Interceptor and Target engagement: System Design, Simulation and Performance evaluation

First year research summary

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Abstract

This report describes the work done so far in the first year of research. The work includes modeling of subsystems and development of a six degree-of-freedom (6DOF) simulation program of the engagement. The key subsystems are various coordinate systems and the transformations between them, the equations of motion of both the interceptor and target, the spheroid earth and atmospheric model, the formulas for computations of such quantities as Mach number, flight path angle and angle of attack, the guidance laws and the control systems, the propulsion model, the actuators, the aerodynamic model and a model that calculates all the physical properties of the interceptor such as mass , moment of inertia, etc. The simulation program is implemented using Simulink, a block diagram based simulation tool. The subsystems that have been developed and implemented will be described in detail, the rest will only be briefly discussed. Completing the modeling and simulation development is the topic of subsequent research.

1 Introduction

The goal of this three-year research is the development of a simulation environment that can be used as a tool for evaluating the performance of interceptors against incoming missiles. The tool will be equipped with basic subsystem modules such as equation of motions and earth model and default modules such as guidance laws and control systems. The modules will have standardized inputs, outputs and parameters. To simulate a particular engagement, the user can easily replace any module with one of his own development. For example, he may replace the guidance law module with one of his own design to test its effectiveness.

This report describes the work done so far in the first year of research. The work includes modeling of subsystems and development of a six degree-of-freedom (6DOF) simulation program of the engagement. The key subsystems are various coordinate systems and the transformations between them, the equations of motion of both the interceptor and target, the spheroid earth and atmospheric model, the formulas for computations of such quantities as Mach number, flight path angle and angle of attack, the guidance laws and the control systems, the propulsion model, the actuators, the aerodynamic model and a model that calculates all the physical properties of the interceptor such as mass , moment of inertia, etc. The simulation program is implemented using Simulink, a block diagram based simulation tool. The subsystems that have been developed and implemented will be described in detail, the rest will only be briefly discussed. Completing the modeling and simulation development is the topic of subsequent research.

The report is organized as follows. Section 2 introduces the coordinate systems used in the simulation. Section 3 describes the equations of motion of the interceptor, the translational equation and the rotational equation. It also defines the Euler angle and the quaternion equation. Section 4 describes the transformation matrices between the various coordinate systems. Section 5 is the detailed earth gravitation model and the calculation of accelerations in inertial coordinate. Section 6 describes the computation of various variables such as Mach number, angle of attack, longitude and latitude, etc. Section 7 is a summary and discussion of future work.

2 The coordinate systems

The simulation uses several coordinate systems to describe the dynamics of the vehicle (the interceptor). They include inertial coordinate systems, earth surface fixed rotating coordinate systems, and vehicle-fixed coordinate systems. The description of these coordinate systems follows.

(I) Earth-centered Inertial Coordinate $\mathbf{S}_I$

- Type: Non-rotating
- Origin: The center of the earth

- The Z_I-axis points to the North pole
- The X_I-axis points to the Greenwich Meridian at start of simulation (time zero) and on the equatorial plane
- The Y_I-axis completes the right-hand system

(II) Inertial Launch Coordinate $\mathbf{S}_L$

- Type: Non-rotating
- Origin: The vehicle's center of gravity at the start of simulation
- The X_L-axis points to the local vertical, positive upward
- The Z_L-axis points to the vehicle's aiming azimuth
- The Y_L-axis completes the the right-hand system

(III) Earth-centered Rotating Coordinate $\mathbf{S}_R$

- Type: Rotating
- Origin: The center of the earth
- The Z_R-axis points to the North pole
- The X_R-axis coincides with X_I-axis at start of simulation (time zero) and rotates with the earth, hence it is always along the Greenwich Meridian
- The Y_R-axis completes the right-hand system

(IV) Geographical Coordinate $\mathbf{S}_G$

- Type: Rotating
- Origin: The vehicle's current subvehicle point on the earth surface
- The X_G-axis is in the local horizontal plane and points north
- The Y_G-axis is in the local horizontal plane and points east
- The Z_G-axis completes the right-hand system

The subvehicle point is defined as the point on the reference ellipsoid (spheroid) closest to the vehicle [8, p.6-56]. This is the so-called north-east-down (NED) coordinate, due to the spheroid earth model the 'down' direction does not necessarily point to the earth center.

(V) Body Coordinate $\mathbf{S}_B$

- Type: Rotating
- Origin: The vehicle's center of gravity
- The X_B-axis coincides with the forward longitudinal axis of the vehicle
- The Y_B-axis points right
- The Z_B-axis points down

3 Equations of motion

The rigid body equations of motion include the translational equations and the rotational equations. Together they are called the six degree-of-freedom (6DOF) equations of motion.

3.1 Translational equations

The translational equations describe the evolution of position and velocity of the center of gravity (c.g.) of the vehicle. In the S_I coordinate, these equations are simply double integrators:

$$\begin{aligned} \dot{v}_{xI} &= a_{xI} \\ \dot{v}_{yI} &= a_{yI} \\ \dot{v}_{zI} &= a_{zI} \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \dot{x}_I &= v_{xI} \\ \dot{y}_I &= v_{yI} \\ \dot{z}_I &= v_{zI} \end{aligned} \tag{3.2}$$

where $\{a_{xI}, a_{yI}, a_{zI}\}$ $\{v_{xI}, v_{yI}, v_{zI}\}$, and $\{x_I, y_I, z_I\}$ are respectively the acceleration components, the velocity components, and the position components of the vehicle in the S_I coordinate. In matrix form we write these equations as

$$\dot{x} = Ax + Bu \tag{3.3}$$

where

$$x = \begin{bmatrix} x_I \\ y_I \\ z_I \\ v_{xI} \\ v_{yI} \\ v_{zI} \end{bmatrix}, \quad u = \begin{bmatrix} a_{xI} \\ a_{yI} \\ a_{zI} \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

3.2 Initial position

The initial position components can be computed from the (geographic) longitude $\lambda(0)$, the (geodetic) latitude $\phi_g(0)$, and the altitude $h(0)$ of the launch site (the vehicle at $t = 0$). We describe two formulas for the computation: one is exact and the other approximate.

(I) Exact formula [1, p. 98]

Given $(\lambda(0), \phi_g(0), h(0))$ as the longitude, geodetic latitude and altitude of the launch site, the initial position components in the $\mathbf{S}_I$ coordinate are

$$\begin{bmatrix} x_I(0) \\ y_I(0) \\ z_I(0) \end{bmatrix} = \begin{bmatrix} \left(\frac{R_e}{\sqrt{1-e^2 \sin^2 \phi_g(0)}} + h(0)\right) \cos \phi_g(0) \cos \lambda(0) \\ \left(\frac{R_e}{\sqrt{1-e^2 \sin^2 \phi_g(0)}} + h(0)\right) \cos \phi_g(0) \sin \lambda(0) \\ \left(\frac{R_e(1-e^2)}{\sqrt{1-e^2 \sin^2 \phi_g(0)}} + h(0)\right) \sin \phi_g(0) \end{bmatrix}. \tag{3.4}$$

where R_e is the equatorial radius of the earth, $e = \sqrt{1 - (R_p/R_e)^2}$ is the eccentricity of the earth and R_p is the polar radius of the earth.

(II) Approximate formula [2]

At time $t = 0$, the inertial longitude $\lambda_I(0)$ is the same as the geographic longitude $\lambda(0)$, that is,

$$\lambda_I(0) = \lambda(0).$$

The geocentric latitude $\phi_c(0)$ can be computed from the geodetic latitude $\phi_g(0)$ by

$$\phi_c(0) = \tan^{-1}\left(\frac{1}{k} \tan \phi_g(0)\right) \tag{3.5}$$

where $k = (R_e/R_p)^2$ the ratio of the equatorial radius and the polar radius squared. The distance, $R_s(0)$, from the center of the earth to the subvehicle point on the surface of the earth (sea level) is given by

$$R_s(0) = \frac{R_e}{\sqrt{1 + (k-1)\sin^2\phi_c(0)}}. \tag{3.6}$$

The distance from the center of the earth to the vehicle is (approximately)

$$r_I(0) = R_s(0) + h(0). \tag{3.7}$$

The initial position components are then given by

$$\begin{bmatrix} x_I(0) \\ y_I(0) \\ z_I(0) \end{bmatrix} = \begin{bmatrix} r_I \cos\phi_c(0)\cos\lambda_I(0) \\ r_I \cos\phi_c(0)\sin\lambda_I(0) \\ r_I \sin\phi_c(0) \end{bmatrix} = \begin{bmatrix} r_I \cos\phi_c(0)\cos\lambda(0) \\ r_I \cos\phi_c(0)\sin\lambda(0) \\ r_I \sin\phi_c(0) \end{bmatrix}. \tag{3.8}$$

3.3 Initial velocity

The initial velocity components are

$$\begin{bmatrix} v_{xI}(0) \\ v_{yI}(0) \\ v_{zI}(0) \end{bmatrix} = \begin{bmatrix} 0 & -\Omega_p & 0 \\ \Omega_p & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_I(0) \\ y_I(0) \\ z_I(0) \end{bmatrix} = \begin{bmatrix} -\Omega_p y_I(0) \\ \Omega_p x_I(0) \\ 0 \end{bmatrix},$$

where Ω_p is the angular rate of the earth. In vector notations, we would write the equation as

$$\overrightarrow{v_I}(0) = \overrightarrow{\Omega_p} \times \overrightarrow{r_I}(0). \tag{3.9}$$

3.4 Rotational equations

The rotational equations consists of two sets of first-order equations: the *moment equation* and the *Euler equation*. The moment equation describes the relation between the angular rate and the external torque applied on the vehicle. The Euler equation describes the relation between the vehicle's attitude, specified by the Euler angles, and the angular rate. The attitude of the vehicle can be equivalently expressed in terms of the Euler parameters, also known as the quaternion elements [4, p. 148]. The differential equations relating the Euler parameters and the angular rates are known as the quaternion equation. From a numerical point of view, the quaternion equation is preferred.

3.5 The Euler angles

We need to define first the Euler angles (ϕ, ψ, θ) that specify the orientation of the vehicle with respect to an inertial coordinate. They are the angles of the three consecutive rotations of $\mathbf{S}_L$ to make it parallel to $\mathbf{S}_B$. After these rotations, the two coordinate systems are said to be translatable meaning that they can be made identical through a translation of the origin of one coordinate. We will denote the axes after the first rotation as $X_L' - Y_L' - Z_L'$ and those after the second rotation as $X_L'' - Y_L'' - Z_L''$. For ease of visualization, we will assume the coordinates have the same origin.

The first rotation ϕ is around X_L so that the projection of Y_L' onto the $Y_B - Z_B$ plane is precisely Y_B; the second rotation ψ is around Z_L' so that $Y_L'' = Y_B$; the third rotation θ is around Y_B so that the remaining two axes coincide as well.

We note that his roll-yaw-pitch sequence is different from the yaw-pitch-roll sequence commonly used in the description of aircraft or missile attitudes. This is primarily to avoid the so-called 'gimbal-lock' problem when the pitch angle approaches $\pm 90°$ [5, p. 206], [7].

3.6 Moment equation

The equations relating moments (M_x, M_y, M_z) to the body rate (p, q, r) are

$$\begin{aligned} \dot{p} &= qr(I_y - I_z)/I_x + M_x/I_x \\ \dot{q} &= rp(I_z - I_x)/I_y + M_y/I_y \\ \dot{r} &= pq(I_x - I_y)/I_z + M_z/I_z \end{aligned} \tag{3.10}$$

where M_x, M_y, M_z are components of the (total) moment, in $\mathbf{S}_B$ coordinate, acting on the vehicle, I_x, I_y, I_z are moments of inertia of the vehicle, and p, q, r are vehicle angular velocity components in $\mathbf{S}_B$ coordinate.

The total moment is the sum of the aerodynamic moment and the moment due to propulsion, that is,

$$\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} M_{Ax} \\ M_{Ay} \\ M_{Az} \end{bmatrix} + \begin{bmatrix} M_{Tx} \\ M_{Ty} \\ M_{Tz} \end{bmatrix} \tag{3.11}$$

where M_{Ax}, M_{Ay}, M_{Az} and M_{Tx}, M_{Ty}, M_{Tz} are respectively the aerodynamic moment components and propulsion moment components in $\mathbf{S}_B$ coordinate.

Initially the vehicle rotates with the earth, so its angular rate has a magnitude Ω_p and is along the Z_I-axis. By coordinate transformation we should have

$$\begin{bmatrix} p(0) \\ q(0) \\ r(0) \end{bmatrix} = C_{BI}(0) \begin{bmatrix} 0 \\ 0 \\ \Omega_p \end{bmatrix},$$

where $C_{BI}(0)$ is the transformation matrix from $\mathbf{S}_I$ to $\mathbf{S}_B$ at time $t = 0$ (see Section 4.)

We have assumed that the cross moment of inertia

$$I_{xy} = I_{yz} = I_{xz} = 0$$

as a result of symmetry of the launch vehicle. The moment equation needs to be modified if these assumptions do not hold [3, p. 144].

3.7 Euler equation

The Euler equation relating the angular rates with the Euler angles are

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} \cos\psi\cos\theta & 0 & -\sin\theta \\ -\sin\psi & 1 & 0 \\ \cos\psi\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{bmatrix}, \tag{3.12}$$

or equivalently,

$$\begin{bmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{bmatrix} = \begin{bmatrix} \sec\psi\cos\theta & 0 & \sin\theta\sec\psi \\ \tan\psi\cos\theta & 1 & \tan\psi\sin\theta \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}. \tag{3.13}$$

Initial conditions are $\phi(0) = \psi(0) = \theta(0) = 0$ for vertically launched vehicle. If the longitudinal axis makes an angle η with the local vertical, then $\theta(0) = -\eta$ and $\phi(0) = \psi(0) = 0$.

Equation (3.13) contains the functions $\tan\psi$ and $\sec\psi$, hence as the yaw angle ψ approaches $\pm\pi/2$ we run into numerical difficulties. Normally for roll-stabilized vehicles, the yaw angle ψ (and the roll angle ϕ) are usually close to zero, so are the rates p and r, while the pitch angle θ may vary substantially. In this case, solving (3.13) poses no numerical problem. However for spinning vehicles, the roll rate p induces strong pitch-yaw coupling in (3.10)(close to a harmonic oscillator) so both ψ and θ can be large. In this case, solving (3.13) for the Euler angles is not practical and the quaternion equation is a standard way to get around the problem [7].

3.8 Quaternion equation

The 4 quaternion elements or the Euler parameters e_0, e_1, e_2, e_3 can be defined as functions of the Euler angles ϕ, ψ, θ (the roll-yaw-pitch sequence) as follows [4, p.155].

$$e_0 = \cos\frac{\psi}{2}\cos\frac{\theta}{2}\cos\frac{\phi}{2} + \sin\frac{\psi}{2}\sin\frac{\theta}{2}\sin\frac{\phi}{2} \tag{3.14}$$

$$e_1 = \cos\frac{\psi}{2}\cos\frac{\theta}{2}\sin\frac{\phi}{2} - \sin\frac{\psi}{2}\sin\frac{\theta}{2}\cos\frac{\phi}{2} \tag{3.15}$$

$$e_2 = \cos\frac{\psi}{2}\sin\frac{\theta}{2}\cos\frac{\phi}{2} - \sin\frac{\psi}{2}\cos\frac{\theta}{2}\sin\frac{\phi}{2} \tag{3.16}$$

$$e_3 = \sin\frac{\psi}{2}\cos\frac{\theta}{2}\cos\frac{\phi}{2} + \cos\frac{\psi}{2}\sin\frac{\theta}{2}\sin\frac{\phi}{2} \tag{3.17}$$

We note that if the sequence of rotations of Euler angles is different, then the relation between Euler angles and the quaternion elements will be different.

The quaternion elements are related to the angular rate components through the quaternion equation [7]:

$$\begin{bmatrix} \dot{e}_0 \\ \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -e_1 & -e_2 & -e_3 \\ e_0 & -e_3 & -e_2 \\ e_3 & e_0 & -e_1 \\ -e_2 & e_1 & e_0 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}. \tag{3.18}$$

The initial conditions: $e_0(0), e_1(0), e_2(0), e_3(0)$ are computed from the initial Euler angles using (3.14), (3.15), (3.16), and (3.17). For example, if the initial Euler angles are $\phi(0) = \psi(0) = \theta(0) = 0$, then

$$e_0(0) = 1, e_1(0) = e_2(0) = e_3(0) = 0.$$

The quaternion elements are not independent and are constrained by

$$e_0^2(t) + e_2^2(t) + e_2^2(t) + e_3^2(t) = 1 \quad \text{for all} \quad t.$$

Clearly the solution of (3.18) has a constant norm, since

$$\dot{e}(t)^T e(t) = 0$$

for all $p, q, r \in \mathbf{R}$ and for all $t \geq 0$.

4 The transformation matrices

We need to express force and velocity vectors in different coordinates. For example, we need to express the aerodynamic and thrust accelerations, which come naturally in $\mathbf{S}_B$ coordinate, in the $\mathbf{S}_I$ coordinate so that they can be used as inputs to the translational equation; we also need to express vehicle velocity and wind velocity in $\mathbf{S}_B$ coordinate for the computation of the aerodynamic angles. The different expressions (of the same vector) are related by transformation matrices. These matrices are orthogonal as each of them is the result of a sequence of simple rotations.

4.1 Transformation from $\mathbf{S}_B$ to $\mathbf{S}_L$

The transformation matrix from $\mathbf{S}_L$ to $\mathbf{S}_B$ is

$$C_{BL} = \begin{bmatrix} c(\theta)c(\psi) & c(\theta)s(\psi)c(\phi) + s(\theta)s(\phi) & c(\theta)s(\psi)s(\phi) - s(\theta)c(\phi) \\ -s(\psi) & c(\psi)c(\phi) & c(\psi)s(\phi) \\ s(\theta)c(\psi) & s(\theta)s(\psi)c(\phi) - c(\theta)s(\phi) & s(\theta)s(\psi)s(\phi) + c(\theta)c(\phi) \end{bmatrix} \tag{4.1}$$

where ϕ, ψ, θ are the Euler angles and we use $c(x)$ for $\cos(x)$ and $s(x)$ for $\sin(x)$. In flight, the Euler angles changes with time so the matrix C_{BL} is a function of time.

The matrix is the result of a sequence of three simple rotations described in Section 3.5 and can be computed as

$$C_{BL} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}$$

The transformation matrix from $\mathbf{S}_B$ to $\mathbf{S}_L$ is

$$C_{LB} = C_{BL}^{-1} = C_{BL}^T. \tag{4.2}$$

The transformation matrix C_{BL} can also be expressed in terms of the quaternion elements:

$$C_{BL} = \begin{bmatrix} e_0^2 + e_1^2 - e_2^2 - e_3^2 & 2(e_1e_2 + e_0e_3) & 2(e_1e_3 - e_0e_2) \\ 2(e_1e_2 - e_0e_3) & e_0^2 - e_1^2 + e_2^2 - e_3^2 & 2(e_0e_1 + e_2e_3) \\ 2(e_1e_3 + e_0e_2) & 2(e_2e_3 - e_0e_1) & e_0^2 - e_1^2 - e_2^2 + e_3^2 \end{bmatrix}. \tag{4.3}$$

It is easy, although tedious, to verify that the two matrices defined in (4.1) and (4.3) are indeed identical.

4.2 Transformation from $\mathbf{S}_L$ to $\mathbf{S}_I$

The transformation matrix from $\mathbf{S}_I$ to $\mathbf{S}_L$ is C_{LI} and is given by

$$C_{LI} = \begin{bmatrix} c(\lambda_0)c(\phi_{c0}) & c(\phi_{c0})s(\lambda_0) & s(\phi_{c0}) \\ s(A_L)s(\phi_{c0})c(\lambda_0) - c(A_L)s(\lambda_0) & s(A_L)s(\phi_{c0})s(\lambda_0) + c(A_L)c(\lambda_0) & -s(A_L)c(\phi_{c0}) \\ -c(A_L)s(\phi_{c0})c(\lambda_0) - s(A_L)s(\lambda_0) & -c(A_L)s(\phi_{c0})s(\lambda_0) + s(A_L)c(\lambda_0) & c(A_L)c(\phi_{c0}) \end{bmatrix} \tag{4.4}$$

where λ_0 and ϕ_{c0} are the longitude and geocentric latitude of the launch site, and A_L is the angle from the aiming azimuth to local north on the local horizontal plane, positive counterclockwise. For example, if the aiming azimuth is local east then $A_L = \pi/2$.

We note that C_{LI} is a constant matrix and does not change with time. The matrix represents the result of three consecutive rotations of $\mathbf{S}_I$ to make it parallel to $\mathbf{S}_L$:

(i) First rotation of λ_0 around Z_I-axis

(ii) Second rotation of ϕ_{c0} around $-Y_I'$-axis

(iii) Third rotation of A_L around $-X_I''$-axis $= X_L$-axis

Hence C_{LI} is the product of three simple rotation matrices:

$$C_{LI} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos A_L & -\sin A_L \\ 0 & \sin A_L & \cos A_L \end{bmatrix} \begin{bmatrix} \cos\phi_{c0} & 0 & \sin\phi_{c0} \\ 0 & 1 & 0 \\ -\sin\phi_{c0} & 0 & \cos\phi_{c0} \end{bmatrix} \begin{bmatrix} \cos\lambda_0 & \sin\lambda_0 & 0 \\ -\sin\lambda_0 & \cos\lambda_0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Again the transformation matrix from $\mathbf{S}_L$ to $\mathbf{S}_I$ is

$$C_{IL} = C_{LI}^{-1} = C_{LI}^T.$$

4.3 Transformation from $\mathbf{S}_B$ to $\mathbf{S}_I$

The transformation matrix from $\mathbf{S}_I$ to $\mathbf{S}_B$, denoted C_{BI}, can be computed as

$$C_{BI} = C_{BL}C_{LI}.$$

The transformation matrix from $\mathbf{S}_B$ to $\mathbf{S}_I$ is

$$C_{IB} = C_{BI}^T.$$

4.4 Transformation from S_I to S_R

The transformation matrix from $\mathbf{S}_I$ to $\mathbf{S}_R$ is given by

$$C_{RI} = \begin{bmatrix} \cos(\Omega_p t) & \sin(\omega_p t) & 0 \\ -\sin(\omega_p t) & \cos(\Omega_p t) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This is a simple rotation around the Z_I-axis.

4.5 Transformation from S_I to S_G

The transformation matrix from $\mathbf{S}_I$ to $\mathbf{S}_G$ is

$$C_{GI} = \begin{bmatrix} -s(\phi_c)c(\lambda_I) & -s(\phi_c)s(\lambda_I) & c(\phi_c) \\ -s(\lambda_I) & c(\lambda_I) & 0 \\ -c(\phi_c)c(\lambda_I) & -c(\phi_c)s(\lambda_I) & -s(\phi_c) \end{bmatrix}$$

where λ_I and ϕ_c are respectively the inertial longitude and geocentric latitude of the vehicle. The matrix C_{GI} is the result of two consecutive rotations: first rotation λ_I around Z_I-axis and the second rotation $\phi_c + 90°$ around $-Y_I'$-axis. More precisely,

$$C_{GI} = \begin{bmatrix} c(\phi_c + \pi/2) & 0 & s(\phi_c + \pi/2) \\ 0 & 1 & 0 \\ -s(\phi_c + \pi/2) & 0 & c(\phi_c + \pi/2) \end{bmatrix} \begin{bmatrix} c(\lambda_I) & s(\lambda_I) & 0 \\ -s(\lambda_I) & c(\lambda_I) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The transformation matrix

$$C_{IG} = C_{GI}^T.$$

5 Computing the acceleration a_{xI}, a_{yI}, a_{zI}

The three forces acting on the vehicle are: the propulsion force, the aerodynamic force, and the gravitation force. The first two forces are naturally expressed in $\mathbf{S}_B$ coordinate and the gravitation force can be conveniently expressed in $\mathbf{S}_I$ coordinate.

Let $F_{TxB}, F_{TyB}, F_{TzB}$ and $F_{AxB}, F_{AyB}, F_{AzB}$ are respectively the propulsion force components and aerodynamic force components in the $\mathbf{S}_B$ coordinate. Then the (total) acceleration components in $\mathbf{S}_I$ coordinate can be computed as

$$\begin{bmatrix} a_{xI} \\ a_{yI} \\ a_{zI} \end{bmatrix} = \left(\frac{1}{m}\right) C_{IB} \begin{bmatrix} F_{TxB} + F_{AxB} \\ F_{TyB} + F_{AyB} \\ F_{TzB} + F_{AzB} \end{bmatrix} + \begin{bmatrix} g_{xI} \\ g_{yI} \\ g_{zI} \end{bmatrix} \tag{5.1}$$

where m is the mass of the vehicle and g_{xI}, g_{yI}, g_{zI} are the components of the gravitational acceleration in $\mathbf{S}_I$ coordinate.

The gravitational acceleration of the vehicle depends on the position (x_I, y_I, z_I), and would be in the radial direction for spherical earth model. The expressions for the components are more complicated if an oblate earth model is adopted. Based on the 1960 Fisher Earth Model, the gravitational acceleration components in the $\mathbf{S}_I$ coordinate are given by [2]

$$\begin{bmatrix} g_{xI} \\ g_{yI} \\ g_{zI} \end{bmatrix} = \begin{bmatrix} -\mu \frac{x_I}{r_I^3} \left[1 + \frac{3}{2} J_2 \left(\frac{R_e}{r_I} \right)^2 \left(1 - 5 \left(\frac{z_I}{r_I} \right)^2 \right) \right] \\ -\mu \frac{y_I}{r_I^3} \left[1 + \frac{3}{2} J_2 \left(\frac{R_e}{r_I} \right)^2 \left(1 - 5 \left(\frac{z_I}{r_I} \right)^2 \right) \right] \\ -\mu \frac{z_I}{r_I^3} \left[1 + \frac{3}{2} J_2 \left(\frac{R_e}{r_I} \right)^2 \left(3 - 5 \left(\frac{z_I}{r_I} \right)^2 \right) \right] \end{bmatrix} \tag{5.2}$$

where J_2 is the second gravitational harmonic, μ is the gravitational constant, and

$$r_I = \sqrt{x_I^2 + y_I^2 + z_I^2} \tag{5.3}$$

is radial distance from the earth center to the vehicle.

6 Auxiliary variables

Many variables of interest can be computed from data obtained from the 6DOF simulation. They include: longitude, latitude, altitude, Mach number, speed, dynamic pressure, aerodynamic angles, Euler angles, flight-path and azimuth angles.

6.1 Longitude, latitude and altitude

Integrating the translational equations, we obtain the inertial position (x_I, y_I, z_I) of the vehicle. From this we need to determine the geocentric longitude λ, the geodetic latitude ϕ_g and the altitude h. We describe two methods for the computation: one approximate and the other exact.

(I) Approximate method [2]

The inertial longitude is computed as

$$\lambda_I = \tan^{-1} \left(\frac{y_I}{x_I} \right). \tag{6.1}$$

The geocentric longitude is then given by

$$\lambda = \lambda_I - \Omega_p t. \tag{6.2}$$

The geocentric latitude is computed as

$$\phi_c' = \tan^{-1}\left(\frac{z_I}{\sqrt{x_I^2 + y_I^2}}\right). \tag{6.3}$$

The geodetic latitude is then given by

$$\phi_g' = \tan^{-1}\left(k \tan \phi_c'\right) \tag{6.4}$$

where $k = (R_e/R_p)^2$. The altitude is computed as

$$h'(t) = r_I - \frac{R_e}{1 + (k-1)\sin^2 \phi_c'} \tag{6.5}$$

where r_I is defined in (5.3).

(II) Exact method

The intersection of the reference spheroid and the Meridian plane containing the vehicle is a two-dimensional ellipsoid, the subvehicle point is the point on the ellipsoid that is closest to the vehicle. The subvehicle point must be solved by an iterative method since an analytical expression does not exist [5, p.227]. We propose here to formulate the problem as a minimization problem.

Let $r_z = \sqrt{x_I^2 + y_I^2}$. The position components of the vehicle on the Meridian plane is (r_z, z_I). A generic point on the ellipsoid can be expressed as

$$(R_e \cos\gamma, R_p \sin\gamma)$$

for some angle γ. Define the distance squared

$$D(\gamma) = \left(r_z - R_e \cos\gamma\right)^2 + \left(z_I - R_p \sin\gamma\right)^2. \tag{6.6}$$

The problem is then to find γ so that D is minimized. Let D^* and γ^* be respectively the minimum value of D and the corresponding angle. Then the altitude

$$h = \begin{cases} \sqrt{D^*}, & r_z \geq R_e \cos\gamma^*; \\ -\sqrt{D^*}, & r_z < R_e \cos\gamma^*. \end{cases} \tag{6.7}$$

and the geodetic latitude is given by [1, p.97]

$$\phi_g = \tan^{-1}\left(\sqrt{k} \tan\gamma^*\right). \tag{6.8}$$

Finally the geocentric longitude can be computed using (6.1) and (6.2).

6.2 Range

The range of the vehicle, $R(t)$, at time t is defined as the distance between the launch site and the subvehicle point at time t. The distance is measured along the earth surface. This distance can be computed if we know the angle between two radial vectors: one from the earth center to the launch site and one from the earth center to the vehicle. If we denote the angle by $\theta_R(t)$, then

$$R(t) = R_a(t)\theta_R(t)$$

where $R_a(t)$ is the average radius of the earth. The angle can be computed from the S_I coordinates of the vehicle and the launch site. Let $x_{ls}(t), y_{ls}(t), z_{ls}(t)$ be the S_I coordinates of the launch site at time t, then

$$\theta_R(t) = \cos^{-1}\left(\frac{x_{ls}(t)x_I(t) + y_{ls}(t)y_I(t) + z_{ls}(t)z_I(t)}{\sqrt{x_{ls}(t)^2 + y_{ls}(t)^2 + z_{ls}(t)^2}\sqrt{x_I(t)^2 + y_I(t)^2 + z_I(t)^2}}\right).$$

The average radius to the surface of the oblate spheroid computed defined as

$$R_a(t) = \frac{R_s(0) + R_s(t)}{2} \tag{6.9}$$

where

$$R_s(t) = \frac{R_e}{1 + (k-1)\sin^2\phi_c'(t)}$$

and ϕ_c' is defined in (6.3).

6.3 Velocity, speed, dynamic pressure and Mach number

The velocity of the vehicle relative to earth is computed as

$$\overrightarrow{V_R} = \overrightarrow{V_I} - \overrightarrow{\Omega_p} \times \overrightarrow{r_I}.$$

In component form this becomes

$$\begin{bmatrix} v_{RxI} \\ v_{RyI} \\ v_{RzI} \end{bmatrix} = \begin{bmatrix} v_{xI} \\ v_{yI} \\ v_{zI} \end{bmatrix} - \begin{bmatrix} 0 & -\Omega_p & 0 \\ \Omega_p & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_I(t) \\ y_I(t) \\ z_I(t) \end{bmatrix}. \tag{6.10}$$

The atmospheric relative velocity in S_I coordinate is

$$\begin{bmatrix} v_{AxI} \\ v_{AyI} \\ v_{AzI} \end{bmatrix} = \begin{bmatrix} v_{RxI} \\ v_{RyI} \\ v_{RzI} \end{bmatrix} - \begin{bmatrix} v_{WxI} \\ v_{WyI} \\ v_{WzI} \end{bmatrix}, \tag{6.11}$$

where

$$\begin{bmatrix} v_{WxI} \\ v_{WyI} \\ v_{WzI} \end{bmatrix} = C_{IG} \begin{bmatrix} v_{WxG} \\ v_{WyG} \\ v_{WzG} \end{bmatrix}$$

is the wind velocity components in $\mathbf{S}_I$ coordinate. The wind velocity is an environmental parameter whose components $(v_{WxG}, v_{WyG}, v_{WzG})$ in $\mathbf{S}_G$ coordinate are specified (usually as tables of values which depend on the altitude.)

The atmospheric relative velocity in $\mathbf{S}_B$ coordinate is compute as

$$\begin{bmatrix} v_{AxB} \\ v_{AyB} \\ v_{AzB} \end{bmatrix} = C_{BI} \begin{bmatrix} v_{AxI} \\ v_{AyI} \\ v_{AzI} \end{bmatrix}. \tag{6.12}$$

The atmospheric relative *speed* V_A is computed as

$$V_A = \sqrt{v_{AxB}^2 + v_{AyB}^2 + v_{AzB}^2} \;\; (= \sqrt{v_{AxI}^2 + v_{AyI}^2 + v_{AzI}^2}.) \tag{6.13}$$

By definition the dynamic pressure is

$$q_d = \frac{1}{2}\rho V_A^2, \tag{6.14}$$

where ρ is the air density. The *Mach number* is

$$M = \frac{V_A}{c_s}. \tag{6.15}$$

where c_s is the speed of sound. We note that c_s is a function of altitude.

6.4 Angle of attack and sideslip angle

The *angle of attack* α is defined (and computed) as

$$\alpha = \tan^{-1}\left(\frac{v_{AzB}}{v_{AxB}}\right). \tag{6.16}$$

The *sideslip angle* β is defined as

$$\beta = \sin^{-1}\left(\frac{v_{AyB}}{v_A}\right). \tag{6.17}$$

6.5 The Euler angles

It follows from (4.1) that

$$\phi = \tan^{-1}\left(\frac{C_{BL}(2,3)}{C_{BL}(2,2)}\right) \tag{6.18}$$

$$\psi = -\sin^{-1}\left(C_{BL}(2,1)\right) \tag{6.19}$$

$$\theta = \tan^{-1}\left(\frac{C_{BL}(3,1)}{C_{BL}(1,1)}\right) \tag{6.20}$$

where $C_{BL}(i,j)$ is the ijth entry of C_{BL}. In terms of the Euler parameters, the equations become

$$\phi = \tan^{-1}\left(\frac{2(e_0e_1 + e_2e_3)}{e_0^2 - e_1^2 + e_2^2 - e_3^2}\right) \tag{6.21}$$

$$\psi = -\sin^{-1}\left(2(e_1e_2 - e_0e_3)\right) \tag{6.22}$$

$$\theta = \tan^{-1}\left(\frac{2(e_1e_3 + e_0e_2)}{e_0^2 + e_1^2 - e_2^2 - e_3^2}\right) \tag{6.23}$$

Remark: If we had solved the Euler equation (3.13), we would have the angles directly from the solution.

6.6 Flight-path and Azimuth angles

The flight-path angle γ of the vehicle is defined as the angle between the velocity vector (relative to earth) and the local horizontal plane. The angle is positive if the velocity vector points 'above' the horizontal plane, thus the vehicle is gaining altitude. The azimuth angle A_Z of the vehicle is the angle from the projection of the velocity vector on the local horizontal plane and the local north, positive counterclockwise.

To compute γ and A_Z we first compute the earth relative velocity components in the $\mathbf{S}_G$ coordinate:

$$\begin{bmatrix} v_{RxG} \\ v_{RyG} \\ v_{RzG} \end{bmatrix} = C_{GI} \begin{bmatrix} v_{RxI} \\ v_{RyI} \\ v_{RzI} \end{bmatrix}. \tag{6.24}$$

Then the flight-path angle is

$$\gamma = -\sin^{-1}\left(\frac{v_{RzG}}{\sqrt{v_{RxG}^2 + v_{RyG}^2 + v_{RzG}^2}}\right) \tag{6.25}$$

and the azimuth angle is

$$A_Z = \tan^{-1}\left(\frac{v_{RyG}}{v_{RxG}}\right). \tag{6.26}$$

Finally we list the constants used in the simulation. Constants such as equatorial radius, polar radius listed in standard textbooks are often different although the differences are not significant. The constants listed here, adopted from the 1960 Fisher Earth Model [2], are used in the simulation.

Symbol	value	unit	description
R_e	6.378165857×10^6	m	equatorial radius
R_p	6.356783832×10^6	m	polar radius
Ω_p	7.29211×10^{-5}	rad/s	earth angular rate
e	0.0818139		eccentricity of earth
μ	$3.9860319541 \times 10^{14}$	m^3/s^2	gravitational constant
J_2	0.0010823		2nd gravitational harmonic
k	1.0067386		$(R_e/R_p)^2$

7 Summary and discussion

We have described in some detail the subsystem models that have been developed and implemented in Simulink. The modules implemented so far have been tested numerically. Subsystem modules currently under development are

- Guidance law: Both mid-course guidance law and terminal guidance law are needed. Mid-course law uses radar measurements for the determination of time-to-go t_{go} and the predicted interception point (PIP) which are needed for the generation of guidance command. A recursive algorithm for compute t_{go} and PIP was proposed in [11] and used in [**?**] to generate a three-dimensional mid-course guidance law for ballistic target interception. We studied and numerically tested the laws proposed in the papers. We proposed modifications to the algorithm and is currently implementing it. The terminal guidance laws we considered are the conventional true proportional navigation (TPN) law [9], which is also currently being implemented.

- Control systems: Control systems are needed during the mid-course and homing phases. The actuators used in these two phases are different. Thrust vectors are used in the mid-course phase for both attitude and directional control, which in the terminal phase on-off thrusters are used. Control system design is the topic of next year's research. We will initially consider simple linear control with rate and angle feedback and move on to more advanced design as needed.

- Aerodynamic model: The aerodynamic forces and moments are computed using the aerodynamic coefficients. We will use tables of coefficients that are available to use and hopefully be able to refine it as the work progresses on.

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