行政院國家科學委員會專題研究計畫 期中進度報告

連結網路上的連通性相關之研究(第 1 年)
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※ 連結網路上連通性之研究 ※

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Lun-Min Shih, Chieh-Feng Chiang, Lih-Hsing Hsu, and Jimmy J.M. Tan "Strong Menger Connectivity with Conditional Faults on the Class of Hypercube-like Networks", Information Processing Letters, (2008), 106, pp. 64-69.

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□國際合作研究計畫國外研究報告書一份

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連結網路上的連通性相關之研究

A Study on the Connected Property of Interconnection Networks

計劃編號：96-2221-E-009-137-MY3
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主持人：譚建民 國立交通大學資訊工程學系 教授

一、 中英文摘要

中文摘要

連結網路一個相當廣泛的研究領域。對連結網路的結構時常以圖形G來表示。連通度是在連結網路的研究領域中重要的研究主題之一；對一個圖形G的連通度我們以κ(G)來表示，其定義為一個連通的圖形中移除最少的節點數後會變為不連通。在古典的定理中，Menger提出局部性的連通度的觀念；在此，我們在特定的N維超立方體家族的網路中，對局部性的連通度也做了更進一步的研究。

N維度的超立方體家族的網路其建構方式，是將兩個N-1維度的超立方體家族的網路以一種完全配對的方式將兩個網路連結起來。在此計劃中，我們研究在N維的超立方體家族網路中，移除N-2個節點之後，對任何兩個節點u,v之間都有min{deg(u),deg(v)}條節點獨立的路徑；其中deg(u)及deg(v)是代表N維超立方體家族網路中移除部分節點數後，u和v各別的分支度數目。這主題在OH及Chen[6]的文章中有相關的研究。我們提出了另外的方法，更簡單地去証明這個特性。不僅如此，如果我們將移除的節點加一些條件，規定移除後的圖形每個節點都至少要有兩個好的節點相鄰。在這個條件之下，所允許可以移除的節點數個數將會提昇到2N-5，而且在移除這些節點之後，圖形仍然具有局部連通度的性質。對於超立方體家族網路而言，移除2N-5的節點數是具有最佳情形的容錯性質了。

關鍵詞：局部連通度、條件式容錯、超立方體家族、連結網路。

英文摘要

Interconnection networks have been widely studied recently. The architecture of an interconnection network is usually denoted as an undirected graph G. Among all fundamental properties for interconnection networks, the (vertex) connectivity is a major parameter widely discussed for the connection status of networks. A basic definition of the connectivity κ(G) of a graph G is defined as the minimum number of vertices whose removal from G produces a disconnected graph. In contrast to this concept, Menger [5] provided a local point of view, and define the connectivity of any two vertices as the minimum number of internally vertex-disjoint paths between them. In this project, we study the Menger property on a class of hypercube-like networks [9], which is a variation of the classical hypercube network by twisting some pairs of links in it. We show that in all n-dimensional hypercube-like networks with some vertices removed, every pair of unremoved vertices u and v are connected by min{deg(u); deg(v)} vertex-disjoint paths, where deg(u) and deg(v) are the remaining degree of vertices u and v, respectively. This concept is firstly applied on hypercubes and stars by Oh and Chen [6, 7, 8]. In this project, we give a simpler proof of this result. Furthermore, if we restrict a condition such that each vertex has at least two fault-free adjacent vertices, all hypercube-like networks still have this strong Menger property, even if there are up to 2n - 5 vertex faults. The bound of 2n - 5 is sharp.

Keywords: strong Menger connectivity, conditional faults, hypercube-like network

二、 計劃緣由及目的

The topology of a multiprocessor system can be modeled as an undirected graph $G=(V,E)$, where $V(G)$ represents the set of all processors and $E(G)$ represents the set of all connecting links between the processors. For a subset of vertices $F \subset V(G)$, the induced graph obtained by deleting the vertices of F from G is denoted by $G - F$. Let u be a vertex, we use $N(u)$ to denote the set of vertices adjacent to u, and use deg(u) to denote the cardinality of $N(u)$. For a set of vertices V', the neighborhood of V' is defined as the set $N(V') = \{\cup_{v\in V'} N(v)\} - V'$. Let G be a graph with a set F of faulty vertices, the number of fault-free neighbors of u in $G-F$ is denoted by $\deg_{G-F}(u)$.

Let $G_0 = (V_0, E_0)$ and $G_1 = (V_1, E_1)$ be two disjoint graphs with the same number of vertices. A one-to one connection between $V(G_0)$ and $V(G_1)$ is defined as an edge set $M= \{(v, \phi(v)) \mid v \in V_0, \phi(v) \in V_1$ and $\phi: V_0 \rightarrow V_1$ is a bijection$\}$. We use $G_0 \oplus_M G_1$ to denote the graph $G = (V_0 \cup V_1, E_0 \cup E_1 \cup M)$. Different bijection functions ϕ lead to different operations $\oplus_M$ and generate different graphs.

The *hypercube* network is one of the popular topologies in interconnection networks. Several variants of hypercubes are proposed by twisting some pairs of links in hypercubes, including twisted cubes [1,4], Möbius cubes [2], and crossed cubes [3], to name a few. To make a unified study on these variants, Vaidya et al. [9] proposed a class of graphs, called *a class of hypercube-like networks.* We now give a recursive definition of the $n$dimensional hypercube-like networks HL_n as follows:

(1) $HL_0 = K_1$, where K_1 is a trivial graph in the sense that it has only one vertex; and (2) $G \in HL_n$ if and only if $G = G_0 \oplus_M G_1$ for some $G_0, G_1 \in HL_{n-1}$. By the definitions above if G is a graph in HL_n, then G is a composition of $G_0 \oplus_M G_1$ with both G_0 and G_1 in HL_{n-1}, $n \geqq 1$. Each vertex in G_0 has exactly one neighbor in G_1.

A graph G is r-regular if the degree of every vertex in G is r. We say that a graph G is *connected* if there is a path between every pair of two distinct vertices. A subset S of $V(G)$ is a *cut set* if $G - S$ is disconnected. The *connectivity* of G, written as $\kappa(G)$, is defined as the minimum size of a vertex cut if G is not a complete graph, and $\kappa(G) = |V(G)| - 1$ if otherwise. We say that a graph G is k-*connected* if $k \leq \kappa(G)$. In addition, a graph has *connectivity* k if it is k-connected but not $(k+1)$-connected.

A classical theorem about connectivity was provided by Menger as follows.

Theorem 1. *(See [5].) Let x and y be two* *distinct vertices of a graph* G *and* (x, y) *not belong to* $E(G)$. *The minimum size of an* x,y*-cut equals the maximum number of pairwise internally disjoint* x,y*-paths.* Following this theorem, Oh and Chen [7] gave a definition to extend the Menger's theorem.

Definition 1. (See [7].) A k-regular graph G is *strongly Menger-connected* if for any subgraph $G-F$ of G with at most $k-2$ vertices removed, each pair of vertices u and v in $G-F$ are connected by $\min\{\deg_{G-F}(u), \deg_{G-F}(v)\}$ vertex-disjoint fault-free paths in $G-F$, where $\deg_{G-F}(u)$ and $\deg_{G-F}(v)$ are the degree of u and v in $G-F$, respectively.

By Definition 1, Oh and Chen [6–8] showed that an n-dimensional star graph S_n (respectively, an n-dimensional hypercube Q_n) with at most $n-3$ (respectively, $n-2$) vertices removed is strongly Menger-connected. In order to be consistent with Definition 1, we say that a graph G possess the strongly Menger-connected property with respect to a vertex set F if, after deleting F from G, there are $\min\{\deg_{G-F}(u), \deg_{G-F}(v)\}$ vertex-disjoint fault-free paths connecting u and v, for each pair of vertices u and v in $G-F$. Throughout this project, we shall call a graph "strongly Menger-connected", and omit the description of the remaining structure $G-F$ of the graph, if there is no ambiguous. It is known that the connectivity of an n-dimensional hypercube-like network *HL*n is n [9]. To extend the connectivity result of *HL*n further, we study the strongly Menger-connected property of *HL*n with at most $n-2$ vertices deleted. Moreover, if we restrict a condition such that each vertex has at least two fault-free adjacent vertices, *HL*n still have the strong Menger property, even if there are up to $2n-5$ vertex faults.

三、 研究方法與成果

這幾年來的研究，我們在連結網路的領域上，對一些著名的網路架構做了深入的探討及研究。連結網路中的一些觀念如 connectivity(連通度)、conditional connectivity(條件式連通度)、local connectivity（局部性連連度）我們都有相關的研究，也將我們的成果投稿，並且持續努力撰寫論文。在實驗室的規範中，每週都定時研討會礒，內容如下：

一、 搜尋及集合文獻

每位學生主動從圖書館、國內外研討會及網際網路等所收集相關主題及所需要的相關文獻。

二、 探討文獻及發現問題

將所收集到的文獻由計畫中的成員將文獻做進一步的分析研究，並且在每週輪流定期報告其文獻的內容、共同分析文獻中的主題內容，再找尋從文獻中發現可以做進一步研究之問題，由主持人帶領成員選定研究之主題。

三、 由指導老師帶領博士班，並由博士班帶領碩士班共同研究，一起解決其問題，在過程中，會需要撰寫程式來輔助定理的證明正確性。我們也有自行發展出一些軟體程式以供測試例子。

四、 成果發表

近年來，我們已有多篇論文被國際知名期刊刊登。而這個計畫其中之一結果已刊登在 2008 年 Information Processing Letters期刊，發表的論文為：

Lun-Min Shih, Chieh-Feng Chiang, Lih-Hsing Hsu, and Jimmy J.M. Tan "Strong Menger Connectivity with Conditional Faults on the Class of Hypercube-like Networks",

Information Processing Letters, (2008), 106, pp. 64-69.

四、 結論與討論

在主持人的帶領下，本計劃每週都會討論所收集的資料，並分析和比較各種方法的可行性。對一些有名連結網路(如：hypercube、matching composition networks、hypercube-like networks、star graph)的連通度相關問題有了更清楚的了解，也因此為本次計劃的執行有更良好的基礎，也順利地完成我們所預定的研究進度。對於局部性連通度及其容錯性質，希望可以在學術上找到新的特性，並且可以比較分析各種不周的連結網路，去探討它們之間的相關連通特性。期望這些相關的問題可以提昇這方面的領域能力，對於日後的研究有更深入的探討。

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Strong Menger connectivity with conditional faults on the class of hypercube-like networks ☆

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Abstract

In this paper, we study the Menger property on a class of hypercube-like networks. We show that in all n-dimensional hypercube-like networks with $n-2$ vertices removed, every pair of unremoved vertices u and v are connected by $\min\{\deg(u), \deg(v)\}$ vertex-disjoint paths, where $\deg(u)$ and $\deg(v)$ are the remaining degree of vertices u and v, respectively. Furthermore, under the restricted condition that each vertex has at least two fault-free adjacent vertices, all hypercube-like networks still have the strong Menger property, even if there are up to $2n-5$ vertex faults.

Keywords: Strong Menger connectivity; Conditional faults; Hypercube-like network; Interconnection networks

1. Introduction

Interconnection networks have been widely studied recently. The architecture of an interconnection network is usually denoted as an undirected graph G. Among all fundamental properties for interconnection networks, the (vertex) connectivity is a major parameter widely discussed for the connection status of networks. A basic definition of the connectivity $\kappa(G)$ of a graph G is defined as the minimum number of vertices whose removal from G produces a disconnected graph. In contrast to this concept, Menger [5] provided a local point of view, and define the connectivity of any two vertices as the minimum number of internally vertex-disjoint paths between them.

In this paper, we study the Menger property on a class of hypercube-like networks [9], which is a variation of the classical hypercube network by twisting some pairs of links in it. We show that in all n-dimensional hypercube-like networks with some vertices removed, every pair of unremoved vertices u and v are connected by $\min\{\deg(u), \deg(v)\}$ vertex-disjoint paths, where $\deg(u)$ and $\deg(v)$ are the remaining degree of vertices u and v, respectively. This concept is firstly applied on hypercubes and stars by Oh and Chen [6–8]. In this paper, we give a simpler proof of this result. Furthermore, if we restrict a condition such that each vertex has at least two fault-free adjacent vertices, all hypercube-like networks still have this strong Menger

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property, even if there are up to $2n-5$ vertex faults. The bound of $2n-5$ is sharp.

2. Preliminary

The topology of a multiprocessor system can be modeled as an undirected graph $G=(V,E)$, where $V(G)$ represents the set of all processors and $E(G)$ represents the set of all connecting links between the processors. For a subset of vertices $F\subset V(G)$, the induced graph obtained by deleting the vertices of F from G is denoted by $G-F$. Let u be a vertex, we use $N(u)$ to denote the set of vertices adjacent to u, and use $\deg(u)$ to denote the cardinality of $N(u)$. For a set of vertices V', the neighborhood of V' is defined as the set $N(V')=\{\bigcup_{v\in V'}N(v)\}-V'$. Let G be a graph with a set F of faulty vertices, the number of fault-free neighbors of u in $G-F$ is denoted by $\deg_{G-F}(u)$.

Let $G_0=(V_0,E_0)$ and $G_1=(V_1,E_1)$ be two disjoint graphs with the same number of vertices. A one-to-one connection between $V(G_0)$ and $V(G_1)$ is defined as an edge set $M=\{(v,\phi(v))\mid v\in V_0,\phi(v)\in V_1$ and $\phi:V_0\to V_1$ is a bijection$\}$. We use $G_0\oplus_M G_1$ to denote the graph $G=(V_0\cup V_1,E_0\cup E_1\cup M)$. Different bijection functions ϕ lead to different operations $\oplus_M$ and generate different graphs.

The *hypercube* network is one of the popular topologies in interconnection networks. Several variants of hypercubes are proposed by twisting some pairs of links in hypercubes, including twisted cubes [1,4], Möbius cubes [2], and crossed cubes [3], to name a few. To make a unified study on these variants, Vaidya et al. [9] proposed a class of graphs, called *a class of hypercube-like networks*. We now give a recursive definition of the n-dimensional hypercube-like networks HL_n as follows: (1) $HL_0=K_1$, where K_1 is a trivial graph in the sense that it has only one vertex; and (2) $G\in HL_n$ if and only if $G=G_0\oplus_M G_1$ for some $G_0,G_1\in HL_{n-1}$. By the definitions above if G is a graph in HL_n, then G is a composition of $G_0\oplus_M G_1$ with both G_0 and G_1 in HL_{n-1}, $n\geqslant 1$. Each vertex in G_0 has exactly one neighbor in G_1.

A graph G is r-regular if the degree of every vertex in G is r. We say that a graph G is *connected* if there is a path between every pair of two distinct vertices. A subset S of $V(G)$ is a *cut set* if $G-S$ is disconnected. The *connectivity* of G, written as $\kappa(G)$, is defined as the minimum size of a vertex cut if G is not a complete graph, and $\kappa(G)=|V(G)|-1$ if otherwise. We say that a graph G is *k-connected* if $k\leqslant\kappa(G)$. In addition, a graph has *connectivity k* if it is k-connected but not $(k+1)$-connected.

A classical theorem about connectivity was provided by Menger as follows.

Theorem 1. *(See [5].) Let x and y be two distinct vertices of a graph G and $(x,y)\notin E(G)$. The minimum size of an x,y-cut equals the maximum number of pairwise internally disjoint x,y-paths.*

Following this theorem, Oh and Chen [7] gave a definition to extend the Menger's theorem.

Definition 1. (See [7].) A k-regular graph G is *strongly Menger-connected* if for any subgraph $G-F$ of G with at most $k-2$ vertices removed, each pair of vertices u and v in $G-F$ are connected by $\min\{\deg_{G-F}(u),\deg_{G-F}(v)\}$ vertex-disjoint fault-free paths in $G-F$, where $\deg_{G-F}(u)$ and $\deg_{G-F}(v)$ are the degree of u and v in $G-F$, respectively.

By Definition 1, Oh and Chen [6–8] showed that an n-dimensional star graph S_n (respectively, an n-dimensional hypercube Q_n) with at most $n-3$ (respectively, $n-2$) vertices removed is strongly Menger-connected. In order to be consistent with Definition 1, we say that a graph G possess the strongly Menger-connected property with respect to a vertex set F if, after deleting F from G, there are $\min\{\deg_{G-F}(u),\deg_{G-F}(v)\}$ vertex-disjoint fault-free paths connecting u and v, for each pair of vertices u and v in $G-F$. Throughout this paper, we shall call a graph "strongly Menger-connected", and omit the description of the remaining structure $G-F$ of the graph, if there is no ambiguous.

It is known that the connectivity of an n-dimensional hypercube-like network HL_n is n [9]. To extend the connectivity result of HL_n further, we study the strongly Menger-connected property of HL_n with at most $n-2$ vertices deleted. Moreover, if we restrict a condition such that each vertex has at least two fault-free adjacent vertices, HL_n still have the strong Menger property, even if there are up to $2n-5$ vertex faults.

3. Strong Menger connectivity

In this section, we will prove that all graphs in the class of n-dimensional hypercube-like networks are strongly Menger-connected if there are at most $n-2$ vertex faults. Before proving this main result, we need the following lemma, essentially it says that every n-dimensional hypercube-like network with no more than $2n-3$ vertex faults, still contains a large connected component.

Lemma 1. *Let $G \in HL_n$ be an n-dimensional hypercube-like network, and S be a set of vertices with $|S| \leqslant 2n-3$, for $n \geqslant 2$. There exists a connected component C in $G-S$ such that $|V(C)| \geqslant 2^n - |S| - 1$.*

Proof. We prove this statement by induction on n. For $n=2$, HL_2 is a cycle of length four, the result is trivially true. Assume this lemma holds for $n-1$, for some $n \geqslant 3$, we will prove that it is true for n.

Let G be an n-dimensional hypercube-like network, $G = G_0 \oplus_M G_1$, and $G_0, G_1 \in HL_{n-1}$. Let S be a set of vertices with $|S| \leqslant 2n-3$, for $n \geqslant 3$, and let S_0 and S_1 be subsets of set S in G_0 and G_1, respectively. Then $|S_0| + |S_1| = |S| \leqslant 2n-3$. Without loss of generality, we assume $|S_0| \leqslant |S_1|$. The proof is divided into two major cases:

Case 1: $0 \leqslant |S_0| \leqslant 1$.

Since G_0 is $(n-1)$-connected, $G_0 - S_0$ is connected, for $n \geqslant 3$. All the vertices in $G_0 - S_0$ are connected and form a connected component C_0 with $|V(C_0)| = 2^{n-1} - S_0$. By definition, all the vertices in $G_1 - S_1$ are adjacent to the vertices in $G_0 = C_0 \cup S_0$. Thus, $G-S$ contains a connected component C such that the number of vertices in C is greater than $|V(G_0) - S_0| + |V(G_1) - S_1| - |S_0| = |V(G)| - |S| - |S_0| \geqslant 2^n - |S| - 1$. (See Fig. 1.)

Case 2: $|S_0| \geqslant 2$ and consequently $|S_1| \leqslant 2n-5$.

Since $2 \leqslant |S_0| \leqslant |S_1| \leqslant 2n-5$, so $|S_0| \leqslant n-2$ and $n \geqslant 4$. By induction hypothesis, there exists a connected component C_1 in $G_1 - S_1$, and $|V(C_1)| \geqslant 2^{n-1} - |S_1| - 1$. Since the connectivity of G_0 is $n-1$ and $|S_0| \leqslant n-2$, $G_0 - S_0$ is connected. Then $G-S$ contains a connected component C such that the number of vertices in C is greater than $|V(G_0) - S_0| + (|V(G_1) - S_1| - 1) = |V(G)| - |S| - 1 = 2^n - |S| - 1$. □

By Lemma 1, we have the following corollary.

Corollary 1. *Let G be an n-dimensional hypercube-like network, $n \geqslant 2$, and let V' be a set of vertices in G with $|V'| = 2$. Then $|N(V')| \geqslant 2n-2$.*

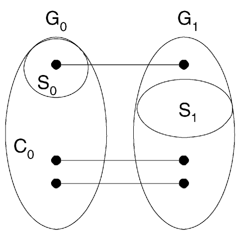

Fig. 1. The illustration of the proof of Case 1 in Lemma 1.

In the following, we show that with up to $n-2$ vertex faults, an n-dimensional hypercube-like network has strongly Menger-connected property. Referring to the relative study proposed by Oh [6], the strong Menger connectivity of regular hypercube networks has been proved. Here we provide a significantly simpler proof for the general hypercube-like networks.

Theorem 2. *Consider an n-dimensional hypercube-like network $G \in HL_n$, for $n \geqslant 2$. Let F be a set of faulty vertices with $|F| \leqslant n-2$. Then each pair of vertices u and v in $G-F$ are connected by* $\min\{\deg_{G-F}(u), \deg_{G-F}(v)\}$ *vertex-disjoint fault-free paths, where* $\deg_{G-F}(u)$ *and* $\deg_{G-F}(v)$ *are the remaining degree of u and v in $G-F$, respectively.*

Proof. Let G be an n-dimensional hypercube-like network, and u and v be two fault-free vertices in $G-F$. We first assume, without loss of generality, that $\deg_{G-F}(u) \leqslant \deg_{G-F}(v)$, so $\min\{\deg_{G-F}(u), \deg_{G-F}(v)\} = \deg_{G-F}(u)$. We now show that u is connected to v if the number of vertices deleted is smaller than $\deg_{G-F}(u) - 1$ in $G-F$. By Theorem 1, this implies that each pair of vertices u and v in $G-F$ are connected by $\deg_{G-F}(u)$ vertex-disjoint fault-free paths, where $|F| \leqslant n-2$.

For the sake of contradiction, suppose that u and v are separated by deleting a set of vertices V_f, where $|V_f| \leqslant \deg_{G-F}(u) - 1$. As a consequence, $|V_f| \leqslant n-1$ because of $\deg_{G-F}(u) \leqslant \deg(u) \leqslant n$. Then, the summation of the cardinality of these two sets F and V_f is $|F| + |V_f| \leqslant 2n-3$. Let $S = F \cup V_f$. By Lemma 1, there exists a connected component C in $G-S$ such that $|V(C)| \geqslant 2^n - |S| - 1$. It means that (i) either $G-S$ is connected, or (ii) $G-S$ has two components, one of which contains only one vertex. If $G-S$ is connected, it contradicts to the assumption that u and v are disconnected. Otherwise, if $G-S$ has two component and one of which contains only one vertex x. Since we assume that u and v are separated, one of u and v is the vertex x, say $u = x$. Thus, the set V_f must be the neighborhood of u and $|V_f| = \deg_{G-F}(u)$, which is also a contradiction. Then, u is connected to v when the number of vertices deleted is smaller than $\deg_{G-F}(u) - 1$ in $G-F$.

The proof is complete. □

4. Strong Menger connectivity with conditional faults

As proved in the last section, an n-dimensional hypercube-like network with at most $n-2$ faulty vertices is strongly Menger-connected. But the result can-not be guaranteed, if there are $n-1$ faulty vertices and all these faulty vertices are adjacent to the same vertex. In most circumstances, the possibility of all the neighbors of a vertex being faulty simultaneously is very small. Motivated by the deficiency of traditional fault tolerance, we consider a measure of conditional faults by restricting that every vertex has at least two fault-free neighboring vertices.

Under this condition, we claim that for every n-dimensional hypercube-like network with at most $2n-5$ faulty vertices and $n \geqslant 5$, the resulting network is still strongly Menger-connected. We have an example to show that this result does not hold for $n=4$. Consider a 4-dimensional HL_4, this network may not be strongly Menger-connected, if the number of conditional faults is 3. (See Fig. 2. The remaining degrees of nodes u and v are both four, with three vertices deleted as indicated in the graph. But the number of vertex-disjoint paths between u and v is three.) So we can only expect the result holds for $n \geqslant 5$.

To prove this result, we need some preliminary lemma. In the following, we show that an n-dimensional hypercube-like network with at most $3n-6$ vertex faults S has a connected component having at least $2^n - |S| - 2$ vertices.

The proof is by induction, and the case for $n = 5$ is proved in the following two lemmas.

Lemma 2. *Let V' be a set of vertices in a 4-dimensional hypercube-like network with $|V'| = 3$. Then, $|N(V')| \geqslant 7$.*

Proof. Let G be a 4-dimensional hypercube-like network. G is a composition of two 3-dimensional hypercube-like networks G_0 and G_1, $G = G_0 \oplus_M G_1$, for a matching operation $\oplus_M$. Without loss of generality, let V' be a subset of $V(G)$ containing three vertices $\{x, y, z\}$. If x, y, z are all in G_0, by Lemma 1, $\{x, y, z\}$ has at least 4 neighboring vertices in G_0. Besides, $\{x, y, z\}$ has 3 neighboring vertices in G_1. Then, $|N(\{x, y, z\})| \geqslant 4 + 3 = 7$. If x, y are in G_0, and z is in G_1, by Lemma 1, $\{x, y\}$ has at least 4 neighboring vertices in G_0. In addition, $\{z\}$ has 3 neighboring vertices in G_1. Then, $|N(\{x, y, z\})| \geqslant 4 + 3 = 7$. □

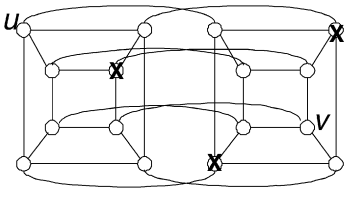

Fig. 2. An example showing that an HL_4 is not strongly Menger-connected.

Lemma 3. *Let G be a 5-dimensional hypercube-like network and S be a set of vertices with $|S| \leqslant 9$. ($3n - 6 = 9$, for $n = 5$.) There exists a connected component C in $G - S$ such that $|V(C)| \geqslant 2^5 - |S| - 2$.*

Proof. Let G be a 5-dimensional hypercube-like network, $G_0, G_1 \in HL_4$, and $G = G_0 \oplus_M G_1$, for a matching operation $\oplus_M$. Let S be a set of vertices with $|S| \leqslant 3n - 6 = 9$, for $n = 5$, and let S_0 and S_1 be subsets of S in G_0 and G_1, respectively. Without loss of generality, we assume $|S_0| \leqslant |S_1|$. (Note that $|S| \leqslant 9$, so $|S_0| \leqslant 4$.) We then consider three cases:

Case 1: $0 \leqslant |S_0| \leqslant 2$.

Since G_0 is $(n-1)$-connected, $G_0 - S_0$ is connected, for $n = 4$. So $G_0 - S_0$ has only one connected component C_0 with $|V(C_0)| = 2^4 - S_0$. By definitions, all vertices in $G_1 - S_1$ are adjacent to the vertices of $G_0 = C \cup S_0$. Let C be the connected component of $G - S$ containing C_0. Then the number of vertices in C is greater than $|V(G_0) - S_0| + |V(G_1) - S_1| - |S_0| = |V(G)| - |S| - |S_0| \geqslant 2^5 - |S| - 2$.

Case 2: $|S_0| = 3$ and therefore $|S_1| \leqslant 6$.

$G_0 - S_0$ is connected by the fact that G_0 is $(n-1)$-connected, for $n \geqslant 4$. Thus, $G_0 - S_0$ has only one connected component C_0 with $|V(C_0)| = 2^4 - S_0$. Then, all vertices in G_1 are connected to component C_0, except for the three vertices in G_1 adjacent to the vertices in S_0. Since $|S_1| \leqslant 6$ and by Lemma 2, at least one of these three vertices is connected to component $G_1 - S_1$. So at least $2^4 - |S_1| - 2$ vertices are connected to component C_0. Let C be the connected component of $G - S$ containing C_0. Then, the number of vertices in C is $|V(C)| \geqslant |V(G_0) - S_0| + |V(G_1) - S_1 - 2| = |V(G)| - |S| - 2 = 2^5 - |S| - 2$.

Case 3: $|S_0| = 4$ and consequently $4 \leqslant |S_1| \leqslant 5$.

Since $5 \leqslant 2n - 3$, for $n \geqslant 4$. By Lemma 1, there exists a connected components C_0 (respectively, C_1) in $G_0 - S_0$ (respectively, $G_1 - S_1$) such that $|V(C_0)| \geqslant 2^4 - |S_0| - 1$ (respectively, $|V(C_1)| \geqslant 2^4 - |S_1| - 1$). Thus, there exists a connected component C in $G - S$ such that $|V(C)| \geqslant |V(G_0) - S_0 - 1| + |V(G_1) - S_1 - 1| = |V(G)| - |S| - 2 = 2^5 - |S| - 2$. □

Based on Lemma 3, the general case for $n \geqslant 5$ is stated as follows.

Lemma 4. *Let G be an n-dimensional hypercube-like network, and S be a set of vertices with $|S| \leqslant 3n - 6$, for*

$n \geqslant 5$. There exists a connected component C in $G - S$ such that $|V(C)| \geqslant 2^n - |S| - 2$.

Proof. We prove this statement by induction on n. By Lemma 3, the result holds for $n = 5$. Assume the lemma holds for $n - 1$, for some $n \geqslant 6$. We now show that it is true for n.

Let G be an n-dimensional hypercube-like network, $G_0, G_1 \in HL_{n-1}$, and $G = G_0 \oplus_M G_1$, for some matching operation $\oplus_M$. Let S be a set of vertices with $|S| \leqslant 3n - 6$, for $n \geqslant 6$, and let S_0 and S_1 be subsets of S in G_0 and G_1, respectively. Therefore, $|S_0| + |S_1| = |S| \leqslant 3n - 6$. Without loss of generality, we assume $|S_0| \leqslant |S_1|$. The proof is divided into two major cases:

Case 1: $0 \leqslant |S_0| \leqslant 2$.

Since G_0 is $(n - 1)$-connected, $G_0 - S_0$ is connected, for $n \geqslant 6$. Let $C_0 = G_0 - S_0$, C_0 is a connected component with $|V(C_0)| \geqslant 2^{n-1} - S_0$. By definitions, all vertices in $G_1 - S_1$ are adjacent to the vertices in $G_0 = C_0 \cup S_0$. Let C be the connected component of $G - S$ containing C_0. The number of vertices in C is greater than $|V(G_0) - S_0| + |V(G_1) - S_1| - |S_0| = |V(G)| - |S| - |S_0| \geqslant 2^n - |S| - 2$.

Case 2: $|S_0| \geqslant 3$ and consequently $|S_1| \leqslant 3n - 9$.

By induction hypothesis, there are two connected components C_0 and C_1 in $G_0 - S_0$ and $G_1 - S_1$, and $|V(C_0)| \geqslant 2^{n-1} - |S_0| - 2$ and $|V(C_1)| \geqslant 2^{n-1} - |S_1| - 2$, respectively. Without loss of generality, we assume that $|V(C_0)| \geqslant |V(C_1)|$. Now we focus on the number of vertices in the component C_1, and discuss two situations. First, suppose $|V(C_1)| = 2^{n-1} - |S_1| - 2$. By Corollary 1, $|S_1| \geqslant 2(n - 1) - 2 = 2n - 4$. So $|S_0| = |S| - |S_1| \leqslant n - 2$. Since G_0 is $(n-1)$-connected, $G_0 - S_0$ is connected. $G_0 - S_0$ has only one connected component C_0 and $|V(C_0)| = 2^{n-1} - |S_0|$. Let C be the connected component containing C_0. Then $|V(C)| = |V(C_0)| + |V(C_1)| \geqslant 2^{n-1} - |S_0| + 2^{n-1} - |S_1| - 2 \geqslant 2^n - |S| - 2$. Second, suppose that $|V(C_1)| \geqslant 2^{n-1} - |S_1| - 1$. Since $|V(C_0)| \geqslant |V(C_1)| \geqslant 2^{n-1} - |S_1| - 1$, there exists a connected component C containing C_0 such that $|V(C)| = |V(C_0)| + |V(C_1)| \geqslant 2^{n-1} - |S_0| - 1 + 2^{n-1} - |S_1| - 1 \geqslant 2^n - |S| - 2$. □

Corollary 2. *Let G be an n-dimensional hypercube-like network, $n \geqslant 5$, and let V' be a set of vertices in G with $|V'| = 3$. Then $|N(V')| \geqslant 3n - 5$.*

As stated in the last section, we showed that every n-dimensional hypercube-like network with at most $n - 2$ vertex faults is strongly Menger-connected. In the following, we will show another main result that, by restricting every vertex having at least two fault-free neighboring vertices, every n-dimensional hypercube-like network with up to $2n - 5$ vertex faults is still strongly Menger-connected.

For the next theorem, we define a set of vertices F_c in graph G to be a *conditional* faulty vertex set if, in the induced subgraph $G - F_c$, every vertex has at least two fault-free neighboring vertices. We also call the subgraph $G - F_c$ a *conditional* faulty graph.

Theorem 3. *Consider an n-dimensional hypercube-like network $G \in HL_n$, for $n \geqslant 5$. Let F_c be a set of conditional faulty vertices with $|F_c| \leqslant 2n - 5$. Then each pair of vertices u and v in $G - F_c$ are connected by $\min\{\deg_{G-F_c}(u), \deg_{G-F_c}(v)\}$ vertex-disjoint fault-free paths, where $\deg_{G-F_c}(u)$ and $\deg_{G-F_c}(v)$ are the degree of u and v in $G - F_c$, respectively.*

Proof. Without loss of generality, we assume $\deg_{G-F_c}(u) \leqslant \deg_{G-F_c}(v)$, and therefore

$$\min\{\deg_{G-F_c}(u), \deg_{G-F_c}(v)\} = \deg_{G-F_c}(u).$$

We want to prove that each pair of vertices u and v in $G - F_c$ are connected by $\deg_{G-F_c}(u)$ vertex-disjoint fault-free paths, for $|F_c| \leqslant 2n - 5$. We are going to show that u is connected to v if the number of vertices deleted is smaller than $\deg_{G-F_c}(u) - 1$ in $G - F_c$, where $|F_c| \leqslant 2n - 5$.

Suppose on the contrary that u and v are separated by deleting a set of vertices V_{f_c}, where $|V_{f_c}| \leqslant \deg_{G-F_c}(u) - 1$. By $\deg_{G-F_c}(u) \leqslant \deg(u) \leqslant n$, we have $|V_{f_c}| \leqslant n - 1$. We sum up the cardinality of these two sets F_c and V_{f_c}. Since $|F_c| \leqslant 2n - 5$ and $|V_{f_c}| \leqslant n - 1$, then $|F_c| + |V_{f_c}| \leqslant 3n - 6$. Let $S = F_c \cup V_{f_c}$. By Lemma 4, there exits a connected component C in $G - S$ such that $|V(C)| \geqslant 2^n - |S| - 2$ and $|S| \leqslant 3n - 6$. It means that there are at most two vertices in $G - S$ not belonging to C. We then consider three cases:

Case 1: $|V(C)| = 2^n - |S|$. It means that all vertices in $G - S$ are connected, which contradicts to the assumption that u and v are disconnected.

Case 2: $|V(C)| = 2^n - |S| - 1$. Only one vertex is disconnected to $G - S$. Since $|V_{f_c}| \leqslant \deg_{G-F_c}(u) - 1 \leqslant \deg_{G-F_c}(v) - 1$, neither u nor v can be the only one disconnected vertex, a contradiction.

Case 3: $|V(C)| = 2^n - |S| - 2$. Let a and b be the two vertices in $G - S$ not belonging to C. We consider two situations. (i) Suppose first that $u \in C$. If $v \in C$, then u and v are connected, a contradiction. If $v \in \{a, b\}$, since $|V_{f_c}| \leqslant \deg_{G-F_c}(v) - 1$, v is connected to at least one vertex in component C, a contradiction. (ii) Suppose $u \in \{a, b\}$. We without loss of generality let $u = a$, and consider the adjacency between a and b.

Subcase 1: Suppose that a is not adjacent to b. By the assumption that u and v are separated by deleting a set of vertices V_{f_c} with $|V_{f_c}| = \deg_{G-F_c}(u) - 1$. Let V_{f_c} be a subset of the neighborhood of u, that is, $V_{f_c} \subset N(u)$. Since $|V_{f_c}| < |N(u)|$, vertex u and component C are connected, which is a contradiction.

Subcase 2: Suppose that a is adjacent to b. Let $V_{f_c} = N(u) - \{b\}$. Since $G - F_c$ is a conditional faulty graph, one of the neighbors of b is in C. Then, b is connected to C, which is a contradiction.

Therefore, vertex u and v are still connected with up to $\deg_{G-F_c}(u) - 1$ vertex faults. By Theorem 1, this implies that each pair of vertices u and v in $G - F_c$ are connected by $\min\{\deg_{G-F_c}(u), \deg_{G-F_c}(v)\}$ vertex-disjoint fault-free paths, where $|F_c| \leqslant 2n-5$. The proof is complete. □

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