = 0.57$, $y_1(0) = -0.2$, $y_2(0) = 0.41$, $y_3(0) = 0.25$, and $y_4(0) = -0.81$, respectively.

Two Quantum-CNN chaotic systems using the unidirectional linear coupling can be written as

$$\begin{cases} \frac{d}{dt}x_1 = -2a_1\sqrt{1-x_1^2}\sin x_2, \\ \frac{d}{dt}x_2 = -\omega_1(x_1-x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2, \\ \frac{d}{dt}x_3 = -2a_2\sqrt{1-x_3^2}\sin x_4, \\ \frac{d}{dt}x_4 = -\omega_2(x_3-x_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4 \end{cases} \tag{10}$$

and

$$\begin{cases} \frac{d}{dt}y_1 = -2a_1\sqrt{1-y_1^2}\sin y_2 + k_1(x_1-y_1), \\ \frac{d}{dt}y_2 = -\omega_1(y_1-y_3) + 2a_1\frac{y_1}{\sqrt{1-y_1^2}}\cos y_2 + k_2(x_2-y_2), \\ \frac{d}{dt}y_3 = -2a_2\sqrt{1-y_3^2}\sin y_4 + k_3(x_3-y_3), \\ \frac{d}{dt}y_4 = -\omega_2(y_3-y_1) + 2a_2\frac{y_3}{\sqrt{1-y_3^2}}\cos y_4 + k_4(x_4-y_4), \end{cases} \tag{11}$$

where the values of k_1, k_2, k_3, and k_4 are to be determined.

Expand the right-hand sides of Eqs. (10) and (11) into power series:

$$\begin{cases} \frac{d}{dt}x_1 = -2a_1\left(-\frac{1}{2}x_1^2x_2 + \frac{1}{12}x_1^2x_2^3 - \frac{1}{8}x_1^4x_2 + x_2 - \frac{1}{6}x_2^3 + \frac{1}{120}x_2^5 + \cdots\right), \\ \frac{d}{dt}x_2 = -\omega_1(x_1-x_3) + 2a_1\left(x_1 - \frac{1}{2}x_1x_2^2 + \frac{1}{24}x_1x_2^4 + \frac{1}{2}x_1^3 - \frac{1}{4}x_1^3x_2^2 + \frac{5}{8}x_1^5 + \cdots\right), \\ \frac{d}{dt}x_3 = -2a_2\left(-\frac{1}{2}x_3^2x_4 + \frac{1}{12}x_3^2x_4^3 - \frac{1}{8}x_3^4x_4 + x_4 - \frac{1}{6}x_4^3 + \frac{1}{120}x_4^5 + \cdots\right), \\ \frac{d}{dt}x_4 = -\omega_2(x_3-x_1) + 2a_2\left(x_3 - \frac{1}{2}x_3x_4^2 + \frac{1}{24}x_3x_4^4 + \frac{1}{2}x_3^3 - \frac{1}{4}x_3^3x_4^2 + \frac{5}{8}x_3^5 + \cdots\right) \end{cases} \tag{12}$$

and

$$\begin{cases} \frac{d}{dt}y_1 = -2a_1\left(-\frac{1}{2}y_1^2y_2 + \frac{1}{12}y_1^2y_2^3 - \frac{1}{8}y_1^4y_2 + y_2 - \frac{1}{6}y_2^3 + \frac{1}{120}y_2^5 + \cdots\right) + k_1(x_1-y_1), \\ \frac{d}{dt}y_2 = -\omega_1(y_1-y_3) + 2a_1\left(y_1 - \frac{1}{2}y_1y_2^2 + \frac{1}{24}y_1y_2^4 + \frac{1}{2}y_1^3 - \frac{1}{4}y_1^3y_2^2 + \frac{5}{8}y_1^5 + \cdots\right) + k_2(x_2-y_2), \\ \frac{d}{dt}y_3 = -2a_2\left(-\frac{1}{2}y_3^2y_4 + \frac{1}{12}y_3^2y_4^3 - \frac{1}{8}y_3^4y_4 + y_4 - \frac{1}{6}y_4^3 + \frac{1}{120}y_4^5 + \cdots\right) + k_3(x_3-y_3), \\ \frac{d}{dt}y_4 = -\omega_2(y_3-y_1) + 2a_2\left(y_3 - \frac{1}{2}y_3y_4^2 + \frac{1}{24}y_3y_4^4 + \frac{1}{2}y_3^3 - \frac{1}{4}y_3^3y_4^2 + \frac{5}{8}y_3^5 + \cdots\right) + k_4(x_4-y_4). \end{cases} \tag{13}$$

From Eqs. (12) and (13), the error dynamics is

$$\dot{e} = Be + H(x,y,e), \tag{14}$$

where $e = (y_1-x_1, y_2-x_2, y_3-x_3, y_4-x_4)^{\mathrm{T}}$ and

$$B = \begin{pmatrix} -k_1 + B_{11} & -2a_1 + B_{21} & 0 & 0 \\ 2a_1 - \omega_1 + B_{12} & -k_2 + B_{22} & \omega_1 & 0 \\ 0 & 0 & -k_3 + B_{33} & -2a_2 + B_{43} \\ \omega_2 & 0 & 2a_2 - \omega_2 + B_{34} & -k_4 + B_{44} \end{pmatrix}$$

in which

$$B_{11} = a_1\left[2x_1y_2 - \frac{1}{6}x_1y_2^3 + \frac{1}{4}\left(x_1y_1^2y_2 + 3x_1^2y_1y_2\right) + \cdots\right],$$
$$B_{12} = a_1\left[3x_1y_1 - y_2^2 + \frac{1}{12}y_2^4 - \frac{1}{2}y_2^2(y_1^2 + 2x_1^2) + \frac{25}{4}x_1^2y_1^2 + \cdots\right],$$
$$B_{21} = a_1\left[x_1^2 + x_2y_2 - \frac{1}{6}\left(x_1^2x_2^2 + x_1y_2^3 + 2x_1^2y_2^2\right) + \frac{1}{4}x_1^4 - \frac{1}{12}x_2^2y_2^2 + \cdots\right],$$
$$B_{22} = a_1\left[2x_1y_2 + \frac{1}{12}x_1x_2y_2(x_2 + 3y_2) - x_1^3y_2 + \cdots\right]$$
$$B_{33} = a_2\left[2x_3y_4 - \frac{1}{6}x_3y_4^3 + \frac{1}{4}\left(x_3y_3^2y_4 + 3x_3^2y_3y_4\right) + \cdots\right],$$
$$B_{34} = a_2\left[3x_3y_3 - y_4^2 + \frac{1}{12}y_4^4 - \frac{1}{2}y_4^2(y_3^2 + 2x_3^2) + \frac{25}{4}x_3^2y_3^2 + \cdots\right],$$
$$B_{43} = a_2\left[x_3^2 + x_4y_4 - \frac{1}{6}\left(x_3^2x_4^2 + x_3y_4^3 + 2x_3^2y_4^2\right) + \frac{1}{4}x_3^4 - \frac{1}{12}x_4^2y_4^2 + \cdots\right],$$
$$B_{44} = a_2\left[2x_3y_4 + \frac{1}{12}x_3x_4y_4(x_4 + 3y_4) - x_3^3y_4 + \cdots\right]$$

and $H(x,y,e)$ contains nonlinear terms of e.

The infinite power series in the first element of B_{11} is

$$2x_1y_2 - \frac{1}{6}x_1y_2^3 + \frac{1}{4}\left(x_1y_1^2y_2 + 3x_1^2y_1y_2\right) + \cdots \tag{15}$$

It is well-known [38] that a necessary and sufficient condition for the convergence of the infinite series

$$u_1 + u_2 + \cdots + u_n + \cdots$$

is that for any previously assigned positive ε there exists an N_1 such that, for any $n > N$ and for positive p,

$$|u_{n+1} + u_{n+2} + \cdots u_{n+p}| < \varepsilon. \tag{16}$$

From Fig. 1, we know that

$$|x_i| < 1, \quad |y_i| < 1 \qquad (i = 1,2,3,4), \tag{17}$$

therefore, series (15) which satisfies condition (16), is convergent and has a bounded sum. For the same reason, the other power series in B_{11}, B_{12}, B_{21}, B_{22}, B_{33}, B_{34}, B_{43}, and B_{44} are all convergent and have bounded sums. As an example, the time history of B_{11} is shown in Fig. 2.

Choose the positive definite symmetric constant matrix $\boldsymbol{P} = \mathrm{diag}(p_1,p_2,p_3,p_4)$, $p_i > 0$, $i = 1,2,3,4$ and any constants $\varepsilon > 0$, then $C = B^{\mathrm{T}}P + PB + \varepsilon I$ is negative definite if and only if

$$(-1)^{i+1}\Delta_i < 0, \quad i = 1,2,3,4, \tag{18}$$

where Δ_i represents the ith order sequential subdeterminant of matrix $\boldsymbol{C}$. Let $\varepsilon = 0.1$, $p_i = 1$, $i = 1,2,\ldots,4$:

$$C = \begin{pmatrix} -2k_1 + 2B_{11} + 0.1 & -\omega_1 + B_{12} + B_{21} & 0 & \omega_2 \\ -\omega_1 + B_{12} + B_{21} & -2k_2 + 2B_{22} + 0.1 & \omega_1 & 0 \\ 0 & \omega_1 & -2k_3 + 2B_{33} + 0.1 & -\omega_2 + B_{34} + B_{43} \\ \omega_2 & 0 & -\omega_2 + B_{34} + B_{43} & -2k_4 + 2B_{44} + 0.1 \end{pmatrix}.$$

Conditions (18) become

$$k_1 > 30.07, \quad k_2 > 10.9, \quad k_3 > 50.35, \quad \text{and} \quad k_4 > 3018. \tag{19}$$

We chose $k_1 = 33.07$, $k_2 = 11.8$, $k_3 = 52$, and $k_4 = 3020$.

The Lyapunov asymptotical local stability theorem is satisfied. This means that the synchronization of two Quantum-CNN systems by unidirectional linear coupling can be achieved. The numerical results are shown in Fig. 3.

Case 2 (*The synchronization by bidirectional linear coupling*). Two Quantum-CNN systems with bidirectional linear coupling are given:

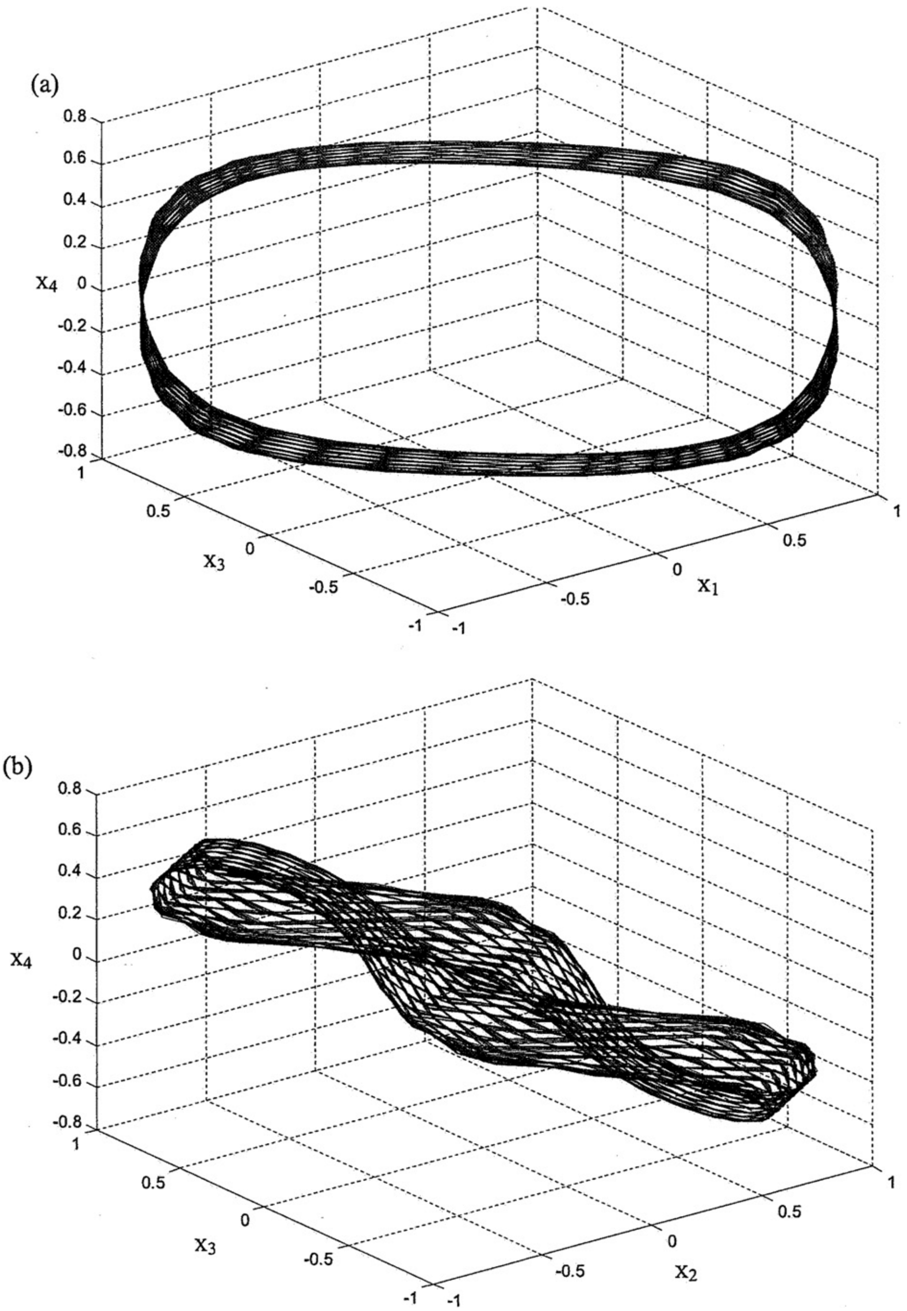

Fig. 1. Phase portrait of master system (10).

$$
\begin{cases}
\frac{\mathrm{d}}{\mathrm{d}t}x_1 = -2a_1\sqrt{1-x_1^2}\sin x_2 + k_{11}(y_1 - x_1), \\
\frac{\mathrm{d}}{\mathrm{d}t}x_2 = -\omega_1(x_1 - x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 + k_{12}(y_2 - x_2), \\
\frac{\mathrm{d}}{\mathrm{d}t}x_3 = -2a_2\sqrt{1-x_3^2}\sin x_4 + k_{13}(y_3 - x_3), \\
\frac{\mathrm{d}}{\mathrm{d}t}x_4 = -\omega_2(x_3 - x_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4 + k_{14}(y_4 - x_4)
\end{cases}
\tag{20}
$$

and

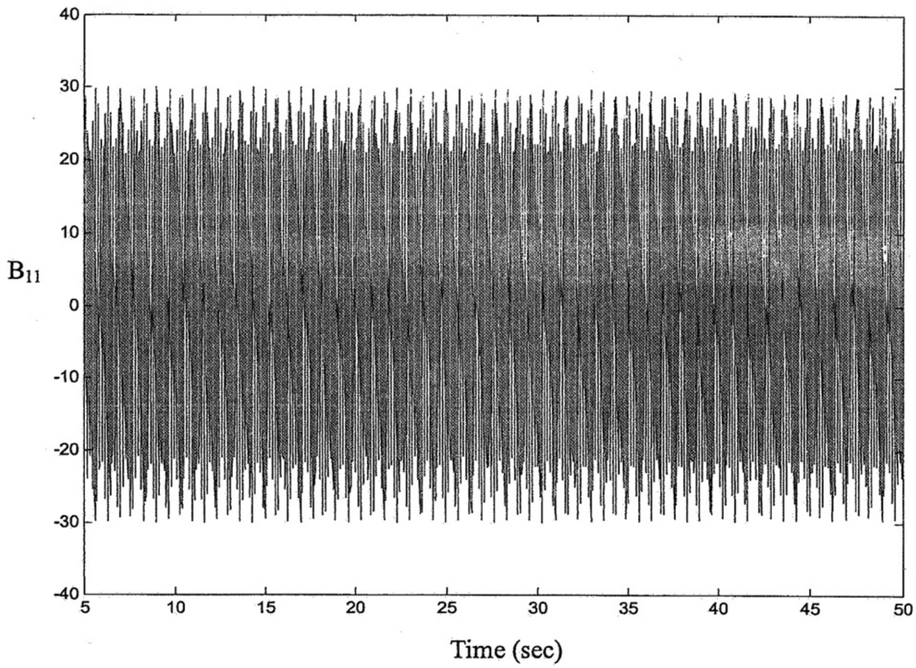

Fig. 2. Time history of B_{11} for Case 1.

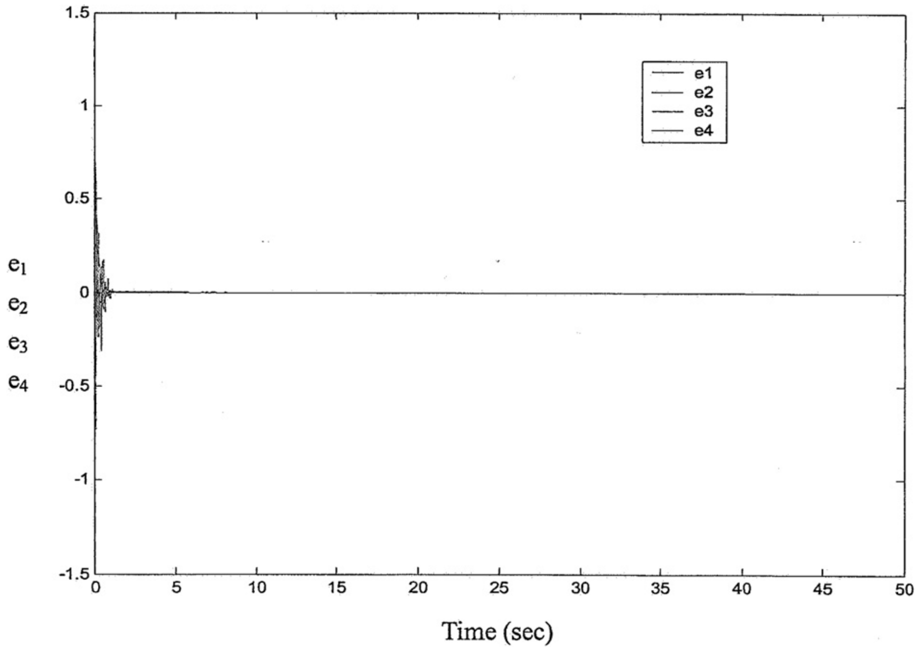

Fig. 3. Time histories of state errors for Case 1.

$$\begin{cases} \frac{d}{dt}y_1 = -2a_1\sqrt{1-y_1^2}\sin y_2 + k_{21}(x_1 - y_1), \\ \frac{d}{dt}y_2 = -\omega_1(y_1 - y_3) + 2a_1\frac{y_1}{\sqrt{1-y_1^2}}\cos y_2 + k_{22}(x_2 - y_2), \\ \frac{d}{dt}y_3 = -2a_2\sqrt{1-y_3^2}\sin y_4 + k_{23}(x_3 - y_3), \\ \frac{d}{dt}y_4 = -\omega_2(y_3 - y_1) + 2a_2\frac{y_3}{\sqrt{1-y_3^2}}\cos y_4 + k_{24}(x_4 - y_4), \end{cases} \tag{21}$$

where the values of k_{11}, k_{12}, k_{13}, k_{14}, k_{21}, k_{22}, k_{23}, and k_{24} are to be determined.

Expand the right-hand sides of Eqs. (20) and (21) into power series:

$$\begin{cases} \frac{d}{dt}x_1 = -2a_1(-\frac{1}{2}x_1^2x_2 + \frac{1}{12}x_1^2x_2^3 - \frac{1}{8}x_1^4x_2 + x_2 - \frac{1}{6}x_2^3 + \frac{1}{120}x_2^5 + \cdots) + k_{11}(y_1 - x_1), \\ \frac{d}{dt}x_2 = -\omega_1(x_1 - x_3) + 2a_1(x_1 - \frac{1}{2}x_1x_2^2 + \frac{1}{24}x_1x_2^4 + \frac{1}{2}x_1^3 - \frac{1}{4}x_1^3x_2^2 + \frac{5}{8}x_1^5 + \cdots) + k_{12}(y_2 - x_2), \\ \frac{d}{dt}x_3 = -2a_2(-\frac{1}{2}x_3^2x_4 + \frac{1}{12}x_3^2x_4^3 - \frac{1}{8}x_3^4x_4 + x_4 - \frac{1}{6}x_4^3 + \frac{1}{120}x_4^5 + \cdots) + k_{13}(y_3 - x_3), \\ \frac{d}{dt}x_4 = -\omega_2(x_3 - x_1) + 2a_2(x_3 - \frac{1}{2}x_3x_4^2 + \frac{1}{24}x_3x_4^4 + \frac{1}{2}x_3^3 - \frac{1}{4}x_3^3x_4^2 + \frac{5}{8}x_3^5 + \cdots) + k_{14}(y_4 - x_4) \end{cases} \tag{22}$$

and

$$\begin{cases} \frac{d}{dt}y_1 = -2a_1(-\frac{1}{2}y_1^2y_2 + \frac{1}{12}y_1^2y_2^3 - \frac{1}{8}y_1^4y_2 + y_2 - \frac{1}{6}y_2^3 + \frac{1}{120}y_2^5 + \cdots) + k_{21}(x_1 - y_1), \\ \frac{d}{dt}y_2 = -\omega_1(y_1 - y_3) + 2a_1(y_1 - \frac{1}{2}y_1y_2^2 + \frac{1}{24}y_1y_2^4 + \frac{1}{2}y_1^3 - \frac{1}{4}y_1^3y_2^2 + \frac{5}{8}y_1^5 + \cdots) + k_{22}(x_2 - y_2), \\ \frac{d}{dt}y_3 = -2a_2(-\frac{1}{2}y_3^2y_4 + \frac{1}{12}y_3^2y_4^3 - \frac{1}{8}y_3^4y_4 + y_4 - \frac{1}{6}y_4^3 + \frac{1}{120}y_4^5 + \cdots) + k_{23}(x_3 - y_3), \\ \frac{d}{dt}y_4 = -\omega_2(y_3 - y_1) + 2a_2(y_3 - \frac{1}{2}y_3y_4^2 + \frac{1}{24}y_3y_4^4 + \frac{1}{2}y_3^3 - \frac{1}{4}y_3^3y_4^2 + \frac{5}{8}y_3^5 + \cdots) + k_{24}(x_4 - y_4). \end{cases} \tag{23}$$

From Eqs. (22) and (23), the error dynamics is

$$\dot{e} = Be + H(x, y, e), \tag{24}$$

where $e = (y_1 - x_1, y_2 - x_2, y_3 - x_3, y_4 - x_4)^{\mathrm{T}}$ and

$$B = \begin{pmatrix} -(k_{11} + k_{21}) + B_{11} & -2a_1 + B_{21} & 0 & 0 \\ 2a_1 - \omega_1 + B_{12} & -(k_{12} + k_{22}) + B_{22} & \omega_1 & 0 \\ 0 & 0 & -(k_{13} + k_{23}) + B_{33} & -2a_2 + B_{43} \\ \omega_2 & 0 & 2a_2 - \omega_2 + B_{34} & -(k_{14} + k_{24}) + B_{44} \end{pmatrix}$$

in which

$$B_{11} = a_1\left[2x_1y_2 - \frac{1}{6}x_1y_2^3 + \frac{1}{4}(x_1y_1^2y_2 + 3x_1^2y_1y_2) + \cdots\right],$$

$$B_{12} = a_1\left[3x_1y_1 - y_2^2 + \frac{1}{12}y_2^4 - \frac{1}{2}y_2^2(y_1^2 + 2x_1^2) + \frac{25}{4}x_1^2y_1^2 + \cdots\right],$$

$$B_{21} = a_1\left[x_1^2 + x_2y_2 - \frac{1}{6}(x_1^2x_2^2 + x_1y_2^3 + 2x_1^2y_2^2) + \frac{1}{4}x_1^4 - \frac{1}{12}x_2^2y_2^2 + \cdots\right],$$

$$B_{22} = a_1\left[2x_1y_2 + \frac{1}{12}x_1x_2y_2(x_2 + 3y_2) - x_1^3y_2 + \cdots\right],$$

$$B_{33} = a_2\left[2x_3y_4 - \frac{1}{6}x_3y_4^3 + \frac{1}{4}(x_3y_3^2y_4 + 3x_3^2y_3y_4) + \cdots\right],$$

$$B_{34} = a\left[3x_3y_3 - y_4^2 + \frac{1}{12}y_4^4 - \frac{1}{2}y_4^2(y_3^2 + 2x_3^2) + \frac{25}{4}x_3^2y_3^2 + \cdots\right],$$

$$B_{43} = a_2\left[x_3^2 + x_4y_4 - \frac{1}{6}(x_3^2x_4^2 + x_3y_4^3 + 2x_3^2y_4^2) + \frac{1}{4}x_3^4 - \frac{1}{12}x_4^2y_4^2 + \cdots\right],$$

$$B_{44} = a_2\left[2x_3y_4 + \frac{1}{12}x_3x_4y_4(x_4 + 3y_4) - x_3^3y_4 + \cdots\right].$$

Similar to Case 1, from Fig. 5, $|x_i| < 1$, $|y_i| < 1$ $(i = 1, 2, 3, 4)$, the infinite power series in B_{11}, B_{12}, B_{21}, B_{22}, B_{33}, B_{34}, B_{43} and B_{44} are all convergent and have bounded sums [38]. As an example, the time history of B_{11} is shown in Fig. 4.

Choose the positive definite symmetric constant matrix $\boldsymbol{P} = \mathrm{diag}(p_1, p_2, p_3, p_4)$, $p_i > 0$, $i = 1, 2, 3, 4$ and any constant $\varepsilon > 0$, then $C = B^{\mathrm{T}}P + PB + \varepsilon I$ is negative definite if and only if

$$(-1)^{i+1}\Delta_i < 0, \quad i = 1, 2, 3, 4, \tag{25}$$

where Δ_i represents the ith order sequential subdeterminant of matrix $\boldsymbol{C}$. Let $\varepsilon = 0.1$, $p_i = 1$, $i = 1, 2, \ldots, 4$:

$$C = \begin{pmatrix} -2(k_{11}+k_{21}+B_{11})+0.1 & -\omega_1+B_{12}+B_{21} & 0 & \omega_2 \\ -\omega_1+B_{12}+B_{21} & -2(k_{12}+k_{22}+B_{22})+0.1 & \omega_1 & 0 \\ 0 & \omega_1 & -2(k_{13}+k_{23}+B_{33})+0.1 & -\omega_2+B_{34}+B_{43} \\ \omega_2 & 0 & -\omega_2+B_{34}+B_{43} & -2(k_{14}+k_{24}+B_{44})+0.1 \end{pmatrix}$$

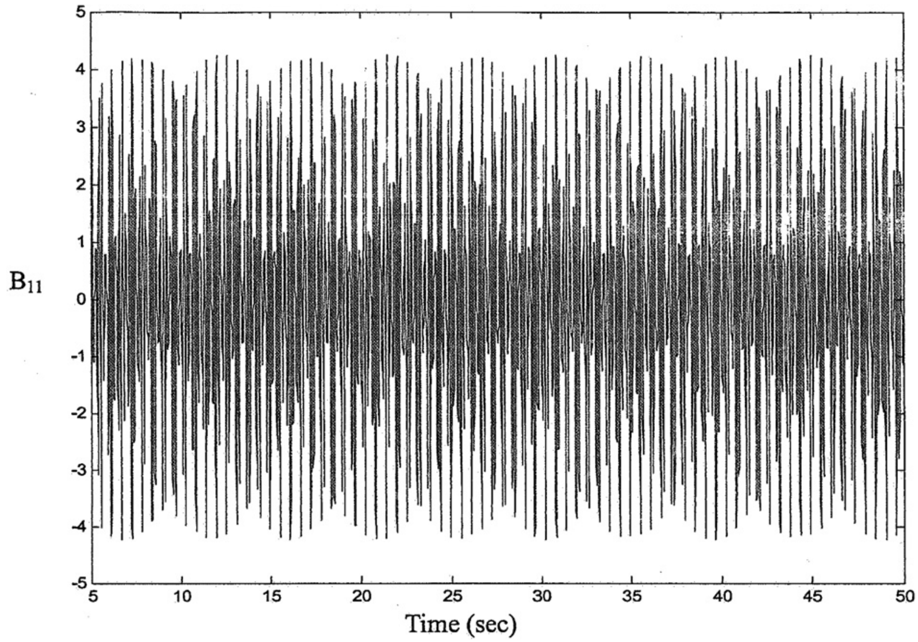

Fig. 4. Time history of B_{11} for Case 2.

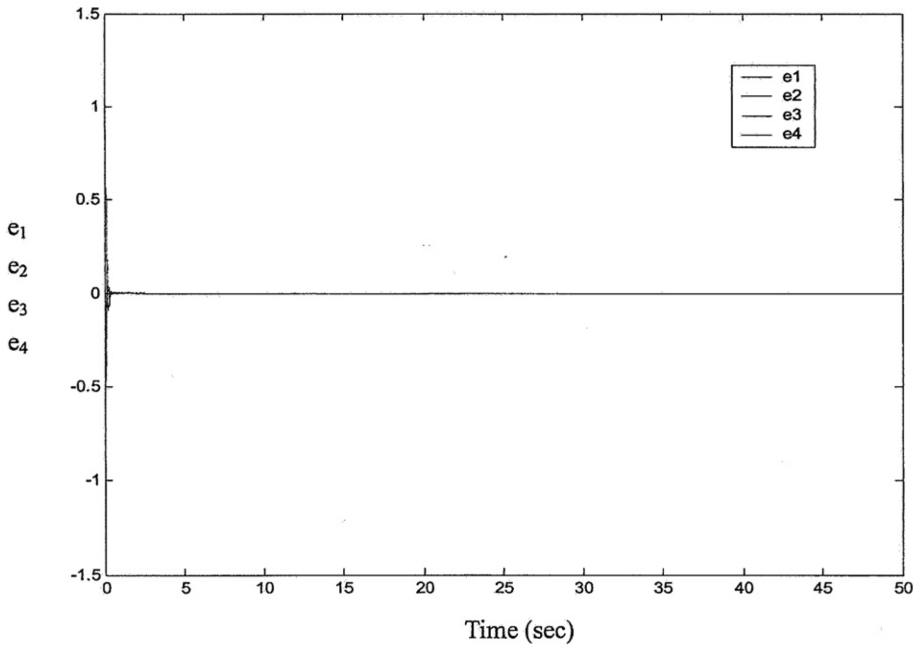

Fig. 5. Time histories of state errors for Case 2.

Conditions (25) become

$$k_{11} + k_{21} > 8.13, \quad k_{12} + k_{22} > 7.62, \quad k_{13} + k_{23} > 20.11, \quad \text{and} \quad k_{14} + k_{24} > 569. \tag{26}$$

We chose $k_{11} = 9.2$, $k_{21} = 8.4$, $k_{12} = 7.6$, $k_{22} = 6.7$, $k_{13} = 13.2$, $k_{23} = 7.6$, $k_{14} = 284$ and $k_{24} = 291$.

The Lyapunov asymptotical local stability theorem is satisfied. This means that the synchronization of two Quantum-CNN systems by bidirectional linear coupling can be achieved. The numerical results are shown in Fig. 5.

4. Conclusions

The synchronization of complex chaotic systems in series expansion form are implemented by the Lyapunov asymptotical stability theorem. Two Quantum-CNN systems are synchronized in two cases: unidirectional linear coupling case and bidirectional linear coupling case. In both cases, by a theorem of convergent series, we prove that all infinite power series in the elements of $B = A + M(x(t), y(t)) - (k_1 + k_2)$ are convergent and have bounded sums.

Acknowledgments

This research was supported by the National Science Council, Republic of China, under Grant Number NSC 94-2212-E-009-013.

References

[1] Pecora L-M, Carroll T-L. Synchronization in chaotic systems. Phys Rev Lett 1990;64:821–4.
[2] Carroll T-L, Pecora L-M. Synchronization chaotic circuits. IEEE Trans Circ Syst 1991;38(4):453–6.
[3] Chen G, Dong X. From chaos to order: methodologies, perspectives and applications. Singapore: World Scientific; 1998.
[4] Curren P-F, Suykens J-A-K, Chua L-O. Absolute stability theory and master–slave synchronization. Int J Bifurcat Chaos 1997;7(12):2891–6.
[5] Suykens J-A-K, Yang T, Chua L-O. Impulsive synchronization of chaotic Lur'e systems by measurement feedback. Int J Bifurcat Chaos 1998;8(6):1371–81.
[6] Boccaletti S, Kurths J, Osipov G, Valladares D-L, Zhou C-S. The synchronization of chaotic systems. Phys Rep 2002;366:1–101.
[7] Tsay S, Huang C, Qiu D, Chen W. Implementation of bidirectional chaotic communication systems based on Lorenz circuits. Chaos, Solitons & Fractals 2004;20(3):567–79.
[8] Suykens J-A-K, Vandewalle J. Master–slave synchronization of Lur'e systems. Int J Bifurcat Chaos 1997;7(3):665–9.
[9] Jiang G-P, Tang W. A global synchronization criterion for coupled chaotic systems via unidirectional linear error feedback approach. Int J Bifurcat Chaos 2002;12(10):2239–53.
[10] Jiang G, Tang W, Chen G. A simple global synchronization criterion for coupled chaotic systems. Chaos, Solitons & Fractals 2003;15(5):925–35.
[11] Luigi Fortuna, Domenico Porto. Quantum-CNN to generate nanoscale chaotic oscillator. Int J Bifurcat Chaos 2004;14(3):1085–9.
[12] Ge Z-M, Yang C-H. Generalized synchronization of quantum-CNN chaotic oscillator with different order system. Chaos, Solitons & Fractals, in press, doi:10.1016/j.chaos.2005.04.108.
[13] Lu J-H, Zhou T-S, Zhang S-C. Chaos synchronization between linearly coupled chaotic systems. Chaos, Solitons & Fractals 2002;14:529–41.
[14] Liao X-X, Chen G. Chaos synchronization of general Lur'e systems via time-delay feedback control. Int J Bifurcat Chaos 2003;13(1):207–13.
[15] Lorenz E-N. Deterministic non-periodic flows. J Atmos Sci 1963;20:130–41.
[16] Chen S, Lu J. Synchronization of uncertain unified chaotic system via adaptive control. Chaos, Solitons & Fractals 2002;14(4):643–7.
[17] Chen S, Zhang Q, Xie J, Wang C. A stable-manifold-based method for chaos control and synchronization. Chaos, Solitons & Fractals 2004;20(5):947–54.
[18] Liao T. Adaptive synchronization of two Lorenz systems. Chaos, Solitons & Fractals 1998;9(9):1555–61.
[19] Bai E, Lonngren K. Synchronization of two Lorenz systems using active control. Chaos, Solitons & Fractals 1997;8(1):51–8.
[20] Bai E, Lonngren K. Sequential synchronization of two Lorenz systems using active control. Chaos, Solitons & Fractals 2000;11(7):1041–4.
[21] Ge Zheng-Ming, Cheng Jui-Wen, Chen Yen-Sheng. Chaos anticontrol and synchronization of three time scales brushless DC motor system. Chaos, Solitons & Fractals 2004;22:1165–82.
[22] Ge Zheng-Ming, Lee Ching-I. Control, anticontrol and synchronization of chaos for an autonomous rotational machine system with time-delay. Chaos, Solitons & Fractals 2004;23:1855–64.

[23] Ge Z-M, Chang C-M. Chaos synchronization and parameters identification of single time scale brushless DC motors. Chaos, Solitons & Fractals 2004;20:883–903.
[24] Ge Zheng-Ming, Chen Chien-Cheng. Phase synchronization of coupled chaotic multiple time scales systems. Chaos, Solitons & Fractals 2004;20:639–47.
[25] Ge Zheng-Ming, Leu Wei-Ying. Chaos synchronization and parameter identification for identical system. Chaos, Solitons & Fractals 2004;21:1231–47.
[26] Ge Zheng-Ming, Leu Wei-Ying. Anti-control of chaos of two-degrees-of-freedom louderspeaker system and chaos synchronization of different order systems. Chaos, Solitons & Fractals 2004;20:503–21.
[27] Ge Zheng-Ming, Chen Yen-Sheng. Synchronization of unidirectional coupled chaotic systems via partial stability. Chaos, Solitons & Fractals 2004;21:101–11.
[28] Jiang G-P, Tang K-S, Chen G-R. A simple global synchronization criterion for coupled chaotic systems. Chaos, Solitons & Fractals 2003;15:925–35.
[29] Liu F, Ren Y, Shan X, Qiu Z. A linear feedback synchronization theorem for a class of chaotic systems. Chaos, Solitons & Fractals 2002;13(4):723–30.
[30] Lü J, Zhou T, Zhang S. Chaos synchronization between linearly coupled chaotic systems. Chaos, Solitons & Fractals 2002;14(4):529–41.
[31] Krawiecki A, Sukiennicki A. Generalizations of the concept of marginal synchronization of chaos. Chaos, Solitons & Fractals 2000;11(9):1445–58.
[32] Yang X, Chen G. Some observer-based criteria for discrete-time generalized chaos synchronization. Chaos, Solitons & Fractals 2002;13(6):1303–8.
[33] Jiang G, Zheng W, Chen G. Global chaos synchronization with channel time-delay. Chaos, Solitons & Fractals 2004;20(2):267–75.
[34] Kyprianidis I, Stouboulos I. Chaotic synchronization of three coupled oscillators with ring connection. Chaos, Solitons & Fractals 2003;17:327–36.
[35] Zhou T, Lu J, Chen G, Tang Y. Synchronization stability of three chaotic systems with linear coupling. Phys Lett A 2002;301:231–40.
[36] Yu Yoongguang, Zhang Suochun. Global synchronization of three coupled chaotic system with ring connection. Chaos, Solitons & Fractals 2005;24:1233–42.
[37] Khalil HK. Nonlinear system. third ed. New Jersey: Prentice Hall; 2002.
[38] Smirnov VI. A course of higher mathematics, vol. 1. Oxford: Pergamon Press; 1964. p. 331.