On the Existence of Rainbows in 1-Factorizations of K_{2n}

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Abstract: A 1-factor of a graph $G = (V, E)$ is a collection of disjoint edges which contain all the vertices of V. Given a $2n - 1$ edge coloring of $K_{2n}, n \geq 3$, we prove there exists a 1-factor of K_{2n} whose edges have distinct colors. Such a 1-factor is called a ''Rainbow.'' © 1998 John Wiley & Sons, Inc. J Combin Designs **6:** 1–20, 1998

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1. INTRODUCTION

A **1-factor** in a graph $G = (V, E)$ is a set of pairwise disjoint edges in E which contain all the vertices in V. A **1-factorization** of G is a partition of the edges in E into 1-factors. These notions can be generalized to hypergraphs: If $V = \{v_1, v_2, \ldots, v_n\}$ is a finite set and $E = \{E_i | i \in I\}$ is a family of nonempty subsets of V such that $\cup_{i \in I} E_i = V$, then the pair (V, E) is a **hypergraph** with vertex set V and edge set E. A **1-factor** of the hypergraph is a collection of pairwise disjoint edges which contain all the vertices of V. A **1-factorization** of the hypergraph is a partition of the edges of E into 1-factors. If $V = \{v_1, v_2, \ldots, v_n\}$ and E is the set of all k-element subsets of V then (V, E) is denoted by K_n^k and is called the

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complete k-uniform hypergraph on n vertices. Zs. Baranyai [1] has proven that there exist 1-factorizations of all $K^k_{kn}, k \geq 2$. The reader will note that if $k = 2$ then K^2_{2n} is simply the complete graph on $2n$ vertices.

In 1977 Alexander Rosa suggested the following interesting problem: Given a 1-factorization, $\mathcal{F}$, of $K^k_{kn}, n \geq 3$, prove there exists a 1-factor in K^k_{kn} whose edges belong to n different 1-factors of $\mathcal{F}$. The first author [2] has investigated this problem and has shown that for any 1-factorization, $\mathcal{F}$, of $K^k_{kn}, k \geq 2$, there exists a 1-factor whose edges belong to at least $n-1$ 1-factors of $\mathcal{F}$. There is a colorful way to think of Rosa's problem. Imagine coloring the edges of K^k_{kn} in such a way that any two edges have the same color if and only if they belong to a common 1-factor of $\mathcal{F}$. Then Rosa's conjecture states that there exists a 1-factor in K^k_{kn} with the property that no two of its edges have the same color—a colorful 1-factor. We will call such a 1-factor a *rainbow.* This is also commonly called an *orthogonal* 1-factor. Formally, an *edge coloring* of a graph $G = (V, E)$, is a function $\phi : E \rightarrow \{1, 2, \cdots\}$ such that adjacent edges have distinct images. A *k-edge coloring* is an edge coloring whose image set is $\{1, 2, \ldots, k\}$. The purpose of this article is to show that Rosa's conjecture is true for certain complete graphs:

Theorem 1.1. *For any $2n-1$ edge coloring of $K_{2n}, n \geq 3$, there exists a 1-factor whose edges have exactly n colors.*

It remains an open problem whether rainbows exist in all 1-factorizations of K^k_{kn} for $k \geq 3$.

2. EXTENDING A PREVIOUS RESULT

We begin the proof of the theorem mentioned above. Let $G = (V, E) = K_{2n}$ be a complete graph on $2n$ vertices, $n \geq 3$, and $\mathcal{F}$ a 1-factorization of G. Assume that ϕ is a $2n-1$ edge coloring of G. It has been shown [2] that there exists a 1-factor, F, in G whose edges are colored with at least $n-1$ colors. A moment's reflection shows that if $G = K_6$, no 1-factor in G can have edges of only two colors. This means that F must have edges of 3 different colors. We have dispatched the case where $G = K_6$ and now consider $G = K_{2n}, n \geq 4$.

For brevity of notation we will denote an edge $\{v_i, v_j\}$ as v_iv_j and define edge sets $F = \{e_1 = v_1v_2, e_2 = v_3v_4, \ldots, e_n = v_{2n-1}v_{2n}\}$ and $F' = F \setminus \{e_1\}$. We can assume

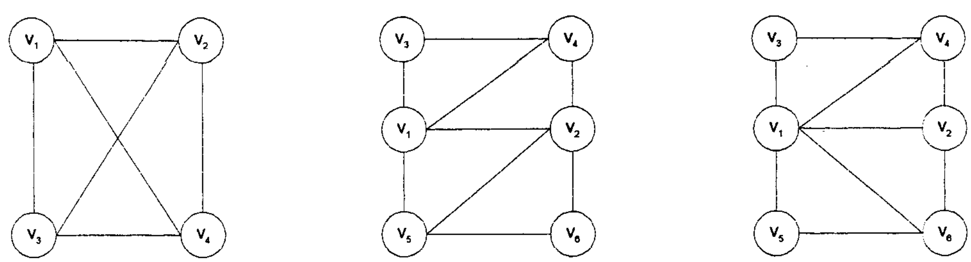


FIG. 1.

that the $n-1$ edges of F' have colors $n+1, n+2, \ldots, 2n-1$, and that edge e_1 has color $n+1$. Let $T = \{e \in E | e$ is incident with v_1 or v_2 and $\phi(e) \leq n\}$. Obviously $|T| = 2n$.

Two cases can occur:

1. one edge in F' is incident with exactly four edges in T, or

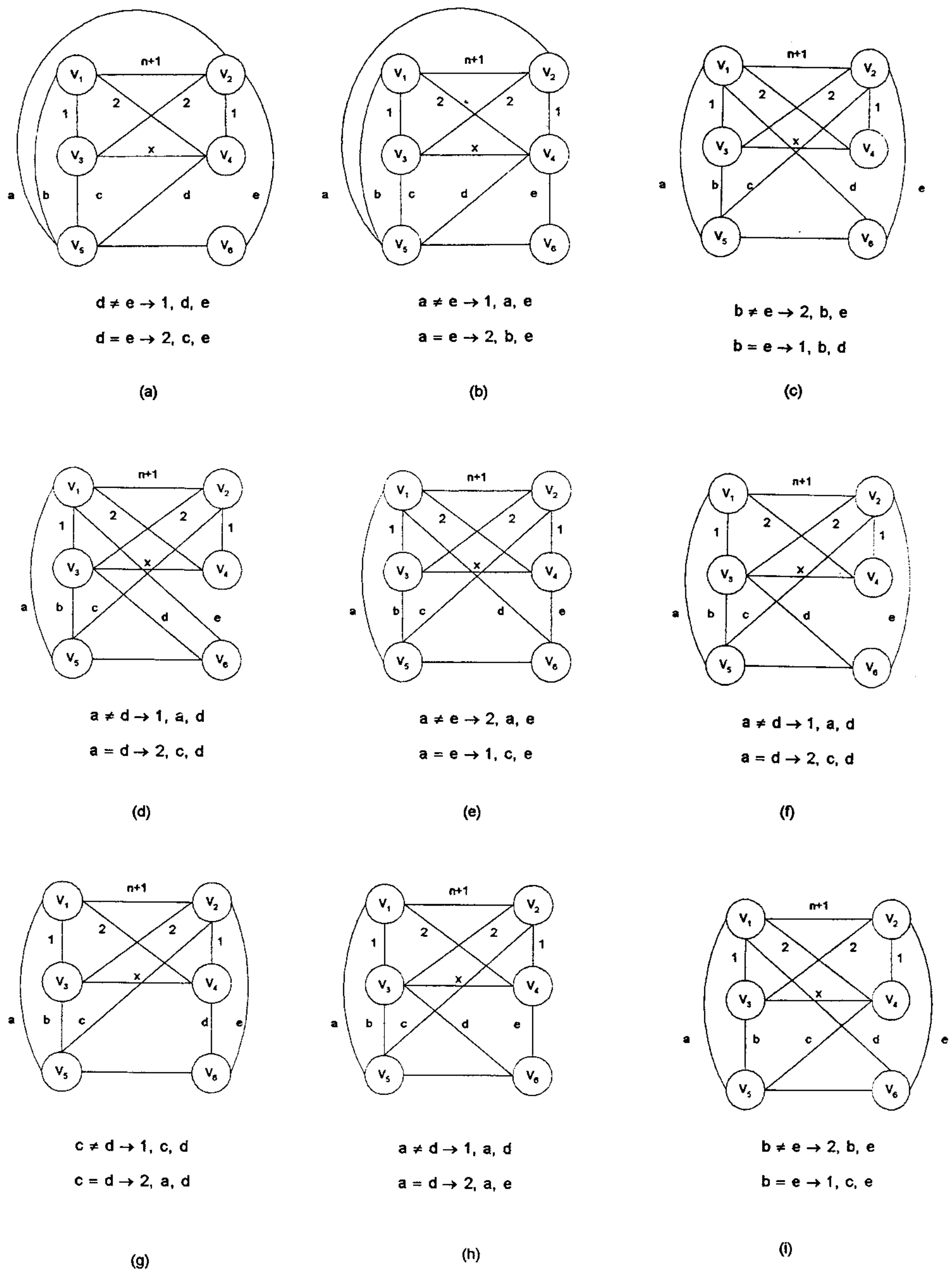


FIG. 2.

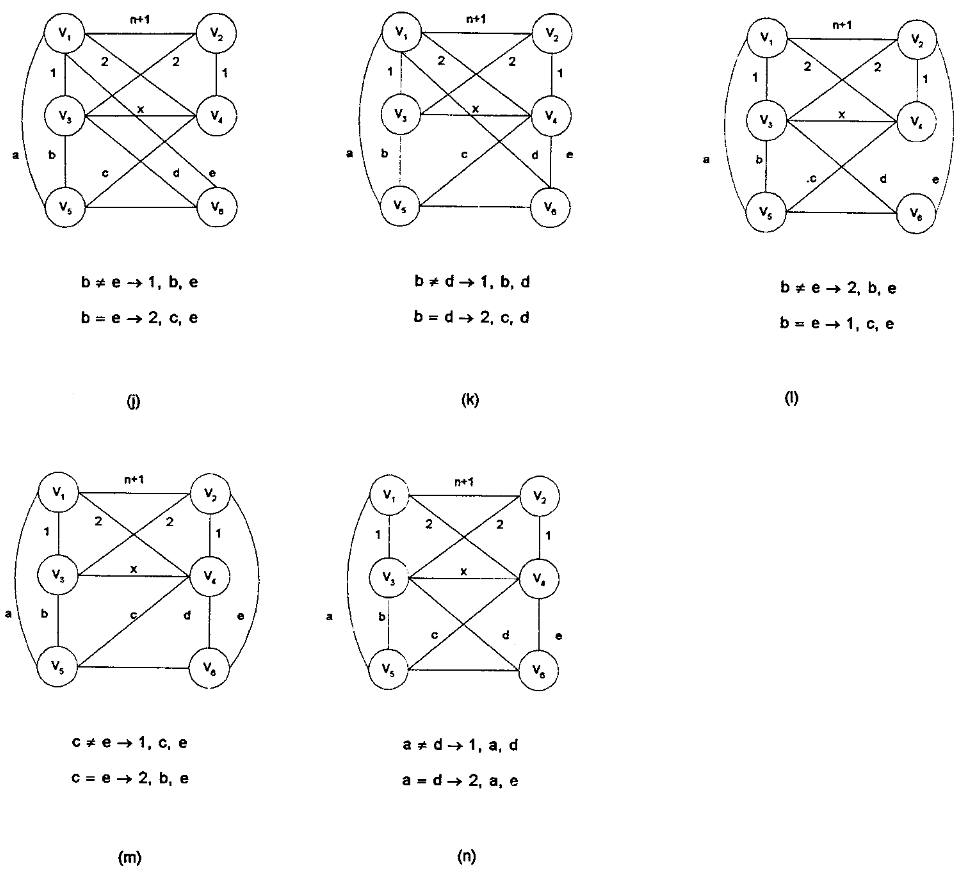


FIG. 2. (continued)

2. each of at least two edges in F' is incident with exactly three edges in T.

Otherwise at most one edge in F' is incident with exactly 3 edges in T, and the other $n-2$ edges in F' are incident with at most two edges in T. This, however, would account for

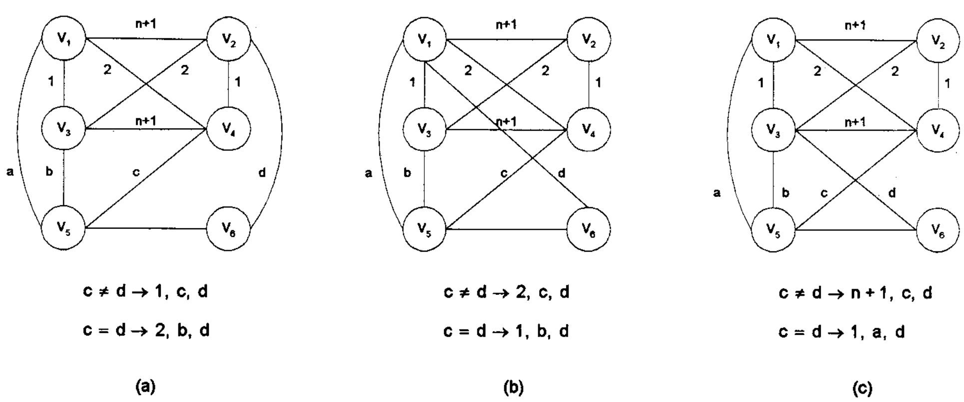
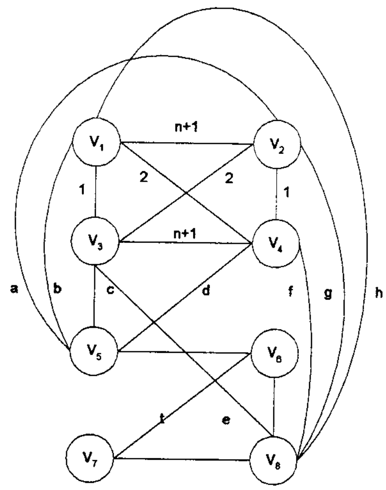


FIG. 3.



$$\begin{pmatrix} t = 1 \\ a \neq e \end{pmatrix} \to t,\ a,\ e,\ 2$$

$$\begin{pmatrix} t = 1 \\ a = e \end{pmatrix} \to t,\ a,\ h,\ n+1$$

$$\begin{pmatrix} t = 2 \\ a \neq f \end{pmatrix} \to t,\ a,\ f,\ 1$$

$$\begin{pmatrix} t = 2 \\ a = f \end{pmatrix} \to t,\ c,\ f,\ n+1$$

$$\begin{pmatrix} a = e = t \\ c \neq f \end{pmatrix} \to t,\ c,\ f,\ n+1$$

$$\begin{pmatrix} a = e = t \\ c = f \end{pmatrix} \to t,\ c,\ g,\ 2$$

$$\begin{pmatrix} a \neq e \\ a \neq t \\ e \neq t \end{pmatrix} \to t,\ a,\ e,\ 2$$

$$\begin{pmatrix} a \neq e \\ a \neq t \\ e = t \\ a = f \end{pmatrix} \to t,\ a,\ h,\ n+1$$

$$\begin{pmatrix} a \neq e \\ a \neq t \\ e = t \\ a \neq f \end{pmatrix} \to t,\ a,\ f,\ 1$$

$$\begin{pmatrix} a \neq e \\ a = t \\ e \neq t \\ b = e \end{pmatrix} \to t,\ e,\ d,\ n+1$$

$$\begin{pmatrix} a \neq e \\ a = t \\ e \neq t \\ b \neq e \end{pmatrix} \to t,\ e,\ b,\ 1$$

$$\begin{pmatrix} a = e \\ a \neq t \\ e \neq t \\ b = t \end{pmatrix} \to t,\ e,\ d,\ n+1$$

$$\begin{pmatrix} a = e \\ a \neq t \\ e \neq t \\ b \neq t \end{pmatrix} \to t,\ e,\ b,\ 1$$

FIG. 4.

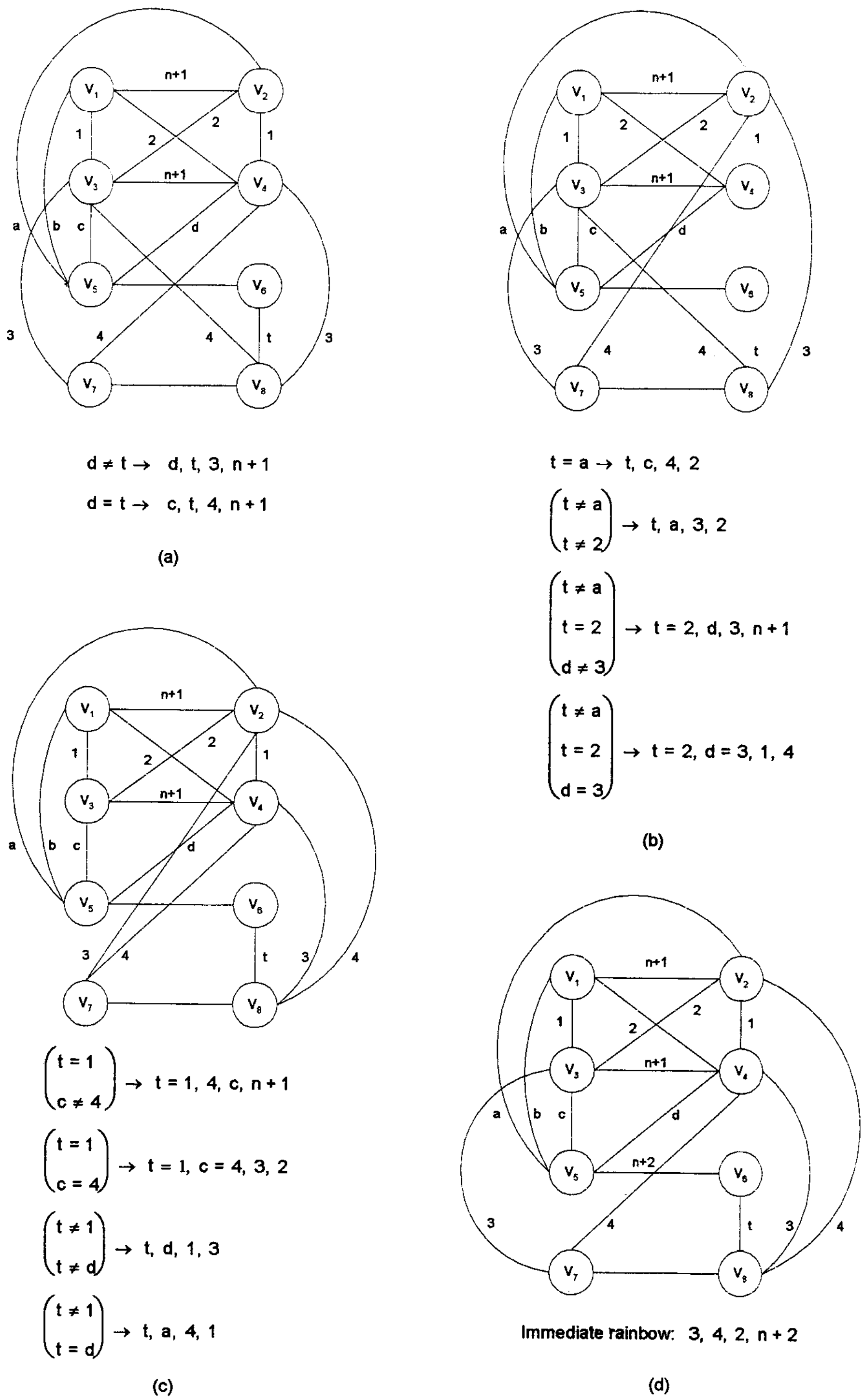
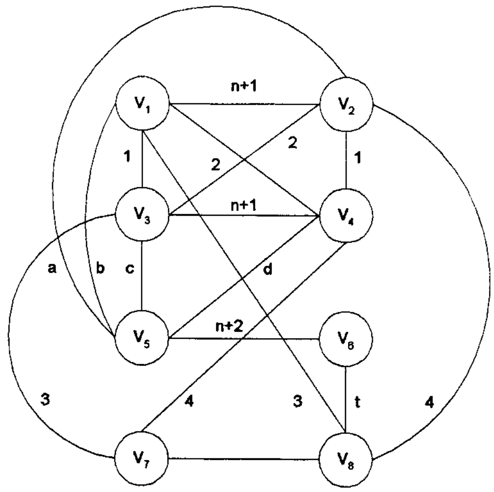

d ≠ t → d, t, 3, n + 1
d = t → c, t, 4, n + 1
(a)
t = a → t, c, 4, 2
(t ≠ a, t ≠ 2) → t, a, 3, 2
(t ≠ a, t = 2, d ≠ 3) → t = 2, d, 3, n + 1
(t ≠ a, t = 2, d = 3) → t = 2, d = 3, 1, 4
(b)
(t = 1, c ≠ 4) → t = 1, 4, c, n + 1
(t = 1, c = 4) → t = 1, c = 4, 3, 2
(t ≠ 1, t ≠ d) → t, d, 1, 3
(t ≠ 1, t = d) → t, a, 4, 1
(c)
Immediate rainbow: 3, 4, 2, n + 2
(d)

FIG. 5.



Immediate rainbow: 3, 4, 2, n + 2

(e)

FIG. 5. (continued)

only $3 + 2(n - 2) = 2n - 1$ of the edges of T, a contradiction. The proof breaks into the two cases stated above:

Case 1. There exists an edge, say $e_2 = v_3v_4$, in F' incident with exactly four edges in T [Fig. 1(a)]. If $\phi(v_1v_3) \neq \phi(v_2v_4)$, then by replacing v_1v_2 and v_3v_4 in F with v_1v_3 and v_2v_4 we have constructed a rainbow in G. A similar argument holds if $\phi(v_1v_4) \neq \phi(v_2v_3)$. Therefore, without loss of generality, we assume $\phi(v_1v_3) = \phi(v_2v_4) = 1$ and $\phi(v_1v_4) = \phi(v_2v_3) = 2$ and consider the two subcases below:

Subcase 1.1. $\phi(v_3v_4) = x > n + 1$.

Let $A = E_{1,2} \cup E_{3,4}$ where $E_{1,2} = \{e \in E | e$ is incident with v_1 or v_2, and $\phi(e) \in \{3, 4, \ldots, n, x\}\}$, and $E_{3,4} = \{e \in E | e$ is incident with v_3 or v_4, and $3 \leq \phi(e) \leq n\}$. Obviously, $|A| = 4(n - 2) + 2$. By the pigeon-hole principle there exists an edge, say v_5v_6, in $F'' = F' \setminus \{v_3v_4\}$ which is incident with at least 5 edges in A. We continue the proof by examining all distinct ways in which 5 edges in A can be incident with the edges in F. In each case we demonstrate how F can be modified (by adding and deleting edges) to form a rainbow in G. Since there will be many cases to consider, we handle each case by drawing a figure of the graph with some accompanying statements that indicate how a rainbow can be constructed. In every figure, each edge is labeled with its color.

For example, omitting symmetric cases, there are two distinct ways [Fig. 2(a) and 2(b)] that 5 edges in A can be incident with edges v_1v_2, v_3v_4, and v_5v_6 where $\deg(v_5) = 4$, and $\deg(v_6) = 1$. Consider Figure 2(a). The statement "$d \neq e \rightarrow 1, d, e$" means if $\phi(v_4v_5) \neq \phi(v_2v_6)$, then we can add to F edges v_1v_3, v_4v_5, and v_2v_6 (which are colored 1, d, and e, respectively) and delete edges v_1v_2, v_3v_4, v_5v_6 from F to obtain a rainbow. On the other hand, "$d = e \rightarrow 2, c, e$" means that if $\phi(v_4v_5) = \phi(v_2v_6)$, then $(F \setminus \{v_1v_2, v_3v_4, v_5v_6\}) \cup \{v_1v_4, v_3v_5, v_2v_6\}$ is a rainbow. The cases where the $\deg(v_5) = 1$

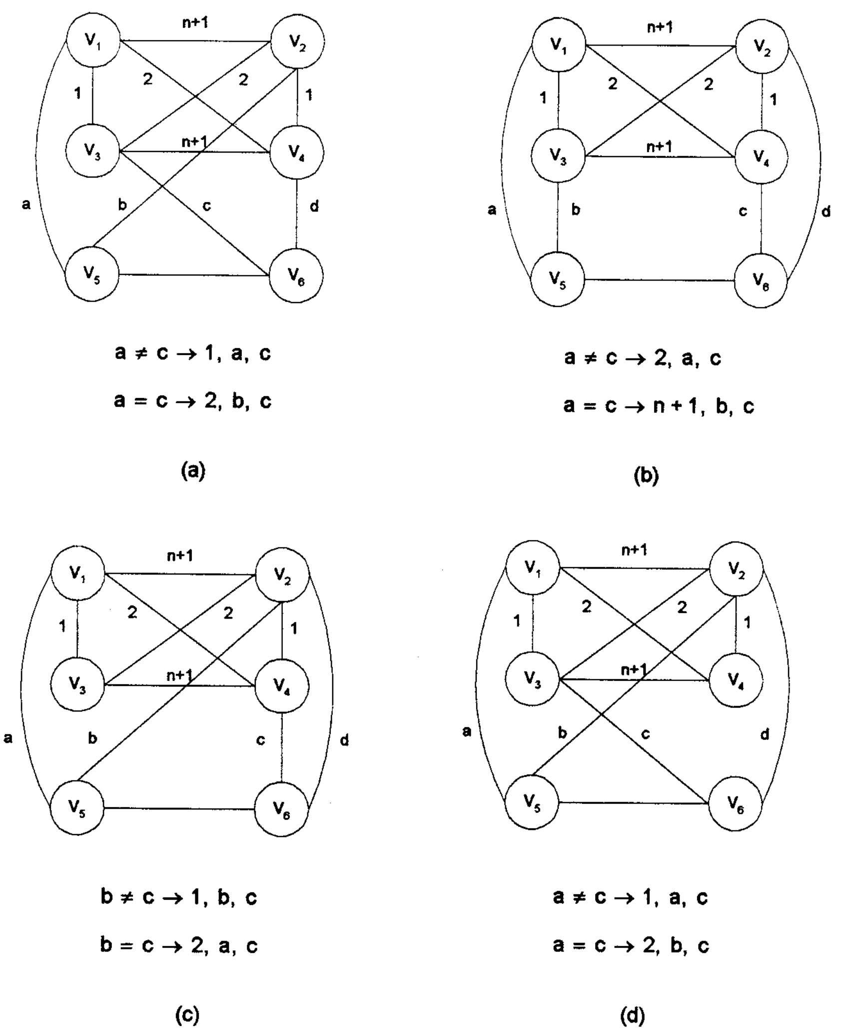


FIG. 6.

and deg$(v_6) = 4$ are analogous to the cases represented by Figures 2(a) and 2(b) and as a result are omitted. Figures 2(c)–2(n) represent distinct ways in which 5 edges in A can be incident with edges v_1v_2, v_3v_4, and v_5v_6 where deg$(v_5) = 3$, and deg$(v_6) = 2$. We consider cases where v_5 is adjacent to v_1, v_2, and v_3 and where v_5 is adjacent to v_1, v_3, and v_4. We omit the symmetric cases where v_5 is adjacent to v_1, v_2, and v_4 and where v_5 is adjacent to v_2, v_3, and v_4. The cases with deg$(v_5) = 2$ and deg$(v_6) = 3$ being analogous, are also omitted.

Subcase 1.2. $\phi(v_3v_4) = n + 1$.

In this case we let $A = \{e \in E | e$ is incident with one of $\{v_1, v_2, v_3, v_4\}$ and $3 \leq \phi(e) \leq n\}$. Clearly, $|A| = 4(n-2)$. By the pigeon-hole principle, either one edge in F'' is incident with at least 5 edges in A, or every edge in F'' is incident with exactly 4 edges in A. In handling subcase 1.1, we never used the edge colored x in constructing any rainbow. As a result, our previous arguments in subcase 1.1 apply here as well when one edge in F'' is incident with 5 edges of A.

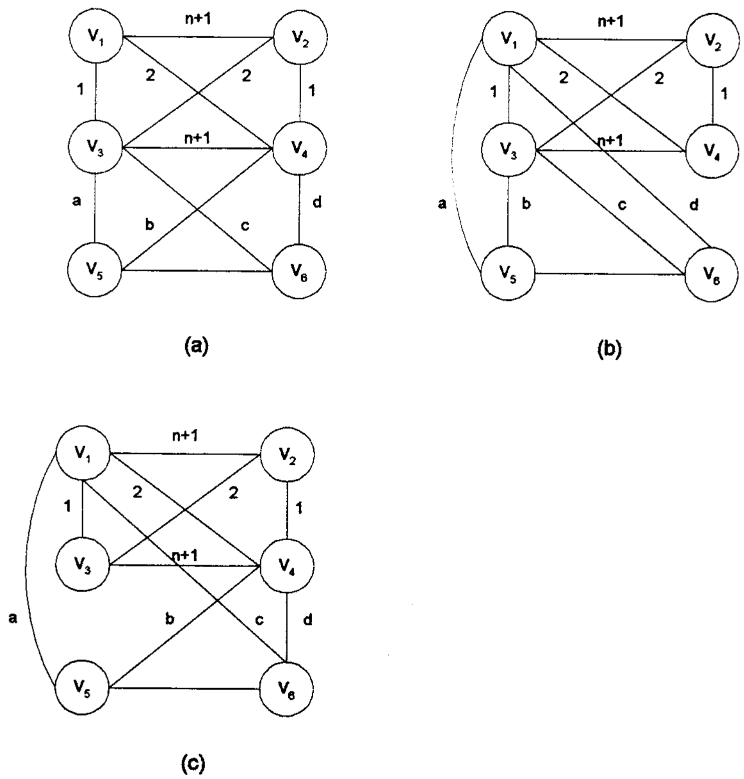


FIG. 7.

We are left to consider the case where every edge in F'' is incident with 4 edges in A. Call an edge in F'' ‘‘type one’’, if one of its vertices is incident with exactly 1 edge in A (and the other vertex with exactly 3). Call the edge ‘‘type two’’ if both vertices are incident with exactly two edges in A, and call it ‘‘type three’’ if one of its vertices is incident with 4 edges in A. Every edge in F'' belongs to one of the three types. Figures 3(a) to 3(c) demonstrate how to obtain a rainbow in the cases where F'' contains a type one edge, and so we proceed with the assumption that F'' contains only edges of types two and three.

Suppose v_5v_6 is a type 3 edge and that v_5 is incident with 4 edges of A. Consider the n edges colored $1, 2, \ldots, n$ which are incident with v_6. None of these edges is incident with v_1v_2 or v_3v_4 since v_5v_6 is a type three edge. By the pigeon-hole principle, there is an edge in F'', say v_7v_8, which is incident with two of the n edges described above. Figure 4 addresses the case in which that edge is type three and Figures 5(a) to 5(e) handle the cases in which that edge is type two. Notice that the edges of A which contain vertices v_7 or v_8 are colored 3 or 4. We can do this without loss of generality since using three or more colors results in an immediate rainbow by appropriately swapping edges. Note that in Figures 5(a) to 5(c), the edges of A which contain vertices v_7 or v_8 are adjacent to exactly two other vertices. In Figure 5(d) these edges contain exactly 3 other vertices, and in Figure 5(e) these edges contain 4 other vertices. Because of the symmetry of the subgraph induced by vertices in $\{v_1, v_2, v_3, v_4\}$ and its edge coloring, many subgraphs similar to 5(d) and 5(e) are omitted.

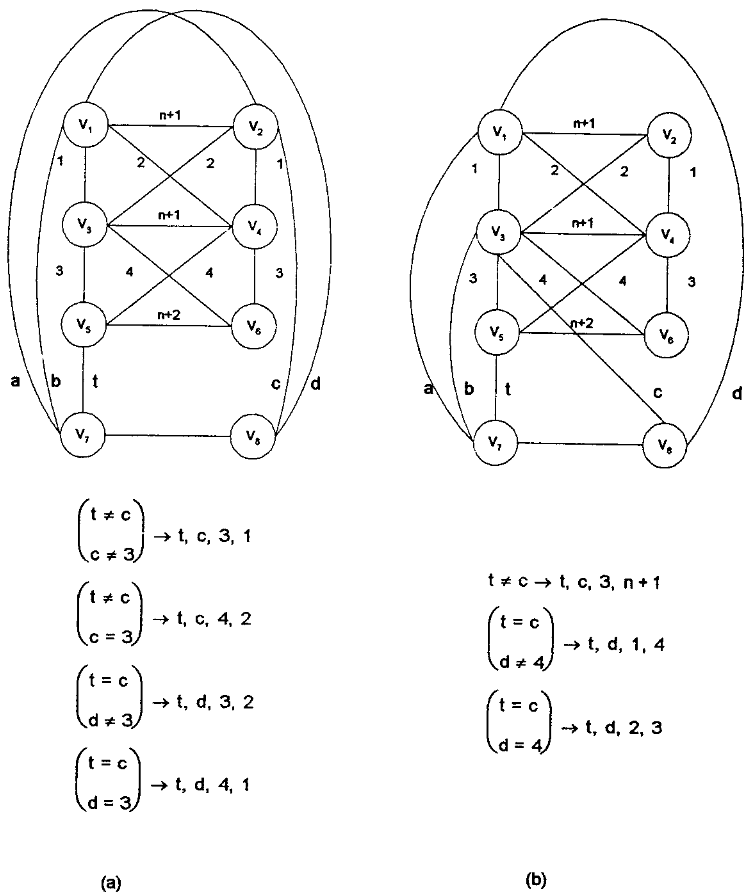


FIG. 8.

As a result of the remarks above, we assume that all edges in F'' are of type two. Let v_5v_6 be a type two edge. Figures 6(a) to 6(d) indicate how to obtain a rainbow in the cases where there are at least 3 vertices in $\{v_1, v_2, v_3, v_4\}$ which are incident with edges of A containing v_5 or v_6. Figures 7(a) to 7(c) illustrate the cases in which exactly two vertices in $\{v_1, v_2, v_3, v_4\}$ are incident with 4 edges of A containing v_5 or v_6. In each of these graphs we assume $a = d$ and $b = c$, otherwise a rainbow is easily found. The reader should note that these subgraphs are isomorphic if you are allowed to relabel the colors. We must show that a rainbow can be constructed in each of these cases. Let $B_{i,j} = \{v_xv_y \in F'' | v_iv_x, v_iv_y, v_jv_x, v_jv_y \in A\}$. Without loss of generality, we assume that $v_5v_6 \in B_{3,4}$. Obviously $|B_{3,4}| \leq \frac{(n-2)}{2}$ since there are $n-2$ edges in A which are incident with v_3 and each edge in $B_{3,4}$ is incident with two of these edges in A.

Suppose $|B_{3,4}| = \frac{(n-2)}{2}$, then it is easy to see that $|B_{1,2}|$ is also $\frac{(n-2)}{2}$. Recall that each edge in F'' is of type 2 and as a result is incident with exactly 4 edges in A. If $|B_{3,4}| = \frac{(n-2)}{2}$, then the edges of $B_{3,4}$ are incident with exactly $2(n-2)$ edges in A. This means there are $2(n-2)$ edges of A not incident with edges in $B_{3,4}$. These edges are all incident with v_1 or v_2. Also there are $n-2$ edges in $F'' \setminus B_{3,4}$. Each of these edges is incident with exactly 4 edges in A. We must have $B_{1,2} = F'' \setminus B_{3,4}$ and $|B_{1,2}| = \frac{(n-2)}{2}$. Let v_5v_6 be an edge

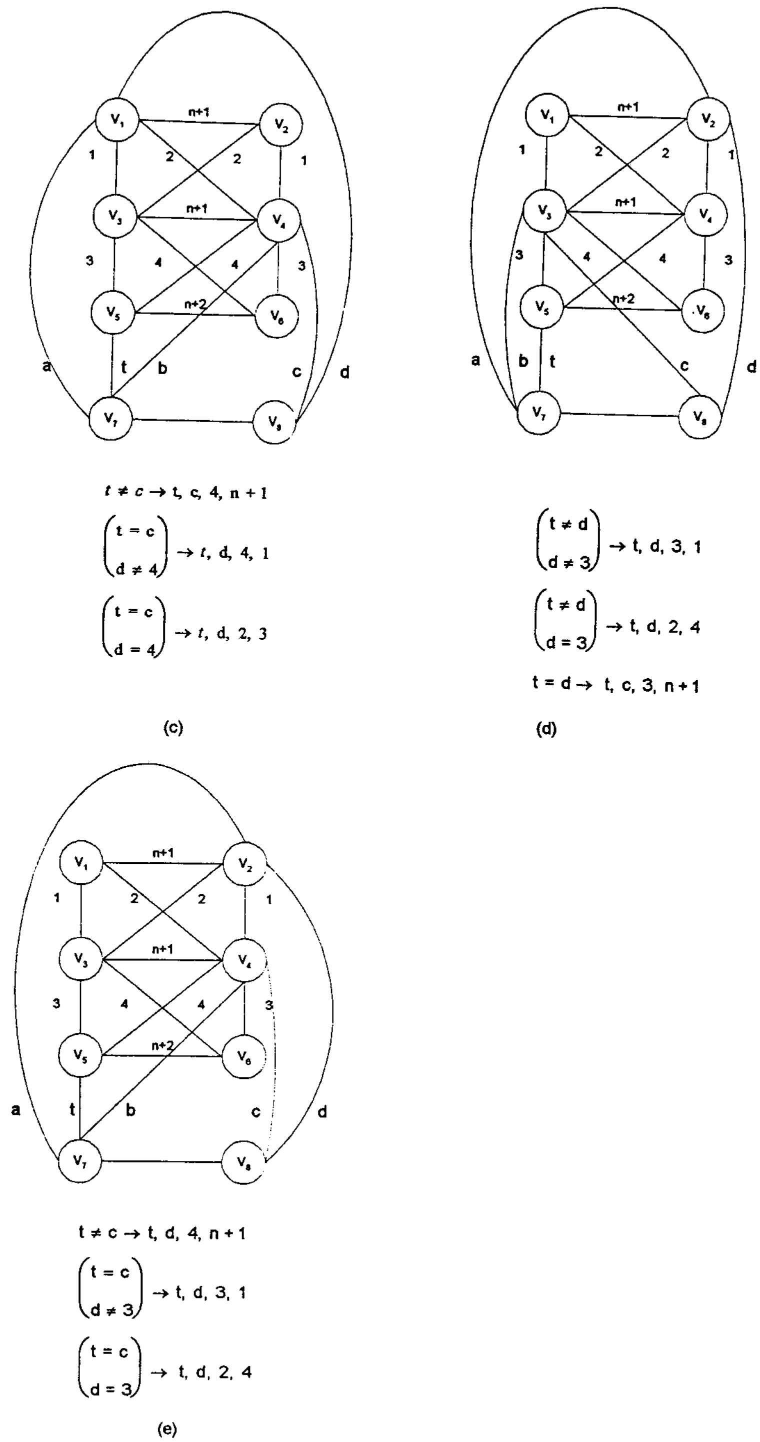


FIG. 8. (continued)

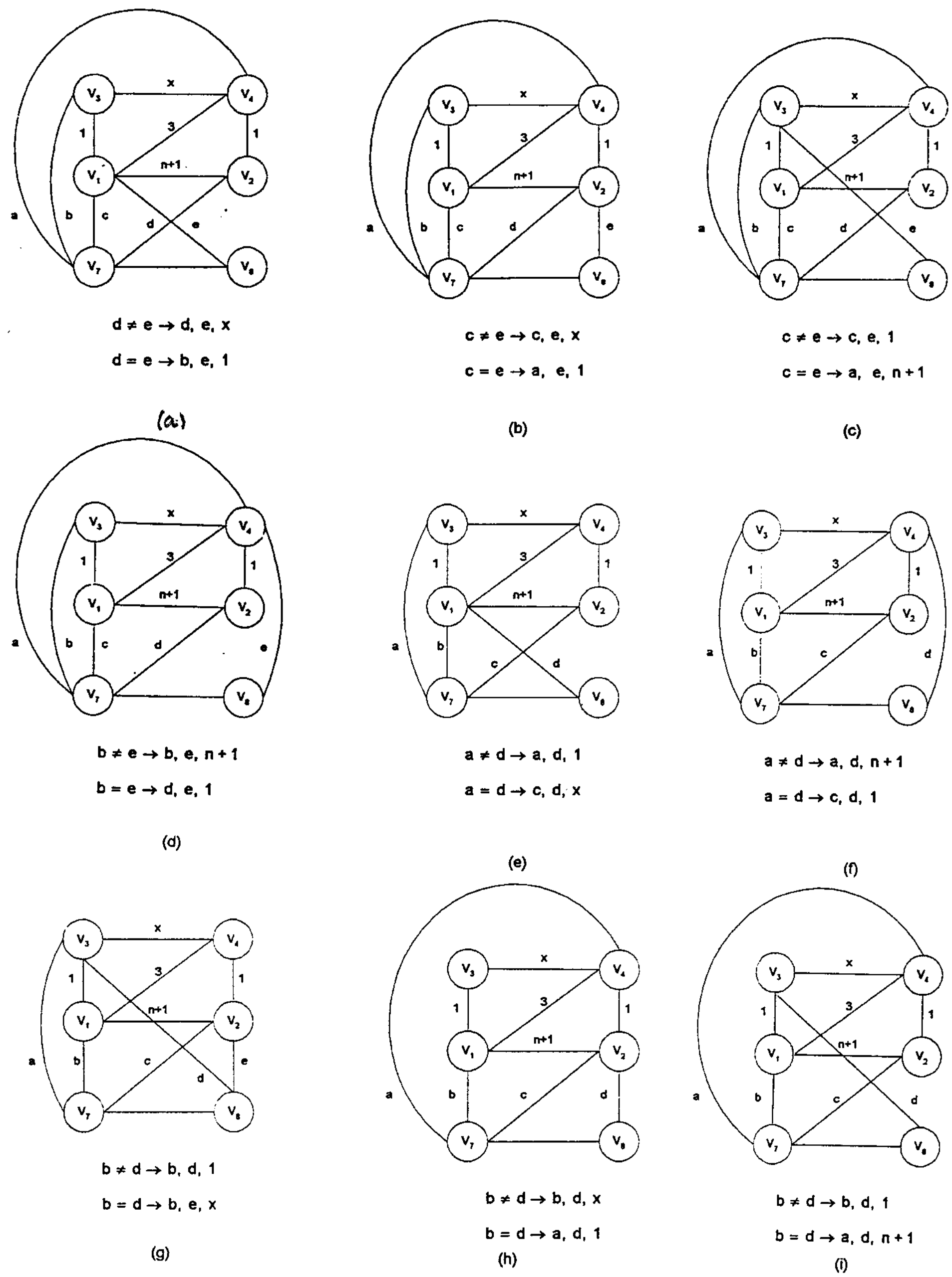


FIG. 9.

in $B_{3,4}$. Without loss of generality we assume $v_3v_5 = v_4v_6 = 3$ and $v_3v_6 = v_4v_5 = 4$, otherwise a rainbow is achieved by swapping edges. For instance, if $v_3v_5 = 3$ and $v_4v_6 = 4$, then $(F \setminus \{v_3v_4, v_5v_6\}) \cup \{v_3v_5, v_4v_6\}$ is a rainbow. Consider the $n-3$ edges colored $5, 6, \ldots, n, n+1$ which are incident with v_5. At most $2(\frac{(n-2)}{2} - 1) = n-4$ of these edges are incident with edges in $B_{3,4}$, other than v_5v_6, and so at least one edge is incident with

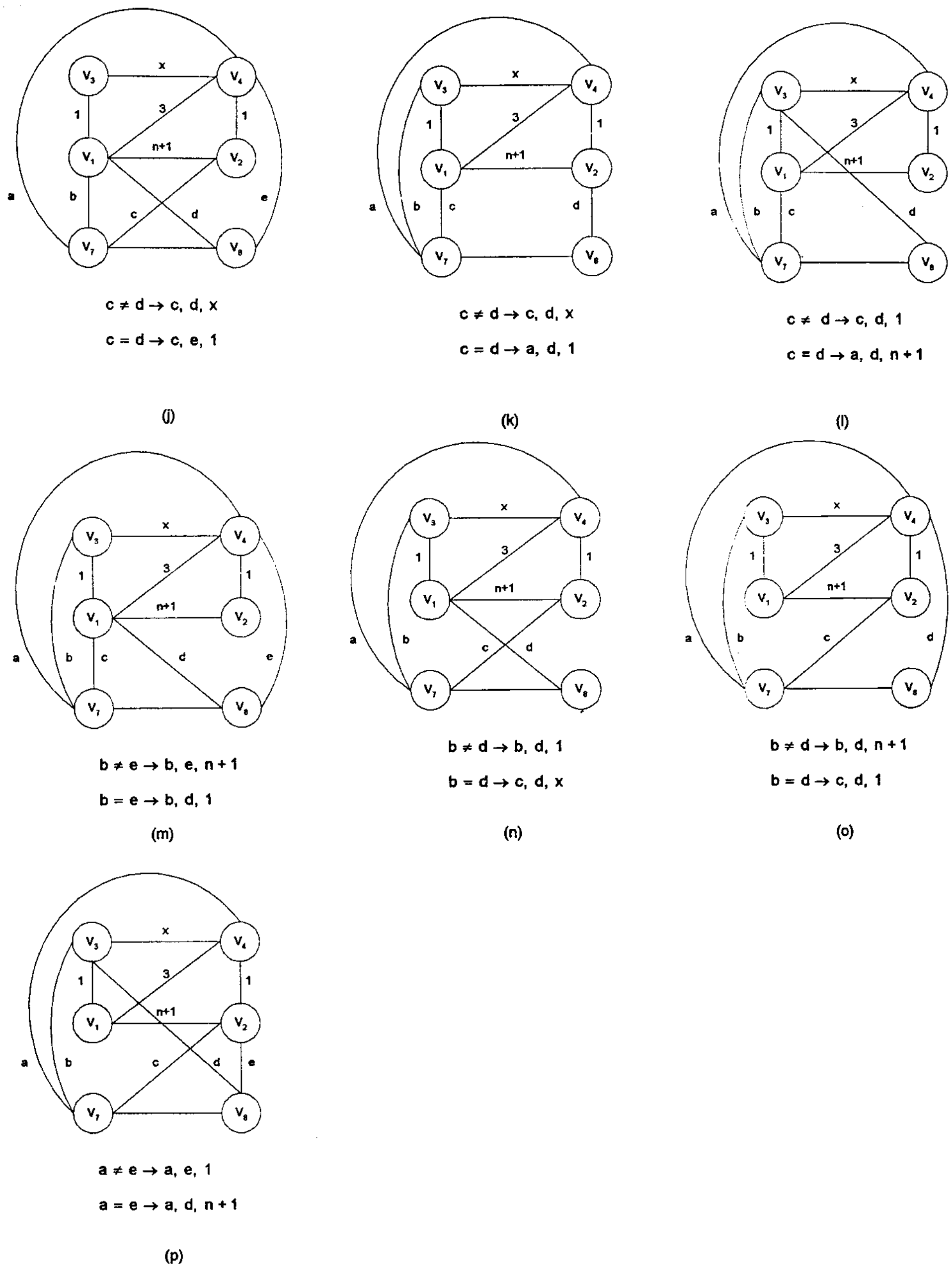


FIG. 9. (continued)

an edge in $B_{1,2}$. Call that edge v_7v_8. Figure 8(a) demonstrates how to construct a rainbow in this case.

On the other hand, suppose we assume that $|B_{3,4}| < \frac{(n-2)}{2}$. In this case we consider the $n-4$ edges incident with v_5 which are colored $5, 6, \ldots, n$. Now $|B_{3,4}| \leq \frac{(n-2)}{2} - 1$ and at most $2(\frac{(n-2)}{2} - 2) = n - 6$ of these $n - 4$ edges are incident with edges in $B_{3,4}$ other than

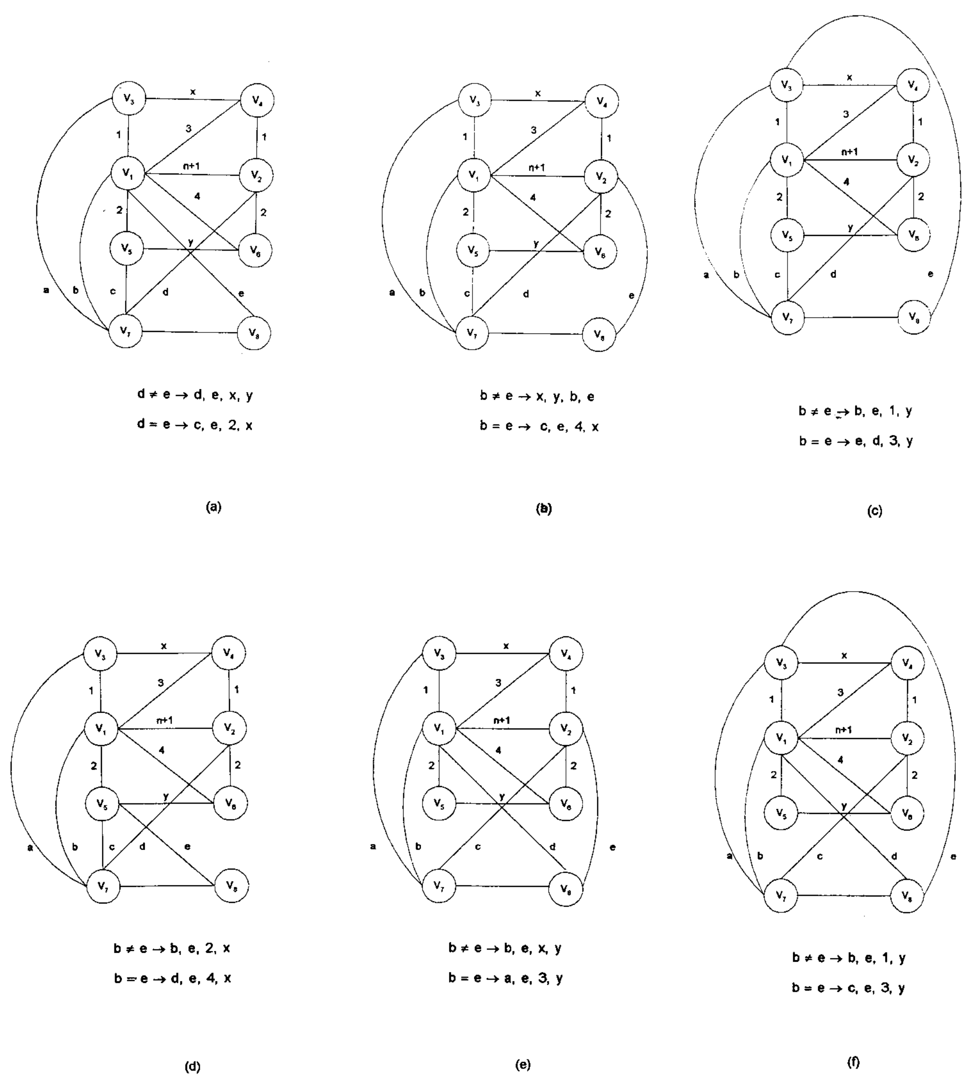


FIG. 10.

v_5v_6. Thus, at least one of the $n-4$ edges, say t, is incident with an edge in $F'' \setminus B_{3,4}$. This means t is incident with an element of either $B_{1,2}, B_{1,3}, B_{1,4}, B_{2,3}$, or $B_{2,4}$. Each of these situations are handled in Figures 8(a) to 8(d).

Case 2. There exist at least two edges, say v_3v_4 and v_5v_6 in F' each of which are incident with exactly 3 edges in T. We examine two subcases [Figure 1(b)–(c)].

Subcase 2.1. Consider Figure 1(b). Without loss of generality we may assume $\phi(v_1v_3) = \phi(v_2v_4) = 1, \phi(v_1v_5) = \phi(v_2v_6) = 2, \phi(v_1v_4) = 3$, and $\phi(v_2v_5) = 3$ or 4. We also let $\phi(v_3v_4) = x$ and $\phi(v_5v_6) = y$ and note that $x \neq y$ and at least one of x and y is not equal to $n+1$. In what follows we assume that $y \neq n+1$ and the strategy will be to find a 1-factor that includes y (we omit this edge in the accompanying diagrams).

If $\phi(v_2v_5) = 3$ let $E_1 = \{e = v_1v_z | \phi(e) = 4, 5, \ldots, n\}, E_2 = \{e = v_2v_z | \phi(e) = 4, 5, \ldots, n\}, E_3 = \{e = v_3v_z | \phi(e) = 2, 4, 5, \ldots, n\}$, and $E_4 = \{e = v_4v_z | \phi(e) =$

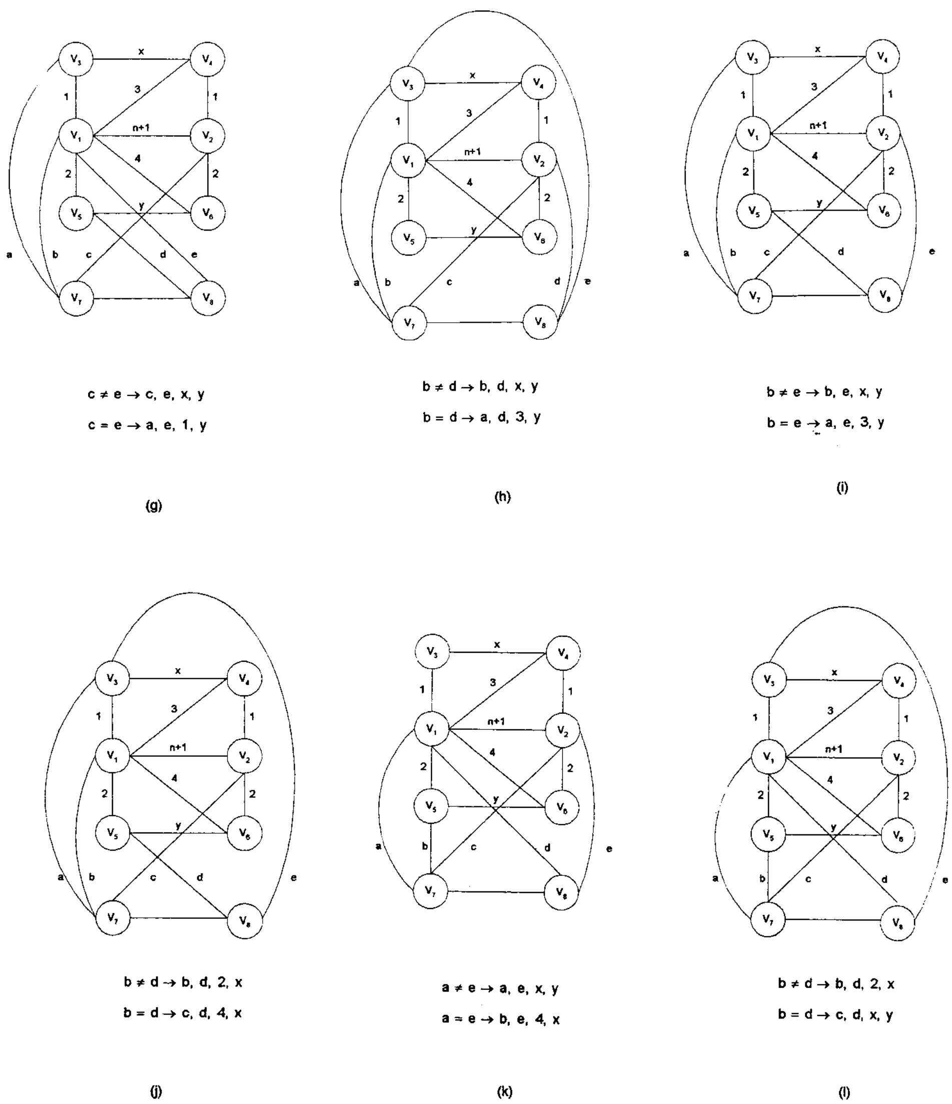


FIG. 10. (continued)

$4, 5, \ldots, n\}$. Let $A = E_1 \cup E_2 \cup E_3 \cup E_4$, then no edge in A has two vertices in $\{v_1, v_2, v_3, v_4\}$ otherwise we can easily find a rainbow. Since the edges described above are distinct, we have $|A| = 3(n-3) + (n-2) = 4(n-3) + 1$ and by the pigeon hole principle there exists an edge v_7v_8 in $F'' \setminus \{v_5v_6\}$ which is incident with 5 edges in A. Without loss of generality we may assume that v_7 is incident with 3 or 4 of the edges of A. Figures 9(a) to (d) handle the cases in which v_7 is incident 4 edges of A. Figures 9(e) to (p) handle the cases in which v_7 is incident 3 edges of A and v_8 is incident 2 edges of A. In Figures 9(e, f, h, i, k, l, n, and o) one of the edges incident with v_8 is omitted. Since the missing edge could be incident with 3 other vertices, these figures represent 3 cases.

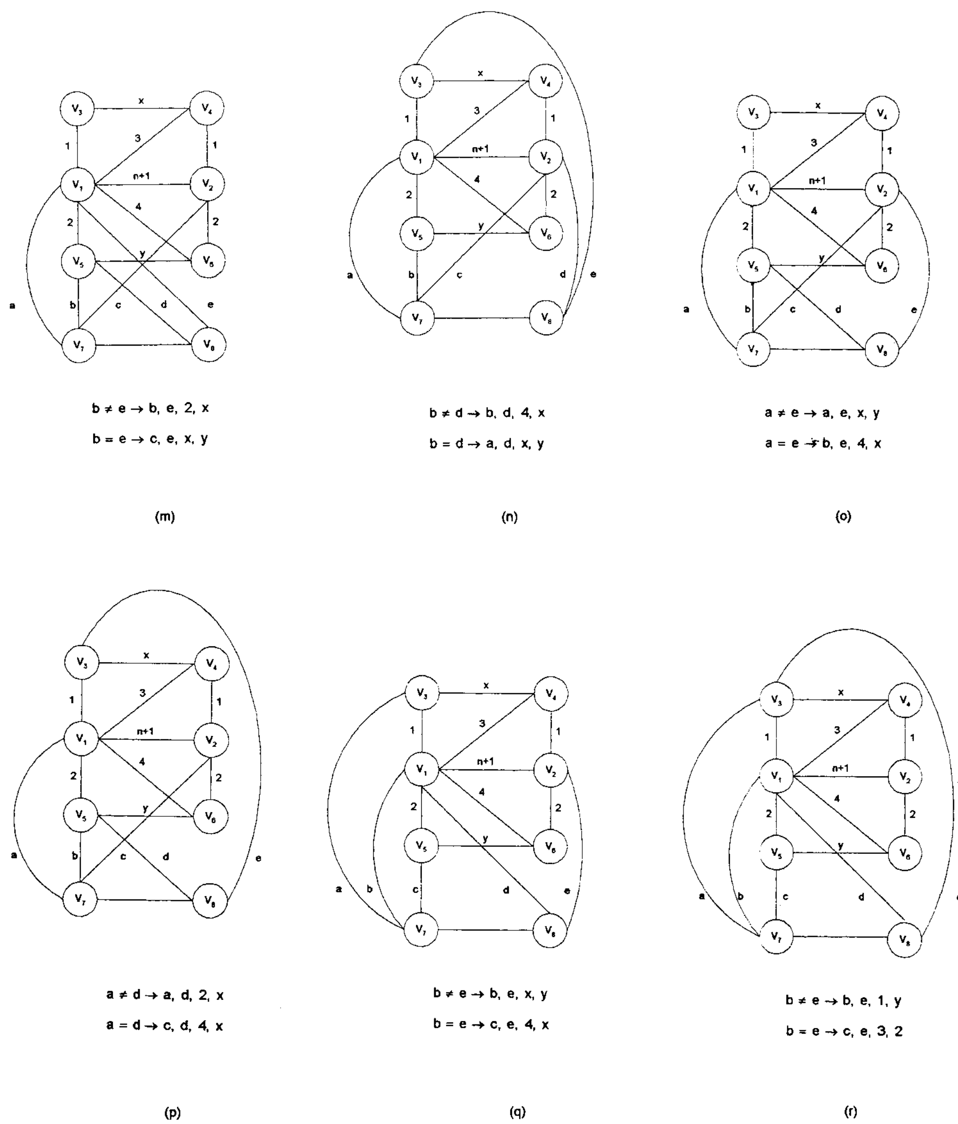


FIG. 10. (continued)

If $\phi(v_2v_5) = 4$ we let $E_1 = \{e = v_1v_z|\phi(e) = 4, 5, \dots, n\}, E_2 = \{e = v_2v_z|\phi(e) = 3, 5, \dots, n\}, E_3 = \{e = v_3v_z|\phi(e) = 2, 4, 5, \dots, n\}$, and $E_4 = \{e = v_4v_z|\phi(e) = 2, 5, \dots, n\}$. Note that none of the 1-factors produced in Figure 9 use an edge with color 3. As a result, an analogous argument to that of the previous paragraph together with the diagrams in Figure 9 apply in this case as well.

Subcase 2.2. Consider Figure 1(c). Without loss of generality we may assume $\phi(v_1v_4) = 3, \phi(v_1v_6) = 4, \phi(v_1v_3) = \phi(v_2v_4) = 1$, and $\phi(v_1v_5) = \phi(v_2v_6) = 2$. Again we let $\phi(v_3v_4) = x$ and $\phi(v_5v_6) = y \neq n+1$. Let $E_1 = \{e = v_1v_z|z \geq 7$ and $\phi(e) = 5, 6, \dots, n\}, E_2 = \{e = v_2v_z|z \geq 7$ and $\phi(e) = 5, 6, \dots, n\}, E_3 = \{e = v_3v_z|z \geq 7$ and $\phi(e) = 4, 5, \dots, n\}$, and $E_5 = \{e = v_4v_z|z \geq 7$ and $\phi(e) = 1, 5, \dots, n\}$. Note that

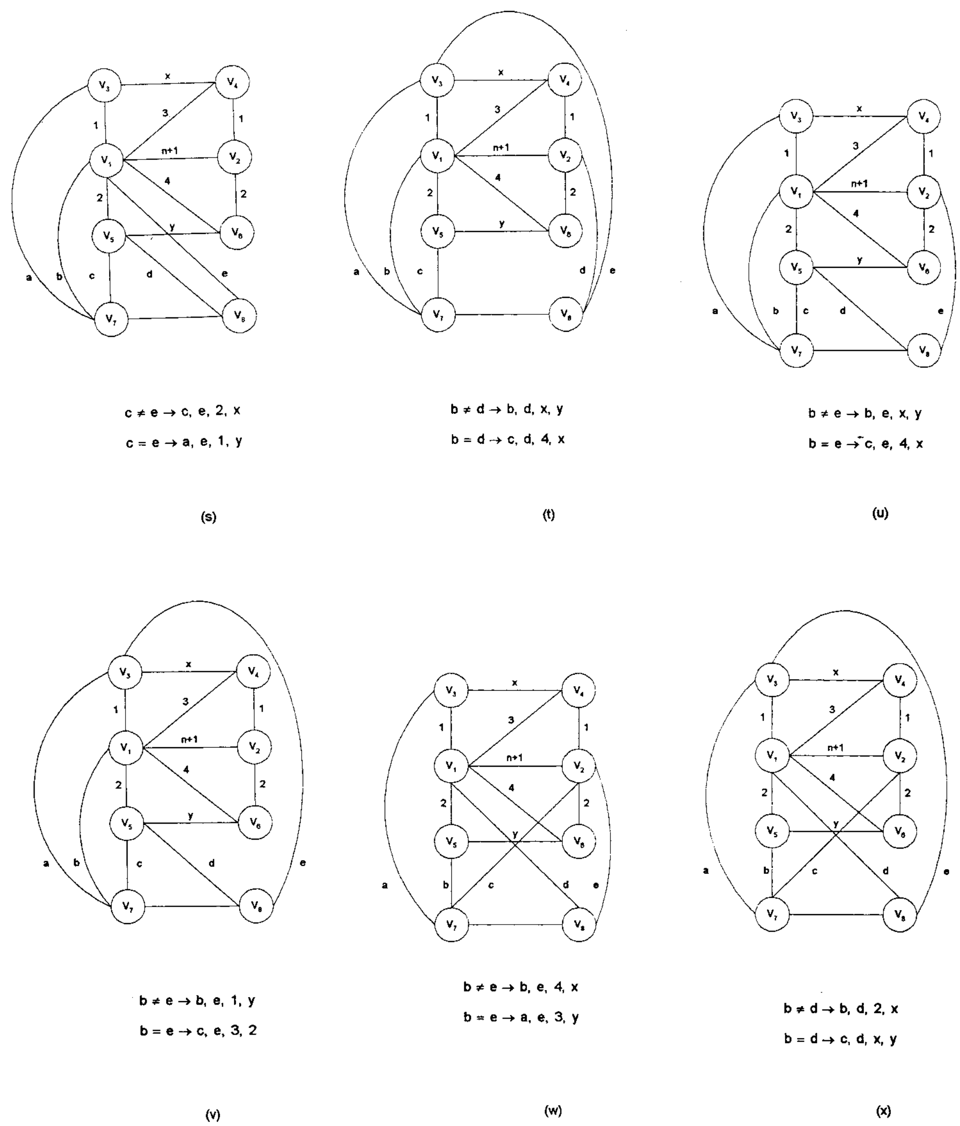


FIG. 10. (continued)

if two vertices in $\{v_1, v_2, v_3, v_5\}$ are joined by an edge colored $1, 4, 5, 6, \ldots, n$, then we can immediately find a rainbow. As a result, If $A = E_1 \cup E_2 \cup E_3 \cup E_5$, then we assume $|A| = 4n - 14$. Let $F''' = F'' \setminus \{v_5v_6\}$. If there is an edge, say v_7v_8, in F''' which is incident with 5 edges in A, then we are able to construct a rainbow using the graphs in Figure 10. Therefore, we assume that each edge in F''' is incident with at most 4 edges in A.

Let v_7v_8 be an edge which is incident with exactly 4 edges in A and let S be the set of 4 edges. If the edges of S are incident with exactly two vertices in $\{v_1, v_2, v_3, v_5\}$ then by exchanging edges we obtain a 1-factor of a type which was handled in Case 1. For example, in Figure 11, the edges of S are incident with only v_1 and v_5. If we remove edges v_1v_2 and

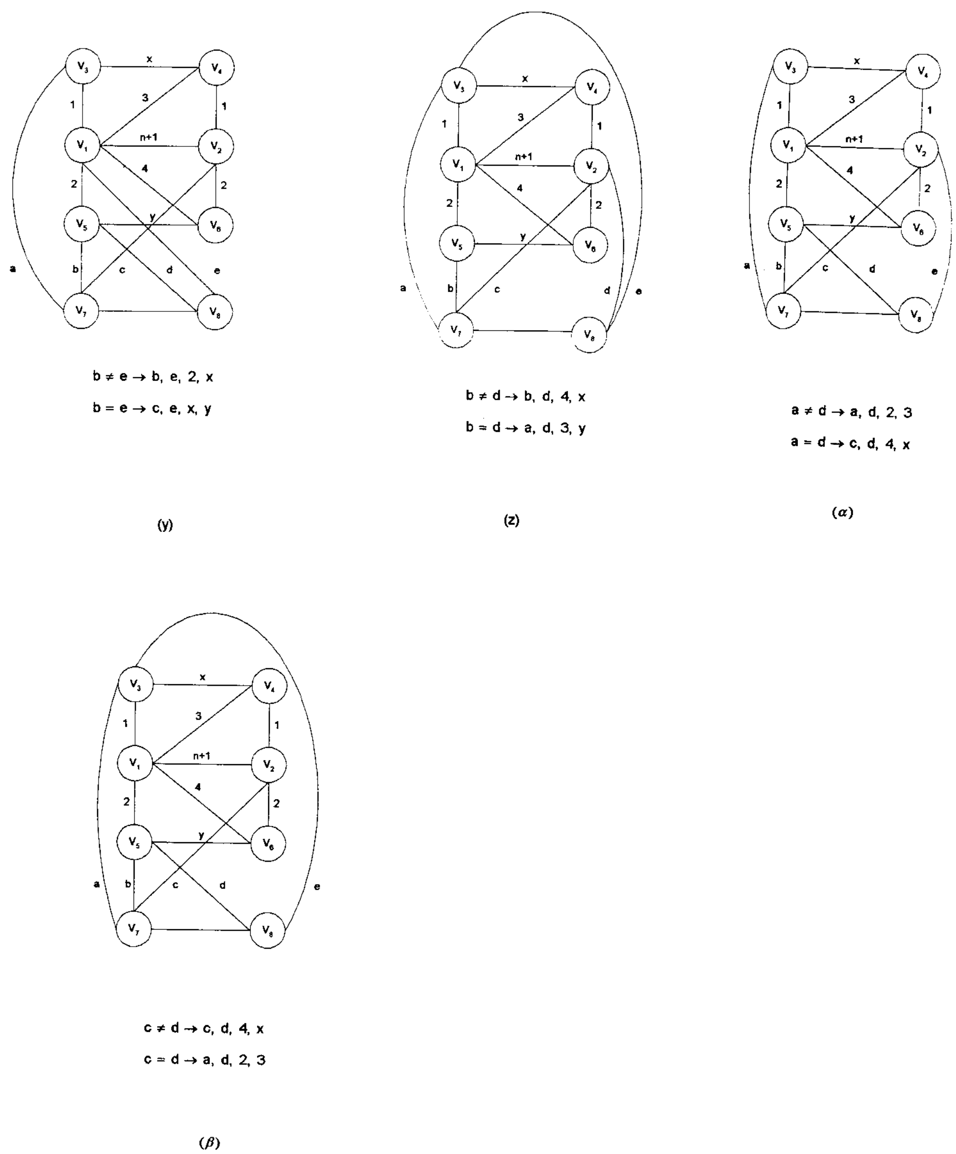


FIG. 10. (continued)

v_5v_6 and add edges v_1v_5 and v_2v_6 we obtain a 1-factor in which an edge with a repeated color (edge v_1v_5) satisfies the conditions in Case 1.

On the other hand, if the edges of S are incident with 3 or 4 of the vertices in $\{v_1, v_2, v_3, v_5\}$, it is easy to see that either v_7 or v_8 is incident with all 4 edges in S, or two of the edges in S are disjoint and have different colors. In the latter case, rainbows are easily found. For example, in Figure 12, we can construct a rainbow by using edges colored $a, b, 2$, and 3. The reader is left to check the 5 other possibilities.

We next consider the former case:

1. Each edge in F''' is incident with at most 4 edges in A, and

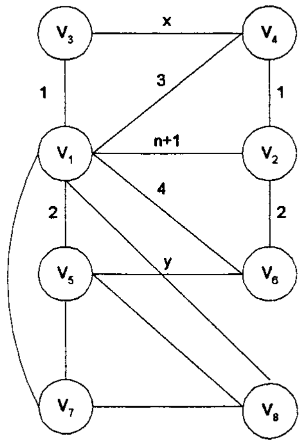


FIG. 11.

2. If an edge in F''', say v_7v_8, is incident with 4 edges in A, then all 4 edges are incident with v_7 or all are incident with v_8.

Most of the edges in F''' are incident with 4 edges in A. To see this, call an edge in F''' ''type one'' if it is incident with at most 3 edges in A, otherwise call the edge ''type two.'' Note that if there are at least 3 type one edges then the largest number of edges in A which are incident with edges in F''' is $(3)(3) + 4(n - 6) = 4n - 15$. Since A contains $4n - 14$ edges, this would not account for all the edges of A. There must be at most two type one edges in F'''. If v_xv_y is a type two edge, we call v_x a ''degree 4'' vertex if 4 edges of A contain v_x. Otherwise, call the vertex a ''degree 0'' vertex. We intend to show that there is an edge e, containing two degree 0 vertices, such that $\varphi(e) \in \{1, 2, \ldots, n\}$.

Suppose $v_7v_8 \in F'''$ and v_7 is a degree 4 vertex. Consider the set of edges $B = \{e = v_8v_x | \phi(e) = 1, 2, \ldots, n\}$. At most 4 of these edges are incident with type one edges since

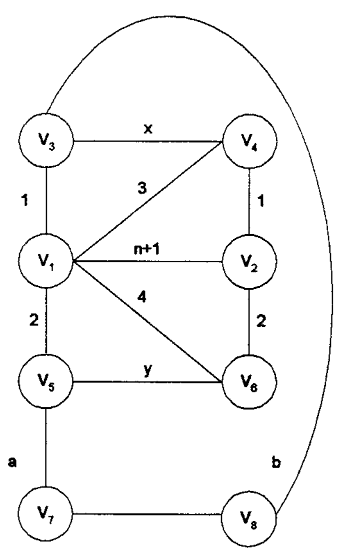


FIG. 12.

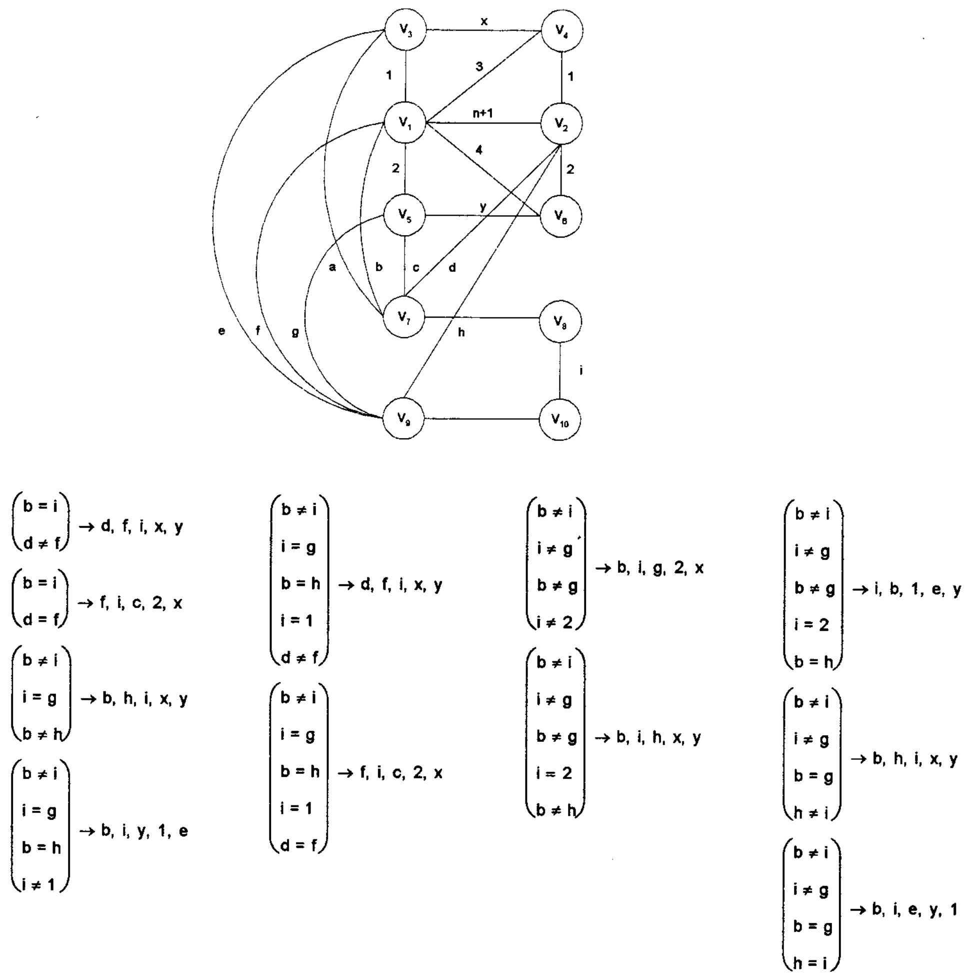


FIG. 13.

there are at most two type 1 edges. It is straightforward to show that if any edge in B is incident with a vertex in $\{v_1, v_2, v_3, v_4, v_5\}$, then F can be modified to produce a rainbow. One edge in B might be incident with v_6 and so there are at least $n-5$ edges in B which are incident with at most $n-6$ edges in F''' other than $v_1v_2, v_3v_4, v_5v_6, v_7v_8$ and any type one edges. We see there is an edge, say v_9v_{10}, which is incident with 2 edges in B, and so one of the two edges is incident with a degree 0 vertex. This is the edge $e(=v_8v_9)$ alluded to in the paragraph above. Figure 13 shows how to construct a rainbow in this case.

REFERENCES

[1] Z. Baranyai, *On the factorization of the complete uniform hypergraph*, Colloq. Math. Soc. János Bolyai, (1973), 91–108.

[2] D. E. Woolbright, *On the size of partial 1-factors of 1-factorizations of the complete k-uniform hypergraph on kn vertices*, Ars Combin. **6** (1978), 185–192.