行政院國家科學委員會補助專題研究計畫 □ **成果報告** ☑**期中進度報告(精簡版)**

六種新渾沌系統及三種新型的渾沌同步之研究(第一年)

計畫類別：☑ 個別型計畫　　□ 整合型計畫
計畫編號：NSC 96－2221－E－009－144－MY3
執行期間：96 年 8 月 1 日至 97 年 7 月 31 日

計畫主持人：戈正銘
共同主持人：
計畫參與人員： 張晉銘，李彥賢，江峻宇，陳聰文

成果報告類型(依經費核定清單規定繳交)：☑期中進度精簡報告□完整報告

本成果報告包括以下應繳交之附件：
□赴國外出差或研習心得報告一份
□赴大陸地區出差或研習心得報告一份
□出席國際學術會議心得報告及發表之論文各一份
□國際合作研究計畫國外研究報告書一份

處理方式：除產學合作研究計畫、提升產業技術及人才培育研究計畫、列管計畫及下列情形者外，得立即公開查詢
□涉及專利或其他智慧財產權，□一年□二年後可公開查詢

執行單位：國立交通大學機械工程學系
中　華　民　國　97　年　5　月　21　日

中文摘要

關鍵詞：Mathieu 系統，新自治雙 Mathieu 系統，新非自治雙 Mathieu 系統，純誤差穩定性，廣義渾沌同步，精緻李雅普諾夫函數

渾沌系統之研究在物理、化學、生物學、生理學、各種工程等方面皆有日益重要之廣泛應用。非線性 Mathieu 系統是典型的重要渾沌系統。本計畫(第一年)採取適當的耦合方式將它們推廣為兩種新雙 Mathieu 系統，即新自治雙 Mathieu 系統與新非自治雙 Mathieu 系統。研究其渾沌性質，從而就典型重要渾沌系統而言，既擴大其研究範圍也深化其研究內容。渾沌同步之研究在秘密通訊、神經網路、自組織、物理系統、生態系統、工程系統等方面有長足之應用。本計畫(第一年)提出一種新型的渾沌同步，具有重要理論及實際意義，即純誤差穩定的廣義同步法。用以改進目前需先由數值計算預先得出誤差方程中渾沌變量之最大值之有缺陷之方法。研究重點為：

1. 兩種雙 Mathieu 系統之渾沌研究。用相圖、分歧圖、功率譜圖、李雅普諾夫指數、碎形維度等分析渾沌之行為。
2. 採用純誤差穩定理論及精緻之李雅普諾夫函數得出純誤差穩定的廣義同步法。並以對兩種雙 Mathieu 系統為例實現此種廣義同步。

英文摘要

key words: Mathieu system, New autonomous double Mathieu system, New nonautonomous double Mathieu system, Chaos, Pure error stability, Generalized chaos synchronization, Elaborate Lyapunov function

The study of chaotic system has found wide applications in physics, chemistry, biology, physiology, and various engineerings. Nonlinear Mathieu system is a paradigmatic important chaotic system. In this project (first year), the study is extended to two kinds of double Mathieu system by suitable coupling. For these paradigmatic and important systems, the study will be extended and deepened.

Chaos synchronizations are applied in various regions, such as secure communication, neural networks, self-organization, physical systems, ecological systems and engineering systems, etc. In this project (first year), a new type of chaos synchronization with theoretical and practical importance are studied, i.e. pure error stability synchronization, to improve the present defective method in which the maximum values of state variables appeared in error dynamics must be preliminarily calculated by simulations. The main parts of our study are:

1. The study of chaos of two kinds of double Mathieu system. By phase portraits, bifurcation diagrams, power spectra, Lyapunov exponents, fractal dimensions, the various chaotic behaviors of these systems are studied.
2. By pure error stability theory and elaborate Lyapunov functions, the pure error generalized synchronization method is given, proved and illustrated by two kinds of double Mathieu systems.

報告內容

(一)前言及研究目的：

渾沌系統之研究除了在理論上的重要價值外，在物理、化學、生理學及各種工程等方面皆有廣泛之應用。非線性 Mathieu 系統是重要的典型渾沌系統。對於這個系統的極重要的渾沌現象及渾沌同步都已有豐富的研究成果，直到現在，這個重要典型系統仍為研究熱點[1-7]。本計畫(第一年)為了對這個著名系統，擴大其研究範圍並深化其研究內容，特提出兩種新系統，即新自治雙 Mathieu 系統與新非自治雙 Mathieu 系統。首先証明其為渾沌系統，其次研究其渾沌行為。渾沌同步之研究在秘密通訊、神經網路、自我組織等方面有長足之應用[8-61]。本計畫(第一年)研究一種新的渾沌同步方式及其對這些新系統的應用。

(二)研究方法及文獻探討：

(a)兩種雙Mathieu系統的渾沌行為

經典的非線性Mathieu系統是

$$\ddot{x}+a(1+\sin\omega t)x+(1+\sin\omega t)x^3+a\dot{x}=0$$

或

$$\begin{aligned}\dot{x}_1&=x_2\\ \dot{x}_2&=-a(1+\sin\omega t)x_1-(1+\sin\omega t)x_1^3-ax_2\end{aligned}$$

其中 a,ω 為常數。本計畫提出的創新系統，其一為自治的雙Mathieu系統

$$\begin{aligned}\dot{x}_1&=x_2\\ \dot{x}_2&=-a(1+x_4)x_1-(1+x_4)x_1^3-ax_2+bx_3\\ \dot{x}_3&=x_4\\ \dot{x}_4&=-(1+x_2)x_3-a(1+x_2)x_3^3-ax_4+bx_1\end{aligned}$$

其中將兩個Mathieu系統的 $\sin\omega t$ 交替換成對方的渾沌狀態 x_4，x_2，並在第二、四式之末加上耦合項 bx_3，bx_1。

其二為非自治的雙Mathieu系統

$$\begin{aligned}\dot{x}_1&=x_2\\ \dot{x}_2&=-a(1+\sin\omega t)x_1-(1+\sin\omega t)x_1^3-ax_2+bx_3\\ \dot{x}_3&=x_4\\ \dot{x}_4&=-(1+\sin\omega t)x_3-a(1+\sin\omega t)x_3^3-ax_4+bx_1\end{aligned}$$

其中保留 $\sin\omega t$，僅在第二、四式之末加上耦合項 bx_3，bx_1。

同樣地，對它們的研究，一方面是對經典的單Mathieu系統研究之擴展及深化，且一方面它們也一定比單Mathieu系統更具複雜性。本計畫研究其週期運動、準週期運動、渾沌運動及超渾沌運動。

(b)純誤差穩定的廣義同步及其對兩種雙Mathieu系統的應用

考慮最一般形式的主從系統

$$\begin{aligned}\dot{x}&=f(t,x)\\ \dot{y}&=f(t,y)+u(t,x,y)\end{aligned}$$

其中 $x,y\in R^n$ 為主從狀態向量，$f:R_+\times R^n\to R^n$ 為非線性向量函數，$u:R_+\times R\times R^n\to R^n$ 是控制向量。廣義同步指 $y=g(t,x)$，其中 g 為指定函數。$e=y-g(t,x)$ 是廣義同步誤差向量。

誤差動力學為

$$\dot{e} = \dot{y} - \dot{g}(t,x)$$
$$= \dot{y} - \frac{\partial g(t,x)}{\partial x}\dot{x} - \frac{\partial g(t,x)}{\partial t}$$

加上控制項後得：

$$\dot{e} = f(t,y) - \frac{\partial g(t,x)}{\partial x} f(t,x) - \frac{\partial g(t,x)}{\partial t} + u(t,x,y)$$

目前文獻中所用之Lyapunov函數千篇一律地採用$V = \frac{1}{2}e^T e$平方和的形式[62-71]，此種做法實為對Lyapunov直接法之極端自我窄化的做法。其實Lyapunov函數之形式千變萬化，運用得法，可得出人意外的佳績。今採用一個精緻的Lyapunov函數

$$V(t,e) = \frac{1}{2}e^T \Lambda(t) e$$
$$= \frac{1}{2}\lambda_{11}(t)e_1^2 + \cdots + \frac{1}{2}\lambda_{nn}(t)e_n^2$$

其中$\Lambda(t) = [\lambda_{ij}(t)] \in R^{n\times n}$為待求之可逆對角矩陣，$\lambda_{ij}(t)$皆為時間之函數。

上式可寫成

$$\dot{V}(t,e) = G_1(\lambda_{11},\dot{\lambda}_{11})e_1^2 + \cdots + G_n(\lambda_{nn},\dot{\lambda}_{nn})e_n^2$$
$$+[H_1(\lambda_{11},\cdots,\lambda_{nn},x_1,\cdots,x_n,y_1,\cdots,y_n,t) + \lambda_{11}u_1]e_1 + \cdots$$
$$+[H_n(\lambda_{11},\cdots,\lambda_{nn},x_1,\cdots,x_n,y_1,\cdots,y_n,t) + \lambda_{nn}u_n]e_n$$

其中G_i, H_i為連續可微函數，u_i為待求之控制器。此式可分為兩類：(1)所有G_i與λ_{ii}及$\dot{\lambda}_{ii}$有關。(2)一些G_j與$\lambda_{jj},\dot{\lambda}_{jj}$有關，其他$G_k$僅與$\dot{\lambda}_{kk}$有關。

對第(1)情況，設計u_i使

$$H_i + \lambda_{ii}u_i = 0 \qquad (i = 1,2,\cdots,n)$$

則$\dot{V}$中之狀態變量x_i, y_i皆不存在，乃得純誤差動力學。現在文獻中採用之$\dot{V}$中皆含x_i, y_i狀態變量，為了保証$\dot{V}$之負定性，或為了得出混沌同步條件，必須依賴數值計算，算出x_i, y_i之最大值[72-78]。此方法有三缺點：1.同步理論其實並不限於兩渾沌系統間之同步，非渾沌系統之同步亦極有研究價值，其中包括x_i或y_i趨於無限大之非週期運動。即此時狀態變量不存在有限之最大值，故現在文獻所用方法成為無效。2.如果$\dot{V}$中出現之狀態變量之最大值很大，則保證$\dot{V}$為負定之條件將變得極為保守，而無足可取。3.需要以數值模擬計算之結果為條件之理論推導為有缺陷之理論(defective theory)，價值較低。

由上式，如令λ_{ii}滿足

$$\forall t \ge 0, \quad 0 < \lambda_{mii} \le \lambda_{ii}(t) \le \lambda_{Mii} \quad (i = 1,\cdots,n)$$

則可得

$$\forall t \ge 0, \quad G_i(\lambda_{ii},\dot{\lambda}_{ii}) < 0 \quad (i = 1,\cdots,n)$$

即$\dot{V}$為負定。Lyapunov函數乃得到。

對第(2)情況，則設

$$\forall k,\quad \lambda_{kk}=1$$
$$\forall k,\quad H_k+\lambda_{kk}u_k=-e_k$$
$$\forall j,\quad H_j+\lambda_{jj}u_j=0$$

則可得純誤差動力學。再巧妙地適當設計u_i及λ_{jj}使

$$\forall t\ge 0,\quad 0<\lambda_{mjj}\le\lambda_{jj}(t)\le\lambda_{Mjj}$$
$$\forall t\ge 0,\quad G_j(\lambda_{jj},\dot{\lambda}_{jj})<0$$

即可得負定之$\dot{V}$，Lyapunov函數乃得到。

本計畫將對兩種雙Mathieu系統給出廣義同步$y_i=\alpha(t)x_i+\beta(t)$，其中$\alpha(t)$，$\beta(t)$為給定時間函數。由於每步設計都有賴於經驗及技巧的發揮，故難度較高。

(三)討論與結果：

Duffing系統，van der Pol系統與線性Mathieu系統原為振動學科之最重要最典型的系統。自渾沌動力學興起後，Duffing系統，van der Pol系統由於其為非線性系統故亦沿習成為渾沌動力學學科中最重要最典型的系統，四十年來對此二系統的渾沌研究之文獻可謂汗牛充棟，至今方興未艾。而線性Mathieu系統，則由於其為線性方程，不具渾沌性質，故在渾沌動力學學科中乃不再提及。人們忽視了非線性Mathieu系統實為Duffing系統中參數由常數轉為時間週期函數之推廣，實亦應成為渾沌動力學科之最重要最典型之系統。本計畫主持人率先研究非線性Mathieu系統之渾沌行為[6]，可謂遲來之補求。眾所週知，此三種典型系統除理論意義外，廣泛應用於機械、電機、物理、化學、生科、奈米系統，本計畫(第一年)研究雙種類型的非線性Mathieu系統，不僅對渾沌動力學學科中最重要最典型的渾沌系統的研究的拓廣與深化，更重要的是它們本身顯然具有更複雜的，未經發現的複雜渾沌行為，本研究對渾沌動力學學科具重大意義。其應用於機械、電機、物理、化學、生科、奈米之耦合系統，具有重要的實用價值。

渾沌同步除本身之重要理論價值外，其研究在秘密通訊、神經網路、自我組織等方面有日益廣泛之應用。廣義渾沌同步則為渾沌同步之進一步發展，其應用亦方興未艾。本計畫(第一年)提出新的純誤差渾沌同步。現行文獻中同步方法之三缺點已如前述，不再多贅。

兩種雙 Mathieu 系統的渾沌行為與純誤差穩定的廣義同步及其對此二系統的應用：

1. 研究獲得諸多相圖、分歧圖、功率譜圖、參數圖及李亞普諾夫指數及碎形維度等研究自治的雙 Mathieu 系統之週期運動、準週期運動、渾沌運動及超渾沌運動各種行為。

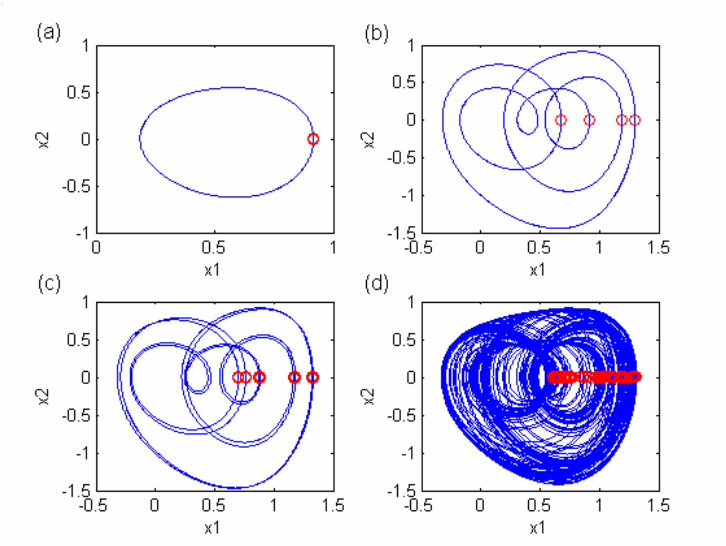

Fig. 1 Phase portraits and Poincaré maps for autonomous double Mathieu system: (a) period 1 for $b = 1.1$, (b) period 4 for $b = 1.243$, (c) period 8 for $b = 1.246$, (d) chaotic for $b = 1.24$.

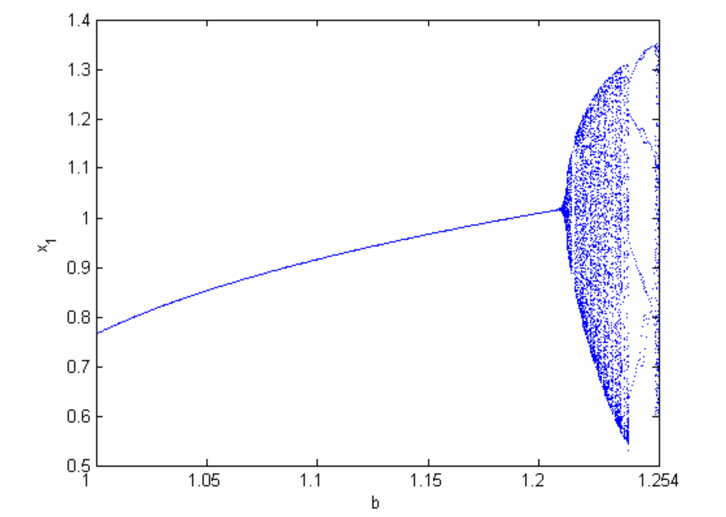

Fig. 2 Bifurcation diagram for autonomous double Mathieu system.

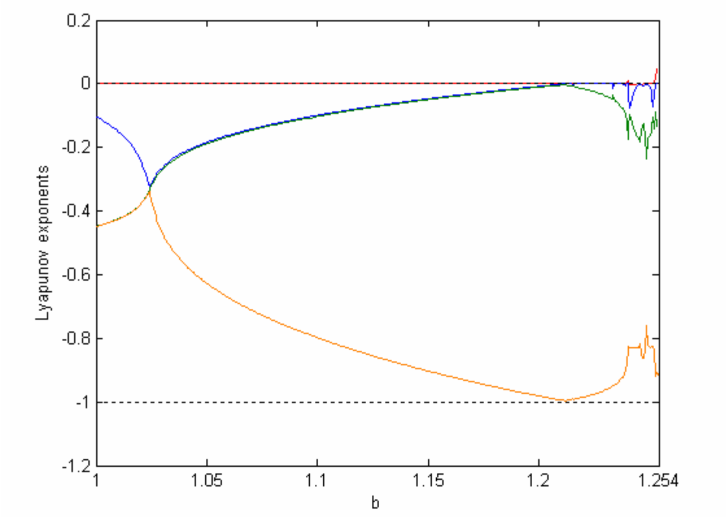

Fig. 3 Lyapunov exponents for autonomous double Mathieu system.

2. 研究獲得諸多相圖、分歧圖、功率譜圖、參數圖及李亞普諾夫指數及碎形維度等研究非自治的雙 Mathieu 系統之週期運動、準週期運動、渾沌運動及超渾沌運動各種行為。

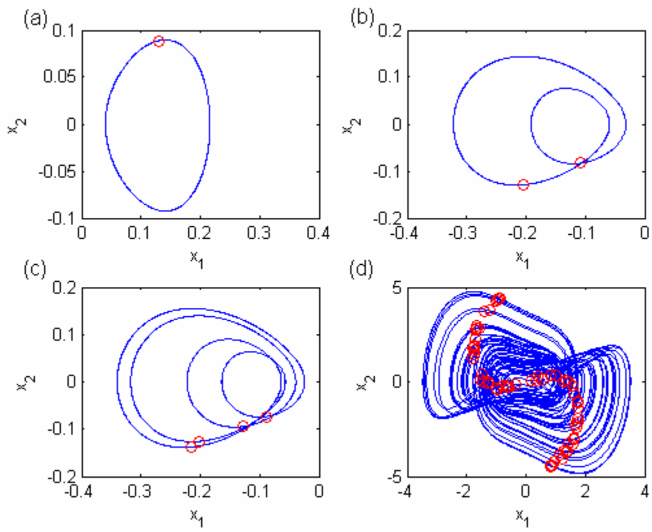

Fig. 4 Phase portraits and Poincaré maps for nonautonomous double Mathieu system: (a) period 1 for $b=0.9$, (b) period 2 for $b=0.93$, (c) period 4 for $b=0.934$, (d) chaotic for $b=1$.

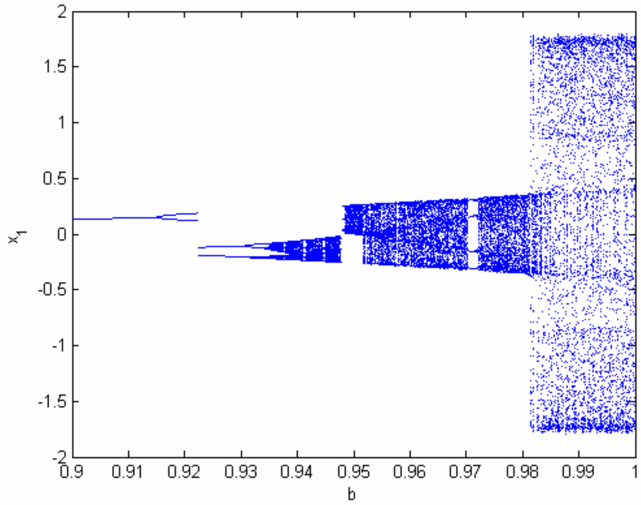

Fig 5. Bifurcation diagram for nonautonomous double Mathieu system.

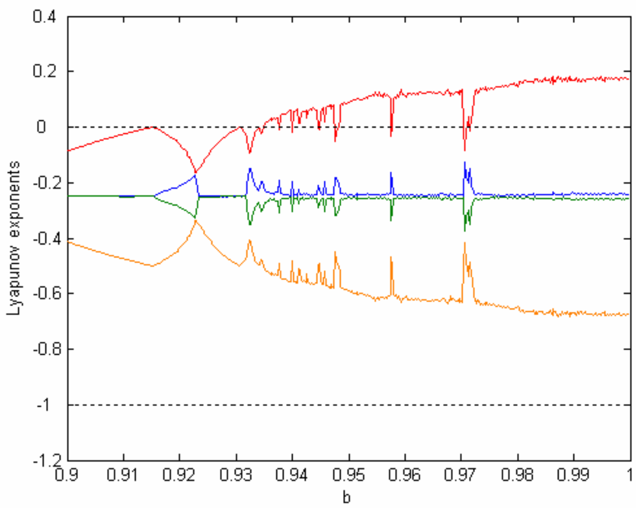

Fig. 6 Lyapunov exponents for nonautonomous double Mathieu system.

3. 研究獲得純誤差穩定的廣義同步法對自治的雙 Mathieu 系統之應用。

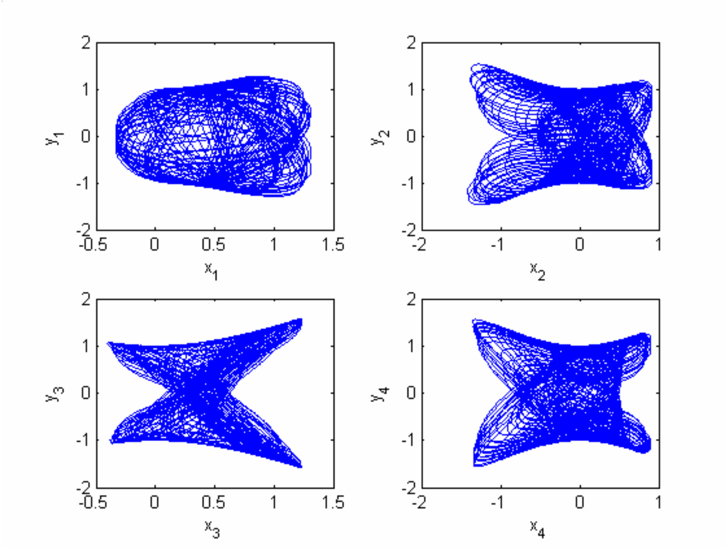

Fig. 7 Phase portraits of x_i to y_i $(i = 1,\cdots,4)$ when generalized synchronization is obtained.

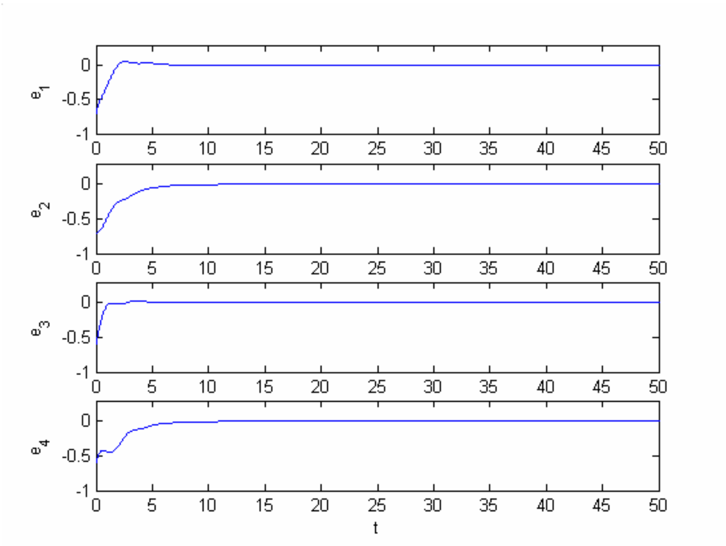

Fig. 8 Time histories of errors.

4. 研究獲得純誤差穩定的廣義同步法對非自治的雙 Mathieu 系統之應用。

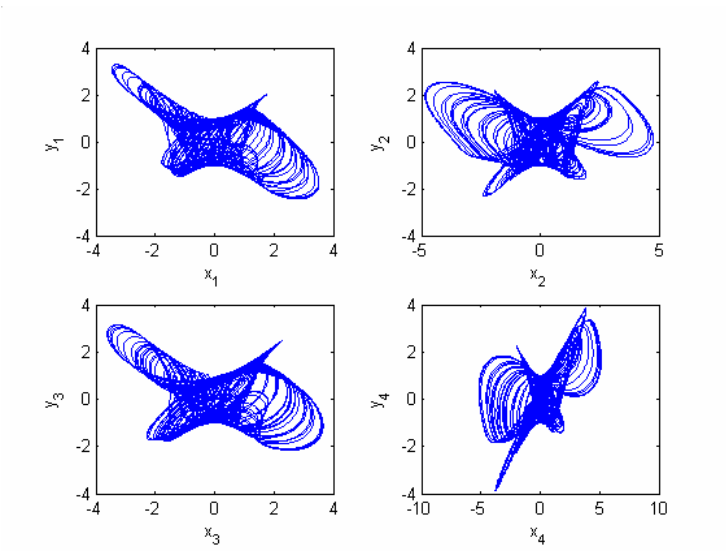

Fig. 9 Phase portraits of x_i to y_i $(i = 1,\cdots,4)$ when generalized synchronization is obtained.

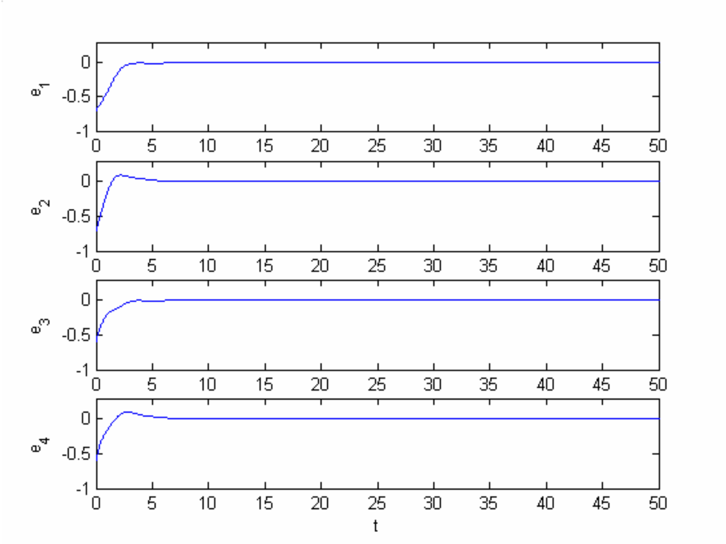

Fig. 10 Time histories of errors.

參考文獻

1. É. L. Mathieu, 1868, "Mémoire sur le mouvement vibratoire d'une membrane de forme elliptique", J. Math. Pures Appl., Vol. 13, pp. 137–203.
2. M. Mond, G. Cederbaum, P. B. Khan, and Y. Zarmi, 1993, "Stability Analysis Of The Non-Linear Mathieu Equation", Journal of Sound and Vibration, Vol. 167, pp. 77-89.
3. J. W. Norris, 1994, "The Nonlinear Mathieu Equation", International Journal of Bifurcation and Chaos, Vol. 4, pp. 71-86.
4. Yusry O. El-Dib, 2001, "Nonlinear Mathieu Equation and Coupled Resonance Mechanism", Chaos, Solitons and Fractals, Vol. 12, pp. 705-720.
5. Leslie Ng and Richard Rand, 2002, "Bifurcations in a Mathieu Equation with Cubic Nonlinearities", Chaos, Solitons and Fractals, Vol. 14, pp. 173-181.
6. Zheng-Ming Ge and Chang-Xian Yi, 2007, "Chaos in a Nonlinear Damped Mathieu System, in a Nano Resonator System and in Its Fractional Order Systems", Chaos, Solitons and Fractals, Vol. 32, pp. 42-61.
7. Zheng-Ming Ge and Chang-Xian Yi, 2006, "Parameter Excited Chaos Synchronization of Integral and Fractional Order Nano Resonator System", accepted by Mathematical Methods, Physical Models and Simulation in Sicience & Technology.
8. G. Álvarez, Shujun Li, F. Montoya, G. Pastor, and M. Romera, 2005, "Breaking Projective Chaos Synchronization Secure Communication Using Filtering and Generalized Synchronization", Chaos, Solitons and Fractals, Vol. 24, pp. 775-783.
9. M. Feki, 2003, "An Adaptive Chaos Synchronization Scheme Applied to Secure Communication", Chaos, Solitons and Fractals, Vol. 18, pp. 141-148.
10. Zhigang Li and Daolin Xu, 2004, "A Secure Communication Scheme Using Projective Chaos Synchronization", Chaos, Solitons and Fractals, Vol. 22, pp. 477-481.
11. Chil-Min Kim, Won-Ho Kye, Sunghwan Rim, and Soo-Young Lee, 2004, "Communication Key Using Delay Times in Time-Delayed Chaos Synchronization", Physics Letters A Vol.

333, pp. 235-240.
12. J. Zhou, H. B. Huang, G. X. Qi, P. Yang, and X. Xie, 2005, “Communication with Spatial Periodic Chaos Synchronization”, Physics Letters A, Vol. 335, pp. 191-196.
13. Atsushi Uchida and Shigeru Yoshimori, 2004, “Synchronization of Chaos in Microchip Lasers and Its Communication Applications”, Comptes Rendus Physique, Vol. 5, pp. 643-656.
14. Qing Yun Wang, Qi Shao Lu, Guan Rong Chen, and Ding Hui Guo, 2006, “Chaos Synchronization of Coupled Neurons with Gap Junctions”, Physics Letters A, Vol. 356, pp. 17-25.
15. E. N. Sanchez, L. J. Ricalde, 2003, “Chaos Control and Synchronization, with Input Saturation, via Recurrent Neural Networks”, Neural Networks, Vol. 16, pp. 711-717.
16. Jinhu Lü, Xinghuo Yu, and Guanrong Chen, 2004, “Chaos Synchronization of General Complex Dynamical Networks”, Physica A, Vol. 334, pp. 281-302.
17. Yao-Chen Hung, Ming-Chung Ho, Jiann-Shing Lih, and I-Min Jiang, 2006, “Chaos Synchronization of Two Stochastically Coupled Random Boolean Networks”, Physics Letters A, Vol. 356, pp. 35-43.
18. Xiaoming Zhang, Maolin Fu, Jinghua Xiao, Gang Hu, 2006, “Self-Organization of Chaos Synchronization and Pattern Formation in Coupled Chaotic Oscillators”, Physical Review E, Vol. 74, pp. 015202-1-4.
19. A. Raffone and C. van Leeuwen, 2003 “Dynamic Synchronization and Chaos in an Associative Neural Network with Multiple Active Memories”, Chaos, Vol. 13, pp. 1090-104.
20. J. M. Liu, H. F. Chen, S. Tang, 2001, “Synchronization of Chaos in Semiconductor Lasers”, Nonlinear Analysis, Vol. 47, pp. 5741-5751.
21. Kanako Suzuki, Yoh Imai, 2004, “Periodic Chaos Synchronization in Slave Subsystems Using Optical Fiber Ring Resonators”, Optics Communications, Vol. 241, pp. 507-512.
22. Jianping Yan and Changpin Li, 2007, “On Chaos Synchronization of Fractional Differential Equations”, Chaos, Solitons and Fractals, Vol. 32, pp. 725-735.
23. Er-Wei Bai, Karl E. Lonngren, and J. C. Sprott, 2002, “On the Synchronization of a Class of Electronic Circuits that Exhibit Chaos”, Chaos, Solitons and Fractals, Vol. 13, pp. 1515-1521.
24. A. Barsella and C. Lepers, 2002, “Chaotic Lag Synchronization and Pulse-Induced Transient Chaos in Lasers Coupled by Saturable Absorber”, Optics Communications, Vol. 205, pp. 397-403.
25. Y. Zhang, S. Q. Hu, and G. H. Du, 1999, “Chaos Synchronization of Two Parametrically Excited Pendulums”, Journal of Sound and Vibration, Vol. 223, pp. 247-254.
26. Liguo Luo, P. L. Chu, T. Whitbread, and R. F. Peng, 2000, “Experimental Observation of Synchronization of Chaos in Erbium-Doped Fiber Lasers”, Optics Communications, Vol. 176, pp. 213-217.
27. A. Uchida, S. Kinugawa, and S. Yoshimori, 2003, “Synchronization of Chaos in Two Microchip Lasers by Using Incoherent Feedback Method”, Chaos, Solitons and Fractals, Vol. 17, pp. 363-368.
28. P. Saha, S. Banerjee, and A. R. Chowdhury, 2002, “Some Aspects of Synchronization and Chaos in a Coupled Laser System”, Chaos, Solitons and Fractals, Vol. 14, pp. 1083-1093.

29. Fan Zhang and Pak L. Chu, 2004, “Effect of Coupling Strength on Chaos Synchronization Generated by Erbium-Doped Fiber Ring Laser”, Optics Communications, Vol. 237, pp. 213-219.
30. Yan-Ni Li, Lan Chen, Zun-Sheng Cai, and Xue-Zhuang Zhao, 2003, “Study on Chaos Synchronization in the Belousov–Zhabotinsky Chemical System”, Chaos, Solitons and Fractals, Vol. 17, pp. 699-707.
31. Yan-Ni Li, Lan Chen, Zun-Sheng Cai, and Xue-zhuang Zhao, 2004, “Experimental Study of Chaos Synchronization in the Belousov–Zhabotinsky chemical system”, Chaos, Solitons and Fractals, Vol. 22, pp. 767-771.
32. A. Kittel, J. Parisi, and K. Pyragas, 1998, “Generalized Synchronization of Chaos in Electronic Circuit Experiments”, Physica D, Vol. 112, pp. 459-471.
33. A. Ucar, K. E. Lonngren, and Er-Wei Bai, 2007, “Chaos Synchronization in RCL-Shunted Josephson Junction via Active Control”, Chaos, Solitons and Fractals, Vol. 31, pp. 105-111.
34. F. T. Arecchi, R. Meucci, A. Di Garbo, and E. Allaria, 2003, “Homoclinic Chaos in a Laser: Synchronization and Its Implications in Biological Systems”, Optics and Lasers in Engineering, Vol. 39, pp. 293-304.
35. Y. Imai, H. Murakawa, and T. Imoto, 2003, “Chaos Synchronization Characteristics in Erbium-Doped Fiber Laser Systems”, Optics Communications, Vol. 217, pp. 415-420.
36. Qingxian Xie, Guanrong Chen, E. M. Bollt, 2002, “Hybrid Chaos Synchronization and Its Application in Information Processing”, Mathematical and Computer Modelling, Vol. 35, pp. 145-163.
37. R. McAllister, A. Uchida, R. Meucci, R. Roy, 2004, “Generalized Synchronization of Chaos: Experiments on a Two-Mode Microchip Laser with Optoelectronic Feedback”, Physica D, Vol. 195, pp. 244-262.
38. A. Shabunin, V. Astakhov, V. Demidov, A. Provata, F. Baras, G. Nicolis, and V. Anishchenko, 2003, “Modeling Chemical Reactions by Forced Limit-Cycle Oscillator: Synchronization Phenomena and Transition to Chaos”, Chaos, Solitons and Fractals, Vol. 15, pp. 395-405.
39. Z.-M. Ge and T.-N. Lin, 2001, “Chaos, Chaos Control and Synchronization of Gyrostat System”, Journal of Sound and Vibration, Vol. 251, pp.519-542.
40. Zheng-Ming Ge, Tsung-Chih Yu and Yen-Sheng Chen, 2003, “Chaos Synchronization of a Horizontal Platform System”, Journal of Sound and Vibration, Vol. 268, pp. 731-749.
41. Z.-M. Ge and T.-N. Lin, 2003, “Chaos, Chaos Control and Synchronization of Electro-Mechanical Gyrostat System”, Journal of Sound and Vibration, Vol.259, pp. 585-603.
42. Z.-M. Ge and Hong-Wen Wu, 2004, "Chaos Synchronization and Chaos Anticontrol of a Suspended Track with Moving Loads", Journal of Sound and Vibration, Vol. 270, pp. 685-712.
43. Zheng-Ming Ge and Yen-Sheng Chen, 2004, “Synchronization of Unidirectional Coupled Chaotic Systems via Partial Stability”, Chaos, Solitons and Fractals, Vol. 21, pp. 101-111.
44. Zheng-Ming Ge, Chia-Yang Yu and Yen-Sheng Chen, 2004, “Chaos Synchronization and Anticontrol of a Rotationally Supported Simple Pendulum”, JSME International Journal, Series C, Vol. 47, No. 1, pp. 233-241.

45. Zheng-Ming Ge and Wei-Ying Leu, 2004, “Anti-Control of Chaos of Two-degrees-of-Freedom Louderspeaker System and Chaos Synchronization of Different Order Systems”, Chaos, Solitons & Fractals, Vol. 20, pp.503-521.
46. Zheng-Ming Ge and Chien-Cheng Chen, 2004, “Phase Synchronization of Coupled Chaotic Multiple Time Scales Systems”, Chaos, Solitons & Fractals, Vol. 20, pp. 639-647.
47. Z.-M. Ge and C.-M. Chang, 2004, “Chaos Synchronization and Parameters Identification of Single Time Scale Brushless DC Motors”, Chaos, Solitons and Fractals, Vol. 20, pp. 883-903.
48. Zheng-Ming Ge and Wei-Ying Leu, 2004, “Chaos Synchronization and Parameter Identification for Loudspeaker System”, Chaos, Solitons & Fractals, Vol. 21, pp. 1231-1247.
49. Zheng-Ming Ge, Chui-Chi Lin and Yen-Sheng Chen, 2004, “Chaos, Chaos Control and Synchronization of Vibrometer System”, Journal of Mechanical Engineering Science, Vol.218, pp.1001-1020.
50. Zheng-Ming Ge, Jui-Wen Cheng and Yen-Sheng Chen, 2004, “Chaos Anticontrol and Synchronization of Three Time Scales Brushless DC Motor System”, Chaos, Solitons & Fractals Vol. 22, pp.1165-1182.
51. Zheng-Ming Ge and Jui-Kai Lee, 2005, “Chaos Synchronization and Parameter Identification for Gyroscope System”, Applied Mathematics and Computation, Vol. 163, pp. 667-682.
52. Z.-M. Ge and C.-I Lee, 2005, “Anticontrol and Synchronization of Chaos for an Autonomous Rotational Machine System with a Hexagonal Centrifugal Governor”, Journal of Sound and Vibration Vol. 282, pp. 635-648.
53. Zheng-Ming Ge and Ching-I Lee, 2005, “Control, Anticontrol and Synchronization of Chaos for an Autonomous Rotational Machine System with Time-Delay”, Chaos, Solitons and Fractals Vol.23, pp.1855-1864.
54. Zheng-Ming Ge and Jui-Wen Cheng, 2005, “Chaos Synchronization and Parameter Identification of Three Time Scales Brushless DC Motor System”, Chaos, Solitons and Fractals Vol. 24, pp.597-616.
55. Zheng-Ming Ge, Cheng-Hsiung Yang, 2005, “The Generalized Synchronization of Quantum-CNN Chaotic Oscillator with Different Order Systems”, accepted by Chaos, Solitons and Fractals.
56. Zheng-Ming Ge, Yen-Sheng Chen, 2005, “Adaptive Synchronization of Unidirectional and Mutual Coupled Chaotic Systems”, Chaos, Solitons and Fractals. Vol. 26, pp. 881-888.
57. Z.-M. Ge, C.-M. Chang, Y.-S. Chen, 2006, “Anti-Control of Chaos of Single Time Scale Brushless DC Motor and Chaos Synchronization of Different Order Systems”, Chaos, Solitons and Fractals, Vol. 27, pp.1298-1315
58. Zheng-Ming Ge and Guo-Hua Lin, 2007, “The Complete, Lag and Anticipated Synchronization of a BLDCM Chaotic System”, Chaos, Solitons and Fractals, Vol. 34, pp. 740-764.
59. Zheng-Ming Ge and Yen-Sheng Chen, 2007, “Synchronization of Mutual Coupled Chaotic Systems via Partial Stability Theory”, Chaos, Solitons and Fractals, Vol. 34, pp. 787-794.
60. Zheng-Ming Ge and Wei-Ren Jhuang, 2007, “Chaos, Control and Synchronization of a Fractional Order Rotational Mechanical System with a Centrifugal Governor”, Chaos,

Solitons and Fractals, Vol. 33, pp. 270-289.
61. Zheng-Ming Ge and Cheng-Hsiung Yang, 2007, "Synchronization of Complex Chaotic Systems in Series Expansion Form", Chaos, Solitons and Fractals, Vol. 34, pp. 1649-1658.
62. Shihua Chen, Feng Wang, and Changping Wang, 2004, "Synchronizing Strict-Feedback and General Strict-Feedback Chaotic Systems via a Single Controller", Chaos, Solitons and Fractals, Vol. 20, pp. 235-243.
63. Wenxiang Xie, Changyun Wen, and Zhengguo Li, 2000, "Impulsive Control for the Stabilization and Synchronization of Lorenz Systems", Physics Letters A, Vol. 275, pp. 67-72.
64. Maoyin Chen, Donghua Zhou, and Yun Shang, 2005, "Synchronizing a Class of Uncertain Chaotic Systems", Physics Letters A, Vol. 337, pp. 384-390.
65. Xiaohui Tan, Jiye Zhang, and Yiren Yang, 2003, "Synchronizing Chaotic Systems Using Backstepping Design", Chaos, Solitons and Fractals, Vol. 16, pp. 37-45.
66. M. T. Yassen, 2007, "Controlling, Synchronization and Tracking Chaotic Liu System Using Active Backstepping Design", Physics Letters A, Vol. 360, pp. 582-587.
67. Changpin Li and Jianping Yan, 2006, "Generalized Projective Synchronization of Chaos: The Cascade Synchronization Approach", Chaos, Solitons and Fractals, Vol. 30, pp. 140-146.
68. Jianping Yan and Changpin Li, 2005, "Generalized Projective Synchronization of a Unified Chaotic System", Chaos, Solitons and Fractals, Vol. 26, pp. 1119-1124.
69. Guo Hui Li, Shi Ping Zhou, and Kui Yang, 2006, "Generalized Projective Synchronization Between Two Different Chaotic Systems Using Active Backstepping Control", Physics Letters A, Vol. 355, pp. 326-330.
70. Gang Zhang, Zengrong Liu, and Zhongjun Ma, 2007, "Generalized Synchronization of Different Dimensional Chaotic Dynamical Systems", Chaos, Solitons and Fractals, Vol. 32, pp. 773-779.
71. I. Belykh, V. Belykh, and M. Hasler, 2006, "Generalized Connection Graph Method for Synchronization in Asymmetrical Networks", Physica D, Vol. 224, pp. 42-51.
72. Guo-Ping Jiang, Wallace Kit-Sang Tang, and Guanrong Chen, 2003, "A Simple Global Synchronization Criterion for Coupled Chaotic Systems", Chaos, Solitons and Fractals, Vol. 15, pp. 925-935.
73. Yanwu Wang, Zhi-Hong Guan, and Xiaojiang Wen, 2004, "Adaptive Synchronization for Chen Chaotic System with Fully Unknown Parameters", Chaos, Solitons and Fractals, Vol. 19, pp. 899-903.
74. Yinping Zhang and Jitao Sun, 2004, "Delay-Dependentent Stability Criterion for Coupled Chaotic Systems via Unidirectional Linear Error Feedback Approach", Chaos, Solitons and Fractals, Vol. 22, pp. 199-205.
75. Yongguang Yu and Suochun Zhang, 2004, "The Synchronization of Linearly Bidirectional Coupled Chaotic Systems", Chaos, Solitons and Fractals, Vol. 22, pp. 189-197.
76. Ju H. Park, 2005, "Stability Criterion for Synchronization of Linearly Coupled Unified Chaotic Systems", Chaos, Solitons and Fractals, Vol. 23, pp. 1319-1325.
77. E. M. Elabbasy, H. N. Agiza, and M. M. El-Dessoky, 2005, "Global Synchronization Criterion and Adaptive Synchronization for New Chaotic System", Chaos, Solitons and

Fractals, Vol. 23, pp. 1299-1309.
78. Damei Li, Jun-An Lu, and Xiaoqun Wu, 2005, “Linearly Coupled Synchronization of the Unified Chaotic Systems and the Lorenz Systems”, Chaos, Solitons and Fractals, Vol. 23, pp. 79-85.

計畫(第一年)成果自評

非線性 Mathieu 系統，作為最典型最重要之渾沌系統，自 Lorenz1963 年發現渾沌現象以來未受到應有之關注與研究，此實為渾沌研究之一大缺憾。計畫主持人繼研究非線性 Mathieu 系統以後，在本計畫(第一年)對新自治雙 Mathieu 系統及新非自治雙 Mathieu 系統之渾沌性質作全面而詳盡之研究，對渾沌研究而言，其重要性不言而喻。純誤差渾沌廣義同步理論之提出糾正了當下流行之廣義同步之三缺點：1.同步理論其實並不限於兩渾沌系統間之同步，非渾沌系統之同步亦極有研究價值，其中包括 x_i 或 y_i 趨於無限大之非週期運動。即此時狀態變量不存在有限之最大值，故現在文獻所用方法成為無效。2.如果 $\dot{V}$ 中出現之狀態變量之最大值很大，則保證 $\dot{V}$ 為負定之條件將變得極為保守，而無足可取。3.需要以數值模擬計算之結果為條件之理論推導為有缺陷之理論(defective theory)，價值較低。故此理論有重大意義。已投出之國際著名論文已達 3 篇，其中已被接受者 1 篇。

1. Z. M. Ge and C. M. Chang, “Nonlinear Generalized Synchronization of Chaotic Systems by Pure Error Dynamics and Elaborate Nondiagonal Lyapunov Function”, 2007, accepted by Chaos, Solitons and Fractals. (SCI, Impact Factor: 2.042)
2. Z. M. Ge and C. M. Chang, “Generalized Synchronization of Chaotic Systems by Pure Error Dynamics and Elaborate Lyapunov Function”, 2007, submitted to Nonlinear Analysis: Theory, Methods, and Applications.
3. Z. M. Ge and C. M. Chang, “Chaos of Nonholonomic Moving Target Tracking Problems”, 2008, submitted to Physics Letters A.

另外由本計畫經費贊助已出版之國際期刊論文 2 篇；見附錄。

附錄

Paper List:

1. Zheng-Ming Ge and Cheng-Hsiung Yang, 2007, “Symplectic Synchronization of Different Chaotic Systems”, accepted by Chaos, Solitons and Fractals. (SCI, Impact factor: 2.042).
2. Zheng-Ming Ge and Pu-Chien Tzen, 2007, “Chaos Synchronization by Variable Strength Linear Coupling and Lyapunov Function Derivative in Series Form”, accepted by Nonlinear Analysis: Theory, Mehtods, and Applications. (SCI, Impact factor: 0.677).

Available online at www.sciencedirect.com
ScienceDirect
Chaos, Solitons and Fractals xxx (2007) xxx–xxx
CHAOS SOLITONS & FRACTALS
www.elsevier.com/locate/chaos

Symplectic synchronization of different chaotic systems

Zheng-Ming Ge *, Cheng-Hsiung Yang

Department of Mechanical Engineering, National Chiao Tung University, Hsinchu 300, Taiwan, ROC

Accepted 29 October 2007

Communicated by Prof. Ji-Huan He

Abstract

In this paper, a new symplectic synchronization of chaotic systems is studied. Traditional generalized synchronizations are special cases of the symplectic synchronization. A sufficient condition is given for the asymptotical stability of the null solution of an error dynamics. The symplectic synchronization may be applied to the design of secure communication. Finally, numerical results are studied for a Quantum-CNN oscillators synchronized with a Rössler system in three different cases.

1. Introduction

Many approaches have been presented for the synchronization of chaotic systems [2–6]. There are a chaotic master system and either an identical or a different slave system. Our goal is the synchronization of the chaotic master and the chaotic slave by coupling or by other methods.

Among many kinds of synchronizations [7], generalized synchronization is investigated [8–12]. There exists a functional relationship between the states of the master and that of the slave. In this paper, a new synchronization

$$y = H(x, y, t) + F(t) \tag{1}$$

is studied, where x, y are the state vectors of the "master" and of the "slave", respectively, $F(t)$ is a given function of time in different form, such as a regular or a chaotic function. When $H(x, y, t) = x$, Eq. (1) reduces to the generalized synchronization given in [1]. Therefore this paper is an extension of [1].

In Eq. (1), the final desired state y of the "slave" system not only depends upon the "master" system state x but also depends upon the "slave" system state y itself. Therefore the "slave" system is not a traditional pure slave obeying the "master" system completely but plays a role to determine the final desired state of the "slave" system. In other words, it plays an "interwined" role, so we call this kind of synchronization "symplectic synchronization"[1], and call the "master" system partner A, the "slave" system partner B.

* Corresponding author. Tel.: +886 3 5712121; fax: +886 3 5720634.
E-mail address: zmg@cc.nctu.edu.tw (Z.-M. Ge).

[1] The term "symplectic" comes from the Greek for "interwined". H. Weyl first introduced the term in 1939 in his book "The Classical Groups" (p. 165 in both the first edition, 1939, and second edition, 1946, Princeton University Press).

0960-0779/$ - see front matter
doi:10.1016/j.chaos.2007.10.055

When $H(x,y,t) = H(x,t)$, Eq. (1) becomes

$$y = H(x,t) + F(t) \tag{2}$$

which reduces to generalized synchronization. Therefore generalized synchronization is a special case of the symplectic synchronization. There exists great potential of the application of the symplectic synchronization. For instance, when the symplectically synchronized chaotic signal is used as a signal carrier, the secure communication is more difficult to be deciphered.

As numerical examples, recently developed Quantum Cellular Neural Network (Quantum-CNN) chaotic oscillator is used to synchronize with different systems, respectively. Quantum-CNN oscillator equations are derived from a Schrödinger equation taking into account quantum dots cellular automata structures to which in the last decade a wide interest has been devoted, with particular attention towards quantum computing [13].

This paper is organized as follows. In Section 2, by the Lyapunov asymptotical stability theorem, a symplectic synchronization scheme is given. In Section 3, various feedback controllers are designed for the symplectic synchronization of the Quantum-CNN oscillator and a Rössler system. Numerical simulations are also given in Section 3. Finally, some concluding remarks are given in Section 4.

2. Symplectic synchronization scheme

There are two different nonlinear chaotic systems. The partner A controls the partner B partially. The partner A is given by

$$\dot{x} = f(x) \tag{3}$$

where $x = [x_1, x_2, \ldots, x_n]^T \in R^n$ is a state vector and f is a vector function.

The partner B is given by

$$\dot{y} = g(y) \tag{4a}$$

where $y = [y_1, y_2, \ldots, y_n]^T \in R^n$ is a state vector, and g is a vector function different from f.

After a controller $u(t)$ is added, partner B becomes

$$\dot{y} = g(y) + u(t) \tag{4b}$$

where $u(t) = [u_1(t), u_2(t), \ldots, u_n(t)]^T \in R^n$ is the control vector.

Our goal is to design the controller $u(t)$ so that the state vector y of the partner B asymptotically approaches $H(x,y,t) + F(t)$, a given function $H(x,y,t)$ plus a given vector function $F(t) = [F_1(t), F_2(t), \ldots, F_n(t)]^T$ which is a regular or a chaotic function of time. Define error vector $e(t) = [e_1, e_2, \ldots, e_n]^T$:

$$e = H(x,y,t) - y + F(t) \tag{5}$$

$$\lim_{t\to\infty} e = 0 \tag{6}$$

is demanded.

From Eq. (5), it is obtained that

$$\dot{e} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} + \frac{\partial H}{\partial t} - \dot{y} + \dot{F}(t) \tag{7}$$

By Eqs. (3), (4a) and (4b), (7) becomes

$$\dot{e} = \frac{\partial H}{\partial x}f(x) + \frac{\partial H}{\partial y}g(y) + \frac{\partial H}{\partial t} - g(y) - u(t) + \dot{F}(t) \tag{8}$$

A positive definite Lyapnuov function $V(e)$ is chosen:

$$V(e) = \frac{1}{2}e^T e \tag{9}$$

Its derivative along any solution of Eq. (8) is

$$\dot{V}(e) = e^T\left\{\frac{\partial H}{\partial x}f(x) + \frac{\partial H}{\partial y}g(y) + \frac{\partial H}{\partial t} - g(y) + \dot{F}(t) - u(t)\right\}. \tag{10}$$

In Eq. (10), $u(t)$ is designed so that $\dot{V} = e^T C_{n\times n} e$ where $C_{n\times n}$ is a diagonal negative definite matrix. $\dot{V}$ is a negative definite function of e. By Lyapunov theorem of asymptotical stability

$$\lim_{t\to\infty} e = 0$$

The symplectic synchronization is obtained [14–16].

3. Numerical results for the symplectic chaos synchronization of Quantum-CNN oscillator and Rössler System

Case I: A cubic symplectic synchronization

For a two-cell Quantum-CNN, following differential equations are obtained [13]

$$\begin{cases} \dot{x}_1 = -2a_1\sqrt{1-x_1^2}\sin x_2 \\ \dot{x}_2 = -\omega_1(x_1-x_3) + 2a_1\dfrac{x_1}{\sqrt{1-x_1^2}}\cos x_2 \\ \dot{x}_3 = -2a_2\sqrt{1-x_3^2}\sin x_4 \\ \dot{x}_4 = -\omega_2(x_3-x_1) + 2a_2\dfrac{x_3}{\sqrt{1-x_3^2}}\cos x_4 \end{cases} \tag{11}$$

where x_1, x_3 are polarizations, x_2, x_4 are quantum phase displacements, a_1 and a_2 are proportional to the inter-dot energy inside each cell and ω_1 and ω_2 are the parameters that weigh the effects on the cell of the difference of polarization of the neighboring cells, like the cloning templates in traditional CNNs. When $a_1 = 19.4$, $a_2 = 13.1$, $\omega_1 = 9.529$ and $\omega_2 = 7.94$, the system is chaotic.

A chaotic Rössler system is described by

$$\begin{cases} \dot{y}_1 = -y_2 - y_3 \\ \dot{y}_2 = y_1 - \alpha y_2 + y_4 \\ \dot{y}_3 = y_1 y_3 + \beta \\ \dot{y}_4 = \gamma y_3 + \sigma y_4 \end{cases} \tag{12}$$

where $\alpha = 0.5$, $\beta = 0.52$, $\gamma = 0.5$, $\sigma = 0.05$.

For symplectic synchronization of these two systems, u_1, u_2, u_3 and u_4 are added to the four equations of Eq. (12), respectively:

$$\begin{cases} \dot{y}_1 = -y_2 - y_3 + u_1 \\ \dot{y}_2 = y_1 - \alpha y_2 + y_4 + u_2 \\ \dot{y}_3 = y_1 y_3 + \beta + u_3 \\ \dot{y}_4 = \gamma y_3 + \sigma y_4 + u_4 \end{cases} \tag{13}$$

The initial values of the states of the Quantum-CNN system and of the Rössler system are taken as $x_1(0) = 0.8$, $x_2(0) = -0.77$, $x_3(0) = -0.72$, $x_4(0) = 0.57$, $y_1(0) = 0.3$, $y_2(0) = -0.4$, $y_3(0) = -0.7$ and $y_4(0) = 0.15$.

We take $F_1(t) = x_4^3(t)$, $F_2(t) = x_1^3(t)$, $F_3(t) = x_2^3(t)$, and $F_4(t) = x_3^3(t)$. They are chaotic functions of time. $H_i(x,y,t) = -x_i^2 y_i$ $(i = 1,2,3,4)$ are given. By Eq. (6) we have

$$\lim_{t\to\infty} e_i = \lim_{t\to\infty}(-x_i^2 y_i - y_i + x_j^3) = 0, \quad i = 1,2,3,4 \quad j = \begin{cases} 4, & i = 1 \\ i-1, & i \neq 1 \end{cases} \tag{14}$$

From Eq. (7) we have

$$\dot{e}_i = -2\dot{x}_i x_i y_i - x_i^2 \dot{y}_i - \dot{y}_i + 3\dot{x}_j x_j^2, \quad i = 1,2,3,4 \quad j = \begin{cases} 4, & i = 1 \\ i-1, & i \neq 1 \end{cases} \tag{15}$$

Eq. (8) can be expressed as

$$\begin{aligned} \dot{e}_1 = {} & 2y_1x_1\left(2a_1\sqrt{1-x_1^2}\sin x_2\right) + (y_2+y_3)x_1^2 + y_2 + y_3 - u_1 \\ & + 3x_4^2\left(-\omega_2(x_3-x_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4\right) \end{aligned}$$

Please cite this article in press as: Ge Z-M, Yang C-H, Symplectic synchronization of different chaotic systems, Chaos, Solitons & Fractals (2007), doi:10.1016/j.chaos.2007.10.055

$$\begin{aligned}
\dot{e}_2 &= -2y_2x_2\left(-\omega_1(x_1-x_3)+2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2\right)-(y_1-\alpha y_2+y_4)x_2^2\\
&\quad -y_1+\alpha y_2-y_4-u_2+3x_1^2\left(-2a_1\sqrt{1-x_1^2}\sin x_2\right)\\
\dot{e}_3 &= 2y_3x_3\left(2a_2\sqrt{1-x_3^2}\sin x_4\right)-(y_1y_3+\beta)x_2^3-y_1y_3-\beta-u_3\\
&\quad +3x_2^2\left(-\omega_1(x_1-x_3)+2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2\right)\\
\dot{e}_4 &= -2y_4x_4\left(-\omega_2(x_3-x_1)+2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4\right)-(\gamma y_3+\sigma y_4)x_4^2-\gamma y_3\\
&\quad -\sigma y_4-u_4+3x_3^2\left(-2a_2\sqrt{1-x_3^2}\sin x_4\right)
\end{aligned}$$

where $e_1=-x_1^2y_1-y_1+x_4^3$, $e_2=-x_2^2y_2-y_2+x_1^3$, $e_3=-x_3^2y_3-y_3+x_2^3$ and $e_4=-x_4^2y_4-y_4+x_3^3$.

Choose a positive definite Lyapunov function:

$$V(e_1,e_2,e_3,e_4)=\frac{1}{2}(e_1^2+e_2^2+e_3^2+e_4^2) \tag{17}$$

Its time derivative along any solution of Eq. (16) is

$$\begin{aligned}
\dot{V} &= e_1\left\{2y_1x_1\left(2a_1\sqrt{1-x_1^2}\sin x_2\right)+(y_2+y_3)x_1^2+y_2+y_3\right.\\
&\quad \left.+3x_4^2\left(-\omega_2(x_3-x_1)+2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4\right)-u_1\right\}\\
&\quad +e_2\left\{-2y_2x_2\left(-\omega_1(x_1-x_3)+2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2\right)-(y_1-\alpha y_2+y_4)x_2^2\right.\\
&\quad \left.-y_1+\alpha y_2-y_4+3x_1^2\left(-2a_1\sqrt{1-x_1^2}\sin x_2\right)-u_2\right\}\\
&\quad +e_3\left\{2y_3x_3\left(2a_2\sqrt{1-x_3^2}\sin x_4\right)-(y_1y_3+\beta)x_2^3-y_1y_3-\beta\right.\\
&\quad \left.+3x_2^2\left(-\omega_1(x_1-x_3)+2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2\right)-u_3\right\}\\
&\quad +e_4\left\{-2y_4x_4\left(-\omega_2(x_3-x_1)+2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4\right)-(\gamma y_3+\sigma y_4)x_4^2-\gamma y_3\right.\\
&\quad \left.-\sigma y_4+3x_3^2\left(-2a_2\sqrt{1-x_3^2}\sin x_4\right)-u_4\right\}
\end{aligned} \tag{18}$$

Choose

$$\begin{aligned}
u_1 &= 2y_1x_1\left(2a_1\sqrt{1-x_1^2}\sin x_2\right)+(y_2+y_3)x_1^2+y_2+y_3\\
&\quad +3x_4^2\left(-\omega_2(x_3-x_1)+2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4\right)-y_1x_1^2-y_1+x_4^3\\
u_2 &= -2y_2x_2\left(-\omega_1(x_1-x_3)+2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2\right)-(y_1-\alpha y_2+y_4)x_2^2\\
&\quad -y_1-y_4+3x_1^2\left(-2a_1\sqrt{1-x_1^2}\sin x_2\right)-\alpha(y_2x_2^2-x_1^3)
\end{aligned}$$

$$
\begin{aligned}
u_3 &= 2y_3x_3\left(2a_2\sqrt{1-x_3^2}\sin x_4\right) - (y_1y_3+\beta)x_2^3 - y_1y_3 - \beta \\
&\quad + 3x_2^2\left(-\omega_1(x_1-x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2\right) - y_3x_3^2 - y_3 + x_2^3 \\
u_4 &= -2y_4x_4\left(-\omega_2(x_3-x_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4\right) - (\gamma y_3+\sigma y_4)x_4^2 - \gamma y_3 \\
&\quad + 3x_3^2\left(-2a_2\sqrt{1-x_3^2}\sin x_4\right) - \sigma(y_4x_4^2 + 2y_4 - x_3^3)
\end{aligned}
$$

Eq. (18) becomes

$$\dot V = -(e_1^2 + \alpha e_2^2 + e_3^2 + \sigma e_4^2) < 0 \tag{19}$$

which is negative definite. The Lyapunov asymptotical stability theorem is satisfied. Cubic symplectic synchronization of the Quantum-CNN system and the Rössler system is achieved. The numerical results are shown in Fig. 1. After 5 s, the motion trajectories enter a chaotic attractor.

Case II: A time delay symplectic synchronization

We take $F_1(t) = x_1(t-T)$, $F_2(t) = x_2(t-T)$, $F_3(t) = x_3(t-T)$ and $F_4(t) = x_4(t-T)$. They are chaotic functions of time, where time delay $T = 1$ s is a positive constant. $H_i(x, y, t) = (x_i^2 + y_i)(e^{-t} + 2)$ $(i = 1, 2, 3, 4)$ are given. By Eq. (6) we have

$$\lim_{t\to\infty} e_i = \lim_{t\to\infty}((x_i^2 + y_i)(e^{-t} + 2) - y_i + x_i(t-T)) = 0, \quad i = 1, 2, 3, 4 \tag{20}$$

From Eq. (7) we have

$$\dot e_i = (2x_i\dot x_i + \dot y_i)(e^{-t} + 2) - e^{-t}(x_i^2 + y_i) - \dot y_i + \dot x_i(t-T), \quad i = 1, 2, 3, 4 \tag{21}$$

Eq. (8) is expressed as

$$
\begin{aligned}
\dot e_1 &= 2x_1\left(-2a_1\sqrt{1-x_1^2}\sin x_2\right)(e^{-t}+2) + (-y_2-y_3)(e^{-t}+2) - (x_1^2+y_1)e^{-t} \\
&\quad + y_2 + y_3 - u_1 - 2a_1\sqrt{1-x_1^2(t-T)}\sin x_2(t-T) \\
\dot e_2 &= 2x_2\left(-\omega_1(x_1-x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2\right)(e^{-t}+2) + (y_1-\alpha y_2+y_4)(e^{-t}+2) \\
&\quad - (x_2^2+y_2)e^{-t} - y_1 + \alpha y_2 - y_4 - u_2 - \omega_1(x_1(t-T) - x_3(t-T)) \\
&\quad + 2a_1\frac{x_1(t-T)}{\sqrt{1-x_1^2(t-T)}}\cos x_2(t-T) \\
\dot e_3 &= 2x_3\left(-2a_2\sqrt{1-x_3^2}\sin x_4\right)(e^{-t}+2) + (y_1y_3+\beta)(e^{-t}+2) - (x_3^2+y_3)e^{-t} \\
&\quad - y_1y_3 - \beta - u_3 - 2a_2\sqrt{1-x_3^2(t-T)}\sin x_4(t-T) \\
\dot e_4 &= 2x_4\left(-\omega_2(x_3-x_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4\right)(e^{-t}+2) + (\gamma y_3+\sigma y_4)(e^{-t}+2) \\
&\quad - (x_4^2+y_4)e^{-t} - \gamma y_3 - \sigma y_4 - u_4 - \omega_2(x_3(t-T) - x_1(t-T)) \\
&\quad + 2a_2\frac{x_3(t-T)}{\sqrt{1-x_3^2(t-T)}}\cos x_4(t-T)
\end{aligned}
\tag{22}
$$

where $e_1 = (x_1^2 + y_1)(e^{-t} + 2) - y_1 + x_1(t-T)$, $e_2 = (x_2^2 + y_2)(e^{-t} + 2) - y_2 + x_2(t-T)$, $e_3 = (x_3^2 + y_3)(e^{-t} + 2) - y_3 + x_3(t-T)$, $e_4 = (x_4^2 + y_4)(e^{-t} + 2) - y_4 + x_4(t-T)$.

Choose a positive definite Lyapunov function:

$$V(e_1, e_2, e_3, e_4) = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + e_4^2) \tag{23}$$

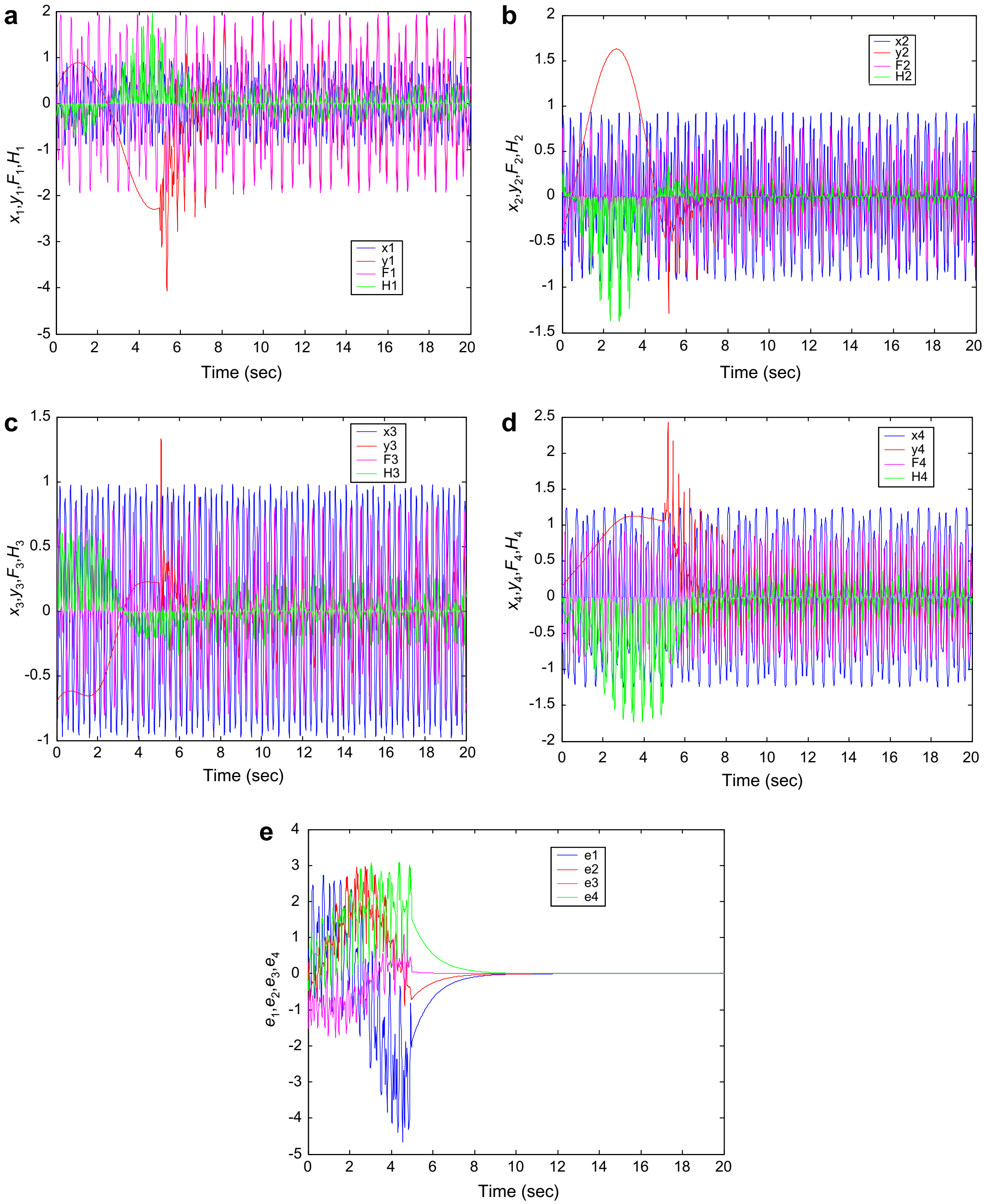

Fig. 1. Time histories of states, state errors, F_1, F_2, F_3, F_4, H_1, H_2, H_3 and H_4 for Case I.

Please cite this article in press as: Ge Z-M, Yang C-H, Symplectic synchronization of different chaotic systems, Chaos, Solitons & Fractals (2007), doi:10.1016/j.chaos.2007.10.055

Its time derivative along any solution of Eq. (22) is

$$\begin{aligned}\dot{V} = {} & e_1\Big\{2x_1\Big(-2a_1\sqrt{1-x_1^2}\sin x_2\Big)(e^{-t}+2)+(-y_2-y_3)(e^{-t}+2)-(x_1^2+y_1)e^{-t}\\ & +y_2+y_3-2a_1\sqrt{1-x_1^2(t-T)}\sin x_2(t-T)-u_1\Big\}\\ & +e_2\Big\{2x_2\Big(-\omega_1(x_1-x_3)+2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2\Big)(e^{-t}+2)+(y_1-\alpha y_2+y_4)(e^{-t}+2)-(x_2^2+y_2)e^{-t}\\ & -y_1+\alpha y_2-y_4-\omega_1(x_1(t-T)-x_3(t-T))+2a_1\frac{x_1(t-T)}{\sqrt{1-x_1^2(t-T)}}\cos x_2(t-T)-u_2\Big\}\\ & +e_3\Big\{2x_3\Big(-2a_2\sqrt{1-x_3^2}\sin x_4\Big)(e^{-t}+2)+(y_1y_3+\beta)(e^{-t}+2)-(x_3^2+y_3)e^{-t}\\ & -y_1y_3-\beta-2a_2\sqrt{1-x_3^2(t-T)}\sin x_4(t-T)-u_3\Big\}\\ & +e_4\Big\{2x_4\Big(-\omega_2(x_3-x_1)+2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4\Big)(e^{-t}+2)+(\gamma y_3+\sigma y_4)(e^{-t}+2)-(x_4^2+y_4)e^{-t}\\ & -\gamma y_3-\sigma y_4-\omega_2(x_3(t-T)-x_1(t-T))+2a_2\frac{x_3(t-T)}{\sqrt{1-x_3^2(t-T)}}\cos x_4(t-T)-u_4\Big\}\end{aligned} \quad (24)$$

Choose

$$\begin{aligned}u_1 = {} & 2x_1\Big(-2a_1\sqrt{1-x_1^2}\sin x_2\Big)(e^{-t}+2)+(-y_2-y_3)(e^{-t}+2)-(x_1^2+y_1)e^{-t}+y_2+y_3\\ & -2a_1\sqrt{1-x_1^2(t-T)}\sin x_2(t-T)+(x_1^2+y_1)(e^{-t}+2)-y_1+x_1(t-T)\\ u_2 = {} & 2x_2\Big(-\omega_1(x_1-x_3)+2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2\Big)(e^{-t}+2)+(y_1-\alpha y_2+y_4)(e^{-t}+2)-(x_2^2+y_2)e^{-t}\\ & -y_1-y_4-\omega_1(x_1(t-T)-x_3(t-T))+2a_1\frac{x_1(t-T)}{\sqrt{1-x_1^2(t-T)}}\cos x_2(t-T)\\ & +\alpha((x_2^2+y_2)(e^{-t}+2)+x_2(t-T))\\ u_3 = {} & 2x_3\Big(-2a_2\sqrt{1-x_3^2}\sin x_4\Big)(e^{-t}+2)+(y_1y_3+\beta)(e^{-t}+2)-(x_3^2+y_3)e^{-t}\\ & -y_1y_3-\beta-2a_2\sqrt{1-x_3^2(t-T)}\sin x_4(t-T)+(x_3^2+y_3)(e^{-t}+2)-y_3+x_3(t-T)\\ u_4 = {} & 2x_4\Big(-\omega_2(x_3-x_1)+2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4\Big)(e^{-t}+2)+(\gamma y_3+\sigma y_4)(e^{-t}+2)-(x_4^2+y_4)e^{-t}\\ & -\gamma y_3-\omega_2(x_3(t-T)-x_1(t-T))+2a_2\frac{x_3(t-T)}{\sqrt{1-x_3^2(t-T)}}\cos x_4(t-T)\\ & +\sigma((x_4^2+y_4)(e^{-t}+2)-2y_4+x_4(t-T))\end{aligned}$$

Eq. (24) becomes

$$\dot{V} = -(e_1^2+\alpha e_2^2+e_3^2+\sigma e_4^2) < 0 \quad (25)$$

which is negative definite. The Lyapunov asymptotical stability theorem is satisfied. Time delay symplectic synchronization of the Quantum-CNN system and the Rössler system is achieved. The numerical results are shown in Fig. 2. After 5 s, the motion trajectories enter a chaotic attractor.

Case III: A cubic time delay symplectic synchronization

We take $F_1(t)=x_4(t)x_1(t-T)$, $F_2(t)=x_1(t)x_2(t-T)$, $F_3(t)=x_2(t)x_3(t-T)$ and $F_4(t)=x_3(t)x_4(t-T)$, where $T=1$ sec is a positive constant time delay. They are chaotic functions of time. $H_i(x,y,t)=x_i^3-(y_i^3\sin\varpi_i t-1)\sin\varpi_i t$ $(i=1,2,3,4)$ are given. By Eq. (5) we have

Please cite this article in press as: Ge Z-M, Yang C-H, Symplectic synchronization of different chaotic systems, Chaos, Solitons & Fractals (2007), doi:10.1016/j.chaos.2007.10.055

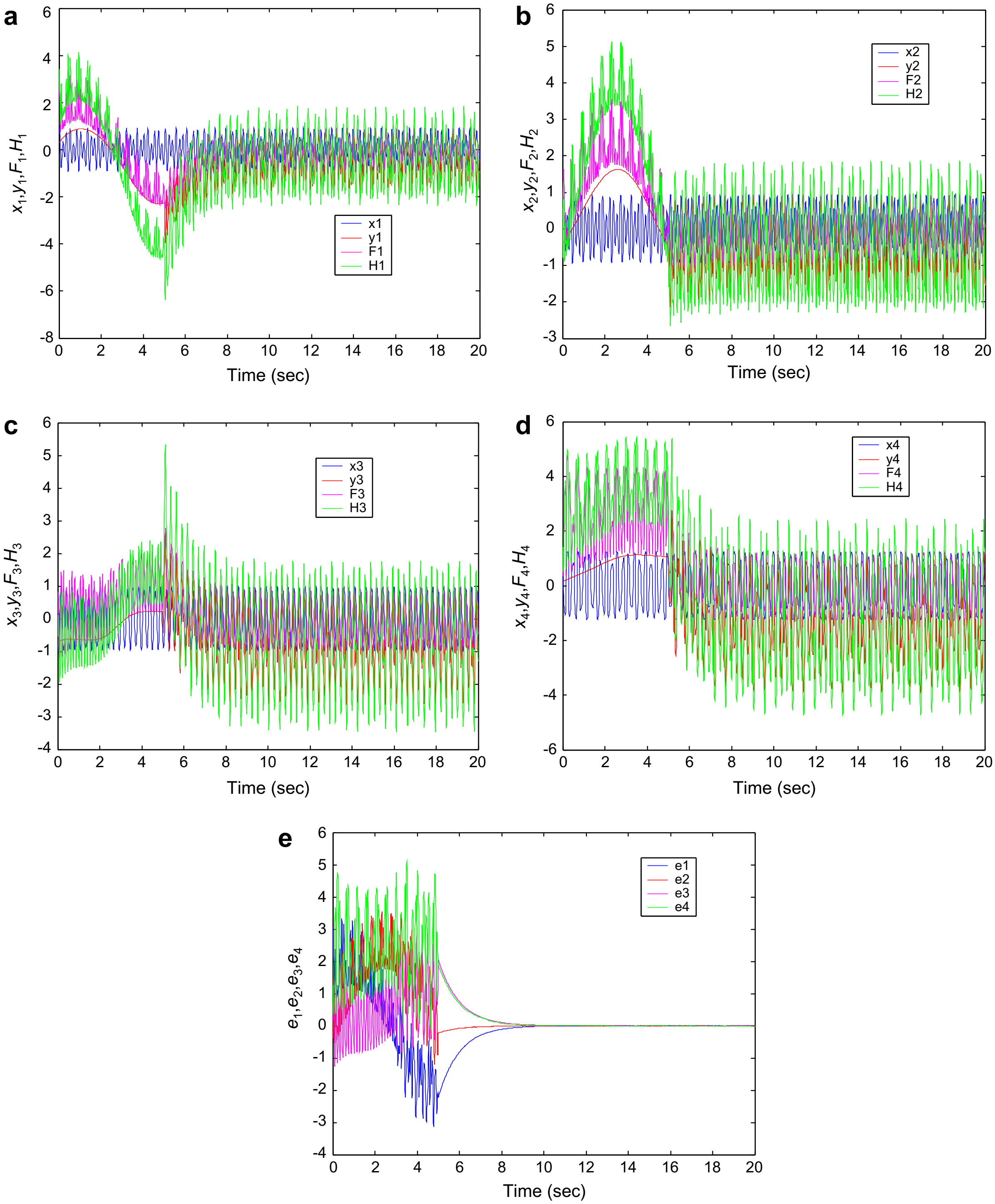

Fig. 2. Time histories of states, state errors, F_1, F_2, F_3, F_4, H_1, H_2, H_3 and H_4 for Case II.

$$\lim_{t\to\infty} e_i = \lim_{t\to\infty}(x_i^3 - (y_i^3 \sin \varpi_i t - 1)\sin \varpi_i t - y_i + x_j x_i(t-T)) = 0, \quad i = 1,2,3,4, \quad j = \begin{cases} 4, & i = 1 \\ i-1, & i \neq 1 \end{cases} \tag{26}$$

Please cite this article in press as: Ge Z-M, Yang C-H, Symplectic synchronization of different chaotic systems, Chaos, Solitons & Fractals (2007), doi:10.1016/j.chaos.2007.10.055

From Eq. (7) we have

$$\dot{e}_i = (3\dot{x}_i x_i^2 - (3\dot{y}_i y_i^2 \sin\varpi_i t + y_i^3 \varpi_i \cos\varpi_i t)\sin\varpi_i t - (y_i^3 \sin\varpi_i t - 1)\varpi_i \cos\varpi_i t - \dot{y}_i + \dot{x}_j x_i(t-T) + x_j \dot{x}_i(t-T),$$
$$i = 1,2,3,4, \quad j = \begin{cases} 4, & i=1 \\ i-1, & i \neq 1 \end{cases} \tag{27}$$

Eq. (8) is expressed as

$$\begin{aligned}
\dot{e}_1 &= 3x_1^2\left(-2a_1\sqrt{1-x_1^2}\sin x_2\right) - 3y_1^2(-y_2-y_3)\sin^2\varpi_1 t - y_1^3\varpi_1 \sin 2\varpi_1 t \\
&\quad + \varpi_1\cos\varpi_1 t + y_2 + y_3 - u_1 + \left(-\omega_2(x_3-x_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4\right)x_1(t-T) \\
&\quad - 2a_1 x_4\sqrt{1-x_1^2(t-T)}\sin x_2(t-T) \\
\dot{e}_2 &= 3x_2^2\left(-\omega_1(x_1-x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2\right) - 3y_2^2(y_1-\alpha y_2+y_4)\sin^2\varpi_2 t \\
&\quad - y_2^3\varpi_2\sin 2\varpi_2 t + \varpi_2\cos\varpi_2 t - y_1 + \alpha y_2 - y_4 - u_2 - 2a_1 x_2(t-T)\sqrt{1-x_1^2}\sin x_2 \\
&\quad + x_1(-\omega_1(x_1(t-T) - x_3(t-T))) + 2a_1\frac{x_1(t-T)}{\sqrt{1-x_1^2(t-T)}}\cos x_2(t-T) \\
\dot{e}_3 &= 3x_3^2\left(-2a_2\sqrt{1-x_3^2}\sin x_4\right) - 3y_3^2(y_1y_3+\beta)\sin^2\varpi_3 t + y_3^3\varpi_3\sin 2\varpi_3 t \\
&\quad + \varpi_3\cos\varpi_3 t - y_1y_3 - \beta - u_3 + \left(-\omega_1(x_1-x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2\right)x_3(t-T) \\
&\quad - 2a_2x_2\sqrt{1-x_3^2(t-T)}\sin x_4(t-T) \\
\dot{e}_4 &= 3x_4^2\left(-\omega_2(x_3-x_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4\right) - 3y_4^2(\gamma y_3+\sigma y_4)\sin^2\varpi_4 t \\
&\quad + y_4^3\varpi_4\sin 2\varpi_4 t + \varpi_4\cos\varpi_4 t - \gamma y_3 - \sigma y_4 - u_4 - 2a_2x_4(t-T)\sqrt{1-x_3^2}\sin x_4 \\
&\quad + x_3\left(-\omega_2(x_3(t-T) - x_1(t-T))) + 2a_2\frac{x_3(t-T)}{\sqrt{1-x_3^2(t-T)}}\cos x_4(t-T)\right)
\end{aligned} \tag{28}$$

where

$$\begin{aligned}
e_1 &= x_1^3 - (y_1^3\sin\varpi_1 t - 1)\sin\varpi_1 t - y_1 + x_4(t)x_1(t-T) \\
e_2 &= x_2^3 - (y_2^3\sin\varpi_2 t - 1)\sin\varpi_2 t - y_2 + x_1(t)x_2(t-T) \\
e_3 &= x_3^3 - (y_3^3\sin\varpi_3 t - 1)\sin\varpi_3 t - y_3 + x_2(t)x_3(t-T) \\
e_4 &= x_4^3 - (y_4^3\sin\varpi_4 t - 1)\sin\varpi_4 t - y_4 + x_3(t)x_4(t-T)
\end{aligned}$$

Choose a positive definite Lyapunov function:

$$V(e_1,e_2,e_3,e_4) = \frac{1}{2}(e_1^2+e_2^2+e_3^2+e_4^2) \tag{29}$$

Its time derivative along any solution of Eq. (28) is

$$\begin{aligned}
\dot{V} &= e_1\left\{3x_1^2\left(-2a_1\sqrt{1-x_1^2}\sin x_2\right) - 3y_1^2(-y_2-y_3)\sin^2\varpi_1 t - y_1^3\varpi_1\sin 2\varpi_1 t + \varpi_1\cos\varpi_1 t + y_2 + y_3\right. \\
&\quad \left. + \left(-\omega_2(x_3-x_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4\right)x_1(t-T) - 2a_1x_4\sqrt{1-x_1^2(t-T)}\sin x_2(t-T) - u_1\right\} \\
&\quad + e_2\left\{3x_2^2\left(-\omega_1(x_1-x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2\right) - 3y_2^2(y_1-\alpha y_2+y_4)\sin^2\varpi_2 t - y_2^3\varpi_2\sin 2\varpi_2 t\right.
\end{aligned}$$

Please cite this article in press as: Ge Z-M, Yang C-H, Symplectic synchronization of different chaotic systems, Chaos, Solitons & Fractals (2007), doi:10.1016/j.chaos.2007.10.055

$$+\varpi_2\cos\varpi_2 t - y_1 + \alpha y_2 - y_4 - 2a_1x_2(t-T)\sqrt{1-x_1^2}\sin x_2 + x_1(-\omega_1(x_1(t-T)-x_3(t-T))$$
$$\left.+2a_1\frac{x_1(t-T)}{\sqrt{1-x_1^2(t-T)}}\cos x_2(t-T) - u_2\right\}$$
$$+e_3\left\{3x_3^2\left(-2a_2\sqrt{1-x_3^2}\sin x_4\right) - 3y_3^2(y_1y_3+\beta)\sin^2\varpi_3 t + y_3^3\varpi_3\sin 2\varpi_3 t + \varpi_3\cos\varpi_3 t - y_1y_3\right.$$
$$\left.-\beta + \left(-\omega_1(x_1-x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2\right)x_3(t-T) - 2a_2x_2\sqrt{1-x_3^2(t-T)}\sin x_4(t-T) - u_3\right\}$$
$$+e_4\left\{3x_4^2\left(-\omega_2(x_3-x_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4\right) - 3y_4^2(\gamma y_3+\sigma y_4)\sin^2\varpi_4 t + y_4^3\varpi_4\sin 2\varpi_4 t\right.$$
$$+\varpi_4\cos\varpi_4 t - \gamma y_3 - \sigma y_4 - 2a_2x_4(t-T)\sqrt{1-x_3^2}\sin x_4 + x_3(-\omega_2(x_3(t-T)-x_1(t-T))$$
$$\left.+2a_2\frac{x_3(t-T)}{\sqrt{1-x_3^2(t-T)}}\cos x_4(t-T)) - u_4\right\}$$

Choose

$$u_1 = 3x_1^2\left(-2a_1\sqrt{1-x_1^2}\sin x_2\right) - 3y_1^2(-y_2-y_3)\sin^2\varpi_1 t - y_1^3\varpi_1\sin 2\varpi_1 t$$
$$+\varpi_1\cos\varpi_1 t + y_2 + y_3 + \left(-\omega_2(x_3-x_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4\right)x_1(t-T)$$
$$-2a_1x_4\sqrt{1-x_1^2(t-T)}\sin x_2(t-T) + x_1^3 - (y_1^3\sin\varpi_1 t - 1)\sin\varpi_1 t - y_1 + x_4(t)x_1(t-T)$$
$$u_2 = 3x_2^2\left(-\omega_1(x_1-x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2\right) - 3y_2^2(y_1-\alpha y_2+y_4)\sin^2\varpi_2 t$$
$$-y_2^3\varpi_2\sin 2\varpi_2 t + \varpi_2\cos\varpi_2 t - y_1 - y_4 - 2a_1x_2(t-T)\sqrt{1-x_1^2}\sin x_2$$
$$+x_1(-\omega_1(x_1(t-T)-x_3(t-T))) + 2a_1\frac{x_1(t-T)}{\sqrt{1-x_1^2(t-T)}}\cos x_2(t-T)$$
$$+\alpha(x_2^3 - (y_2^3\sin\varpi_2 t - 1)\sin\varpi_2 t + x_1(t)x_2(t-T))$$
$$u_3 = 3x_3^2\left(-2a_2\sqrt{1-x_3^2}\sin x_4\right) - 3y_3^2(y_1y_3+\beta)\sin^2\varpi_3 t$$
$$+y_3^3\varpi_3\sin 2\varpi_3 t + \varpi_3\cos\varpi_3 t - y_1y_3 - \beta + \left(-\omega_1(x_1-x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2\right)x_3(t-T)$$
$$-2a_2x_2\sqrt{1-x_3^2(t-T)}\sin x_4(t-T) + x_3^3 - (y_3^3\sin\varpi_3 t - 1)\sin\varpi_3 t - y_3 + x_2(t)x_3(t-T)$$
$$u_4 = 3x_4^2\left(-\omega_2(x_3-x_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4\right) - 3y_4^2(\gamma y_3-\sigma y_4)\sin^2\varpi_4 t$$
$$+y_4^3\varpi_4\sin 2\varpi_4 t + \varpi_4\cos\varpi_4 t - \gamma y_3 - 2a_2x_4(t-T)\sqrt{1-x_3^2}\sin x_4$$
$$+x_3\left(-\omega_2(x_3(t-T)-x_1(t-T)) + 2a_2\frac{x_3(t-T)}{\sqrt{1-x_3^2(t-T)}}\cos x_4(t-T)\right)$$
$$+\sigma(x_4^3 - (y_4^3\sin\varpi_4 t - 1)\sin\varpi_4 t - 2y_4 + x_3(t)x_4(t-T))$$

Eq. (30) becomes

$$\dot{V} = -(e_1^2 + \alpha e_2^2 + e_3^2 + \sigma e_4^2) < 0 \tag{31}$$

which is negative definite. The Lyapunov asymptotical stability theorem is satisfied. Cubic time delay symplectic synchronization of the Quantum-CNN system and the Rössler system is achieved. The numerical results are shown in Fig. 3. After 5 s, the motion trajectories enter a chaotic attractor.

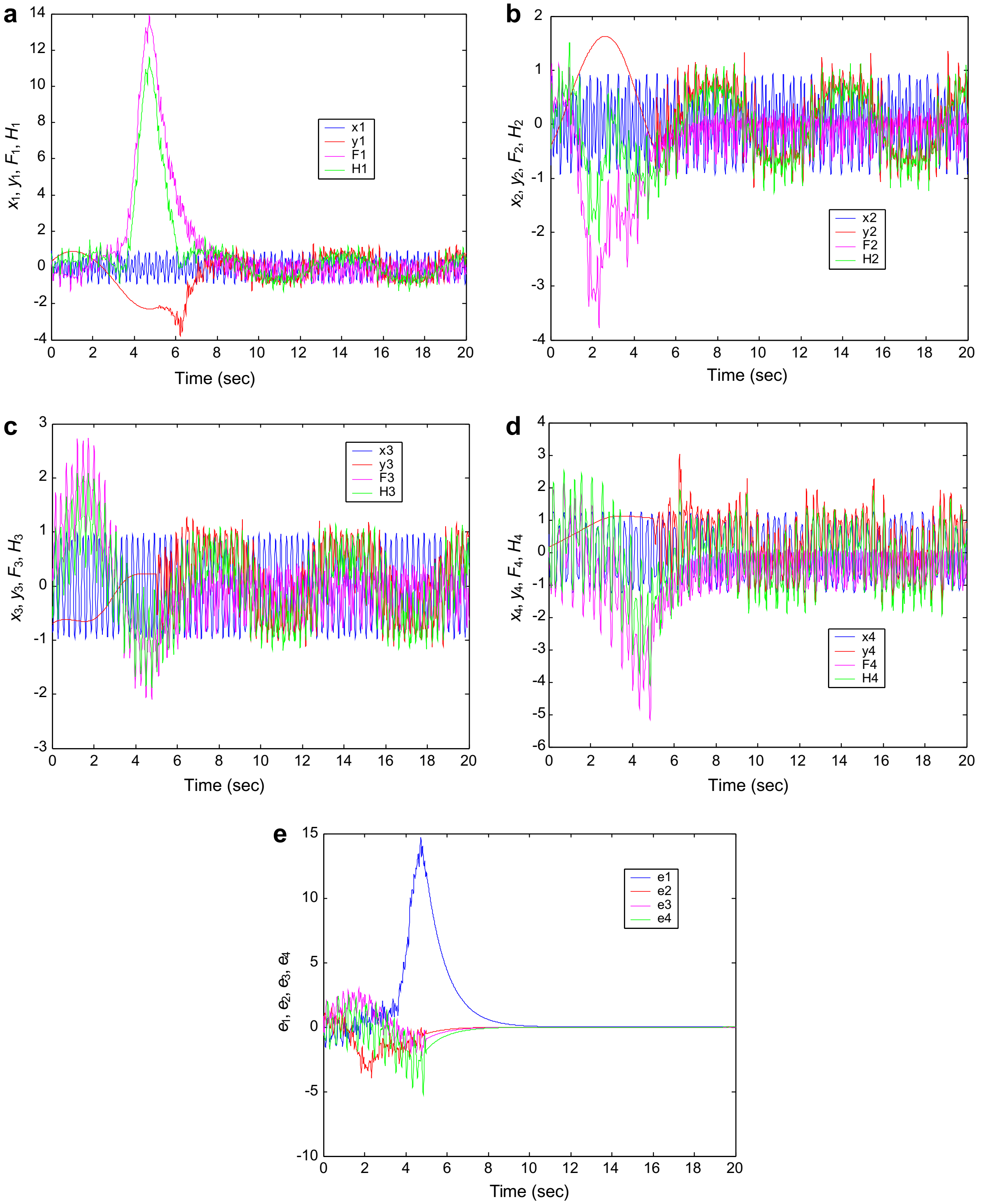

Fig. 3. Time histories of states, state errors, F_1, F_2, F_3, F_4, H_1, H_2, H_3 and H_4 for Case III.

4. Conclusions

A new symplectic synchronization of a Quantum-CNN chaotic oscillator and a Rössler system is obtained by the Lyapunov asymptotical stability theorem. Two different chaotic dynamical systems, the Quantum-CNN system and the Rössler system, are in symplectic synchronization for three cases: the cubic symplectic synchronization, the time delay symplectic synchronization and the cubic time delay symplectic synchronization. Symplectic synchronization of chaotic systems can be used to increase the security of secret communication.

Acknowledgement

This research was supported by the National Science Council, Republic of China, under Grant Number 96-2221-E-009-144-MY3.

References

[1] Ge Z-M, Yang C-H. The generalized synchronization of a Quantum-CNN chaotic oscillator with different order systems. Chaos, Solitons & Fractals 2008;35:980–90.
[2] Ge Z-M, Yang C-H. Synchronization of complex chaotic systems in series expansion form. Chaos, Solitons & Fractals 2007;34:1649–58.
[3] Pecora L-M, Carroll T-L. Synchronization in chaotic system. Phys Rev Lett 1990;64:821–4.
[4] Ge Zheng-Ming, Leu Wei-Ying. Anti-control of chaos of two-degrees-of-freedom louderspeaker system and chaos synchronization of different order systems. Chaos, Solitons & Fractals 2004;20:503–21.
[5] Femat R, Ramirez J-A, Anaya G-F. Adaptive synchronization of high-order chaotic systems: a feedback with low-order parameterization. Physica D 2000;139:231–46.
[6] Ge Z-M, Chang C-M. Chaos synchronization and parameters identification of single time scale brushless DC motors. Chaos, Solitons & Fractals 2004;20:883–903.
[7] Femat R, Perales G-S. On the chaos synchronization phenomenon. Phys Lett A 1999;262:50–60.
[8] Ge Z-M, Yang C-H. Pragmatical generalized synchronization of chaotic systems with uncertain parameters by adaptive control. Physica D: Nonlinear Phenomena 2007;231:87–94.
[9] Yang S-S, Duan C-K. Generalized synchronization in chaotic systems. Chaos, Solitons & Fractals 1998;9:1703–7.
[10] Krawiecki A, Sukiennicki A. Generalizations of the concept of marginal synchronization of chaos. Chaos, Solitons & Fractals 2000;11(9):1445–58.
[11] Ge Z-M, Yang C-H, Chen H-H, Lee S-C. Non-linear dynamics and chaos control of a physical pendulum with vibrating and rotation support. J Sound Vib 2001;242(2):247–64.
[12] Chen M-Y, Han Z-Z, Shang Y. General synchronization of Genesio–Tesi system. Int J Bifurcat Chaos 2004;14(1):347–54.
[13] Fortuna Luigi, Porto Domenico. Quantum-CNN to generate nanoscale chaotic oscillator. Int J Bifurcat Chaos 2004;14(3):1085–9.
[14] Ge Zheng-Ming, Chen Yen-Sheng. Synchronization of unidirectional coupled chaotic systems via partial stability. Chaos, Solitons & Fractals 2004;21:101–11.
[15] Chen S, Lu J. Synchronization of uncertain unified chaotic system via adaptive control. Chaos, Solitons & Fractals 2002;14(4):643–7.
[16] Ge Zheng-Ming, Chen Chien-Cheng. Phase synchronization of coupled chaotic multiple time scales systems. Chaos, Solitons & Fractals 2004;20:639–47.

Please cite this article in press as: Ge Z-M, Yang C-H, Symplectic synchronization of different chaotic systems, Chaos, Solitons & Fractals (2007), doi:10.1016/j.chaos.2007.10.055

ARTICLE IN PRESS

Available online at www.sciencedirect.com

ScienceDirect

Nonlinear Analysis ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Nonlinear Analysis

www.elsevier.com/locate/na

Chaos synchronization by variable strength linear coupling and Lyapunov function derivative in series form

Zheng-Ming Ge*, Pu-Chien Tsen

Department of Mechanical Engineering, National Chiao Tung University, Hsinchu, Taiwan, ROC

Received 14 June 2007; accepted 9 November 2007

Abstract

A new general strategy to achieve chaos synchronization by variable strength linear coupling without another active control is proposed. They give the criteria of chaos synchronization for two identical chaotic systems and two different chaotic dynamic systems with variable strength linear coupling. In this method, the time derivative of Lyapunov function in series form is firstly used. Lorenz system, Duffing system, Rössler system and Hyper-Rössler system are presented as simulated examples.

Keywords: Chaos; Synchronization; Linear coupling; Coupled chaotic systems

1. Introduction

In recent years, synchronization in chaotic dynamic system has been a very interesting problem and has been widely studied [1–3]. Synchronization means that the state variables of a response system approach eventually to that of a drive system. There are many control techniques to synchronize chaotic systems, such as linear error feedback control, adaptive control, active control [2–17].

In this paper, a new general strategy to achieve chaos synchronization by variable strength linear coupling is proposed. This method, in which the time derivative of Lyapunov function in series form is firstly used, can give either local synchronization which is usually good enough or global synchronization which is usually an unnecessary high demand [18–21].

This paper is organized as follows. In Section 2, synchronization strategy by variable strength linear coupling without another active control is proposed, in which the Lyapunov function derivative in series form is first used. In Section 3, Lorenz system, Duffing system, Rössler system and Hyper-Rössler system are presented as simulated examples. In Section 4, conclusions are given.

* Corresponding address: Department of Mechanical Engineering, National Chiao Tung University, 1001 Ta Hsueh Road, Hsinchu 30050, Taiwan, ROC. Tel.: +886 3 5712121x55119; fax: +886 3 5720634.

E-mail address: zmg@cc.nctu.edu.tw (Z.-M. Ge).

0362-546X/$ - see front matter © 2007 Elsevier Ltd. All rights reserved.
doi:10.1016/j.na.2007.11.018

Please cite this article in press as: Z.-M. Ge, P.-C. Tsen, Chaos synchronization by variable strength linear coupling and Lyapunov function derivative in series form, Nonlinear Analysis (2007), doi:10.1016/j.na.2007.11.018

2. Synchronization strategy by variable strength linear coupling and Lyapunov function derivative in series form

(a) Consider the following unidirectional coupled identical chaotic systems

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{f}(\mathbf{x}) \\ \dot{\mathbf{y}} &= \mathbf{A}\mathbf{y} + \mathbf{f}(\mathbf{y}) + \boldsymbol{\Gamma}(\mathbf{y}-\mathbf{x}),\end{aligned} \tag{1}$$

where $\mathbf{x} = [x_1, x_2, \ldots, x_n]^{\mathrm{T}} \in R^n$, $\mathbf{y} = [y_1, y_2, \ldots, y_n]^{\mathrm{T}} \in R^n$ denote two state vectors, $\mathbf{A}$ is an $n \times n$ constant coefficient matrix, $\mathbf{f}$ is a nonlinear vector function, and $\boldsymbol{\Gamma}$ is an $n \times n$ matrix which gives the variable strength of the linear coupling term $(\mathbf{y} - \mathbf{x})$.

In order to study the synchronization of $\mathbf{x}$ and $\mathbf{y}$, define $\mathbf{e} = \mathbf{y} - \mathbf{x}$ as the state error. Error equation can be written as

$$\dot{\mathbf{e}} = \mathbf{A}\mathbf{y} + \mathbf{f}(\mathbf{y}) + \boldsymbol{\Gamma}(\mathbf{y}-\mathbf{x}) - \mathbf{A}\mathbf{x} - \mathbf{f}(\mathbf{x}). \tag{2}$$

By Taylor expansion

$$\begin{aligned}\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x}) &= \mathbf{f}(\mathbf{x}+\mathbf{e}) - \mathbf{f}(\mathbf{x}) = \mathbf{f}'(\mathbf{x})\mathbf{e} + \text{HOT of } \mathbf{e} \\ &= \mathbf{F}(\mathbf{x})\mathbf{e} + \text{HOT of } \mathbf{e},\end{aligned} \tag{3}$$

where $\mathbf{f}'(\mathbf{x})$ is the time derivative $\mathbf{f}(\mathbf{x})$, and $\mathbf{F}(\mathbf{x}) = \mathbf{f}'(\mathbf{x})$.

Theorem 1. *The chaotic systems in Eq. (1) can be locally completely synchronized, if $\|\mathbf{e}\|^2$ is smaller than a bounded value and $\boldsymbol{\Gamma}$ is chosen such that $\mathbf{A} + \boldsymbol{\Gamma} + \mathbf{F} = -\mathbf{C}$, where $\mathbf{C}$ is a positive definite diagonal matrix.*

Proof. Choose a positive definite function as

$$V(\mathbf{e}) = \frac{1}{2}\mathbf{e}^{\mathrm{T}}\mathbf{e}. \tag{4}$$

Then

$$\begin{aligned}\dot{V}(\mathbf{e}) &= \mathbf{e}^{\mathrm{T}}\dot{\mathbf{e}} \\ &= \mathbf{e}^{\mathrm{T}}(\mathbf{A}\mathbf{y} + \mathbf{f}(\mathbf{y}) + \boldsymbol{\Gamma}(\mathbf{y}-\mathbf{x}) - \mathbf{A}\mathbf{x} - \mathbf{f}(\mathbf{x})) \\ &= \mathbf{e}^{\mathrm{T}}(\mathbf{A}\mathbf{e} + \boldsymbol{\Gamma}\mathbf{e} + \mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})) \\ &= \mathbf{e}^{\mathrm{T}}(\mathbf{A} + \boldsymbol{\Gamma} + \mathbf{F})\mathbf{e} + \text{HOT of } \mathbf{e}.\end{aligned} \tag{5}$$

Since $\|\mathbf{e}\|^2$ is smaller than a bounded value and $\boldsymbol{\Gamma}$ is chosen such that $\mathbf{A} + \boldsymbol{\Gamma} + \mathbf{F} = -\mathbf{C}$, Eq. (5) becomes $\dot{V}(\mathbf{e}) = -\mathbf{e}^{\mathrm{T}}\mathbf{C}\mathbf{e} + \text{HOT of } \mathbf{e} < 0$, since $-\mathbf{e}^{\mathrm{T}}\mathbf{C}\mathbf{e}$ is a definite form, the higher-order terms of $\mathbf{e}$ have no influence on the definiteness of $\dot{V}$, provided that $\|\mathbf{e}\|^2$ is smaller than a bounded value. The proof of this theorem can be found in [22,23], which is used extensively in the theory of stability of motion. By the Lyapunov asymptotical stability theorem, the origin of error equation (2) is locally asymptotically stable and the chaotic systems in Eq. (1) are locally completely synchronized. □

Corollary 1. *If $\mathbf{f}(\mathbf{x}+\mathbf{e}) - \mathbf{f}(\mathbf{x})$ is a linear function of $\mathbf{e}, \mathbf{D}\mathbf{e}$, Eq. (5) becomes $\dot{V}(\mathbf{e}) = \mathbf{e}^{\mathrm{T}}(\mathbf{A} + \boldsymbol{\Gamma} + \mathbf{D})\mathbf{e}$. Let $\mathbf{A} + \boldsymbol{\Gamma} + \mathbf{D} = -\mathbf{C}$, then $\dot{V}(\mathbf{e}) = -\mathbf{e}^{\mathrm{T}}\mathbf{C}\mathbf{e} < 0$. By the Lyapunov asymptotical stability theorem, the origin of error equation (2) is globally asymptotically stable. Hence, the chaotic systems in Eq. (1) are globally completely synchronized.* □

(b) Consider the following two unidirectional coupled different chaotic systems

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{f}(\mathbf{x}) \\ \dot{\mathbf{y}} &= \hat{\mathbf{A}}\mathbf{y} + \mathbf{f}(\mathbf{y}) + \mathbf{u},\end{aligned} \tag{6}$$

where $\mathbf{x} = [x_1, x_2, \ldots, x_n]^{\mathrm{T}} \in R^n$, $\mathbf{y} = [y_1, y_2, \ldots, y_n]^{\mathrm{T}} \in R^n$ denote two state vectors, $\mathbf{A}$ and $\hat{\mathbf{A}}$ are two different $n \times n$ constant coefficient matrices, $\mathbf{f}$ is a nonlinear vector function, and $\mathbf{u}$ is the coupling vector of which the elements are functions of $\mathbf{x}$ and $\mathbf{y}$.

Please cite this article in press as: Z.-M. Ge, P.-C. Tsen, Chaos synchronization by variable strength linear coupling and Lyapunov function derivative in series form, Nonlinear Analysis (2007), doi:10.1016/j.na.2007.11.018

In order to study the synchronization of $\mathbf{x}$ and $\mathbf{y}$, define $\mathbf{e} = \mathbf{y} - \mathbf{x}$ as the state error. Error equation can be written as

$$\dot{\mathbf{e}} = \hat{\mathbf{A}}\mathbf{y} + \mathbf{f}(\mathbf{y}) + \mathbf{u} - \mathbf{A}\mathbf{x} - \mathbf{f}(\mathbf{x}). \tag{7}$$

By Taylor expansion

$$\begin{aligned} \mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x}) &= \mathbf{f}(\mathbf{x}+\mathbf{e}) - \mathbf{f}(\mathbf{x}) = \mathbf{f}'(\mathbf{x})\mathbf{e} + \text{HOT of } \mathbf{e} \\ &= \mathbf{F}(\mathbf{x})\mathbf{e} + \text{HOT of } \mathbf{e}. \end{aligned} \tag{8}$$

Theorem 2. *Choose* $\boldsymbol{\Gamma} = -\mathbf{C} - \mathbf{A} - \mathbf{F}$ *and* $\mathbf{B} = -\tilde{\mathbf{A}}$, *where* $\mathbf{C}$ *is positive definite diagonal matrix and* $\tilde{\mathbf{A}} = \hat{\mathbf{A}} - \mathbf{A}$. *The chaotic systems in Eq.* (6) *can be locally completely synchronized, if* $\|\mathbf{e}\|^2$ *is smaller than a bounded value and* $\mathbf{u} = \boldsymbol{\Gamma}\mathbf{e} + \mathbf{B}\mathbf{y}$.

Proof. Choose a positive definite function as

$$V(\mathbf{e}) = \frac{1}{2}\mathbf{e}^{\mathrm{T}}\mathbf{e}. \tag{9}$$

Then

$$\begin{aligned} \dot{V}(\mathbf{e}) &= \mathbf{e}^{\mathrm{T}}\dot{\mathbf{e}} \\ &= \mathbf{e}^{\mathrm{T}}(\hat{\mathbf{A}}y + \mathbf{f}(\mathbf{y}) + \mathbf{u} - \mathbf{A}\mathbf{x} - \mathbf{f}(\mathbf{x})) \\ &= \mathbf{e}^{\mathrm{T}}(\tilde{\mathbf{A}}y + \mathbf{A}\mathbf{e} + \mathbf{u} + \mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})). \end{aligned} \tag{10}$$

Let $\mathbf{u} = \boldsymbol{\Gamma}\mathbf{e} + \mathbf{B}\mathbf{y}$, Eq. (10) becomes

$$\begin{aligned} \dot{V}(\mathbf{e}) &= \mathbf{e}^{\mathrm{T}}(\tilde{\mathbf{A}}\mathbf{y} + \mathbf{A}\mathbf{e} + \boldsymbol{\Gamma}\mathbf{e} + \mathbf{B}\mathbf{y} + \mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})) \\ &= \mathbf{e}^{\mathrm{T}}(\mathbf{A} + \boldsymbol{\Gamma} + \mathbf{F})\mathbf{e} + \mathbf{e}^{\mathrm{T}}(\tilde{\mathbf{A}} + \mathbf{B})\mathbf{y} + \text{HOT of } \mathbf{e}. \end{aligned} \tag{11}$$

Since $\|\mathbf{e}\|^2$ is smaller than a bounded value, $\boldsymbol{\Gamma}$ and $\mathbf{B}$ are chosen such that $\mathbf{A} + \boldsymbol{\Gamma} + \mathbf{F} = -\mathbf{C}$ and $\mathbf{B} = -\tilde{\mathbf{A}}$, Eq. (10) becomes $\dot{V}(\mathbf{e}) = -\mathbf{e}^{\mathrm{T}}\mathbf{C}\mathbf{e} + \text{HOT of } \mathbf{e} < 0$. By the Lyapunov asymptotical stability theorem, the origin of error equation (7) is locally asymptotically stable and the chaotic systems in Eq. (6) are locally completely synchronized. □

Corollary 2. *If* $\mathbf{f}(\mathbf{x}+\mathbf{e}) - \mathbf{f}(\mathbf{x})$ *is a linear function of* $\mathbf{e}$, $\mathbf{D}\mathbf{e}$, *Eq.* (11) *becomes* $\dot{V}(\mathbf{e}) = \mathbf{e}^{\mathrm{T}}(\mathbf{A} + \boldsymbol{\Gamma} + \mathbf{D})\mathbf{e} + \mathbf{e}^{\mathrm{T}}(\tilde{\mathbf{A}} + \mathbf{B})\mathbf{y}$. *Let* $\mathbf{A} + \boldsymbol{\Gamma} + \mathbf{D} = -\mathbf{C}$ *and* $\mathbf{B} = -\tilde{\mathbf{A}}$, *then* $\dot{V}(\mathbf{e}) = -\mathbf{e}^{\mathrm{T}}\mathbf{C}\mathbf{e} < 0$. *By the Lyapunov asymptotical stability theorem, the origin of error equation* (7) *is globally asymptotically stable, and the chaotic systems in Eq.* (6) *are globally completely synchronized.* □

3. Numerical results for typical chaotic systems

First example for Theorem 1 is the Rössler system. Consider the following two unidirectional coupled chaotic Rössler systems:

$$\begin{aligned} \dot{x}_1 &= -y_1 - z_1 \\ \dot{y}_1 &= x_1 + ay_1 \\ \dot{z}_1 &= b + z_1(x_1 - c) \\ \dot{x}_2 &= -y_2 - z_2 + \Gamma_{11}e_1 + \Gamma_{12}e_2 + \Gamma_{13}e_3 \\ \dot{y}_2 &= x_2 + ay_2 + \Gamma_{21}e_1 + \Gamma_{22}e_2 + \Gamma_{23}e_3 \\ \dot{z}_2 &= b + z_2(x_2 - c) + \Gamma_{31}e_1 + \Gamma_{32}e_2 + \Gamma_{33}e_3, \end{aligned} \tag{12}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ 0 & 0 & -c \end{bmatrix}. \tag{13}$$

Please cite this article in press as: Z.-M. Ge, P.-C. Tsen, Chaos synchronization by variable strength linear coupling and Lyapunov function derivative in series form, Nonlinear Analysis (2007), doi:10.1016/j.na.2007.11.018

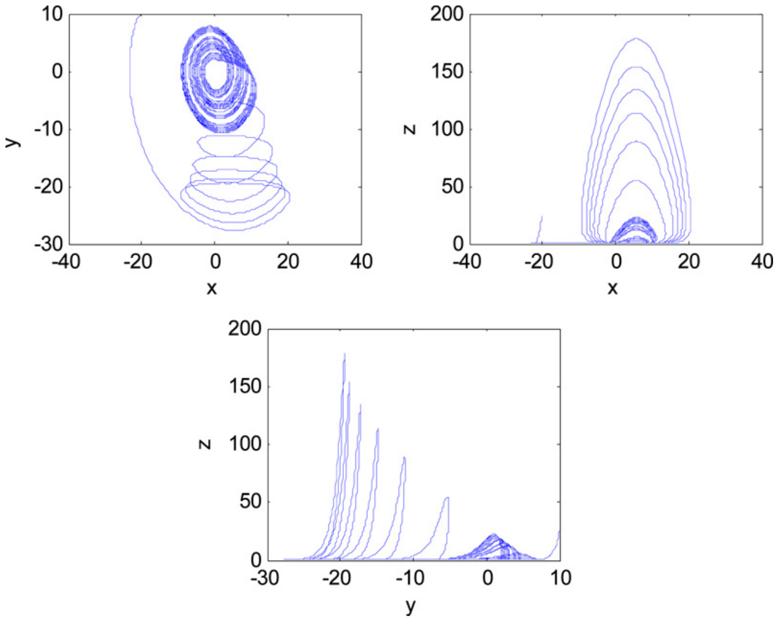

Fig. 1. Chaotic phase portraits for the Rössler system.

Choose a Lyapunov function in the form of a positive definite function:

$$V(e_1, e_2, e_3) = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2) \tag{14}$$

by Taylor Formula

$$\begin{aligned}\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x}) &= \begin{bmatrix} 0 \\ 0 \\ z_1 e_1 + x_1 e_3 + e_1 e_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ z_1 & 0 & x_1 \end{bmatrix} \mathbf{e} + \begin{bmatrix} 0 \\ 0 \\ e_1 e_3 \end{bmatrix} \\ &= \mathbf{F}\mathbf{e} + \cdots. \end{aligned} \tag{15}$$

Let

$$\mathbf{\Gamma} = -\mathbf{I} - \mathbf{A} - \mathbf{F} = \begin{bmatrix} -1 & 1 & 1 \\ -1 & -1-a & 0 \\ -z_1 & 0 & -1+c-x_1 \end{bmatrix}. \tag{16}$$

According to Theorem 1, we obtain that

$$\dot{V} = -e_1^2 - e_2^2 - e_3^2 + \text{HOT of } \mathbf{e} < 0 \tag{17}$$

is negative definite when $\|\mathbf{e}\|^2$ is smaller than a bounded value. The Rössler systems in Eq. (12) are locally synchronized. For the initial states (−20, 10, 25), (−21, 10.5, 25) and system parameters $a = 0.2$, $b = 0.2$, $c = 5.7$, the chaotic phase portraits and state errors versus time are shown in Figs. 1 and 2.

Please cite this article in press as: Z.-M. Ge, P.-C. Tsen, Chaos synchronization by variable strength linear coupling and Lyapunov function derivative in series form, Nonlinear Analysis (2007), doi:10.1016/j.na.2007.11.018

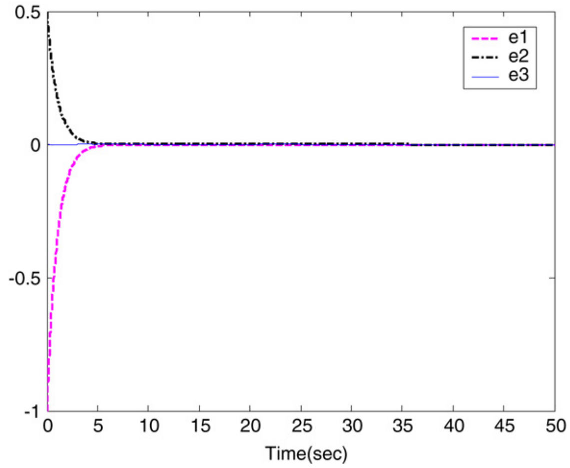

Fig. 2. Time histories of errors for two Rössler systems.

Second example for Corollary 1 is the Hyper-Rössler system. Consider the following two unidirectional coupled chaotic Hyper-Rössler systems:

$$\begin{aligned}
\dot{x}_1 &= -x_2 - x_3 \\
\dot{x}_2 &= x_1 + ax_2 + x_4 \\
\dot{x}_3 &= b + x_1 x_3 \\
\dot{x}_4 &= cx_4 - dx_3 \\
\dot{y}_1 &= -y_2 - y_3 + \Gamma_{11}e_1 + \Gamma_{12}e_2 + \Gamma_{13}e_3 + \Gamma_{14}e_4 \\
\dot{y}_2 &= y_1 + ay_2 + y_4 + \Gamma_{21}e_1 + \Gamma_{22}e_2 + \Gamma_{23}e_3 + \Gamma_{24}e_4 \\
\dot{y}_3 &= b + y_1 y_3 + \Gamma_{31}e_1 + \Gamma_{32}e_2 + \Gamma_{33}e_3 + \Gamma_{34}e_4 \\
\dot{y}_4 &= cy_4 - dy_3 + \Gamma_{41}e_1 + \Gamma_{42}e_2 + \Gamma_{43}e_3 + \Gamma_{44}e_4,
\end{aligned} \tag{18}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & -1 & -1 & 0 \\ 1 & a & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -d & c \end{bmatrix}. \tag{19}$$

Choose a Lyapunov function in the form of a positive definite function:

$$V(e_1, e_2, e_3, e_4) = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + e_4^2) \tag{20}$$

$$\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x}) = \begin{bmatrix} 0 \\ 0 \\ y_1 y_3 - x_1 x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} y_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & x_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{e} = \mathbf{De}. \tag{21}$$

Let

$$\boldsymbol{\Gamma} = -\mathbf{C} - \mathbf{A} - \mathbf{D} = \begin{bmatrix} -1 - y_3 & 1 & 1 & 0 \\ -1 & -1 - a & 0 & -1 \\ 0 & 0 & -1 - x_1 & 0 \\ 0 & 0 & d & -1 - c \end{bmatrix}. \tag{22}$$

Please cite this article in press as: Z.-M. Ge, P.-C. Tsen, Chaos synchronization by variable strength linear coupling and Lyapunov function derivative in series form, Nonlinear Analysis (2007), doi:10.1016/j.na.2007.11.018

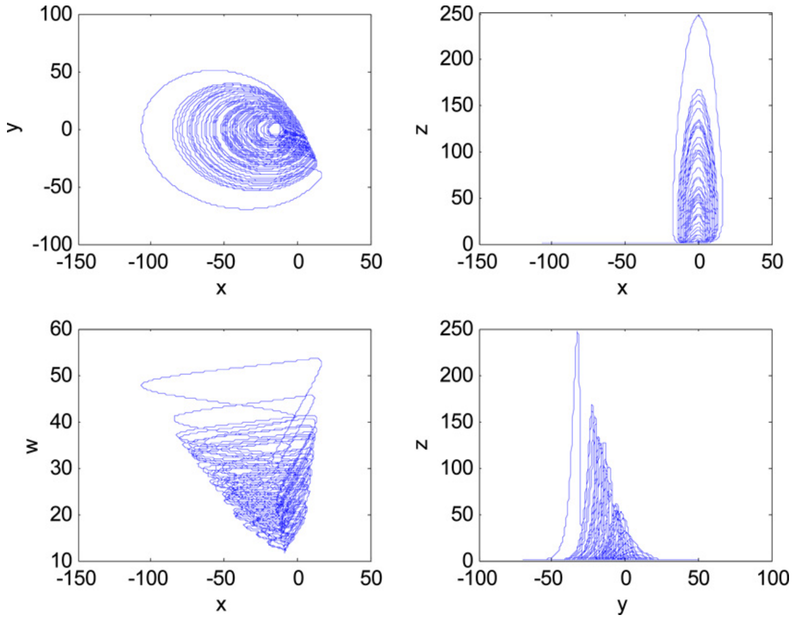

Fig. 3. Chaotic phase portraits for the Hyper-Rössler system.

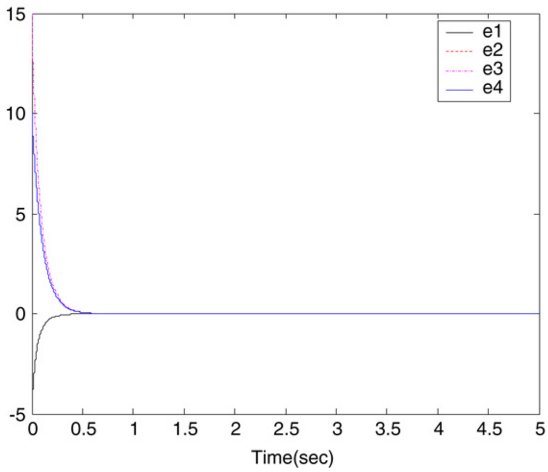

Fig. 4. Time histories of errors for two synchronized Hyper-Rössler systems.

According to Corollary 1, we obtain

$$\dot{V} = -e_1^2 - e_2^2 - e_3^2 - e_4^2 < 0. \tag{23}$$

The Hyper-Rössler systems in Eq. (18) are globally synchronized. For the initial states (−20, 0, 0, 15), (−20, 10.15, 15) and system parameters $a = 0.25$, $b = 3$, $c = 0.05$, $d = 0.5$, the chaotic phase portraits and state errors versus time are shown in Figs. 3 and 4.

Third example for Theorem 2 is the Duffing system. Consider the following two unidirectional coupled chaotic Duffing systems:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\delta x_2 + \alpha x_1 - \beta x_1^3 + a\cos\omega t \end{aligned} \tag{24}$$

Please cite this article in press as: Z.-M. Ge, P.-C. Tsen, Chaos synchronization by variable strength linear coupling and Lyapunov function derivative in series form, Nonlinear Analysis (2007), doi:10.1016/j.na.2007.11.018

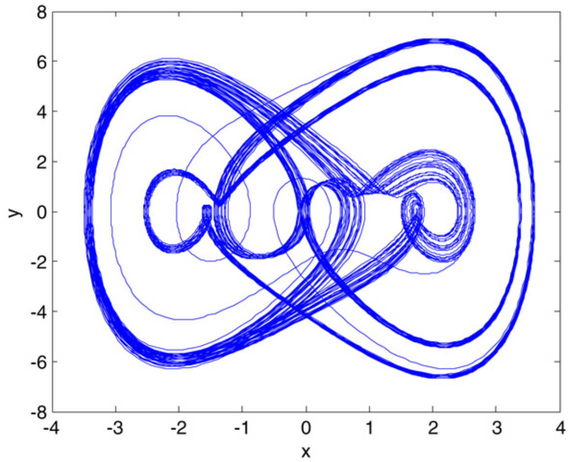

Fig. 5. Chaotic phase portrait for the Duffing system.

$$\dot{y}_1 = y_2 + u_1$$
$$\dot{y}_2 = -\hat{\delta} y_2 + \hat{\alpha} y_1 - \beta y_1^3 + a\cos\omega t + u_2,$$

where $\mathbf{u} = [u_1, u_2]^{\mathrm{T}}$ is the coupling term.

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ \alpha & -\delta \end{bmatrix}. \tag{25}$$

Choose a Lyapunov function in the form of a positive definite function:

$$V(e_1, e_2) = \frac{1}{2}(e_1^2 + e_2^2). \tag{26}$$

By Taylor expansion

$$\begin{aligned} \mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x}) &= \begin{bmatrix} 0 \\ -\beta y_1^3 + \beta x_1^3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -3\beta x_1^2 & 0 \end{bmatrix} \mathbf{e} + \begin{bmatrix} 0 \\ -6\beta x_1 e_1^2 + \cdots \end{bmatrix} \\ &= \mathbf{F}\mathbf{e} + \text{H.O.T. of } \mathbf{e}. \end{aligned} \tag{27}$$

Let $\mathbf{u} = \mathbf{\Gamma}\mathbf{e} + \mathbf{B}\mathbf{y}$

$$\mathbf{\Gamma} = -\mathbf{I} - \mathbf{A} - \mathbf{F} = \begin{bmatrix} -1 & -1 \\ -\alpha + 3\beta x_1^2 & -1 + \delta \end{bmatrix} \tag{28}$$

$$\mathbf{B} = -\tilde{\mathbf{A}} = \begin{bmatrix} 0 & 0 \\ \hat{\alpha} - \alpha & -\hat{\delta} + \delta \end{bmatrix}. \tag{29}$$

According to Theorem 2, we obtain that

$$\dot{V} = -e_1^2 - e_2^2 + \text{HOT of } \mathbf{e} < 0 \tag{30}$$

is negative definite when $\|\mathbf{e}\|^2$ is smaller than a bounded value. The Duffing systems (24) are locally synchronized. For the initial states $(2, 2)$, $(5, 5)$ and system parameters $\alpha = -0.01$, $\delta = 0.1$, $\beta = \omega = 1$, $a = 10$, $\hat{\alpha} = 1$ and $\hat{\delta} = 0.15$, the chaotic phase portrait and state errors versus time are shown in Figs. 5 and 6.

Please cite this article in press as: Z.-M. Ge, P.-C. Tsen, Chaos synchronization by variable strength linear coupling and Lyapunov function derivative in series form, Nonlinear Analysis (2007), doi:10.1016/j.na.2007.11.018

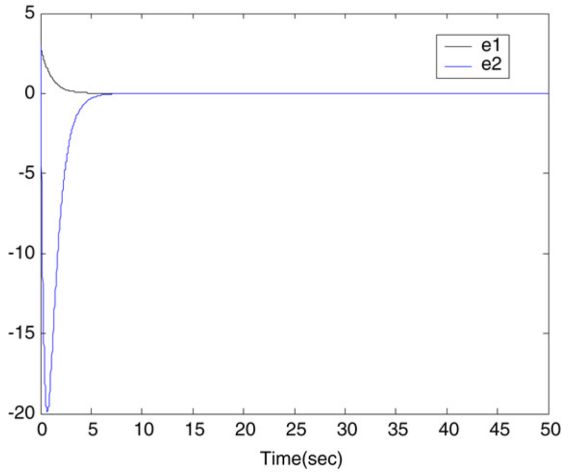

Fig. 6. Time histories of errors for two synchronized Duffing systems.

Last example for Corollary 2 is the Lorenz system. Consider the following two unidirectional coupled chaotic Lorenz systems:

$$\begin{aligned}
\dot{x}_1 &= \sigma(y_1 - x_1)\\
\dot{y}_1 &= \gamma x_1 - x_1 z_1 - y_1\\
\dot{z}_1 &= x_1 y_1 - \beta z_1\\
\dot{x}_2 &= \hat{\sigma}(y_2 - x_2) + u_1\\
\dot{y}_2 &= \hat{\gamma} x_2 - x_2 z_2 - y_2 + u_2\\
\dot{z}_2 &= x_2 y_2 - \hat{\beta} z_2 + u_3,
\end{aligned} \tag{31}$$

where $\mathbf{u} = [u_1, u_2, u_3]^{\mathrm{T}}$ is the coupling term.

$$\mathbf{A} = \begin{bmatrix} -\sigma & \sigma & 0 \\ \gamma & -1 & 0 \\ 0 & 0 & -\beta \end{bmatrix}. \tag{32}$$

Choose a Lyapunov function in the form of a positive definite function:

$$V(e_1, e_2, e_3) = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2) \tag{33}$$

$$\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x}) = \begin{bmatrix} 0 \\ -x_2 z_2 + x_1 z_1 \\ x_2 y_2 - x_1 y_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -z_2 & 0 & -x_1 \\ y_2 & x_1 & 0 \end{bmatrix} \mathbf{e} = \mathbf{De}. \tag{34}$$

Let $\mathbf{u} = \mathbf{\Gamma e} + \mathbf{By}$

$$\mathbf{\Gamma} = -\mathbf{I} - \mathbf{A} - \mathbf{D} = \begin{bmatrix} \sigma - 1 & -\sigma & 0 \\ -\gamma + z_2 & 0 & x_1 \\ -y_2 & -x_1 & \beta - 1 \end{bmatrix} \tag{35}$$

$$\boldsymbol{B} = -\tilde{\mathbf{A}} = \begin{bmatrix} 6 & -6 & 0 \\ -17.92 & 0 & 0 \\ 0 & 0 & 4/3 \end{bmatrix}. \tag{36}$$

According to Corollary 2, we obtain that

$$\dot{V} = -e_1^2 - e_2^2 - e_3^2 < 0 \tag{37}$$

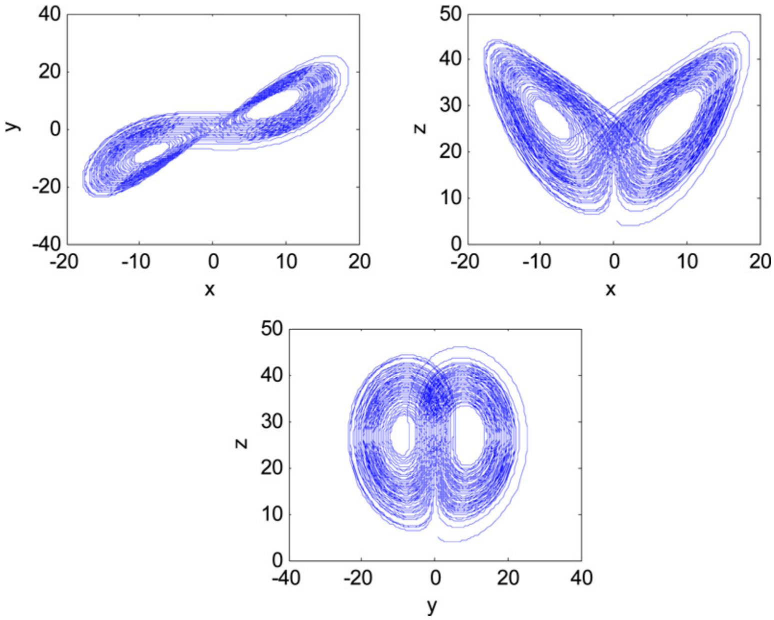

Fig. 7. Chaotic phase portraits for the Lorenz system.

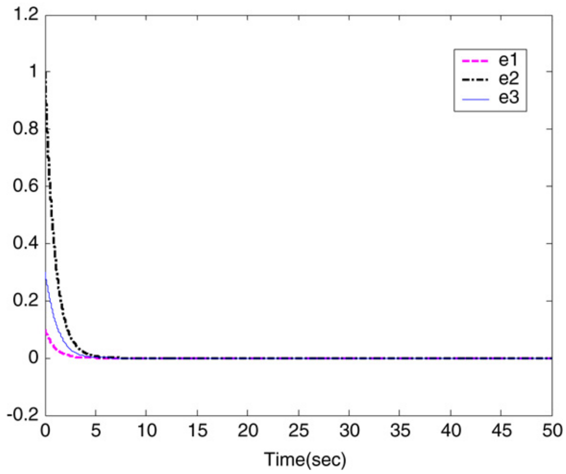

Fig. 8. Time histories of errors for two synchronized Lorenz systems.

is negative definite. The Lorenz systems (31) are global synchronized. For the initial states (0.5, 1, 5), (0.6, 2, 5.3) and system parameters $\sigma = 10$, $\gamma = 28$, $\beta = 8/3$, $\hat{\sigma} = 16$, $\hat{\gamma} = 45.92$ and $\hat{\beta} = 4$, the chaotic phase portraits and state errors versus time are shown in Figs. 7 and 8.

4. Conclusions

In this paper, two theorems for chaos synchronization are proposed by using variable strength linear coupling without another active control, while the time derivative of the Lyapunov function in series form is firstly used, which makes the demand for the Lyapunov function derivative as negative sum of the square of state variables, lower. They give the criteria of chaos synchronization for two identical chaotic systems and for two different chaotic dynamic systems. Either local synchronization which is mostly good enough or global synchronization which is mostly an unnecessary high demand, can be obtained. Lorenz system, Duffing system, Rössler system and Hyper-Rössler system are used as simulation examples which effectively confirm the scheme.

Please cite this article in press as: Z.-M. Ge, P.-C. Tsen, Chaos synchronization by variable strength linear coupling and Lyapunov function derivative in series form, Nonlinear Analysis (2007), doi:10.1016/j.na.2007.11.018

Acknowledgment

This research was supported by the National Science Council, Republic of China, under Grant Number NSC96-2212-E-144-MY3.

References

[1] L.M. Pecora, T.L. Carroll, Synchronization in chaotic system, Physical Review Letters 64 (1990) 821–824.
[2] R. Femat, G.S. Perales, On the chaos synchronization phenomenon, Physics Letters A262 (1999) 50–60.
[3] A. Krawiecki, A. Sukiennicki, Generalizations of the concept of marginal synchronization of chaos, Chaos, Solitons and Fractals 11 (9) (2000) 1445–1458.
[4] C. Wang, S.S. Ge, Adaptive synchronization of uncertain chaotic systems via backstepping design, Chaos, Solitons and Fractals 12 (2001) 1199–1206.
[5] R. Femat, J.A. Ramirez, G.F. Anaya, Adaptive synchronization of high-order chaotic systems: A feedback with low-order parameterization, Physica D 139 (2000) 231–246.
[6] O. Morgul, M. Feki, A chaotic masking scheme by using synchronized chaotic systems, Physics Letters A251 (1999) 169–176.
[7] S. Chen, J. Lu, Synchronization of uncertain unified chaotic system via adaptive control, Chaos, Solitons and Fractals 14 (4) (2002) 643–647.
[8] Ju H. Park, Adaptive synchronization of hyperchaotic Chen system with uncertain parameters, Chaos, Solitons and Fractals 26 (2005) 959–964.
[9] Ju H. Park, Adaptive synchronization of rossler system with uncertain parameters, Chaos, Solitons and Fractals 25 (2005) 333–338.
[10] E.M. Elabbasy, H.N. Agiza, M.M. El-Desoky, Adaptive synchronization of a hyperchaotic system with uncertain parameter, Chaos, Solitons and Fractals 30 (2006) 1133–1142.
[11] Z.-M. Ge, C.-C. Chen, Phase synchronization of coupled chaotic multiple time scales systems, Chaos, Solitons and Fractals 20 (2004) 639–647.
[12] Z.-M. Ge, W.-Y. Leu, Chaos synchronization and parameter identification for identical system, Chaos, Solitons and Fractals 21 (2004) 1231–1247.
[13] Z.-M. Ge, W.-Y. Leu, Anti-control of chaos of two-degrees-of- freedom louderspeaker system and chaos synchronization of different order systems, Chaos, Solitons and Fractals 20 (2004) 503–521.
[14] Z.-M. Ge, Y.-S. Chen, Synchronization of unidirectional coupled chaotic systems via partial stability, Chaos, Solitons and Fractals 21 (2004) 101–111.
[15] Z.-M. Ge, J.-K. Yu, Pragmatical asymptotical stability theorem on partial region and for partial variable with applications to gyroscopic systems, The Chinese Journal of Mechanics 16 (4) (2000) 179–187.
[16] Z.-M. Ge, C.-M. Chang, Chaos synchronization and parameters identification of single time scale brushless dc motors, Chaos, Solitons and Fractals 20 (2004) 883–903.
[17] Zheng-Ming Ge, Yen-Sheng Chen, Synchronization of unidirectional coupled chaotic systems via partial stability, Chaos, Solitons and Fractals 21 (2004) 101.
[18] F. Liu, Y. Ren, X. Shan, Z. Qiu, A linear feedback synchronization theorem for a class of chaotic systems, Chaos, Solitons and Fractals 13 (4) (2002) 723–730.
[19] J. Lü, T. Zhou, S. Zhang, Chaos synchronization between linearly coupled chaotic systems, Chaos, Solitons and Fractals 14 (4) (2002) 529–541.
[20] J. Lu, Y. Xi, Linear generalized synchronization of continuous-time chaotic systems, Chaos, Solitons and Fractals 17 (2003) 825–831.
[21] Z.-M. Ge, C.-H. Yang, Synchronization of complex chaotic systems in series expansion form, Chaos, Solitons, and Fractals 34 (2007) 1649–1658.
[22] I.G. Malkin, Theory of Stability of Motion, The State Publishing House of Technical–Theoretical Literature, Moscow-Leningrad, 1952. Translated by the Language Service Bureau, Washington, D.C., published by Office of Technical Services, Dept. of Commerce, Washington 25, DC, p. 21.
[23] Zheng-Ming Ge, Developing Theory of Motion Stability, Gaulih Book Company, Taipei, 2001, pp. 10–11.