

## 期中報告

### Inner functions of numerical contractions

#### 數值收縮算子之內函數

吳培元

#### 中文摘要：

我們驗證 S. W. Drury 在 2008 年一篇論文中的臆測，即證明了如下的結果：設  $f$  為定義於平面上單位開圓上的解析函數，且其值恆小於或等於 1，則下列二條件為等價：

(一)  $f$  為一內函數且  $f(0)=0$ ;

(二) 存在一個在複數希伯特空間  $H$  上的數值收縮算子  $T$  使得，對某個  $H$  中的單位向量  $x$ ， $\|f(T)x\|=2$  成立。此外，我們還證明了任何滿足上述條件(二)之算子可以表成  $T_1 \oplus T_2$ ，其中  $T_1$  相似於移位算子的壓縮  $S(\phi)$ ，其間  $\phi(z) = zf(z)$ ， $|z| < 1$ 。當  $f(z) = z$  時，這是 J. P. Williams 和 T. Crimmins 在 1967 年證明的定理；當  $f(z) = z^n$ ， $n \geq 2$ ，這是 M. J. Crabb 在 1971 年證的結果。

**關鍵字：**數值域、數值半徑、數值收縮算子、移位算子之壓縮。

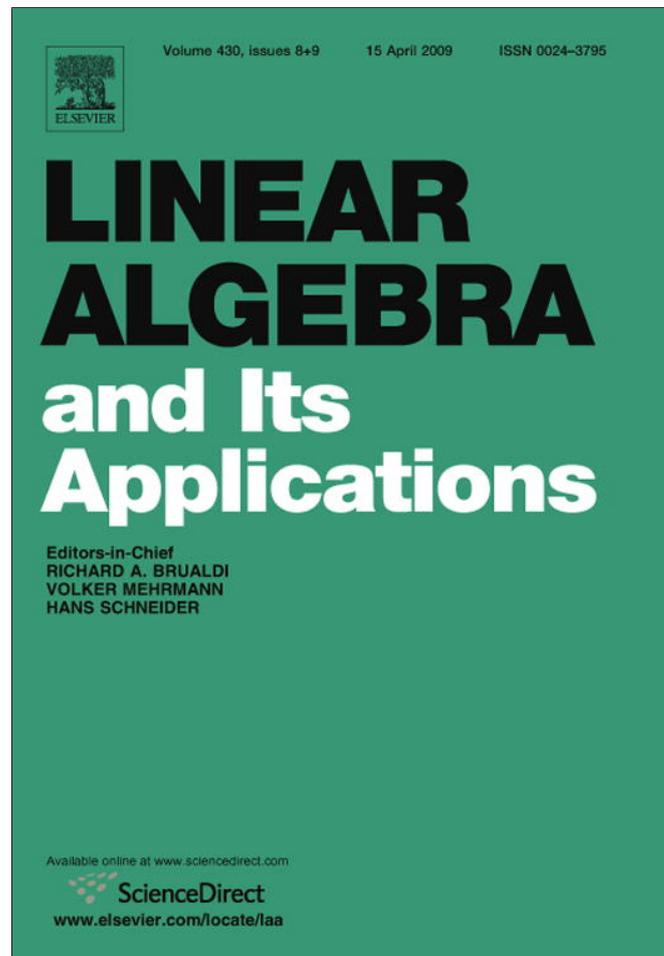
#### 英文摘要：

We prove that, for a function  $f$  in  $H^\infty$  of the unit disc with  $\|f\|_\infty \leq 1$ , the existence of an operator  $T$  on a complex Hilbert space  $H$  with its numerical radius at most one and with  $\|f(T)x\|=2$  for some unit vector  $x$  in  $H$  is equivalent to that  $f$  be an inner function with  $f(0) = 0$ . This confirms a conjecture of Drury [S.W. Drury, Symbolic calculus of operators with unit numerical radius, Linear Algebra Appl. 428 (2008) 2061–2069]. Moreover, we also show that any operator  $T$  satisfying the above conditions has a direct summand similar to the compression of the shift  $S(\phi)$ , where  $\phi(z) = zf(z)$  for  $|z| < 1$ . This generalizes the result of Williams and Crimmins [J.P. Williams, T. Crimmins, On the numerical radius of a linear operator, Amer. Math. Monthly 74 (1967) 832–833] for  $f(z) = z$  and of Crabb [M.J. Crabb, The powers of an operator of numerical radius

one, Michigan Math. J. 18 (1971) 253–256] for  $f(z) = z^n (n \geq 2)$ .

**Keywords:** Numerical range, numerical radius, numerical contraction, compression of the shift.

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## Inner functions of numerical contractions

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## ABSTRACT

We prove that, for a function  $f$  in  $H^\infty$  of the unit disc with  $\|f\|_\infty \leq 1$ , the existence of an operator  $T$  on a complex Hilbert space  $H$  with its numerical radius at most one and with  $\|f(T)x\| = 2$  for some unit vector  $x$  in  $H$  is equivalent to that  $f$  be an inner function with  $f(0) = 0$ . This confirms a conjecture of Drury [S.W. Drury, Symbolic calculus of operators with unit numerical radius, *Linear Algebra Appl.* 428 (2008) 2061–2069]. Moreover, we also show that any operator  $T$  satisfying the above conditions has a direct summand similar to the compression of the shift  $S(\phi)$ , where  $\phi(z) = zf(z)$  for  $|z| < 1$ . This generalizes the result of Williams and Crimmins [J.P. Williams, T. Crimmins, On the numerical radius of a linear operator, *Amer. Math. Monthly* 74 (1967) 832–833] for  $f(z) = z$  and of Crabb [M.J. Crabb, The powers of an operator of numerical radius one, *Michigan Math. J.* 18 (1971) 253–256] for  $f(z) = z^n$  ( $n \geq 2$ ).

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For a bounded linear operator  $A$  on a complex Hilbert space  $H$ , its *numerical range* and *numerical radius* are

$$W(A) = \{ \langle Ax, x \rangle : x \in H, \|x\| = 1 \}$$

and

$$w(A) = \sup\{|z| : z \in W(A)\},$$

respectively, where  $\langle \cdot, \cdot \rangle$  and  $\|\cdot\|$  denote the inner product and its associated norm in  $H$ . It is known that  $W(A)$  is a bounded convex subset of the plane. When  $H$  is finite dimensional, it is even compact.

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Its closure  $\overline{W(A)}$  contains the spectrum  $\sigma(A)$  of  $A$ . For other properties of the numerical range and numerical radius, the reader may consult [11, Chapter 22] or [10].

An operator  $A$  is a *numerical contraction* (resp., *contraction*) if  $w(A) \leq 1$  (resp.,  $\|A\| \leq 1$ ). In 1967, Sz.-Nagy and Foiaş [16] proved that every numerical contraction is similar to a contraction. Some years later, Okubo and Ando [13] gave another proof basing it on a factorization of the numerical contraction by Ando [1], which has the advantage of a sharp control on the invertible operator implementing the similarity. As a consequence, an estimate on the norm of a function of a numerical contraction can easily be obtained.

**Theorem 1.** (a) *An operator  $A$  is a numerical contraction if and only if  $A = 2(I - B^*B)^{1/2}B$  for some contraction  $B$ .*

(b) *If  $A$  is a numerical contraction, then  $A = XCX^{-1}$  for some invertible operator  $X$  with  $\|X\|, \|X^{-1}\| \leq \sqrt{2}$  and some contraction  $C$ .*

(c) *If  $A$  is a numerical contraction and  $f : \mathbb{D} \rightarrow \mathbb{C}$  is a function analytic on  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  and continuous on  $\overline{\mathbb{D}}$ , then  $\|f(A)\| \leq 2\|f\|_\infty$ , where  $\|f\|_\infty = \sup\{|f(z)| : z \in \overline{\mathbb{D}}\}$ .*

For our later use, we briefly sketch a proof of Theorem 1(b) based on (a), which is slightly different from the one in [13, Theorem 2]. Let  $A$  be factored as in (a). If

$$g(t) = \begin{cases} \sqrt{2(1-t)} & \text{if } 0 \leq t \leq 1/2, \\ 1/\sqrt{2t} & \text{if } 1/2 \leq t \leq 1, \end{cases} \tag{1}$$

then both  $g$  and  $1/g$  are continuous functions on  $[0, 1]$  with  $\|g\|_\infty = \|1/g\|_\infty = \sqrt{2}$ , where  $\|\cdot\|_\infty$  denotes the supremum of a function over  $[0, 1]$ . It is easily seen that  $X \equiv g(B^*B)$  is invertible,  $\|X\|, \|X^{-1}\| \leq \sqrt{2}$  and

$$\begin{aligned} \|X^{-1}AX\| &\leq 2\|g(B^*B)^{-1}(I - B^*B)^{1/2}\| \cdot \|(B^*B)^{1/2}g(B^*B)\| \\ &\leq 2 \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = 1. \end{aligned}$$

More recently, Drury [7] in studying the norm and numerical radius of  $f(A)$  proposed a conjecture on the sharpness of the inequality in Theorem 1(c). The purpose of this paper is to confirm this conjecture with a more detailed information on the structure of  $A$ .

In the following, we will consider a more general functional calculus than the one in Theorem 1(c) for numerical contractions. Indeed, if  $A$  is a numerical contraction on  $H$ , then the Berger dilation theorem [3] says that there is a unitary operator  $U$  on a space  $K$  containing  $H$  such that  $A^n = 2P_H U^n |H$  for all  $n \geq 1$ , where  $P_H$  denotes the (orthogonal) projection from  $K$  onto  $H$ . Such a unitary 2-dilation  $U$  of  $A$  can be taken to be minimal in the sense that  $K = \bigvee\{U^n H : n = 0, \pm 1, \pm 2, \dots\}$ . In this case,  $U$  is uniquely determined up to isomorphism, and, moreover, if  $A$  is completely nonunitary, that is, if  $A$  has no unitary direct summand, then  $U$  is absolutely continuous (cf. [8, Theorem 1] and [14, Proposition 2]). (We thank G. Cassier and L. Kérchy for providing us the relevant references on this subject.) Hence if  $A' = U' \oplus A$  on  $L \oplus H$  is a numerical contraction, where  $U'$  is absolutely continuous unitary and  $A$  is completely nonunitary, then  $f(A') \equiv f(U') \oplus ((2P_H f(U)|H) - f(0)I)$  for  $f$  in  $H^\infty$  is well-defined, where  $U$  is the minimal unitary 2-dilation of  $A$ . Note that Theorem 1(c) is obviously true for  $A$  a numerical contraction with no singular unitary part and  $f$  in  $H^\infty$ .

For an inner function  $\phi$  ( $\phi$  bounded analytic on  $\mathbb{D}$  with  $|\phi| = 1$  almost everywhere on  $\partial\mathbb{D}$ ), the *compression of the shift*  $S(\phi)$  is defined on  $H(\phi) = H^2 \ominus \phi H^2$  by

$$S(\phi)f = P_{H(\phi)}(zf(z)|H(\phi)) \quad \text{for } f \in H(\phi).$$

Such operators have been studied extensively since the 1960s starting with the work of Sarason [15]. A nice account of their properties together with those of the more general  $C_0$  contractions can be found in [2]. Sz.-Nagy and Foiaş [17] is the classical treatise on further developments of this subject. In particular, if  $\phi$  is a Blaschke product with  $n$  zeros (counting multiplicity), then  $H(\phi)$  is  $n$ -dimensional.



An operator  $A$  on  $H$  is cyclic with cyclic vector  $x$  if  $H = \vee\{A^n x : n \geq 0\}$ . It is easily seen that for a cyclic  $A$  the dimension of  $\ker A^*$  is at most one.

An operator is irreducible if it is not unitarily equivalent to the direct sum of two other operators. To prove the irreducibility of the operator  $A$  in Theorem 5, we need the following lemma.

**Lemma 7.** *If  $A$  is cyclic with a cyclic vector in  $\ker A^*$ , then  $A$  is irreducible.*

**Proof.** Assume that  $A = A_1 \oplus A_2$  on  $H = H_1 \oplus H_2$ . Let  $x = x_1 \oplus x_2$ , where  $x_j \in H_j, j = 1, 2$ , be a cyclic vector of  $A$  in  $\ker A^*$ . Then  $A^*x = (A_1 \oplus A_2)^*(x_1 \oplus x_2) = 0$  implies that  $(A_1 \oplus A_2)^*(x_1 \oplus 0) = (A_1 \oplus A_2)^*(0 \oplus x_2) = 0$ . On the other hand, since  $H_1 \oplus H_2 = \vee\{A_1^n x_1 \oplus A_2^n x_2 : n \geq 0\}$ , we infer that  $x_j \neq 0$  for  $j = 1, 2$ . Thus  $x_1 \oplus 0$  and  $0 \oplus x_2$  are linearly independent, and, therefore,  $\dim \ker A^* = \dim \ker (A_1 \oplus A_2)^* \geq 2$ , a contradiction. This proves our assertion.  $\square$

**Proof of Theorem 5.** Since  $\phi(0) = 0$ , the function  $g \equiv 1$  is in  $H(\phi)$ . It is a unit cyclic vector for  $S(\phi)$  and generates the one-dimensional subspace  $H_3$ . On the other hand, from the facts that  $f$  is inner and  $\phi(z) = zf(z)$  on  $\mathbb{D}$  we can easily check that  $f = P_{H(\phi)}f = f(S(\phi))g$  and  $S(\phi)f = 0$ . Thus  $f$  is a unit vector which generates the one-dimensional  $H_1$ . That  $f$  and  $g$  are orthogonal follows from a simple computation using  $f(0) = 0$ . Note also that

$$f(A)g = Xf(S(\phi))X^{-1}g = \sqrt{2}Xf(S(\phi))g = \sqrt{2}Xf = 2f,$$

which shows that  $\|f(A)g\| = 2$ . Since  $g \in \ker S(\phi)^*$  is a cyclic vector for  $S(\phi)$ ,  $Xg = g/\sqrt{2} \in \ker A^*$  is cyclic for  $A = XS(\phi)X^{-1}$ . The irreducibility of  $A$  then follows from Lemma 7. Moreover, since  $S(\phi)^n$  converges to 0 in the strong operator topology (SOT), the same is true for  $A^n$ . Hence  $A$  has no unitary part.

To prove that  $\overline{W(A)} \subseteq \mathbb{D}$ , let  $B = S(\phi)X^{-1}/\sqrt{2}$ . Since  $\text{rank}(I - S(\phi)^*S(\phi)) = 1$  and  $S(\phi)^*S(\phi)f = 0$ , we have  $S(\phi)^*S(\phi) = 0 \oplus I \oplus 1$  and hence  $B^*B = 0 \oplus (1/2)I \oplus 1$  on  $H(\phi) = H_1 \oplus H_2 \oplus H_3$ . Therefore,  $B$  is a contraction and

$$\begin{aligned} 2(I - B^*B)^{1/2}B &= 2 \left( I \oplus \frac{1}{\sqrt{2}}I \oplus 0 \right) \frac{1}{\sqrt{2}}S(\phi)X^{-1} \\ &= XS(\phi)X^{-1} = A. \end{aligned}$$

Theorem 1(a) then implies that  $\overline{W(A)} \subseteq \mathbb{D}$ .

To prove the converse, let  $\lambda$  be any point in  $\mathbb{D}$ . Then the operator  $I - \bar{\lambda}S(\phi)$  is invertible and  $u = (I - \bar{\lambda}S(\phi))^{-1}g - g = \sum_{n=1}^{\infty} (\bar{\lambda}S(\phi))^n g$  in norm. Let  $v = u - \langle u, f \rangle f$ . Note that

$$\begin{aligned} \langle v, g \rangle &= \sum_{n=1}^{\infty} \bar{\lambda}^n \langle S(\phi)^n g, g \rangle - \langle u, f \rangle \langle f, g \rangle \\ &= 0 - \langle u, f \rangle \cdot 0 = 0 \end{aligned}$$

and

$$\begin{aligned} \langle v, f \rangle &= \langle u, f \rangle - \langle u, f \rangle \langle f, f \rangle \\ &= \langle u, f \rangle - \langle u, f \rangle = 0. \end{aligned}$$

Hence  $v$  is in  $H_2$ . Finally, letting  $y = \langle u, f \rangle f \oplus \sqrt{2}v \oplus g$  in  $H(\phi) = H_1 \oplus H_2 \oplus H_3$ , we show that  $\bar{\lambda}By = (I - B^*B)^{1/2}y$ . Indeed, on the one hand, we have

$$\begin{aligned} \bar{\lambda}By &= \bar{\lambda}S(\phi) \left( \frac{1}{2} \oplus \frac{1}{\sqrt{2}}I \oplus 1 \right) (\langle u, f \rangle f \oplus \sqrt{2}v \oplus g) \\ &= \bar{\lambda} \left( \frac{1}{2} \langle u, f \rangle S(\phi)f + S(\phi)v + S(\phi)g \right) \\ &= \bar{\lambda}[S(\phi)(I - \bar{\lambda}S(\phi))^{-1}g - S(\phi)g] + \bar{\lambda}S(\phi)g \\ &= \bar{\lambda}S(\phi)(I - \bar{\lambda}S(\phi))^{-1}g. \end{aligned}$$

On the other hand,

$$\begin{aligned} (I - B^*B)^{1/2}y &= \left(1 \oplus \frac{1}{\sqrt{2}}I \oplus 0\right) (\langle u, f \rangle f \oplus \sqrt{2}v \oplus g) \\ &= \langle u, f \rangle f + v \\ &= u \\ &= (I - \bar{\lambda}S(\phi))^{-1}g - g \\ &= \bar{\lambda}S(\phi)(I - \bar{\lambda}S(\phi))^{-1}g. \end{aligned}$$

Thus  $\bar{\lambda}By = (I - B^*B)^{1/2}y$  holds. Hence

$$|\lambda|^2 \|By\|^2 = \|(I - B^*B)^{1/2}y\|^2 = \|y\|^2 - \|By\|^2,$$

which implies that  $\|By\|^2 = \|y\|^2 / (1 + |\lambda|^2)$ . Therefore,

$$\begin{aligned} \langle Ay, y \rangle &= \langle 2(I - B^*B)^{1/2}By, y \rangle \\ &= 2\langle By, (I - B^*B)^{1/2}y \rangle = 2\langle By, \bar{\lambda}By \rangle \\ &= 2\lambda \|By\|^2 = \frac{2\lambda}{1 + |\lambda|^2} \|y\|^2. \end{aligned}$$

This shows that  $2\lambda / (1 + |\lambda|^2)$  is in  $W(A)$  for any  $\lambda$  in  $\mathbb{D}$ . Hence  $\mathbb{D} \subseteq W(A)$  and thus  $\overline{W(A)} = \overline{\mathbb{D}}$  as asserted. This completes the proof.  $\square$

We now proceed to prove the necessity part of Theorem 2.

**Theorem 8.** Let  $f$  be a function in  $H^\infty$  with  $\|f\|_\infty \leq 1$ . If  $T$  is a numerical contraction with no singular unitary part such that  $\|f(T)x\| = 2$  for some unit vector  $x$ , then

- (a)  $f$  is inner with  $f(0) = 0$ , and
- (b)  $T$  is unitarily equivalent to an operator of the form  $XS(\phi)X^{-1} \oplus A'$ , where  $\phi(z) = zf(z)$  for  $|z| < 1$  and  $X = \sqrt{2} \oplus I \oplus (1/\sqrt{2})$  on  $H(\phi) = H_1 \oplus H_2 \oplus H_3$  ( $H_1 = \ker S(\phi)$  and  $H_3 = \ker S(\phi)^*$ ).

For the proof of its part (b), we need the following lemma.

**Lemma 9.** Let  $A$  be a  $C_0$  contraction on  $H$  with minimal function  $\phi$ . Then there is an operator  $\tilde{A}$  on  $\tilde{H} \supseteq H$  of class  $C_0$  such that (a)  $\tilde{A}H \subseteq H$ , (b)  $A = \tilde{A}|_H$ , and (c)  $\tilde{A}$  is unitarily equivalent to  $\sum_{n=1}^d \oplus S(\phi)$ , where  $d = \text{rank}(I - A^*A)^{1/2} \leq \infty$ .

This appeared in [12, Lemma 4] (with  $T$  there replaced by  $A^*$ ) and is dependent on the Sz.-Nagy–Foiaş contraction theory.

**Proof of Theorem 8.** (a) That  $f(0) = 0$  follows from Drury [7, Theorem 4]. Indeed, since the latter is also valid for functions  $f$  in  $H^\infty$  with  $\|f\|_\infty \leq 1$ , we have  $\|f(T)\| \leq v(|f(0)|)$ , where

$$v(t) = (2 - 3t^2 + 2t^4 + 2(1 - t^2)(1 - t^2 + t^4)^{1/2})^{1/2} \quad \text{for } 0 \leq t \leq 1.$$

Our assumption yields that

$$2 = \|f(T)x\| \leq \|f(T)\| \leq v(|f(0)|) \leq 2$$

or  $v(|f(0)|) = 2$ . This is equivalent to  $f(0) = 0$ .

Let  $M = \bigvee \{T^n x : n \geq 0\}$  and  $A = T|M$ . Then  $w(A) \leq 1$  and  $\|f(A)x\| = \|f(T)x\| = 2$ . By Theorem 1(a),  $A = 2(I - B^*B)^{1/2}B$  for some contraction  $B$ . Let  $g$  be as in (1) and  $X = g(B^*B)$ . Then, as indicated before,  $X$  is positive definite and invertible with  $\|X\|, \|X^{-1}\| \leq \sqrt{2}$  and  $C \equiv X^{-1}AX$  is a contraction. It is easily seen that  $C$ , being similar to the operator  $A$  with no singular unitary part, is itself without singular unitary part. Thus  $f(C)$  is well-defined. The chain of inequalities

$$\begin{aligned} 2 &= \|f(A)x\| = \|Xf(C)X^{-1}x\| \\ &\leq \|X\| \|f(C)X^{-1}x\| \leq \|X\| \|f(C)\| \|X^{-1}x\| \\ &\leq \|X\| \|f(C)\| \|X^{-1}\| \leq \sqrt{2} \|f\|_\infty \sqrt{2} \leq 2, \end{aligned}$$

where  $\|f(C)\| \leq \|f\|_\infty$  is by the von Neumann inequality, yields equalities throughout. In particular, we have

$$\|X\| = \|X^{-1}\| = \|f(C)X^{-1}x\| = \|X^{-1}x\| = \sqrt{2}$$

and  $\|f(C)\| = \|f\|_\infty = 1$ . Note that for a positive semidefinite operator  $Y$  and vector  $u$ , the equalities  $\|Yu\| = \|Y\| \|u\|$  and  $Yu = \|Y\|u$  are equivalent. Thus from  $\|X^{-1}x\| = \sqrt{2} = \|X^{-1}\| \|x\|$ , we infer that  $X^{-1}x = \sqrt{2}x$  or  $Xx = (1/\sqrt{2})x$ . Similarly, for  $y \equiv f(C)x$ , we have

$$\|y\| = \|f(C)x\| = \frac{1}{\sqrt{2}} \|f(C)X^{-1}x\| = 1$$

and

$$\|Xy\| = \|Xf(C)x\| = \frac{1}{\sqrt{2}} \|Xf(C)X^{-1}x\| = \frac{1}{\sqrt{2}} \|f(A)x\| = \sqrt{2} = \|X\| \|y\|.$$

As above, this yields  $Xy = \sqrt{2}y$ . Thus  $x$  and  $y$  are eigenvectors associated with the eigenvalues  $1/\sqrt{2}$  and  $\sqrt{2}$  of the positive definite  $X$ , respectively. Hence they are orthogonal to each other. Since  $X = g(B^*B)$  with  $g$  defined in (1), we infer that 1 and 0 are eigenvalues of  $B^*B$  with corresponding eigenvectors  $x$  and  $y$ , respectively. We also have

$$f(A)x = Xf(C)X^{-1}x = \sqrt{2}Xf(C)x = \sqrt{2}Xy = 2y. \tag{2}$$

From  $B^*By = 0$ , we obtain  $By = 0$ . Thus

$$Af(A)x = 2Ay = 2(I - B^*B)^{1/2}By = 0$$

and, consequently,

$$Af(A)A^n x = A^n (Af(A)x) = 0$$

for all  $n \geq 0$ . Since  $M$  is generated by  $A^n x$ ,  $n \geq 0$ , this yields  $Af(A) = 0$ . Hence  $Cf(C) = X^{-1}Af(A)X = 0$ , which shows that  $C$  is a  $C_0$  contraction. Let  $\psi$  be its minimal (inner) function, and let  $\phi(z) = zf(z)$ . Then  $\psi$  divides  $\phi$ . We necessarily have  $\psi(0) = 0$  for otherwise  $\psi$  would divide  $f$ , which would imply  $f(C) = 0$ , contradicting  $\|f(C)\| = 1$ . Hence  $\psi(z) = z\eta(z)$  for some inner function  $\eta$  and  $f(z) = \xi(z)\eta(z)$  for some  $\xi$  in  $H^\infty$  with  $\|\xi\|_\infty = 1$ . Let  $\xi(z) = \xi(0) + z\zeta(z)$  for  $\zeta$  in  $H^\infty$ . We have  $f(z) = \xi(0)\eta(z) + \zeta(z)\psi(z)$  and thus  $f(C) = \xi(0)\eta(C)$ . From

$$1 = \|f(C)\| = |\xi(0)| \|\eta(C)\| \leq \|\eta(C)\| \leq 1,$$

we obtain  $|\xi(0)| = 1$ . Therefore,  $\xi(z) = \xi(0)$  is constant and  $f = \xi(0)\eta$  is inner.

(b) We first show that  $C$  is unitarily equivalent to  $S(\phi)$ , where  $\phi(z) = zf(z)$ . Note that, from the proof of (a),  $\phi$  is the minimal function of  $C$ . By Lemma 9,  $C$  can be extended to (an operator unitarily equivalent to)  $\sum_{n=1}^\infty \oplus S(\phi)$ . Hence  $f(C)$  extends to  $\sum_{n=1}^\infty \oplus f(S(\phi))$ . Let  $x = \sum_{n=1}^\infty \oplus g_n$  with  $g_n$  in  $H(\phi)$  for all  $n$ . We infer from

$$1 = \|y\|^2 = \|f(C)x\|^2 = \sum_{n=1}^\infty \|f(S(\phi))g_n\|^2 \leq \sum_{n=1}^\infty \|g_n\|^2 = \|x\|^2 = 1$$

that  $\|f(S(\phi))g_n\| = \|g_n\|$  for all  $n$ . Since  $f(S(\phi))$  is a contraction, we have  $f(S(\phi))^*f(S(\phi))g_n = g_n$ . Thus  $g_n$  is in  $\text{ran } f(S(\phi))^*$ , a one-dimensional space generated by the function  $g \equiv 1$ . Hence, for each  $n \geq 1$ ,  $g_n = a_n g$  for some scalar  $a_n$ . Define the operator  $V : M \rightarrow H(\phi)$  by

$$V(p(C)x) = p(S(\phi))g$$

for any polynomial  $p$ . Since  $p(C)x = \sum_{n=1}^{\infty} \oplus p(S(\phi))g_n$ , we have

$$\begin{aligned} \|p(C)x\| &= \left( \sum_{n=1}^{\infty} \|p(S(\phi))g_n\|^2 \right)^{1/2} = \|p(S(\phi))g\| \left( \sum_{n=1}^{\infty} |a_n|^2 \right)^{1/2} \\ &= \|p(S(\phi))g\| \|x\| = \|p(S(\phi))g\|. \end{aligned}$$

Note that  $M$  being generated by  $A^n x, n \geq 0$ , is also generated by  $C^n X^{-1}x = \sqrt{2}C^n x, n \geq 0$ . Thus the set of vectors  $p(C)x, p$  polynomial, is dense in  $M$ . From above, we obtain that  $V$  is an isometry with  $VC = S(\phi)V$ . Since  $\phi$  is the minimal function of  $C$ , the unitary equivalence of  $C$  and  $S(\phi)$  follows.

Let  $H_1$  and  $H_3$  be the one-dimensional subspaces of  $M$  which are generated by  $y$  and  $x$ , respectively, and let  $H_2 = M \ominus (H_1 \oplus H_3)$ . On  $M = H_1 \oplus H_2 \oplus H_3$ , the operators  $X$  and  $B^*B$  can be decomposed as  $X = \sqrt{2} \oplus X_1 \oplus (1/\sqrt{2})$  and  $B^*B = 0 \oplus D \oplus 1$ . Let  $B = [B_{ij}]_{i,j=1}^3$  on  $M = H_1 \oplus H_2 \oplus H_3$ . From  $B^*B = 0 \oplus D \oplus 1$ , we obtain  $B_{11}^*B_{11} + B_{21}^*B_{21} + B_{31}^*B_{31} = 0$ , which implies that  $B_{11}, B_{21}$  and  $B_{31}$  are all zero operators. Hence

$$\begin{aligned} A &= 2(I - B^*B)^{1/2}B \\ &= 2 \begin{bmatrix} 1 & & \\ & (I - D)^{1/2} & \\ & & 0 \end{bmatrix} \begin{bmatrix} 0 & B_{12} & B_{13} \\ 0 & B_{22} & B_{23} \\ 0 & B_{32} & B_{33} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 2B_{12} & 2B_{13} \\ 0 & 2(I - D)^{1/2}B_{22} & 2(I - D)^{1/2}B_{23} \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned} \tag{3}$$

We now show that  $X_1 = I$ . This is done by proving  $DB_{22} = B_{22}/2$  and  $DB_{23} = B_{23}/2$ . Note that

$$\begin{aligned} C &= X^{-1}AX \\ &= \begin{bmatrix} 1/\sqrt{2} & & \\ & X_1^{-1} & \\ & & \sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & 2B_{12} & 2B_{13} \\ 0 & 2(I - D)^{1/2}B_{22} & 2(I - D)^{1/2}B_{23} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & & \\ & X_1 & \\ & & 1/\sqrt{2} \end{bmatrix} \\ &= \begin{bmatrix} 0 & \sqrt{2}B_{12}X_1 & B_{13} \\ 0 & 2X_1^{-1}(I - D)^{1/2}B_{22}X_1 & \sqrt{2}X_1^{-1}(I - D)^{1/2}B_{23} \\ 0 & 0 & 0 \end{bmatrix} \equiv \begin{bmatrix} 0 & C_{12} & C_{13} \\ 0 & C_{22} & C_{23} \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Since

$$I - C^*C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & I - C_{12}^*C_{12} - C_{22}^*C_{22} & * \\ 0 & * & 1 - |C_{13}|^2 - C_{23}^*C_{23} \end{bmatrix}$$

has rank one, we have

$$C_{12}^*C_{12} + C_{22}^*C_{22} = I \tag{4}$$

and

$$|C_{13}|^2 + C_{23}^*C_{23} = 1. \tag{5}$$

From (4), we obtain

$$\begin{aligned} I &= 2X_1^*B_{12}^*B_{12}X_1 + 4X_1^*B_{22}^*(I - D)^{1/2}X_1^{-1}X_1^{-1}(I - D)^{1/2}B_{22}X_1 \\ &= 2X_1(B_{12}^*B_{12} + 2B_{22}^*X_1^{-2}(I - D)B_{22})X_1. \end{aligned} \tag{6}$$

Note that

$$B^*B = \begin{bmatrix} 0 & 0 & 0 \\ B_{12}^* & B_{22}^* & B_{32}^* \\ B_{13}^* & B_{23}^* & B_{33}^* \end{bmatrix} \begin{bmatrix} 0 & B_{12} & B_{13} \\ 0 & B_{22} & B_{23} \\ 0 & B_{32} & B_{33} \end{bmatrix} = \begin{bmatrix} 0 & & \\ & D & \\ & & 1 \end{bmatrix}$$

yields  $B_{12}^*B_{12} + B_{22}^*B_{22} + B_{32}^*B_{32} = D$ . We derive from (6) that  $(1/2)X_1^{-2} = D - B_{22}^*B_{22} - B_{32}^*B_{32} + 2B_{22}^*X_1^{-2}(I - D)B_{22}$  or

$$B_{32}^*B_{32} + B_{22}^*(I - 2X_1^{-2}(I - D))B_{22} = D - \frac{1}{2}X_1^{-2}. \tag{7}$$

Since  $X_1 = g(D)$ , a simple computation involving the expression of  $g$  in (1) yields that  $I - 2X_1^{-2}(I - D) \geq 0$ . Hence (7) gives  $D \geq X_1^{-2}/2 = g(D)^{-2}/2$ . Again, from the expression of  $g$  in (1), we derive that  $D \geq I/2$  and thus

$$X_1 = g(D) = \frac{1}{\sqrt{2}}D^{-1/2}. \tag{8}$$

It follows from (7) that  $B_{22}^*(I - 4D(I - D))B_{22} = 0$ , which is the same as

$$0 = B_{22}^*(I - 4D + 4D^2)B_{22} = B_{22}^*(I - 2D)^2B_{22}.$$

We thus obtain  $(I - 2D)B_{22} = 0$  or  $DB_{22} = B_{22}/2$  as asserted.

To prove  $DB_{23} = B_{23}/2$ , we use (5) to derive that

$$\begin{aligned} 1 &= |B_{13}|^2 + 2B_{23}^*(I - D)^{1/2}X_1^{-2}(I - D)^{1/2}B_{23} \\ &= |B_{13}|^2 + 2B_{23}^*X_1^{-2}(I - D)B_{23}. \end{aligned}$$

Since  $B$  is a contraction, we have  $|B_{13}|^2 + B_{23}^*B_{23} \leq 1$ . These two together yield  $1 \leq 1 - B_{23}^*B_{23} + 2B_{23}^*X_1^{-2}(I - D)B_{23}$  or  $B_{23}^*(I - 2X_1^{-2}(I - D))B_{23} \leq 0$ . Since  $I - 2X_1^{-2}(I - D) \geq 0$  as was noted before, we obtain  $B_{23}^*(I - 2X_1^{-2}(I - D))B_{23} = 0$  and thus

$$0 = B_{23}^*(I - 4D(I - D))B_{23} = B_{23}^*(I - 2D)^2B_{23}$$

by (8). Therefore,  $(I - 2D)B_{23} = 0$  or  $DB_{23} = B_{23}/2$  as required.

From  $DB_{22} = B_{22}/2$  and  $DB_{23} = B_{23}/2$ , we have  $(I - D)B_{22} = B_{22}/2$  and  $(I - D)B_{23} = B_{23}/2$  and thus  $(I - D)^{1/2}B_{22} = B_{22}/\sqrt{2}$  and  $(I - D)^{1/2}B_{23} = B_{23}/\sqrt{2}$ . It follows from (3) that

$$A = \begin{bmatrix} 0 & 2B_{12} & 2B_{13} \\ 0 & \sqrt{2}B_{22} & \sqrt{2}B_{23} \\ 0 & 0 & 0 \end{bmatrix} \text{ on } M = H_1 \oplus H_2 \oplus H_3. \tag{9}$$

On the other hand, since  $M = \vee\{A^n x : n \geq 0\}$  and  $H_2 = M \ominus (\vee\{x, y\})$ , we have  $H_2 = \vee\{P_2 A^n x : n \geq 1\}$ , where  $P_2$  denotes the (orthogonal) projection from  $M$  onto  $H_2$ . A simple computation with (9) shows that  $P_2 A^n x = (\sqrt{2}B_{22})^{n-1}(\sqrt{2}B_{23})x$  for all  $n \geq 1$ . Therefore,

$$\begin{aligned} D(P_2 A^n x) &= D \left( \sqrt{2}B_{22} \right)^{n-1} \left( \sqrt{2}B_{23} \right) x \\ &= \frac{1}{2} \left( \sqrt{2}B_{22} \right)^{n-1} \left( \sqrt{2}B_{23} \right) x = \frac{1}{2} P_2 A^n x \end{aligned}$$

if  $n \geq 2$ , and

$$D(P_2 A x) = D \left( \sqrt{2}B_{23} \right) x = \frac{1}{2} \left( \sqrt{2}B_{23} \right) x = \frac{1}{2} P_2 A x.$$

These show that  $D = I/2$  and hence  $X_1 = D^{-1/2}/\sqrt{2} = I$  by (8) or  $X = \sqrt{2} \oplus I \oplus (1/\sqrt{2})$ .

Finally, we prove that  $M$  is a reducing subspace of  $T$ . Since  $f$  is inner with  $f(0) = 0$ , we have  $w(f(T)) \leq 1$  (cf. [4, Theorem 4]). This, together with  $\|f(T)x\| = 2$ , yields that the subspace  $K \equiv H_1 \oplus H_3$  reduces

$f(T)$  and  $f(T)|_K$  has the matrix representation  $\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$  relative to the orthonormal basis  $\{y, x\}$  of  $K$  (cf. Corollary 4 or [18]). In particular, this gives  $f(T)^*x = 0$  and  $f(T)^*y = 2x$ . Now we repeat these with  $T$  and  $f$  replaced by  $T^*$  and  $\tilde{f}$ , where  $\tilde{f}$  is the inner function  $\tilde{f}(z) = \overline{f(\bar{z})}$ ,  $|z| < 1$ . Since  $\tilde{f}(T^*) = f(T)^*$ , we have  $w(\tilde{f}(T^*)) \leq 1$  and  $\|\tilde{f}(T^*)y\| = 2$ . Letting  $\tilde{M} = \vee\{T^{*n}y : n \geq 0\}$ , we infer from what were proved before for  $T$  and  $f$  that  $\tilde{A} \equiv T^*|\tilde{M} = \tilde{X}\tilde{C}\tilde{X}^{-1}$  for some operator

$$\tilde{C} = \begin{bmatrix} 0 & \tilde{C}_{12} & \tilde{C}_{13} \\ 0 & \tilde{C}_{22} & \tilde{C}_{23} \\ 0 & 0 & 0 \end{bmatrix} \text{ on } \tilde{M} = \tilde{H}_1 \oplus \tilde{H}_2 \oplus \tilde{H}_3$$

( $\tilde{H}_1 = \vee\{\tilde{f}(\tilde{C})y\}$  and  $\tilde{H}_3 = \vee\{y\}$ ) which is unitarily equivalent to  $S(\tilde{\phi})$  ( $\tilde{\phi}(z) = z\tilde{f}(z)$  on  $\mathbb{D}$ ), and  $\tilde{X} = \sqrt{2} \oplus I \oplus (1/\sqrt{2})$  on  $\tilde{M} = \tilde{H}_1 \oplus \tilde{H}_2 \oplus \tilde{H}_3$ . We check that  $\tilde{A}$  is unitarily equivalent to  $A^*$ . Indeed, since  $C^*$  is unitarily equivalent to  $S(\tilde{\phi})$  and the latter is in turn unitarily equivalent to  $\tilde{C}$ , there is a unitary operator  $U$  mapping  $M$  onto  $\tilde{M}$  such that  $UC^* = \tilde{C}U$ . In particular, we have  $U(\ker C^*) = \ker \tilde{C}$  and  $U(\ker C) = \ker \tilde{C}^*$ . Note that

$$\tilde{f}(\tilde{C})y = \frac{1}{2}\tilde{f}(\tilde{A})y = \frac{1}{2}\tilde{f}(T^*)y = \frac{1}{2}f(T)^*y = x$$

by the analogue of (2). Hence  $\ker C^* = \ker \tilde{C} = \vee\{x\}$  and also  $\ker C = \ker \tilde{C}^* = \vee\{y\}$ . Therefore,  $Ux = \lambda_1x$  and  $Uy = \lambda_2y$  for some scalars  $\lambda_1$  and  $\lambda_2$  of modulus one. Thus  $U$  is of the form

$$U = \begin{bmatrix} & & \lambda_1 \\ & U_1 & \\ \lambda_2 & & \end{bmatrix}$$

from  $M = H_1 \oplus H_2 \oplus H_3$  to  $\tilde{M} = \tilde{H}_1 \oplus \tilde{H}_2 \oplus \tilde{H}_3$  and hence

$$\begin{aligned} U^*\tilde{A}U &= U^*\tilde{X}\tilde{C}\tilde{X}^{-1}U \\ &= \begin{bmatrix} & U_1^* & \lambda_2 \\ \bar{\lambda}_1 & & \end{bmatrix} \begin{bmatrix} \sqrt{2} & & \\ & I & \\ & & 1/\sqrt{2} \end{bmatrix} \tilde{C} \begin{bmatrix} 1/\sqrt{2} & & \\ & I & \\ & & \sqrt{2} \end{bmatrix} \begin{bmatrix} & U_1 & \lambda_1 \\ \lambda_2 & & \end{bmatrix} \\ &= \begin{bmatrix} 1/\sqrt{2} & & \\ & I & \\ & & \sqrt{2} \end{bmatrix} \begin{bmatrix} & U_1^* & \lambda_2 \\ \bar{\lambda}_1 & & \end{bmatrix} \tilde{C} \begin{bmatrix} & U_1 & \lambda_1 \\ \lambda_2 & & \end{bmatrix} \begin{bmatrix} \sqrt{2} & & \\ & I & \\ & & 1/\sqrt{2} \end{bmatrix} \\ &= X^{-1}U^*\tilde{C}UX = X^{-1}C^*X = A^*. \end{aligned}$$

Finally, we check that  $\tilde{M}$  is contained in  $M$ . This is because, for any  $n \geq 0$ , the equalities

$$\begin{aligned} \|T^{*n}y\| &= \|\tilde{A}^n y\| = \|UA^{*n}U^*y\| \\ &= \|UA^{*n}(\bar{\lambda}_2 y)\| = \|A^{*n}y\| = \|(T|M)^{*n}y\| \end{aligned}$$

hold, which yields that  $T^{*n}y$  belongs to  $M$ . Similarly, we can show that  $M \subseteq \tilde{M}$ . Hence  $M = \tilde{M}$  and  $T^*M = T^*\tilde{M} \subseteq \tilde{M} = M$ . Thus  $M$  reduces  $T$ . This completes the proof.  $\square$

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