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摘要

一個圖形 G 的漢米爾頓圈 C，是為點的有序集合 $\langle u_1, u_2, \ldots, u_{n(G)}, u_1\rangle$，使得 $u_i \neq u_j$ 對於 $i \neq j$ 以及當 $i \in \{1, 2, \ldots, n(G)-1\}$時，$u_i$ 是與 u_{i+1} 相連接的且 $u_{n(G)}$是與 u_1 相連接的，其中 $n(G)$是代表圖形 G 中的點數。點 u_1 是起始點且 u_i 代表是第 i 個點。如果稱圖形 G 中的兩個漢米爾頓圈 $C_1 = \langle u_1, u_2, \ldots, u_{n(G)}, u_1\rangle$ 和 $C_2 = \langle v_1, v_2, \ldots, v_{n(G)}, v_1\rangle$ 是獨立的，是因為 $u_1 = v_1$ 且對於 $i \in \{2, 3, \ldots, n(G)\}$，會使得 $u_i \neq v_i$。如果稱圖形 G 中的一個漢米爾頓圈之集合 $\{C_1, C_2, \ldots, C_k\}$為相互獨立的，是因為集合中的元素都是兩兩互相獨立的。圖形 G 中相互獨立的漢米爾頓圈 $IHC(G)$，是一個最大整數 k，使得對於圖形 G 中的任意一個點 u，存在 k 個相互獨立的漢米爾頓圈，起始點為 u。

如果稱一個二部圖 B 為二部泛圈性，是圖形 B 中包含所有偶數長度從 4 到 $|V(B)|$。如果稱一個二部漢米爾頓圖形 B 為二部泛定位性，是因為對於任意兩個相異點 x 和 y 在圖形 B，存在一個漢米爾頓圈 C，使得對於任意整數 k 介於 $d_B(x,y) \leq k \leq |V(B)|/2$ 且$(k - d_B(x,y))$為偶數下，$d_C(x,y) = k$。如果稱一個二部圖 B 為 k-迴圈二部泛定位性，是因為對於任兩個相異點 x 和 y，在圖形 B 中存在一個迴圈，其中 k 為任意整數，使得 $d_C(x,y)= l$ 以及$|V(C)| = k$，且 l 為任意整數介於 $d_B(x,y) \leq l \leq k/2$，$(l - d_B(x,y))$是為偶數。如果稱一個二部圖 B 為二部泛定位二部泛圈性，是因對於任意偶數整數 k 介於 $4 \leq k \leq |V(B)|$中，圖形 B 中有 k 迴圈二部泛定位性。

在這個計劃中，相互獨立的漢密爾頓圈是被考慮在 Cayley 圖形中的兩個家族，n-維度的鬆餅圖(Pancake graphs P_n)以及 n-維度的星狀圖(Star graphs S_n)，而二部泛定位二部泛圈性是被考慮在 n-維度的超立方體(Hypercube graph Q_n)。我們在這個計劃中，已經證明出來 $IHC(P_3) = 1$，且當 $n \geq 4$，$IHC(P_n) = n - 1$ 又當 $n \in \{3, 4\}$，$IHC(S_n) = n - 2$，且當 $n \geq 5$，$IHC(S_n) = n - 1$，最後當 $n \geq 2$ 的超立方體 Q_n 是有二部泛定位二部泛圈性。

關鍵字：二部泛定位性、二部泛圈性、超立方體、漢米爾頓、鬆餅圖、星狀圖。

Abstract

A *hamiltonian cycle* C of a graph G is an ordered set $\langle u_1, u_2, \ldots, u_{n(G)}, u_1\rangle$ of vertices such that $u_i \neq u_j$ for $i \neq j$ and u_i is adjacent to u_{i+1} for every $i \in \{1, 2, \ldots, n(G) - 1\}$ and $u_{n(G)}$ is adjacent to u_1, where $n(G)$ is the order of G. The vertex u_1 is the starting vertex and u_i is the ith vertex of C. Two hamiltonian cycles $C_1 = \langle u_1, u_2, \ldots, u_{n(G)}, u_1\rangle$ and $C_2 = \langle v_1, v_2, \ldots, v_{n(G)}, v_1\rangle$ of G are *independent* if $u_1 = v_1$ and $u_i \neq v_i$ for every $i \in \{2, 3, \ldots, n(G)\}$. A set of hamiltonian cycles $\{C_1, C_2, \ldots, C_k\}$ of G is *mutually independent* if its elements are pairwise independent. The *mutually independent hamiltonicity* $IHC(G)$ of a graph G is the maximum integer k such that for any vertex u of G there exist k mutually independent hamiltonian cycles of G starting at u.

A bipartite graph B is *bipancyclic* if it contains a cycle of every even length from 4 to $|V(B)|$ inclusive. A hamiltonian bipartite graph B is *bipanpositionable* if, for any two different vertices x and y, there exists a hamiltonian cycle C of B such that $d_C(x,y) = k$ for any integer k with $d_B(x,y) \leq k \leq |V(B)|/2$ and $(k - d_B(x,y))$ being even. A bipartite graph B is *k-cycle bipanpositionable* if, for any two different vertices x and y, there exists a cycle of B with $d_C(x,y)= l$ and $|V(C)| = k$ for any integer l with $d_B(x,y) \leq l \leq k/2$ and $(l - d_B(x,y))$ being even. A bipartite graph B is *bipanpositionable bipancyclic* if B is k-cycle bipanpositionable for every even integer k, $4 \leq k \leq |V(B)|$.

In this project, the mutually independent hamiltonicity is considered for two families of Cayley graphs, the n-dimensional pancake graphs P_n and the n-dimensional star graphs S_n, and the bipanpositionable bipancyclicity is considered the n-dimensional hypercube graph Q_n. We have proven that $IHC(P_3) = 1$, $IHC(P_n) = n - 1$ if $n \geq 4$, $IHC(S_n) = n - 2$ if $n \in \{3, 4\}$ and $IHC(S_n) = n - 1$ if $n \geq 5$, and the hypercube Q_n is bipanpositionable bipancyclic for $n \geq 2$.

Keywords: Bipanpositionable, bipancyclic, hypercube, hamiltonian, pancake networks, star networks.

1. 前言、研究目的及文獻探討

An interconnection network connects the processors of parallel computers. Its architecture can be represented as a graph in which the vertices correspond to processors and the edges correspond to connections. Hence, we use graphs and networks interchangeably. There are many mutually conflicting requirements in designing the topology for computer networks.

In 1969, Lovász [1] asked whether every finite connected vertex transitive graph has a hamiltonian path, that is, a simple path that traverses every vertex exactly once. All known vertex transitive graphs have a hamiltonian path and moreover, only four vertex transitive graphs without a hamiltonian cycle are known. Since none of these four graphs is a Cayley graph there is a folklore conjecture [2] that every Cayley graph with more than two vertices has a hamiltonian cycle. In the last decades this problem was extensively studied (see [3-13]) and for those Cayley graphs for which the existence of hamiltonian cycles is already proven, further properties related to this problem, such as edge-hamiltonicity, Hamiltonconnectivity and Hamilton-laceability, are investigated (see [5,14]). In this project, we introduce one of such properties, the concept of mutually independent hamiltonian cycles which is related to the number of hamiltonian cycles in a given graph. In particular, mutually independent hamiltonian cycles of pancake graphs P_n and star graphs S_n.

The concept of mutually independent hamiltonian arises from the following application. If there are k pieces of data needed to be sent from u to v, and the data needed to be processed at every node (and the process takes times), then we want mutually independent hamiltonian paths so that there will be no waiting time at a processor. The existence of mutually independent hamiltonian paths is useful for communication algorithms. Motivated by this result, we begin the study on graphs with mutually independent hamiltonian paths between every pair of distinct vertices.

The n-dimensional star network S_n was proposed in [15] as n attractive alternative to the n-cube topology for interconnecting processors in parallel computers. Since its introduction, the network has received considerable attention. Akers and Krishnameurthy [15] showed that the star graphs are vertex transitive and edge transitive. The diameter and fault diameters were computed in [15-17]. The hamiltonian and hamiltonian laceability of star graphs are studied in [18-22]. The spanning container of star graph is studied in [23].

Akers and Krishnameurthy [15] proposed another family of interesting interconnection networks, the n-dimensional pancake graph P_n. Hung et al. [24] studied the hamiltonian connectivity on the faulty pancake graphs. The embedding of cycles and trees into the pancake graphs were discussed in [24-27]. The spanning container of pancake graph is studied in [28]. Gates and Papadimitriou [29] studied the diameter of the pancake graphs. Up to now, we do not know the exact value of the diameter of the pancake graphs [30].

The n-dimensional hypercube, Q_n, consists of all n-bit binary strings as its vertices and two vertices u and v are adjacent if and only if their binary labels are different in exactly one bit position. Therefore, Q_n can be constructed recursively by taking two copies of Q_{n-1}, $Q^0{}_n$ and $Q^1{}_n$, and adding a perfect matching between these two copies. The hypercube is a widely used topology in computer architecture, see Leighton [31].

A graph G is pancyclic if it contains a cycle of every length from 3 to $|V(G)|$ inclusive. The concept of pancyclic graphs wasproposed by Bondy [32]. Since there is no odd cycle in bipartite graph, the concept of a bipancyclic graph was proposed by Mitchem and Schmeichel [33]. A bipartite graph is bipancyclic if it contains a cycle of every even length from 4 to $|V(G)|$ inclusive. It is proved that the hypercube Q_n is bipancyclic if $n \geq 2$ [34,35]. A graph is panconnected if, for any two different vertices x and y, there exists a path of length l joining x and y for every l with $d_G(x, y) \leq l \leq |V(G)| - 1$. The concept of panconnected graphs was proposed by Alavi and Williamson [36]. It is easy to see that any bipartite graph with at least 3 vertices is not panconnected. Therefore, the concept of bipanconnected graphs is proposed. A bipartite graph is bipanconnected if, for any two different vertices x and y, there exists a path of length l joining x and y for every l with $d_G(x, y) \leq l \leq |V(G)| - 1$ and $(l - d_G(x, y))$ being even. It is proved that the hypercube is bipanconnected [34]. A hamiltonian graph G is panpositionable if for any two different vertices x and y of G and for any integer k with $d_G(x, y) \leq k \leq |V(G)|/2$, there exists a hamiltonian cycle C of G such that $d_C(x, y) = k$. A hamiltonian bipartite graph G is bipanpositionable if for any two different vertices x and y of G and for any integer k with $d_G(x, y) \leq k \leq |V(G)|/2$ and $(k - d_G(x, y))$ being even, there exists a hamiltonian cycle C of G such that $d_C(x, y) = k$. The concepts of panpositionable and bipanpositionable were proposed by Kao et al. [37]. They proved that the hypercube Q_n is bipanpositionable if $n \geq 2$ [37]. A bipartite graph G is edge-bipancyclic if for any edge in G, there is a cycle of every even length from 4 to $|V(G)|$ traversing through this edge. The concept of edge-bipancyclic was proposed by Alspach and Hare [38]. A bipartite graph G is vertex-bipancyclic if for any vertex in G, there is a cycle of every even length from 4 to $|V(G)|$ going through this vertex. The concept of vertex-bipancyclic was proposed by Hobbs [39]. Obviously, every edge-bipancyclic graph is vertex-bipancyclic. It is proved that the hypercube Q_n is edge-bipancyclic if $n \geq 2$ [34].

We propose a more interesting property about hypercubes. A k-cycle is a cycle of length k. A bipartite graph G is k-cycle bipanpositionable if for every different vertices x and y of G and for any integer l with $d_G(x, y) \leq l \leq k/2$ and $(l - d_G(x, y))$ being even, there exists a k-cycle C of G such that $d_C(x, y) = l$. (Note that $d_C(x, y) \leq k/2$ for every cycle C of length k.) A bipartite graph G is bipanpositionable bipancyclic if G is k-cycle bipanpositionable for every even integer k with $4 \leq k \leq |V(G)|$.

In this project, we have proven that $IHC(P_3) = 1$, $IHC(P_n) = n - 1$ if $n \geq 4$, $IHC(S_n) = n - 2$ if $n \in \{3, 4\}$ and $IHC(S_n) = n - 1$ if $n \geq 5$, and the hypercube Q_n is bipanpositionable bipancyclic for $n \geq 2$.

2. 研究方法

我們知道在目前有許多著名的連結網路抑或是多處理器架構中，都有存在許多的好性質，例如 bipancyclic, bipanconnected, bipanpositionable, edge-bipancyclic and vertex-bipancyclic。但是這麼多的好性質通常需要分開的驗證，或者是有其它的好性質沒有被發現出來，例如，mutually independent hamiltonian cycles。所以在本次的計劃中，我們將深入去探討這些著名的連結網路或是多處理架構中，是否存在著更好的性質。

我們研究的過程分為以下四個步驟：

一. 收集文獻：

我們善加利用學校的圖書館藏的資源、參與國內外重要的演討會以及網路上相關學術網站上的資料，來充實我們對於研究題材本身的知識，以及知道學術界上相關領域的主流發展。

二. 探討文獻及發現問題：

我們利用收集到的資料，請計劃中的參與人員詳細閱讀，並在每週固定時間的討論會中發表心得與感想，並藉由討論過程中，激發出相關議題與我們可再繼續探討研究之主題。

三. 解決問題：

在主持人確定主題與研究方向之後，由主持人帶領著參與計劃的博士生來研究並解決問題。在過程中，有需要利用電腦程式的執行來加快我們驗證的速度，也有需要利用理論及數學方法的推導，加以證明我們所提出的研究議題之正確性。並在每週固定的討論會中，鉅細靡遺的說明解釋給主持人及其他計劃的參與人員知道，以保證不會因個人小部分的觀念偏差，造成有錯誤的解果產生。

四. 成過發表：

當有研究主題被驗證為正確之時，我們會將其撰寫成論文，並發表在國際期刊以及國際研討會中。其中本計劃相關的論文也已經有發表在國際著名 SCI 期刊以及國際研討會中。

Cheng-Kuan Lin, Jimmy J.M. Tan, Hua-Min Huang, D. Frank Hsu, and Lih-Hsing Hsu, "Mutually independent hamiltonian cycles for the pancake graphs and the star graphs," Discrete Mathematics, 309 (2009) 5474-5483.

Yuan-Kang Shih, Cheng-Kuan Lin, Jimmy J.M. Tan, and Lih-Hsing Hsu, "The bipanpositionable bipancyclic property of the hypercube," Computers and Mathematics with Applications, 58 (2009) 1722-1724.

Tung-Yang Ho, Yuan-Kang Shih, Jimmy J.M. Tan, and Lih-Hsing Hsu, "Conditional fault hamiltonian connectivity of the complete graph," Information Processing Letters, 109 (2009) 585-588.

3. 結果與討論

In this project, the mutually independent hamiltonicity is considered for two families of Cayley graphs, the n-dimensional pancake graphs P_n and the n-dimensional star graphs S_n, and the bipanpositionable bipancyclicity is considered the n-dimensional hypercube graph Q_n. We have proven that $IHC(P_3) = 1$, $IHC(P_n) = n - 1$ if $n \geq 4$, $IHC(S_n) = n - 2$ if $n \in \{3, 4\}$ and $IHC(S_n) = n - 1$ if $n \geq 5$, and the hypercube Q_n is bipanpositionable bipancyclic for $n \geq 2$.

We discuss the mutually independent hamiltonian cycles for the pancake graphs and the star graphs. The concept of mutually independent hamiltonian cycle can be viewed as a generalization of Latin rectangles. Perhaps one of the most interesting topics in Latin square is orthogonal Latin square. Two Latin squares of order n are orthogonal if the n-squared pairs formed by juxtaposing the two arrays are all distinct. Similarly, two Latin rectangles of order $n \times m$ are orthogonal if the $n \times m$ pairs formed by juxtaposing the two arrays are all distinct. With this in mind, let G be a Hamiltonian graph and C_1 and C_2 be two sets of mutually independent hamiltonian cycles of G from a given vertex x. We say C_1 and C_2 are orthogonal if their corresponding Latin rectangles are orthogonal.

We can also discuss mutually independent hamiltonian paths for some graphs. Let $P_1 = \langle v_1, v_2, \ldots, v_n\rangle$ and $P_2 = \langle u_1, u_2, \ldots, u_n\rangle$ be two hamiltonian paths of a graph G. We say that P_1 and P_2 are independent if $u_1 = v_1$, $u_n = v_n$, and $u_i \neq v_i$ for $1 < i < n$. We say a set of hamiltonian paths $\{P_1, P_2, \ldots, P_s\}$ of G between two distinct vertices are mutually independent if any two distinct paths in the set are independent. There are some study on mutually independent Hamiltonian paths [40, 41].

Recently, people are interested in a mathematical puzzle, called Sudoku [42]. Sudoku can be viewed as a 9×9 Latin square with some constraints. There are several variations of Sudoku have been introduced. Mutually independent Hamiltonian cycles can also be considered as a variation of Sudoku.

On the other hand, we prove that the hypercube Q_n is bipanpositionable bipancyclic for $n \geq 2$. As a consequence of this result, we can see that many previous results on hypercubes follow directly from ours. For example, the family of the hypercube is bipancyclic, bipanconnected, bipanpositionable, edge-bipancyclic and vertex-bipancyclic. Therefore, our result unifies these results in a general sense.

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附件二

國科會補助專題研究計畫成果報告自評表

請就研究內容與原計畫相符程度、達成預期目標情況、研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性)、是否適合在學術期刊發表或申請專利、主要發現或其他有關價值等，作一綜合評估。

1. 請就研究內容與原計畫相符程度、達成預期目標情況作一綜合評估

■ 達成目標

□ 未達成目標（請說明，以 100 字為限）

□ 實驗失敗

□ 因故實驗中斷

□ 其他原因

說明：

2. 研究成果在學術期刊發表或申請專利等情形：

論文：■已發表 □未發表之文稿 □撰寫中 □無

專利：□已獲得 □申請中 □無

技轉：□已技轉 □洽談中 □無

其他：(以 100 字為限)

3. 請依學術成就、技術創新、社會影響等方面，評估研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性)(以 500 字為限)

這次的計劃執行可以說是非常的成功，目前已經發表且刊登出來的論文已經有三篇，還有其它的論文也已經被接受。總括來說，這次三年計劃可以說完全達到我們預期的目標，更可說是已經超越我們預設的目標。

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Mutually independent hamiltonian cycles for the pancake graphs and the star graphs

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ABSTRACT

A *hamiltonian cycle* C of a graph G is an ordered set $\langle u_1, u_2, \ldots, u_{n(G)}, u_1\rangle$ of vertices such that $u_i \neq u_j$ for $i \neq j$ and u_i is adjacent to u_{i+1} for every $i \in \{1, 2, \ldots, n(G) - 1\}$ and $u_{n(G)}$ is adjacent to u_1, where $n(G)$ is the order of G. The vertex u_1 is the starting vertex and u_i is the ith vertex of C. Two hamiltonian cycles $C_1 = \langle u_1, u_2, \ldots, u_{n(G)}, u_1\rangle$ and $C_2 = \langle v_1, v_2, \ldots, v_{n(G)}, v_1\rangle$ of G are *independent* if $u_1 = v_1$ and $u_i \neq v_i$ for every $i \in \{2, 3, \ldots, n(G)\}$. A set of hamiltonian cycles $\{C_1, C_2, \ldots, C_k\}$ of G is *mutually independent* if its elements are pairwise independent. The *mutually independent hamiltonicity* $IHC(G)$ of a graph G is the maximum integer k such that for any vertex u of G there exist k mutually independent hamiltonian cycles of G starting at u.

In this paper, the mutually independent hamiltonicity is considered for two families of Cayley graphs, the n-dimensional pancake graphs P_n and the n-dimensional star graphs S_n. It is proven that $IHC(P_3) = 1$, $IHC(P_n) = n - 1$ if $n \geq 4$, $IHC(S_n) = n - 2$ if $n \in \{3, 4\}$ and $IHC(S_n) = n - 1$ if $n \geq 5$.

1. Introduction

In 1969, Lovász [32] asked whether every finite connected vertex transitive graph has a hamiltonian path, that is, a simple path that traverses every vertex exactly once. All known vertex transitive graphs have a hamiltonian path and moreover, only four vertex transitive graphs without a hamiltonian cycle are known. Since none of these four graph is a Cayley graph there is a folklore conjecture [9] that every Cayley graph with more than two vertices has a hamiltonian cycle. In the last decades this problem was extensively studied (see [2–5,7,12,19,33–36]) and for those Cayley graphs for which the existence of hamiltonian cycles is already proven, further properties related to this problem, such as edge-hamiltonicity, Hamilton-connectivity and Hamilton-laceability, are investigated (see [4,8]). In this paper we introduce one of such properties, the concept of mutually independent hamiltonian cycles which is related to the number of hamiltonian cycles in a given graph. In particular, mutually independent hamiltonian cycles of pancake graphs P_n and star graphs S_n (for definitions see Sections 4 and 5) are studied.

The paper is organized as follows. In Section 2 definitions and notations needed in the subsequent sections are introduced. In Section 3 applications of the mutually independent hamiltonicity concept are given. In Sections 4 and 5 the mutually independent hamiltonicity of pancake graphs P_n and star graphs S_n, respectively, is computed. And in the last section, Section 6, directions for further research on this topic are discussed.

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2. Definitions

For definitions and notations not defined here see [6]. Let V be a finite set and E a subset of $\{(u, v) \mid (u, v)$ is an unordered pair of $V\}$. Then $G = (V, E)$ is a *graph* with *vertex set* V and *edge set* E. The *order* of G, that is, the cardinality of the set V, is denoted by $n(G)$. For a subset S of V the graph $G[S]$ induced by S is a graph with vertex set $V(G[S]) = S$ and edge set $E(G[S]) = \{(x, y) \mid (x, y) \in E(G)$ and $x, y \in S\}$. Two vertices u and v are *adjacent* if (u, v) is an edge of G. For a vertex u the set $N_G(u) = \{v \mid (u, v) \in E\}$ is called the *set of neighbors* of u. The *degree* $\deg_G(u)$ of a vertex u in G, is the cardinality of the set $N_G(u)$. The *minimum degree* of G, $\delta(G)$, is $\min\{\deg_G(x) \mid x \in V\}$. A graph G is *k-regular* if $\deg_G(u) = k$ for every vertex u in G. The *connectivity* of G is the minimum number of vertices whose removal leaves the remaining graph disconnected or trivial. A *path* between vertices v_0 and v_k is a sequence of vertices represented by $\langle v_0, v_1, \dots, v_k\rangle$ such that there is no repeated vertex and (v_i, v_{i+1}) is an edge of G for every $i \in \{0 \dots k-1\}$. We use $Q(i)$ to denote the ith vertex v_i of $Q = \langle v_1, v_2, \dots, v_k\rangle$. We also write the path $\langle v_0, v_1, \dots, v_k\rangle$ as $\langle v_0, \dots, v_i, Q, v_j, \dots, v_k\rangle$, where Q is a path form v_i to v_j. A path is a *hamiltonian path* if it contains all vertices of G. A graph G is *hamiltonian connected* if there exists a hamiltonian path joining any two distinct vertices of G. A *cycle* is a sequence of vertices represented by $\langle v_0, v_1, \dots, v_k, v_0\rangle$ such that $v_i \neq v_j$ for all $i \neq j$, (v_0, v_k) is an edge of G, and (v_i, v_{i+1}) is an edge of G for every $i \in \{0, \dots, k-1\}$. A *hamiltonian cycle* of G is a cycle that traverses every vertex of G. A graph is *hamiltonian* if it has a hamiltonian cycle.

A hamiltonian cycle C of graph G is described as $\langle u_1, u_2, \dots, u_{n(G)}, u_1\rangle$ to emphasize the order of vertices in C. Thus, u_1 is the starting vertex and u_i is the ith vertex in C. Two hamiltonian cycles $C_1 = \langle u_1, u_2, \dots, u_{n(G)}, u_1\rangle$ and $C_2 = \langle v_1, v_2, \dots, v_{n(G)}, v_1\rangle$ of G are *independent* if $u_1 = v_1$ and $u_i \neq v_i$ for every $i \in \{2, \dots, n(G)\}$. A set of hamiltonian cycles $\{C_1, C_2, \dots, C_k\}$ of G are *mutually independent* if its elements are pairwise independent. The *mutually independent hamiltonicity* $IHC(G)$ of graph G the maximum integer k such that for any vertex u of G there exist k mutually independent hamiltonian cycles of G starting at u. Obviously, $IHC(G) \leq \delta(G)$ if G is a hamiltonian graph.

The mutually independent hamiltonicity of a graph can be interpreted as a Latin rectangle. A *Latin square* of order n is an $n \times n$ array made from the integers 1 to n with the property that any integer occurs once in each row and column. If we delete some rows from a Latin square, we will get a Latin rectangle. Let K_5 be the complete graph with vertex set $\{0, 1, 2, 3, 4\}$ and let $C_1 = \langle 0, 1, 2, 3, 4, 0\rangle$, $C_2 = \langle 0, 2, 3, 4, 1, 0\rangle$, $C_3 = \langle 0, 3, 4, 1, 2, 0\rangle$, and $C_4 = \langle 0, 4, 1, 2, 3, 0\rangle$. Obviously, C_1, C_2, C_3, and C_4 are mutually independent. Thus, $IHC(K_5) = 4$. We rewrite C_1, C_2, C_3, and C_4 into the following Latin square:

1	2	3	4
2	3	4	1
3	4	1	2
4	1	2	3

In general, a Latin square of order n can be viewed as n mutually independent hamiltonian cycles with respect to the complete graph K_{n+1}.

Let H be a group and let S be the generating set of H such that $S^{-1} = S$. Then the *Cayley graph Cayley*$(S; H)$ of the group H with respect to the generating set S is the graph with vertex set H and two vertex u and v are adjacent in *Cayley*$(S; H)$ if and only if $u^{-1}v \in S$. Hamiltonian cycles in Cayley graphs naturally arise in computer science [25], in the study of word-hyperbolic groups and automatic groups [14], in changing-ringing [40], in creating Escher-like repeating patterns in hyperbolic plane [13], and in combinatorial designs [11].

3. Applications of the concept of mutually independent hamiltonian cycles

Mutually independent hamiltonicity of graphs can be applied to many areas. Consider the following scenario. In Christmas, we have a holiday of 10-days. A tour agency will organize a 10-day tour to Italy. Suppose that there will be a lot of people joining this tour. However, the maximum number of people stay in each local area is limited, say 100 people, for the sake of hotel contract. One trivial solution is on the First-Come-First-Serve basis. So only 100 people can attend this tour. (Note that we cannot schedule the tour in a pipelined manner because the holiday period is fixed.) Nonetheless, we observe that a tour is like a hamiltonian cycle based on a graph, in which a vertex is denoted as a hotel and any two vertices are joined with an edge if the associated two hotels can be traveled in a reasonable time. Therefore, we can organize several subgroups, that is, each subgroup has its own tour. In this way, we do not allow two subgroups stay in the same area during the same time period. In other words, any two different tours are indeed independent hamiltonian cycles. Suppose that there are 10 mutually independent hamiltonian cycles. Then we may allow 1000 people to visit Italy on Christmas vacation. For this reason, we would like to find the maximum number of mutually independent hamiltonian cycles. Such applications are useful for task scheduling and resource placement, which are also important for compiler optimization to exploit parallelism.

An interconnection network connects the processors of parallel computers. Its architecture can be represented as a graph in which the vertices correspond to processors and the edges correspond to connections. Hence, we use graphs and networks interchangeably. There are many mutually conflicting requirements in designing the topology for computer networks. The n-cube is one of the most popular topologies [27]. The n-dimensional star network S_n was proposed in [1] as n attractive alternative to the n-cube topology for interconnecting processors in parallel computers. Since its introduction, the network

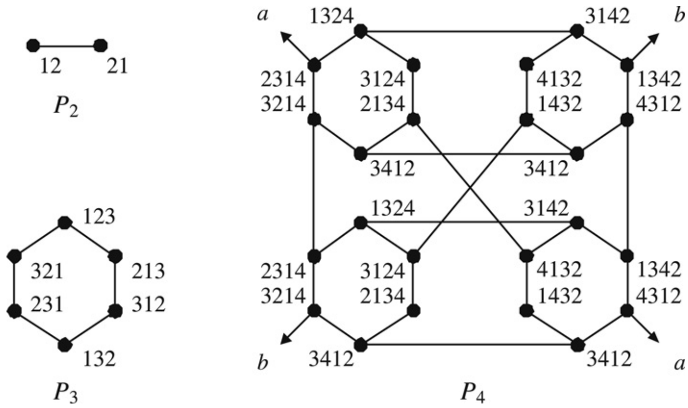

Fig. 1. The pancake graphs P_2, P_3, and P_4.

has received considerable attention. Akers and Krishnameurthy [1] showed that the star graphs are vertex transitive and edge transitive. The diameter and fault diameters were computed in [1,26,37]. The hamiltonian and hamiltonian laceability of star graphs are studied in [16,17,21,23,31]. The spanning container of star graph is studied in [28].

Akers and Krishnameurthy [1] proposed another family of interesting interconnection networks, the n-dimensional pancake graph P_n. Hung et al. [22] studied the hamiltonian connectivity on the faulty pancake graphs. The embedding of cycles and trees into the pancake graphs were discussed in [10,15,22,24]. The spanning container of pancake graph is studied in [28]. Gates and Papadimitriou [18] studied the diameter of the pancake graphs. Up to now, we do not know the exact value of the diameter of the pancake graphs [20].

4. The pancake graphs

Let n be a positive integer. We use $\langle n\rangle$ to denote the set $\{1, 2, \ldots, n\}$. The *n-dimensional pancake graph*, P_n, is a graph with the vertex set $V(P_n) = \{u_1u_2\ldots u_n \mid u_i \in \langle n\rangle \text{ and } u_j \neq u_k \text{ for } j \neq k\}$. The adjacency is defined as follows: $u_1u_2\ldots u_i\ldots u_n$ is adjacent to $v_1v_2\ldots v_i\ldots v_n$ through an edge of dimension i with $2 \le i \le n$ if $v_j = u_{i-j+1}$ for all $1 \le j \le i$ and $v_j = u_j$ for all $i < j \le n$. We will use boldface to denote a vertex of P_n. Hence, $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}$ denote a sequence of vertices in P_n. In particular, $\mathbf{e}$ denotes the vertex $12\ldots n$. The pancake graphs P_2, P_3, and P_4 are illustrated in Fig. 1.

By definition, P_n is an $(n-1)$-regular graph with $n!$ vertices. Akers and Krishnameurthy [1] showed that the connectivity of P_n is $(n-1)$. Let $\mathbf{u} = u_1u_2\ldots u_n$ be an arbitrary vertex of P_n. We use $(\mathbf{u})_i$ to denote the ith component u_i of $\mathbf{u}$, and use $P_n^{\{i\}}$ to denote the ith subgraph of P_n induced by those vertices $\mathbf{u}$ with $(\mathbf{u})_n = i$. Then P_n can be decomposed into n vertex disjoint subgraphs $P_n^{\{i\}}$, $1 \le i \le n$, and each $P_n^{\{i\}}$ is isomorphic to P_{n-1} for all i, $i \le n$. Thus, the pancake graph can be constructed recursively. Let H be any subset of $\langle n\rangle$. We use P_n^H to denote the subgraph of P_n induced by $\cup_{i\in H} V(P_n^{\{i\}})$. By definition, there is exactly one neighbor $\mathbf{v}$ of $\mathbf{u}$ such that $\mathbf{u}$ and $\mathbf{v}$ are adjacent through an i-dimensional edge with $2 \le i \le n$. We use $(\mathbf{u})^i$ to denote the unique i-neighbor of $\mathbf{u}$. We have $((\mathbf{u})^i)^i = \mathbf{u}$ and $(\mathbf{u})^n \in P_n^{\{(\mathbf{u})_1\}}$. For any two distinct elements i and j in $\langle n\rangle$, we use $E_n^{i,j}$ to denote the set of edges between $P_n^{\{i\}}$ and $P_n^{\{j\}}$.

Lemma 1. *Let i and j be any two distinct elements in $\langle n\rangle$ with $n \ge 3$. Then $|E_n^{i,j}| = (n-2)!$.*

Lemma 2. *Let $\mathbf{u}$ and $\mathbf{v}$ be any two distinct vertices of P_n with $d(\mathbf{u}, \mathbf{v}) \le 2$. Then $(\mathbf{u})_1 \neq (\mathbf{v})_1$.*

Theorem 1 *([22]). Suppose that F is a subset of $V(P_n)$ with $|F| \le n-4$. Then $P_n - F$ is hamiltonian connected.*

Theorem 2. *Let $\{a_1, a_2, \ldots, a_r\}$ be a subset of $\langle n\rangle$ for some positive integer $r \in \langle n\rangle$ with $n \ge 5$. Assume that $\mathbf{u}$ and $\mathbf{v}$ are two distinct vertices of P_n with $\mathbf{u} \in P_n^{\{a_1\}}$ and $\mathbf{v} \in P_n^{\{a_r\}}$. Then there is a hamiltonian path $\langle \mathbf{u} = \mathbf{x_1}, H_1, \mathbf{y_1}, \mathbf{x_2}, H_2, \mathbf{y_2}, \ldots, \mathbf{x_r}, H_r, \mathbf{y_r} = \mathbf{v}\rangle$ of $\cup_{i=1}^r P_n^{\{a_i\}}$ joining $\mathbf{u}$ to $\mathbf{v}$ such that $\mathbf{x_1} = \mathbf{u}$, $\mathbf{y_r} = \mathbf{v}$, and H_i is a hamiltonian path of $P_n^{\{a_i\}}$ joining $\mathbf{x_i}$ to $\mathbf{y_i}$ for every i, $1 \le i \le r$.*

Proof. We set $\mathbf{x_1}$ as $\mathbf{u}$ and $\mathbf{y_r}$ as $\mathbf{v}$. We know that $P_n^{\{a_i\}}$ is isomorphic to P_{n-1} for every $i \in \langle r\rangle$. By Theorem 1, this statement holds for $r = 1$. Thus, we assume that $r \ge 2$. By Lemma 1, $|E_n^{a_i,a_{i+1}}| = (n-2)! \ge 6$ for every $i \in \langle r-1\rangle$. We choose $(\mathbf{y_i}, \mathbf{x_{i+1}}) \in E_n^{a_i,a_{i+1}}$ for every $i \in \langle r-1\rangle$ with $\mathbf{y_1} \neq \mathbf{x_1}$ and $\mathbf{x_r} \neq \mathbf{y_r}$. By Theorem 1, there is a hamiltonian path H_i of $P_n^{\{a_i\}}$ joining $\mathbf{x_i}$ to $\mathbf{y_i}$ for every $i \in \langle r\rangle$. Then $\langle \mathbf{u} = \mathbf{x_1}, H_1, \mathbf{y_1}, \mathbf{x_2}, H_2, \mathbf{y_2}, \ldots, \mathbf{x_r}, H_r, \mathbf{y_r} = \mathbf{v}\rangle$ is the desired path. See Fig. 2 for illustration on P_n. $\square$

Lemma 3. *Let $k \in \langle n\rangle$ with $n \ge 4$, and let $\mathbf{x}$ be a vertex of P_n. There is a hamiltonian path P of $P_n - \{\mathbf{x}\}$ joining the vertex $(\mathbf{x})^n$ to some vertex $\mathbf{v}$ with $(\mathbf{v})_1 = k$.*

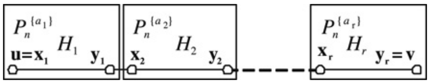

Fig. 2. Illustration for Theorem 2 on P_n.

Proof. Suppose that $n = 4$. Since P_4 is vertex transitive, we may assume that $\mathbf{x} = 1234$. The required paths of $P_4 - \{1234\}$ are listed below:

$k = 1$	⟨4321, 3421, 2431, 4231, 1324, 3124, 2134, 4312, 1342, 2143, 4132, 2314, 3214, 4123, 2143, 3412, 1432, 2341, 3241, 1423, 2413, 4213, 1243⟩
$k = 2$	⟨4321, 3421, 2431, 4231, 1324, 3124, 2134, 4312, 1342, 3142, 2413, 4213, 1243, 2143, 3412, 1432, 4132, 2314, 3214, 4123, 1423, 3241, 2341⟩
$k = 3$	⟨4321, 3421, 2431, 4231, 1324, 3124, 2134, 4312, 1342, 3142, 4132, 2314, 3214, 4123, 1423, 2413, 4213, 1342, 2143, 3412, 1432, 2341, 3241⟩
$k = 4$	⟨4321, 3421, 2431, 1342, 3142, 4132, 2314, 3214, 4123, 2143, 1243, 4213, 2413, 1423, 3241, 2341, 1432, 3412, 4312, 2134, 3124, 1324, 4231⟩

With Theorem 1, we can find the required hamiltonian path in P_n for every n, $n \geq 5$. □

Lemma 4. *Let a and b be any two distinct elements in $\langle n \rangle$ with $n \geq 4$, and let $\mathbf{x}$ be a vertex of P_n. There is a hamiltonian path P of $P_n - \{\mathbf{x}\}$ joining a vertex $\mathbf{u}$ with $(\mathbf{u})_1 = a$ to a vertex $\mathbf{v}$ with $(\mathbf{v})_1 = b$.*

Proof. Suppose that $n = 4$. Since P_4 is vertex transitive, we may assume that $\mathbf{x} = 1234$. Without loss of generality, we may assume that $a < b$. The required paths of $P_4 - \{1234\}$ are listed below:

$a = 1$ and $b = 2$ ⟨1423, 4123, 3214, 2314, 1324, 3124, 4213, 2413, 3142, 4132, 1432, 3412, 2143, 1243, 3421, 4321, 2341, 3241, 4231, 2431, 1342, 4312, 2134⟩
$a = 1$ and $b = 3$ ⟨1423, 4123, 2143, 1243, 4213, 2413, 3142, 1342, 2431, 3421, 4321, 2341, 3241, 4231, 1324, 3124, 2134, 4312, 3412, 1432, 4132, 2314, 3214⟩
$a = 1$ and $b = 4$ ⟨1423, 2413, 3142, 1342, 2431, 3421, 4321, 2341, 3241, 4231, 1324, 2314, 3214, 4123, 2143, 1243, 4213, 3124, 2134, 4312, 3412, 1432, 4132⟩
$a = 2$ and $b = 3$ ⟨2134, 4312, 1342, 3142, 2413, 4213, 1243, 2143, 3412, 1432, 4132, 2314, 3214, 4123, 1423, 3241, 2341, 4321, 3421, 2431, 4231, 1324, 3124⟩
$a = 2$ and $b = 4$ ⟨2134, 3124, 1324, 2314, 3214, 4123, 2143, 1243, 4213, 2413, 1423, 3241, 4231, 2431, 3421, 4321, 2341, 1432, 3412, 4312, 1342, 3142, 4132⟩
$a = 3$ and $b = 4$ ⟨3214, 4123, 2143, 1243, 4213, 3124, 2134, 4312, 3412, 1432, 2341, 4321, 3421, 2431, 1342, 3142, 2413, 1423, 3241, 4231, 1324, 2314, 4132⟩

With Theorem 1, we can find the required hamiltonian path on P_n for every n, $n \geq 5$. □

Lemma 5. *Let a and b be any two distinct elements in $\langle n \rangle$ with $n \geq 4$. Assume that $\mathbf{x}$ and $\mathbf{y}$ are two adjacent vertices of P_n. There is a hamiltonian path P of $P_n - \{\mathbf{x}, \mathbf{y}\}$ joining a vertex $\mathbf{u}$ with $(\mathbf{u})_1 = a$ to a vertex $\mathbf{v}$ with $(\mathbf{v})_1 = b$.*

Proof. Since P_n is vertex transitive, we may assume that $\mathbf{x} = \mathbf{e}$ and $\mathbf{y} = (\mathbf{e})^i$ for some $i \in \{2, 3, \dots, n\}$. Without loss of generality, we assume that $a < b$. Thus, $a \neq n$ and $b \neq 1$. We prove this statement by induction on n. For $n = 4$, the required paths of $P_4 - \{1234, (1234)^i\}$ are listed below:

$\mathbf{y} = 2134$
$a = 1$ and $b = 2$ ⟨1432, 2413, 3142, 4132, 1432, 3412, 4312, 1342, 2431, 3421, 4321, 2341, 3241, 4231, 1324, 3124, 4213, 1243, 2143, 4123, 3214, 2314⟩
$a = 1$ and $b = 3$ ⟨1432, 4123, 2143, 1243, 3421, 4321, 2341, 3241, 4231, 2431, 1342, 4312, 3412, 1432, 4132, 3142, 2413, 4213, 3124, 1324, 2314, 3214⟩
$a = 1$ and $b = 4$ ⟨1432, 4123, 3214, 2314, 1324, 3124, 4213, 2413, 3142, 4132, 1432, 2341, 3241, 4231, 2431, 1342, 4312, 3412, 2143, 1243, 3421, 4321⟩
$a = 2$ and $b = 3$ ⟨2314, 3214, 4123, 2143, 1243, 4213, 3124, 1324, 4231, 2431, 1342, 4312, 3412, 1432, 4132, 3142, 2413, 1423, 3241, 2341, 4321, 3421⟩
$a = 2$ and $b = 4$ ⟨2314, 3214, 4123, 2143, 3412, 4312, 1342, 2431, 3421, 1243, 4213, 3124, 1324, 4231, 3241, 1423, 2413, 3142, 4132, 1432, 2341, 4321⟩
$a = 3$ and $b = 4$ ⟨3214, 4123, 2143, 1243, 3421, 2431, 4231, 3241, 1423, 2413, 4213, 3124, 1324, 2314, 4132, 3142, 1342, 4312, 3412, 1432, 2341, 4321⟩

$\mathbf{y} = 3214$
$a = 1$ and $b = 2$ ⟨1423, 4123, 2143, 1243, 3421, 4321, 2341, 3241, 4231, 2431, 1342, 3142, 2413, 4213, 3124, 1324, 2314, 4132, 1432, 3412, 4312, 2134⟩
$a = 1$ and $b = 3$ ⟨1423, 4123, 2143, 1243, 4213, 2413, 3142, 1342, 2431, 3421, 4321, 2341, 3241, 4231, 1324, 2314, 4132, 1432, 3412, 4312, 2134, 3124⟩
$a = 1$ and $b = 4$ ⟨1423, 4123, 2143, 1243, 3421, 2431, 1342, 3142, 2413, 4213, 3124, 2134, 4312, 3412, 1432, 4132, 2314, 1324, 4231, 3241, 2341, 4321⟩
$a = 2$ and $b = 3$ ⟨2134, 4312, 1342, 2431, 4231, 3241, 1423, 4123, 2143, 3412, 1432, 2341, 4321, 3421, 1243, 4213, 2413, 3142, 4132, 2314, 1324, 3124⟩
$a = 2$ and $b = 4$ ⟨2134, 3124, 4213, 2413, 3142, 1342, 4312, 3412, 1432, 4132, 2314, 1324, 4231, 2431, 3421, 1243, 2143, 4123, 1423, 3241, 2341, 4321⟩
$a = 3$ and $b = 4$ ⟨3124, 2134, 4312, 1342, 3142, 2413, 4213, 1243, 3421, 2431, 4231, 1324, 2314, 4132, 1432, 3412, 2143, 4123, 1423, 3241, 2341, 4321⟩

$\mathbf{y} = 4321$
$a = 1$ and $b = 2$
⟨1423, 4123, 3214, 2314, 4132, 3142, 2413, 4213, 3124, 1324, 4231, 3241, 2341, 1432, 3412, 2143, 1243, 3421, 2431, 1342, 4312, 2134⟩
$a = 1$ and $b = 3$
⟨1423, 4123, 2143, 3412, 1432, 2341, 3241, 4231, 1324, 3124, 2134, 4312, 1342, 2431, 2431, 1243, 4213, 2413, 3142, 4132, 2314, 3214⟩
$a = 1$ and $b = 4$
⟨1423, 2413, 4213, 3124, 2134, 4312, 3412, 2143, 1243, 3421, 2431, 1342, 3142, 4132, 1432, 2341, 3241, 4231, 1324, 2314, 3214, 4123⟩
$a = 2$ and $b = 3$
⟨2134, 4312, 1342, 3142, 4132, 2314, 3214, 4123, 2143, 3412, 1432, 2341, 3241, 1423, 2413, 4213, 1243, 3421, 2431, 4231, 1324, 3124⟩
$a = 2$ and $b = 4$
⟨2134, 3124, 4213, 2413, 1423, 3241, 2341, 1432, 4132, 3142, 1342, 4312, 3412, 2143, 1243, 3421, 2431, 4231, 1324, 2314, 3214, 4123⟩
$a = 3$ and $b = 4$
⟨3214, 2314, 1324, 4231, 3241, 2341, 1432, 4132, 3142, 1342, 2431, 3421, 1243, 2143, 3412, 4312, 2134, 3124, 4213, 2413, 1423, 4123⟩

Suppose that this statement holds for P_k for every k, $4 \leq k < n$. We have the following cases:

Case 1. $\mathbf{y} = (\mathbf{e})^i$ for some $i \neq 1$ and $i \neq n$, that is, $\mathbf{y} \in P_n^{\{n\}}$. Let c be an element in $\langle n-1 \rangle - \{a\}$. By induction, there is a hamiltonian path R of $P_n^{\{n\}} - \{\mathbf{e}, (\mathbf{e})^i\}$ joining a vertex $\mathbf{u}$ with $(\mathbf{u})_1 = a$ to a vertex $\mathbf{z}$ with $(\mathbf{z})_1 = c$. We choose a vertex $\mathbf{v}$ in $P_n^{\langle n-1 \rangle - \{c\}}$ with $(\mathbf{v})_1 = b$. By Theorem 2, there is a hamiltonian path H of $P_n^{\langle n-1 \rangle}$ joining the vertex $(\mathbf{z})^n$ to $\mathbf{v}$. Then $\langle \mathbf{u}, R, \mathbf{z}, (\mathbf{z})^n, H, \mathbf{v} \rangle$ is the desired path.

Case 2. $\mathbf{y} = (\mathbf{e})^n$, that is, $\mathbf{y} \in P_n^{\{1\}}$. Let c be an element in $\langle n-1 \rangle - \{1, a\}$, and let d be an element in $\langle n-1 \rangle - \{1, b, c\}$. By Lemma 4, there is a hamiltonian path R of $P_n^{\{n\}} - \{\mathbf{e}\}$ joining a vertex $\mathbf{u}$ with $(\mathbf{u})_1 = a$ to a vertex $\mathbf{w}$ with $(\mathbf{w})_1 = c$. Again, there is a hamiltonian path H of $P_n^{\{1\}} - \{(\mathbf{e})^n\}$ joining a vertex $\mathbf{z}$ with $(\mathbf{z})_1 = d$ to a vertex $\mathbf{v}$ with $(\mathbf{v})_1 = b$. By Theorem 2, there is a hamiltonian path Q of $P_n^{\langle n-1 \rangle - \{1\}}$ joining the vertex $(\mathbf{w})^n$ to the vertex $(\mathbf{z})^n$. Then $\langle \mathbf{u}, R, \mathbf{w}, (\mathbf{w})^n, Q, (\mathbf{z})^n, \mathbf{z}, H, \mathbf{v} \rangle$ is the desired path. □

Lemma 6. *Let a and b be any two distinct elements in $\langle n \rangle$ with $n \geq 4$. Let $\mathbf{x}$ be any vertex of P_n. Assume that $\mathbf{x_1}$ and $\mathbf{x_2}$ are two distinct neighbors of $\mathbf{x}$. There is a hamiltonian path P of $P_n - \{\mathbf{x}, \mathbf{x_1}, \mathbf{x_2}\}$ joining a vertex $\mathbf{u}$ with $(\mathbf{u})_1 = a$ to a vertex $\mathbf{v}$ with $(\mathbf{v})_1 = b$.*

Proof. Since P_n is vertex transitive, we may assume that $\mathbf{x} = \mathbf{e}$. Moreover, we assume that $\mathbf{x_1} = (\mathbf{e})^i$ and $\mathbf{x_2} = (\mathbf{e})^j$ for some $\{i, j\} \subset \langle n \rangle - \{1\}$ with $i < j$. Without loss of generality, we assume that $a < b$. Thus, $a \neq n$ and $b \neq 1$. We prove this lemma by induction on n. For $n = 4$, the required paths of $P_4 - \{1234, (1234)^i, (1234)^j\}$ are listed below:

$\mathbf{x_1} = 2134$ and $\mathbf{x_2} = 3214$
$a = 1$ and $b = 2$
⟨1423, 4123, 2143, 1243, 3421, 4321, 2341, 3241, 4231, 2431, 1342, 4312, 3412, 1432, 4132, 3142, 2413, 4213, 3124, 1324, 2314⟩
$a = 1$ and $b = 3$
⟨1423, 4123, 2143, 1243, 3421, 4321, 2341, 3241, 4231, 2431, 1342, 4312, 3412, 1432, 4132, 2314, 1324, 3124, 4213, 2413, 3142⟩
$a = 1$ and $b = 4$
⟨1423, 4123, 2143, 1243, 3421, 4321, 2341, 3241, 4231, 2431, 1342, 3142, 2413, 4213, 3124, 1324, 2314, 4132, 1432, 3412, 4312⟩
$a = 2$ and $b = 3$
⟨2143, 4123, 1423, 3241, 4231, 2431, 1342, 4312, 3412, 1432, 2341, 4321, 3421, 1243, 4213, 2413, 3142, 4132, 2314, 1324, 3124⟩
$a = 2$ and $b = 4$
⟨2143, 4123, 1423, 2413, 3142, 1342, 4312, 3412, 1432, 4132, 2314, 1324, 3124, 4213, 1243, 3421, 2431, 4231, 3241, 2341, 4321⟩
$a = 3$ and $b = 4$
⟨3124, 4213, 2413, 3142, 1342, 4312, 3412, 1432, 4132, 2314, 1324, 4231, 2431, 3421, 1243, 2143, 4123, 1423, 3241, 2341, 4321⟩

$\mathbf{x_1} = 2134$ and $\mathbf{x_2} = 4321$
$a = 1$ and $b = 2$
⟨1423, 2413, 3142, 4132, 1432, 2341, 3241, 4231, 1324, 3124, 4213, 1243, 3421, 2431, 1342, 4312, 3412, 2143, 4123, 3214, 2314⟩
$a = 1$ and $b = 3$
⟨1423, 4123, 2143, 1243, 3421, 2431, 1342, 4312, 3412, 1432, 2341, 3241, 4231, 1342, 3124, 4213, 2413, 3142, 4132, 2314, 3214⟩
$a = 1$ and $b = 4$
⟨1423, 4123, 3214, 2314, 1324, 3124, 4213, 2413, 3142, 1342, 4312, 3412, 2143, 1243, 3421, 2431, 4231, 3241, 2341, 1432, 4132⟩
$a = 2$ and $b = 3$
⟨2314, 3214, 4123, 2143, 3412, 4312, 1342, 3142, 4132, 1432, 2341, 3241, 1423, 2413, 4213, 1243, 3421, 2431, 4231, 1324, 3124⟩
$a = 2$ and $b = 4$
⟨2314, 3214, 4123, 2143, 1243, 3421, 2431, 4231, 1324, 3124, 4213, 2413, 1423, 3241, 2341, 1432, 3412, 4312, 1342, 3142, 4132⟩
$a = 3$ and $b = 4$
⟨3214, 2314, 4132, 3142, 1342, 4312, 3412, 1432, 2341, 3241, 1423, 2413, 4213, 3124, 1324, 4231, 2431, 3421, 1243, 2143, 4123⟩

$\mathbf{x_1} = 3214$ and $\mathbf{x_2} = 4321$
$a = 1$ and $b = 2$ ⟨1423, 4123, 2143, 1243, 3421, 2431, 1342, 4312, 3412, 1432, 2341, 3241, 4231, 1324, 2314, 4132, 3142, 2413, 4213, 3124, 2134⟩
$a = 1$ and $b = 3$ ⟨1423, 4123, 2143, 3412, 1432, 2341, 3241, 4231, 1324, 2314, 4132, 3142, 2413, 4213, 1243, 3421, 2431, 1342, 4312, 2134, 3124⟩
$a = 1$ and $b = 4$ ⟨1423, 2413, 4213, 3124, 2134, 4312, 3412, 1432, 2341, 3241, 4231, 1324, 2314, 4132, 3142, 1342, 2431, 3421, 1243, 2143, 4123⟩
$a = 2$ and $b = 3$ ⟨2134, 4312, 3412, 1432, 2341, 3241, 4231, 1324, 2314, 4132, 3142, 1341, 2431, 3421, 1243, 2143, 4123, 1423, 2413, 4213, 3124⟩
$a = 2$ and $b = 4$ ⟨2134, 3124, 4213, 2413, 3142, 1342, 4312, 3412, 1432, 2341, 3241, 1423, 4123, 2143, 1243, 3421, 2431, 4231, 1324, 2314, 4132⟩
$a = 3$ and $b = 4$ ⟨3124, 2134, 4312, 3412, 1432, 2341, 3241, 4231, 1324, 2314, 4132, 3142, 1342, 2431, 3421, 1243, 2143, 4123, 1423, 2413, 4213⟩

Suppose that this statement holds for P_k for every k, $4 \leq k < n$. We have the following cases:

Case 1. $j \neq n$, that is, $\mathbf{x_1} \in P_n^{\{n\}}$ and $\mathbf{x_2} \in P_n^{\{n\}}$. Let $c \in \langle n-1\rangle - \{1, a\}$. By induction, there is a hamiltonian path R of $P_n^{\{n\}} - \{\mathbf{e}, \mathbf{x_1}, \mathbf{x_2}\}$ joining a vertex $\mathbf{u}$ with $(\mathbf{u})_1 = a$ to a vertex $\mathbf{z}$ with $(\mathbf{z})_1 = c$. We choose a vertex $\mathbf{v}$ in $P_n^{\{1\}}$ with $(\mathbf{v})_1 = b$. By Theorem 2, there is a hamiltonian path H of $P_n^{\langle n-1\rangle}$ joining the vertex $(\mathbf{z})^n$ to $\mathbf{v}$. We set $P = \langle \mathbf{u}, R, \mathbf{z}, (\mathbf{z})^n, H, \mathbf{v}\rangle$. Then P is the desired path.

Case 2. $j = n$, that is, $\mathbf{x_1} \in P_n^{\{n\}}$ and $\mathbf{x_2} \in P_n^{\{1\}}$. Let $c \in \langle n-1\rangle - \{1, a\}$ and $d \in \langle n-1\rangle - \{1, b, c\}$. By Lemma 5, there is a hamiltonian path R of $P_n^{\{n\}} - \{\mathbf{e}, \mathbf{x_1}\}$ joining a vertex $\mathbf{u}$ with $(\mathbf{u})_1 = a$ to a vertex $\mathbf{z}$ with $(\mathbf{z})_1 = c$. By Lemma 4, there is a hamiltonian path H of $P_n^{\{1\}} - \{\mathbf{x_2}\}$ joining a vertex $\mathbf{w}$ with $(\mathbf{w})_1 = d$ to a vertex $\mathbf{v}$ with $(\mathbf{v})_1 = b$. By Theorem 2, there is a hamiltonian Q of $P_n^{\langle n-1\rangle - \{1\}}$ joining the vertex $(\mathbf{z})^n$ to the vertex $(\mathbf{w})^n$. We set $P = \langle \mathbf{u}, R, \mathbf{z}, (\mathbf{z})^n, Q, (\mathbf{w})^n, \mathbf{w}, H, \mathbf{v}\rangle$. Then P is the desired path. $\square$

Our main result for the pancake graph P_n is stated in the following theorem.

Theorem 3. *$IHC(P_3) = 1$ and $IHC(P_n) = n - 1$ if $n \geq 4$.*

Proof. It is easy to see that P_3 is isomorphic to a cycle with six vertices. Thus, $IHC(P_3) = 1$. Since P_n is $(n-1)$-regular graph, it is clear that $IHC(P_n) \leq n - 1$. Since P_n is vertex transitive, we only need to show that there exist $(n-1)$ mutually independent hamiltonian cycles of P_n starting form the vertex $\mathbf{e}$. For $n = 4$, we prove that $IHC(P_4) \geq 3$ by listing the required hamiltonian cycles as follows:

$C_1 =$ ⟨1234, 2134, 4312, 3412, 2143, 1243, 4213, 3124, 1324, 4231, 3241, 2341, 1432, 4132, 2314, 3214, 4123, 1423, 2413, 3142, 1342, 2431, 3421, 4321, 1234⟩
$C_2 =$ ⟨1234, 3214, 2314, 1324, 3124, 4213, 2413, 1423, 4123, 2143, 1243, 3421, 4321, 2341, 3241, 4231, 2431, 1342, 3142, 4132, 1432, 3412, 4312, 2134, 1234⟩
$C_3 =$ ⟨1234, 4321, 2341, 1432, 4132, 2314, 1324, 4231, 3241, 1423, 2413, 3142, 1342, 2431, 3421, 1243, 4213, 3124, 2134, 4312, 3412, 2143, 4123, 3214, 1234⟩

Suppose that $n \geq 5$. Let B be the $(n-1) \times n$ matrix with

$$b_{i,j} = \begin{cases} i+j-1 & \text{if } i+j-1 \leq n, \\ i+j-n+1 & \text{if } n \geq i+j. \end{cases}$$

More precisely,

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 & \cdots & n-1 & n \\ 2 & 3 & 4 & 5 & \cdots & n & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n-1 & n & 1 & 2 & \cdots & n-3 & n-2 \end{bmatrix}.$$

It is not hard to see that $b_{i,1}b_{i,2}\ldots b_{i,n}$ forms a permutation of $\{1, 2, \ldots, n\}$ for every i with $1 \leq i \leq n-1$. Moreover, $b_{i,j} \neq b_{i',j}$ for any $1 \leq i < i' \leq n-1$ and $1 \leq j \leq n$. In other words, B forms a Latin rectangle with entries in $\{1, 2, \ldots, n\}$.

For every $k \in \langle n-1\rangle$, we construct C_k as follows:

(1) $k = 1$. By Lemma 3, there is a hamiltonian path H_1 of $P_n^{\{b_{1,n}\}} - \{\mathbf{e}\}$ joining a vertex $\mathbf{x}$ with $\mathbf{x} \neq (\mathbf{e})^{n-1}$ and $(\mathbf{x})_1 = n-1$ to the vertex $(\mathbf{e})^{n-1}$. By Theorem 2, there is a hamiltonian path H_2 of $\cup_{t=1}^{n-1} P_n^{\{b_{1,t}\}}$ joining the vertex $(\mathbf{e})^n$ to the vertex $(\mathbf{x})^n$ with $H_2(i+(j-1)(n-1)!) \in P_n^{\{b_{1,j}\}}$ for every $i \in \langle (n-1)!\rangle$ and for every $j \in \langle n-1\rangle$. We set $C_1 = \langle \mathbf{e}, (\mathbf{e})^n, H_2, (\mathbf{x})^n, \mathbf{x}, H_1, (\mathbf{e})^{n-1}, \mathbf{e}\rangle$.

(2) $k = 2$. By Lemma 5, there is a hamiltonian path Q_1 of $P_n^{\{b_{2,n-1}\}} - \{\mathbf{e}, (\mathbf{e})^2\}$ joining a vertex $\mathbf{y}$ with $(\mathbf{y})_1 = n-1$ to a vertex $\mathbf{z}$ with $(\mathbf{z})_1 = 1$. By Theorem 2, there is a hamiltonian Q_2 of $\cup_{t=1}^{n-2} P_n^{\{b_{2,t}\}}$ joining the vertex $((\mathbf{e})^2)^n$ to the vertex $(\mathbf{y})^n$ such that $Q_2(i+(j-1)(n-1)!) \in P_n^{\{b_{2,j}\}}$ for every $i \in \langle (n-1)!\rangle$ and for every $j \in \langle n-2\rangle$. By Theorem 1, there is a hamiltonian path Q_3 of $P_n^{\{b_{2,n}\}}$ joining the vertex $(\mathbf{z})^n$ to the vertex $(\mathbf{e})^n$. We set $C_2 = \langle \mathbf{e}, (\mathbf{e})^2, ((\mathbf{e})^2)^n, Q_2, (\mathbf{y})^n, \mathbf{y}, Q_1, \mathbf{z}, (\mathbf{z})^n, Q_3, (\mathbf{e})^n, \mathbf{e}\rangle$.

(3) $k \in \langle n-1\rangle - \{1, 2\}$. By Lemma 6, there is a hamiltonian path R_1^k of $P_n^{\{b_{k,n-k+1}\}} - \{\mathbf{e}, (\mathbf{e})^{k-1}, (\mathbf{e})^k\}$ joining a vertex $\mathbf{w_k}$ with $(\mathbf{w_k})_1 = n-1$ to a vertex $\mathbf{v_k}$ with $(\mathbf{v_k})_1 = 1$. By Theorem 2, there is a hamiltonian path R_2^k of $\cup_{t=1}^{n-k} P_n^{\{b_{k,t}\}}$

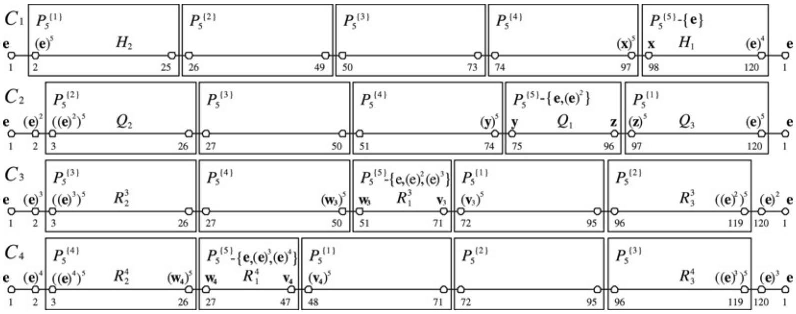

Fig. 3. Illustration for Theorem 3 on P_5.

joining the vertex $((\mathbf{e})^k)^n$ to the vertex $(\mathbf{w_k})^n$ such that $R_2^k(i+(j-1)(n-1)!) \in P_n^{\{b_{k,j}\}}$ for every $i \in \langle (n-1)! \rangle$ and for every $j \in \langle n-k \rangle$. Again, there is a hamiltonian path R_3^k of $\cup_{t=n-k+2}^{n} P_n^{\{b_{k,t}\}}$ joining the vertex $(\mathbf{v_k})^n$ to the vertex $((\mathbf{e})^{k-1})^n$ such that $R_3^k(i+(j-1)(n-1)!) \in P_n^{\{b_{k,n-k+j+1}\}}$ for every $i \in \langle (n-1)! \rangle$ and for every $j \in \langle k-1 \rangle$. We set $C_k = \langle \mathbf{e}, (\mathbf{e})^k, ((\mathbf{e})^k)^n, R_2^k, (\mathbf{w_k})^n, \mathbf{w_k}, R_1^k, \mathbf{v_k}, (\mathbf{v_k})^n, R_3^k, ((\mathbf{e})^{k-1})^n, (\mathbf{e})^{k-1}, \mathbf{e} \rangle$.

Then $\{C_1, C_2, \ldots, C_{n-1}\}$ forms a set of $(n-1)$ mutually independent hamiltonian cycles of P_n starting from the vertex $\mathbf{e}$. □

Example. We illustrate the proof of Theorem 3 with $n = 5$ as follows:

We set

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \end{bmatrix}.$$

Then we construct $\{C_1, C_2, C_3, C_4\}$ as follows:

(1)$k = 1$. By Lemma 3, there is a hamiltonian path H_1 of $P_5^{\{b_{1,5}\}} - \{\mathbf{e}\}$ joining a vertex $\mathbf{x}$ with $\mathbf{x} \neq (\mathbf{e})^4$ and $(\mathbf{x})_1 = 4$ to the vertex $(\mathbf{e})^4$. By Theorem 2, there is a hamiltonian path H_2 of $\cup_{t=1}^{4} P_5^{\{b_{1,t}\}}$ joining the vertex $(\mathbf{e})^5$ to the vertex $(\mathbf{x})^5$ with $H_2(i+24(j-1)) \in P_5^{\{b_{1,j}\}}$ for every $i \in \langle 24 \rangle$ and for every $j \in \langle 4 \rangle$. We set $C_1 = \langle \mathbf{e}, (\mathbf{e})^5, H_2, (\mathbf{x})^5, \mathbf{x}, H_1, (\mathbf{e})^4, \mathbf{e} \rangle$.

(2)$k = 2$. By Lemma 5, there is a hamiltonian path Q_1 of $P_5^{\{b_{2,4}\}} - \{\mathbf{e}, (\mathbf{e})^2\}$ joining a vertex $\mathbf{y}$ with $(\mathbf{y})_1 = 4$ to a vertex $\mathbf{z}$ with $(\mathbf{z})_1 = 1$. By Theorem 2, there is a hamiltonian Q_2 of $\cup_{t=1}^{3} P_5^{\{b_{2,t}\}}$ joining the vertex $((\mathbf{e})^2)^5$ to the vertex $(\mathbf{y})^5$ such that $Q_2(i+24(j-1)) \in P_5^{\{b_{2,j}\}}$ for every $i \in \langle 24 \rangle$ and for every $j \in \langle 3 \rangle$. By Theorem 1, there is a hamiltonian path Q_3 of $P_5^{\{b_{2,5}\}}$ joining the vertex $(\mathbf{z})^5$ to the vertex $(\mathbf{e})^5$. We set $C_2 = \langle \mathbf{e}, (\mathbf{e})^2, ((\mathbf{e})^2)^5, Q_2, (\mathbf{y})^5, \mathbf{y}, Q_1, \mathbf{z}, (\mathbf{z})^5, Q_3, (\mathbf{e})^5, \mathbf{e} \rangle$.

(3)$k \in \{3, 4\}$. By Lemma 6, there is a hamiltonian path R_1^k of $P_5^{\{b_{k,6-k}\}} - \{\mathbf{e}, (\mathbf{e})^{k-1}, (\mathbf{e})^k\}$ joining a vertex $\mathbf{w_k}$ with $(\mathbf{w_k})_1 = 4$ to a vertex $\mathbf{v_k}$ with $(\mathbf{v_k})_1 = 1$. By Theorem 2, there is a hamiltonian path R_2^k of $\cup_{t=1}^{5-k} P_5^{\{b_{k,t}\}}$ joining the vertex $((\mathbf{e})^k)^5$ to the vertex $(\mathbf{w_k})^5$ such that $R_2^k(i+24(j-1)) \in P_5^{\{b_{k,j}\}}$ for every $i \in \langle 24 \rangle$ and for every $j \in \langle 5-k \rangle$. Again, there is a hamiltonian path R_3^k of $\cup_{t=7-k}^{5} P_5^{\{b_{k,t}\}}$ joining the vertex $(\mathbf{v_k})^5$ to the vertex $((\mathbf{e})^{k-1})^5$ such that $R_3^k(i+24(j-1)) \in P_5^{\{b_{k,6-k+j}\}}$ for every $i \in \langle 24 \rangle$ and for every $j \in \langle k-1 \rangle$. We set $C_k = \langle \mathbf{e}, (\mathbf{e})^k, ((\mathbf{e})^k)^5, R_2^k, (\mathbf{w_k})^5, \mathbf{w_k}, R_1^k, \mathbf{v_k}, (\mathbf{v_k})^5, R_3^k, ((\mathbf{e})^{k-1})^5, (\mathbf{e})^{k-1}, \mathbf{e} \rangle$.

Then $\{C_1, C_2, C_3, C_4\}$ forms a set of 4 mutually independent hamiltonian cycles of P_5 starting from the vertex $\mathbf{e}$. See Fig. 3 for illustration.

5. The star graphs

Let n be a positive integer. The *n-dimensional star graph*, S_n, is a graph with the vertex set $V(S_n) = \{u_1 \ldots u_n \mid u_i \in \langle n \rangle$ and $u_j \neq u_k$ for $j \neq k\}$. The adjacency is defined as follows: $u_1 \ldots u_i \ldots u_n$ is adjacent to $v_1 \ldots v_i \ldots v_n$ through an edge of dimension i with $2 \leq i \leq n$ if $v_j = u_j$ for every $j \in \langle n \rangle - \{1, i\}$, $v_1 = u_i$, and $v_i = u_1$. The star graphs S_2, S_3, and S_4 are illustrated in Fig. 4. In [1], it showed that the connectivity of S_n is $(n-1)$. We use boldface to denote vertices in S_n. Hence, $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}$ denotes a sequence of vertices in S_n.

By definition, S_n is an $(n-1)$-regular graph with $n!$ vertices. We use $\mathbf{e}$ to denote the vertex $12 \ldots n$. It is known that S_n is a bipartite graph with one partite set containing the vertices corresponding to odd permutations and the other partite set containing those vertices correspond to even permutations. We use white vertices to represent those even permutation vertices and we use black vertices to represent those odd permutation vertices. Let $\mathbf{u} = u_1 u_2 \ldots u_n$ be an arbitrary vertex of the star graph S_n. We say that u_i is the ith coordinate of $\mathbf{u}$, $(\mathbf{u})_i$, for $1 \leq i \leq n$. For $1 \leq i \leq n$, let $S_n^{\{i\}}$ be the subgraph of S_n induced by those vertices $\mathbf{u}$ with $(\mathbf{u})_n = i$. Then S_n can be decomposed into n subgraph $S_n^{\{i\}}$, $1 \leq i \leq n$, and each $S_n^{\{i\}}$ is isomorphic to S_{n-1}. Thus, the star graph can also be constructed recursively. Let I be any subset of $\langle n \rangle$. We use S_n^I to denote

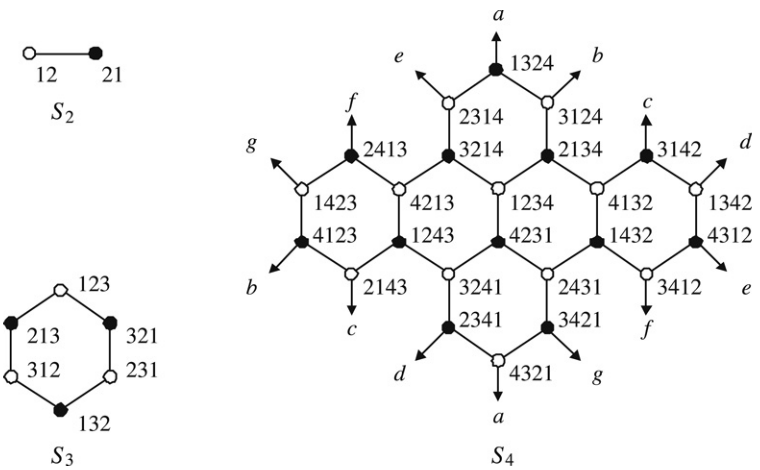

Fig. 4. The star graphs S_2, S_3, and S_4.

the subgraph of S_n induced by $\cup_{i\in I} V(S_n^{\{i\}})$. For any two distinct elements i and j in $\langle n\rangle$, we use $E_n^{i,j}$ to denote the set of edges between $S_n^{\{i\}}$ and $S_n^{\{j\}}$. By the definition of S_n, there is exactly one neighbor $\mathbf{v}$ of $\mathbf{u}$ such that $\mathbf{u}$ and $\mathbf{v}$ are adjacent through an i-dimensional edge with $2 \le i \le n$. For this reason, we use $(\mathbf{u})^i$ to denote the unique i-neighbor of $\mathbf{u}$. We have $((\mathbf{u})^i)^i = \mathbf{u}$ and $(\mathbf{u})^n \in S_n^{\{(\mathbf{u})_1\}}$.

Lemma 7. *Let i and j be any two distinct elements in $\langle n\rangle$ with $n \ge 3$. Then $|E_n^{i,j}| = (n-2)!$. Moreover, there are $(n-2)!/2$ edges joining black vertices of $S_n^{\{i\}}$ to white vertices of $S_n^{\{j\}}$.*

Lemma 8. *Let $\mathbf{u}$ and $\mathbf{v}$ be two distinct vertices of S_n with $d(\mathbf{u},\mathbf{v}) \le 2$. Then $(\mathbf{u})_1 \ne (\mathbf{v})_1$.*

Theorem 4 ([21]). *Let $n \ge 4$. Suppose that $\mathbf{u}$ is a white vertex of S_n and $\mathbf{v}$ is a black vertex of S_n. Then there is a hamiltonian path of S_n joining $\mathbf{u}$ to $\mathbf{v}$.*

Theorem 5. *Let $\{a_1, a_2, \dots, a_r\}$ be a subset of $\langle n\rangle$ for some $r \in \langle n\rangle$ with $n \ge 5$. Assume that $\mathbf{u}$ is a white vertex in $S_n^{\{a_1\}}$ and $\mathbf{v}$ is a black vertex in $S_n^{\{a_r\}}$. Then there is a hamiltonian path $\langle \mathbf{u} = \mathbf{x_1}, H_1, \mathbf{y_1}, \mathbf{x_2}, H_2, \mathbf{y_2}, \dots, \mathbf{x_r}, H_r, \mathbf{y_r} = \mathbf{v}\rangle$ of $\cup_{i=1}^{r} S_n^{\{a_i\}}$ joining $\mathbf{u}$ to $\mathbf{v}$ such that $\mathbf{x_1} = \mathbf{u}$, $\mathbf{y_r} = \mathbf{v}$, and H_i is a hamiltonian path of $S_n^{\{a_i\}}$ joining $\mathbf{x_i}$ to $\mathbf{y_i}$ for every i, $1 \le i \le r$.*

Proof. We set $\mathbf{x_1}$ as $\mathbf{u}$ and $\mathbf{y_r}$ as $\mathbf{v}$. By Theorem 4, this theorem holds on $r = 1$. Suppose that $r \ge 2$. By Lemma 7, there are $(n-2)!/2 \ge 3$ edges joining black vertices of $S_n^{\{a_i\}}$ to white vertices of $S_n^{\{a_{i+1}\}}$ for every $i \in \langle r-1\rangle$. We can choose an edge $(\mathbf{y_i}, \mathbf{x_{i+1}}) \in E_n^{a_i, a_{i+1}}$ with $\mathbf{y_i}$ being a black vertex and $\mathbf{x_{i+1}}$ being a white vertex for every $i \in \langle r-1\rangle$. By Theorem 4, there is a hamiltonian path H_i of $S_n^{\{a_i\}}$ joining $\mathbf{x_i}$ to $\mathbf{y_i}$ for every $i \in \langle r\rangle$. Then the path $\langle \mathbf{u} = \mathbf{x_1}, H_1, \mathbf{y_1}, \mathbf{x_2}, H_2, \mathbf{y_2}, \dots, \mathbf{x_r}, H_r, \mathbf{y_r} = \mathbf{v}\rangle$ is the desired path. □

Theorem 6 ([21]). *Let $\mathbf{w}$ be a black vertex of S_n with $n \ge 4$. Assume that $\mathbf{u}$ and $\mathbf{v}$ are two distinct white vertices of $S_n - \{\mathbf{w}\}$. Then there is a hamiltonian path H of $S_n - \{\mathbf{w}\}$ joining $\mathbf{u}$ to $\mathbf{v}$.*

Lemma 9 ([30]). *Let i be any element in $\langle n\rangle$ with $n \ge 4$. Assume that $\mathbf{r}$ and $\mathbf{s}$ are two adjacent vertices of S_n and $\mathbf{u}$ is a white vertex of $S_n - \{\mathbf{r}, \mathbf{s}\}$. Then there is a hamiltonian path of $S_n - \{\mathbf{r}, \mathbf{s}\}$ joining $\mathbf{u}$ to some black vertex $\mathbf{v}$ with $(\mathbf{v})_1 = i$.*

Lemma 10. *Let a and b be any two distinct elements in $\langle n\rangle$ with $n \ge 4$. Assume that $\mathbf{x}$ is a white vertex of S_n, and assume that $\mathbf{x_1}$ and $\mathbf{x_2}$ are two distinct neighbors of $\mathbf{x}$. Then there is a hamiltonian path P of $S_n - \{\mathbf{x}, \mathbf{x_1}, \mathbf{x_2}\}$ joining a white vertex $\mathbf{u}$ with $(\mathbf{u})_1 = a$ to a white vertex $\mathbf{v}$ with $(\mathbf{v})_1 = b$.*

Proof. Since S_n is vertex transitive and edge transitive, we may assume that $\mathbf{x} = \mathbf{e}$, $\mathbf{x_1} = (\mathbf{e})^2$, and $\mathbf{x_2} = (\mathbf{e})^3$. Without loss of generality, we may also assume that $a < b$. We have $a \ne n$ and $b \ne 1$. We prove this statement by induction on n. For $n = 4$, the required paths of $S_4 - \{1234, 2134, 3214\}$ are listed below:

$a = 1$ and $b = 2$	⟨1324, 3142, 4132, 1432, 3412, 4312, 2314, 1324, 3124, 4123, 2143, 1243, 4213, 2413, 1423, 3421, 4321, 2341, 3241, 4231, 2431⟩
$a = 1$ and $b = 3$	⟨1423, 2413, 4213, 1243, 2143, 4123, 3124, 1324, 2314, 4312, 3412, 1432, 4132, 3142, 1342, 2341, 4321, 3421, 2431, 4231, 3241⟩
$a = 1$ and $b = 4$	⟨1324, 3142, 4132, 1432, 3412, 4312, 2314, 1324, 3124, 4123, 2143, 1243, 4213, 2413, 1423, 3421, 2431, 4231, 3241, 2341, 4321⟩
$a = 2$ and $b = 3$	⟨2314, 1324, 3124, 4123, 2143, 1243, 4213, 2413, 1423, 3421, 4321, 2341, 3241, 4231, 2431, 1432, 4132, 3142, 1342, 4312, 3412⟩
$a = 2$ and $b = 4$	⟨2314, 1324, 3124, 4123, 2143, 1243, 4213, 2413, 1423, 3421, 4321, 2341, 3241, 4231, 2431, 1432, 3412, 4312, 1342, 3142, 4132⟩
$a = 3$ and $b = 4$	⟨3124, 1324, 2314, 4312, 3412, 1432, 4132, 3142, 1342, 2341, 4321, 3421, 2431, 4231, 3241, 1243, 2143, 4123, 1423, 2413, 4213⟩

Suppose that this statement holds for S_k for every k, $4 \le k \le n-1$. Let c be any element in $\langle n-1\rangle - \{1, a\}$. By induction, there is a hamiltonian path H of $S_n^{\{n\}} - \{\mathbf{e}, (\mathbf{e})^2, (\mathbf{e})^3\}$ joining a white vertex $\mathbf{u}$ with $(\mathbf{u})_1 = a$ to a white vertex $\mathbf{z}$ with $(\mathbf{z})_1 = c$. We choose a white vertex $\mathbf{v}$ in $S_n^{\{1\}}$ with $(\mathbf{v})_1 = b$. By Theorem 5, there is a hamiltonian path R of $S_n^{\langle n-1\rangle}$ joining the black vertex $(\mathbf{z})^n$ to $\mathbf{v}$. Then $\langle \mathbf{u}, H, \mathbf{z}, (\mathbf{z})^n, R, \mathbf{v}\rangle$ is the desired path of $S_n - \{\mathbf{e}, (\mathbf{e})^2, (\mathbf{e})^3\}$. □

The following theorem is our main result for the star graph S_n.

Theorem 7. *$IHC(S_3) = 1$, $IHC(S_4) = 2$, and $IHC(S_n) = n-1$ if $n \ge 5$.*

Proof. It is easy to see that S_3 is isomorphic to a cycle with six vertices. Thus, $IHC(S_3) = 1$. Using a computer, we have $IHC(S_4) = 2$ by brute force checking. Thus, we assume that $n \ge 5$. We know that S_n is $(n-1)$-regular graph. Hence, $IHC(S_n) \le n-1$. Since S_n is vertex transitive, we only need to show that there are $(n-1)$ mutually independent hamiltonian cycles of S_n starting from $\mathbf{e}$. Let B be the $(n-1) \times n$ matrix with

$$b_{i,j} = \begin{cases} i+j-1 & \text{if } i+j-1 \le n, \\ i+j-n+1 & \text{if } n < i+j-1. \end{cases}$$

We construct $\{C_1, C_2, \dots, C_{n-1}\}$ as follows:

(1) $k = 1$. We choose a black vertex $\mathbf{x}$ in $S_n^{\{b_{1,n}\}} - \{(\mathbf{e})^{n-1}\}$ with $(\mathbf{x})_1 = n-1$. By Theorem 6, there is a hamiltonian path H_1 of $S_n^{\{b_{1,n}\}} - \{\mathbf{e}\}$ joining $\mathbf{x}$ to the black vertex $(\mathbf{e})^{n-1}$. By Theorem 5, there is a hamiltonian path H_2 of $\cup_{t=1}^{n-1} S_n^{\{b_{1,t}\}}$ joining the black vertex $(\mathbf{e})^n$ to the white vertex $(\mathbf{x})^n$ with $H_2(i + (j-1)(n-1)!) \in S_n^{\{b_{1,j}\}}$ for every $i \in \langle (n-1)!\rangle$ and for every $j \in \langle n-1\rangle$. We set $C_1 = \langle \mathbf{e}, (\mathbf{e})^n, H_2, (\mathbf{x})^n, \mathbf{x}, H_1, (\mathbf{e})^{n-1}, \mathbf{e}\rangle$.

(2) $k = 2$. We choose a white vertex $\mathbf{y}$ in $S_n^{\{b_{2,n-1}\}} - \{\mathbf{e}, (\mathbf{e})^2\}$ with $(\mathbf{y})_1 = n-1$. By Lemma 9, there is a hamiltonian path Q_1 of $S_n^{\{b_{2,j}\}} - \{\mathbf{e}, (\mathbf{e})^2\}$ joining $\mathbf{y}$ to a black vertex $\mathbf{z}$ with $(\mathbf{z})_1 = 1$. By Theorem 5, there is a hamiltonian Q_2 of $\cup_{t=1}^{n-2} S_n^{\{b_{2,t}\}}$ joining the white vertex $((\mathbf{e})^2)^n$ to the black vertex $(\mathbf{y})^n$ such that $Q_2(i + (j-1)(n-1)!) \in S_n^{\{b_{2,j}\}}$ for every $i \in \langle (n-1)!\rangle$ and for every $j \in \langle n-2\rangle$. Again, there is a hamiltonian path Q_3 of $S_n^{\{b_{2,n}\}}$ joining the white vertex $(\mathbf{z})^n$ to the black vertex $(\mathbf{e})^n$. We set $C_2 = \langle \mathbf{e}, (\mathbf{e})^2, ((\mathbf{e})^2)^n, Q_2, (\mathbf{y})^n, \mathbf{y}, Q_1, \mathbf{z}, (\mathbf{z})^n, Q_3, (\mathbf{e})^n, \mathbf{e}\rangle$.

(3) $3 \le k \le n-1$. By Lemma 10, there is a hamiltonian path R_1^k of $S_n^{\{b_{k,n-k+1}\}} - \{\mathbf{e}, (\mathbf{e})^{k-1}, (\mathbf{e})^k\}$ joining a white vertex $\mathbf{w_k}$ with $(\mathbf{w_k})_1 = n-1$ to a white vertex $\mathbf{v_k}$ with $(\mathbf{v_k})_1 = 1$. By Theorem 5, there is a hamiltonian path R_2^k of $\cup_{t=1}^{n-k} S_n^{\{b_{k,t}\}}$ joining the white vertex $((\mathbf{e})^k)^n$ to the black vertex $(\mathbf{w_k})^n$ such that $R_2^k(i + (j-1)(n-1)!) \in S_n^{\{b_{k,j}\}}$ for every $i \in \langle (n-1)!\rangle$ and for every $j \in \langle n-k-1\rangle$. Again, there is a hamiltonian path R_3^k of $\cup_{t=n-k+2}^{n} S_n^{\{b_{k,t}\}}$ joining the black vertex $(\mathbf{v_k})^n$ to the black vertex $((\mathbf{e})^{k-1})^n$ such that $R_3^k(i + (j-1)(n-1)!) \in S_n^{\{b_{k,n-k+j+1}\}}$ for every $i \in \langle (n-1)!\rangle$ and for every $j \in \langle k-1\rangle$. We set $C_k = \langle \mathbf{e}, (\mathbf{e})^k, ((\mathbf{e})^k)^n, R_2^k, (\mathbf{w_k})^n, \mathbf{w_k}, R_1^k, \mathbf{v_k}, (\mathbf{v_k})^n, R_3^k, ((\mathbf{e})^{k-1})^n, (\mathbf{e})^{k-1}, \mathbf{e}\rangle$.

Then $\{C_1, C_2, \dots, C_{n-1}\}$ forms a set of $(n-1)$ mutually independent hamiltonian cycles of S_n starting form the vertex $\mathbf{e}$. □

6. Discussion

In this paper, we discuss the mutually independent hamiltonian cycles for the pancake graphs and the star graphs. The concept of mutually independent hamiltonian cycle can be viewed as a generalization of Latin rectangles. Perhaps one of the most interesting topics in Latin square is orthogonal Latin square. Two Latin squares of order n are *orthogonal* if the n-squared pairs formed by juxtaposing the two arrays are all distinct. Similarly, two Latin rectangles of order $n \times m$ are *orthogonal* if the $n \times m$ pairs formed by juxtaposing the two arrays are all distinct. With this in mind, let G be a hamiltonian graph and C_1 and C_2 be two sets of mutually independent hamiltonian cycles of G from a given vertex x. We say C_1 and C_2 are *orthogonal* if their corresponding Latin rectangles are orthogonal. For example, we know that $IHC(P_4) = 3$. The following Latin rectangle represents three mutually independent hamiltonian cycles beginning at 1234.

2134, 4312, 1342, 2431, 3421, 1243, 4213, 3124, 1324, 4231, 3241, 1423, 2413, 3142, 4132, 2314, 3214, 4123, 2143, 3412, 1432, 2341, 4321
3214, 2314, 4132, 1432, 3412, 4312, 1342, 3142, 2413, 4213, 1243, 2143, 4123, 1423, 3241, 2341, 4321, 3421, 2431, 4231, 1324, 3124, 2134
4321, 2341, 1432, 3412, 2143, 4123, 1423, 3241, 4231, 1324, 3124, 2134, 4312, 1342, 2431, 3421, 1243, 4213, 2413, 3142, 4132, 2314, 3214

Yet, the following Latin rectangle also represents three mutually independent hamiltonian cycles beginning at 1234.

2134, 3124, 4213, 1243, 2143, 4123, 1423, 2413, 3142, 4132, 1432, 3412, 4312, 1342, 2431, 3421, 4321, 2341, 3241, 4231, 1324, 2314, 3214
3214, 2314, 4132, 3142, 2413, 4213, 1243, 3421, 2431, 1342, 4312, 2134, 3124, 1324, 4231, 3241, 1423, 4123, 2143, 3412, 1432, 2341, 4321
4321, 3421, 1243, 2143, 3412, 4312, 1342, 2431, 4231, 1324, 2314, 3214, 4123, 1423, 3241, 2341, 1432, 4132, 3142, 2413, 4213, 3124, 2134

We can check that these two Latin rectangles are orthogonal. Thus, we have two sets of three mutually independent hamiltonian cycles that are orthogonal. With this example in mind, we can consider the following problem. Let G be any hamiltonian graph. We can define $MOMH(G)$ as the largest integer k such that there exist k sets of mutually independent hamiltonian cycle of G beginning from any vertex x such that each set contains exactly $IHC(G)$ hamiltonian cycles and any two different sets are orthogonal. It would be interesting to study the value of $MOMH(G)$ for some hamiltonian graphs G.

We can also discuss mutually independent hamiltonian paths for some graphs. Let $P_1 = \langle v_1, v_2, \ldots, v_n\rangle$ and $P_2 = \langle u_1, u_2, \ldots, u_n\rangle$ be two hamiltonian paths of a graph G. We say that P_1 and P_2 are independent if $u_1 = v_1$, $u_n = v_n$, and $u_i \neq v_i$ for $1 < i < n$. We say a set of hamiltonian paths $\{P_1, P_2, \ldots, P_s\}$ of G between two distinct vertices are mutually independent if any two distinct paths in the set are independent. There are some study on mutually independent hamiltonian paths [29,39].

Recently, people are interested in a mathematical puzzle, called Sudoku [38]. Sudoku can be viewed as a 9×9 Latin square with some constraints. There are several variations of Sudoku have been introduced. Mutually independent hamiltonian cycles can also be considered as a variation of Sudoku.

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The bipanpositionable bipancyclic property of the hypercube[☆,☆☆]

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ABSTRACT

A bipartite graph is *bipancyclic* if it contains a cycle of every even length from 4 to $|V(G)|$ inclusive. A hamiltonian bipartite graph G is *bipanpositionable* if, for any two different vertices x and y, there exists a hamiltonian cycle C of G such that $d_C(x, y) = k$ for any integer k with $d_G(x, y) \leq k \leq |V(G)|/2$ and $(k - d_G(x, y))$ being even. A bipartite graph G is *k-cycle bipanpositionable* if, for any two different vertices x and y, there exists a cycle of G with $d_C(x, y) = l$ and $|V(C)| = k$ for any integer l with $d_G(x, y) \leq l \leq \frac{k}{2}$ and $(l - d_G(x, y))$ being even. A bipartite graph G is *bipanpositionable bipancyclic* if G is k-cycle bipanpositionable for every even integer k, $4 \leq k \leq |V(G)|$. We prove that the hypercube Q_n is bipanpositionable bipancyclic for $n \geq 2$.

1. Introduction

For the graph definitions and notations we follow Bondy and Murty [1]. Let $G = (V, E)$ be a *graph*, where V is a finite set and E is a subset of $\{(u, v) \mid (u, v)$ is an unordered pair of $V\}$. We say that V is the *vertex set* and E is the *edge set* of G. Two vertices u and v are *adjacent* if $(u, v) \in E$. A *path* is represented by $\langle v_0, v_1, v_2, \ldots, v_k\rangle$, where all vertices are distinct except possibly $v_0 = v_k$. The *length* of a path Q is the number of edges in Q. We also write the path $\langle v_0, v_1, v_2, \ldots, v_k\rangle$ as $\langle v_0, Q_1, v_i, v_{i+1} \ldots, v_j, Q_2, v_t, \ldots, v_k\rangle$, where Q_1 is the path $\langle v_0, v_1, \ldots, v_{i-1}, v_i\rangle$ and Q_2 is the path $\langle v_j, v_{j+1}, \ldots, v_{t-1}, v_t\rangle$. We use $d_G(u, v)$ to denote the distance between u and v in G, i.e., the length of the shortest path joining u to v in G. A *cycle* is a path of at least three vertices such that the first vertex is the same as the last vertex. We use $d_c(u, v)$ to denote the distance between u and v in a cycle C, i.e., the length of the shortest path joining u to v in C. A *hamiltonian cycle* of G is a cycle that traverses every vertex of G exactly once. A *hamiltonian graph* is a graph with a hamiltonian cycle. A graph $G = (V_0 \cup V_1, E)$ is *bipartite* if $V(G) = V_0 \cup V_1$ and $E(G)$ is a subset of $\{(u, v) \mid u \in V_0$ and $v \in V_1\}$.

The *n-dimensional hypercube*, Q_n, consists of all n-bit binary strings as its vertices and two vertices $\mathbf{u}$ and $\mathbf{v}$ are adjacent if and only if their binary labels are different in exactly one bit position. Let $\mathbf{u} = u_{n-1}u_{n-2}\ldots u_1u_0$ and $\mathbf{v} = v_{n-1}v_{n-2}\ldots v_1v_0$ be two n-bit binary strings. The Hamming distance $h(u, v)$ between two vertices u and v is the number of different bits in the corresponding strings of both vertices. Let Q_n^i be the subgraph of Q_n induced by $\{u_{n-1}u_{n-2}\cdots u_1u_0 \mid u_{n-1} = i\}$ for $i = 0, 1$. Therefore, Q_n can be constructed recursively by taking two copies of Q_{n-1}, Q_n^0 and Q_n^1, and adding a perfect matching between these two copies. For a vertex $\mathbf{u}$ in Q_n^0 (resp. Q_n^1), we use $\bar{\mathbf{u}}$ to denote the unique neighbor of $\mathbf{u}$ in Q_n^1 (resp. Q_n^0). The *hypercube* is a widely used topology in computer architecture, see Leighton [2].

A graph is *pancyclic* if it contains a cycle of every length from 3 to $|V(G)|$ inclusive. The concept of pancyclic graphs was proposed by Bondy [3]. Since there is no odd cycle in bipartite graph, the concept of a bipancyclic graph was proposed

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0898-1221/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
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by Mitchem and Schmeichel [4]. A bipartite graph is *bipancyclic* if it contains a cycle of every even length from 4 to $|V(G)|$ inclusive. It is proved that the hypercube Q_n is bipancyclic if $n \geq 2$ [5,6]. A graph is *panconnected* if, for any two different vertices x and y, there exists a path of length l joining x and y for every l with $d_G(x, y) \leq l \leq |V(G)| - 1$. The concept of panconnected graphs was proposed by Alavi and Williamson [7]. It is easy to see that any bipartite graph with at least 3 vertices is not panconnected. Therefore, the concept of bipanconnected graphs is proposed. A bipartite graph is *bipanconnected* if, for any two different vertices x and y, there exists a path of length l joining x and y for every l with $d_G(x, y) \leq l \leq |V(G)| - 1$ and $(l - d_G(x, y))$ being even. It is proved that the hypercube is bipanconnected [5]. A hamiltonian graph G is *panpositionable* if for any two different vertices x and y of G and for any integer k with $d_G(x, y) \leq k \leq |V(G)|/2$, there exists a hamiltonian cycle C of G such that $d_C(x, y) = k$. A hamiltonian bipartite graph G is *bipanpositionable* if for any two different vertices x and y of G and for any integer k with $d_G(x, y) \leq k \leq |V(G)|/2$ and $(k - d_G(x, y))$ being even, there exists a hamiltonian cycle C of G such that $d_C(x, y) = k$. The concepts of panpositionable and bipanpositionable were proposed by Kao et al. [8]. They proved that the hypercube Q_n is bipanpositionable if $n \geq 2$ [8]. A bipartite graph G is *edge-bipancyclic* if for any edge in G, there is a cycle of every even length from 4 to $|V(G)|$ traversing through this edge. The concept of edge-bipancyclic was proposed by Alspach and Hare [9]. A bipartite graph G is *vertex-bipancyclic* if for any vertex in G, there is a cycle of every even length from 4 to $|V(G)|$ going through this vertex. The concept of vertex-bipancyclic was proposed by Hobbs [10]. Obviously, every edge-bipancyclic graph is vertex-bipancyclic. It is proved that the hypercube Q_n is edge-bipancyclic if $n \geq 2$ [5].

In this paper, we propose a more interesting property about hypercubes. A k-cycle is a cycle of length k. A bipartite graph G is *k-cycle bipanpositionable* if for every different vertices x and y of G and for any integer l with $d_G(x, y) \leq l \leq \frac{k}{2}$ and $(l - d_G(x, y))$ being even, there exists a k-cycle C of G such that $d_C(x, y) = l$. (Note that $d_C(x, y) \leq \frac{k}{2}$ for every cycle C of length k.) A bipartite graph G is *bipanpositionable bipancyclic* if G is k-cycle bipanpositionable for every even integer k with $4 \leq k \leq |V(G)|$. In this paper, we prove that the hypercube Q_n is bipanpositionable bipancyclic for $n \geq 2$. As a consequence of this result, we can see that many previous results on hypercubes follows directly from ours. For example, the hypercube is bipancyclic, bipanconnected, bipanpositionable, edge-bipancyclic and vertex-bipancyclic. Therefore, our result unifies these results in a general sense.

2. The bipanpositionable bipancyclic property

We prove our main result by induction as stated in Lemma 1 and Theorem 1 below.

Lemma 1. *The hypercube Q_3 is bipanpositionable bipancyclic.*

Proof. Let $\mathbf{x}$ and $\mathbf{y}$ be two different vertices in Q_3. Obviously, $d_{Q_3}(\mathbf{x}, \mathbf{y}) = 1, 2$ or 3. Since the hypercube is vertex symmetric, without loss of generality, we may assume that $\mathbf{x} = 000$.

Case 1: Suppose that $d_{Q_3}(\mathbf{x}, \mathbf{y}) = 1$. Since Q_3 is edge symmetric, we assume that $\mathbf{y} = 001$.

$\mathbf{y} = 001$	4-cycle	$d_C(\mathbf{x}, \mathbf{y}) = 1$	⟨000, 001, 011, 010, 000⟩
	6-cycle	$d_C(\mathbf{x}, \mathbf{y}) = 1$	⟨000, 001, 101, 111, 110, 100, 000⟩
		$d_C(\mathbf{x}, \mathbf{y}) = 3$	⟨000, 100, 101, 001, 011, 010, 000⟩
	8-cycle	$d_C(\mathbf{x}, \mathbf{y}) = 1$	⟨000, 001, 101, 111, 011, 010, 110, 100, 000⟩
		$d_C(\mathbf{x}, \mathbf{y}) = 3$	⟨000, 100, 101, 001, 011, 111, 110, 010, 000⟩

Case 2: Suppose that $d_{Q_3}(\mathbf{x}, \mathbf{y}) = 2$. By symmetry, we assume that $\mathbf{y} = 011$.

$\mathbf{y} = 011$	4-cycle	$d_C(\mathbf{x}, \mathbf{y}) = 2$	⟨000, 001, 011, 010, 000⟩
	6-cycle	$d_C(\mathbf{x}, \mathbf{y}) = 2$	⟨000, 001, 011, 010, 110, 100, 000⟩
	8-cycle	$d_C(\mathbf{x}, \mathbf{y}) = 2$	⟨000, 001, 011, 010, 110, 111, 101, 100, 000⟩
		$d_C(\mathbf{x}, \mathbf{y}) = 4$	⟨000, 001, 101, 111, 011, 010, 110, 100, 000⟩

Case 3: Suppose that $d_{Q_3}(\mathbf{x}, \mathbf{y}) = 3$. We have $\mathbf{y} = 111$.

$\mathbf{y} = 111$	6-cycle	$d_C(\mathbf{x}, \mathbf{y}) = 3$	⟨000, 001, 011, 111, 110, 100, 000⟩
	8-cycle	$d_C(\mathbf{x}, \mathbf{y}) = 3$	⟨000, 001, 011, 111, 101, 100, 110, 010, 000⟩

Thus, Q_3 is bipanpositionable bipancyclic. □

Theorem 1. *The hypercube Q_n is bipanpositionable bipancyclic for $n \geq 2$.*

Proof. We observe that Q_1 is not bipanpositionable bipancyclic. So we start with $n \geq 2$. We prove Q_n is bipanpositionable bipancyclic by induction on n. It is easy to see that Q_2 is bipanpositionable bipancyclic. By Lemma 1, this statement holds for $n = 3$. Suppose that Q_{n-1} is bipanpositionable bipancyclic for some $n \geq 4$. Let $\mathbf{x}$ and $\mathbf{y}$ be two distinct vertices in Q_n, and let k be an even integer with $k \geq \max\{4, 2d_{Q_n}(\mathbf{x}, \mathbf{y})\}$ and $k \leq 2^n$. For every integer l with $d_{Q_n}(\mathbf{x}, \mathbf{y}) \leq l \leq \frac{k}{2}$ and $(l - d_{Q_n}(\mathbf{x}, \mathbf{y}))$ being even, we need to construct a k-cycle C of Q_n with $d_C(\mathbf{x}, \mathbf{y}) = l$.

Case 1: $d_{Q_n}(\mathbf{x}, \mathbf{y}) = 1$. Without loss of generality, we may assume that both $\mathbf{x}$ and $\mathbf{y}$ are in Q_n^0. $(l - d_{Q_n}(\mathbf{x}, \mathbf{y}))$ is even, so l is an odd number. Since Q_n^0 is isomorphic to Q_{n-1}, by introduction, there is a k-cycle of Q_n^0 with $d_C(\mathbf{x}, \mathbf{y}) = l$ for every $4 \leq k \leq 2^{n-1}$. Thus, we consider that $k \geq 2^{n-1} + 2$.

Case 1.1: $l = 1$. By induction, there is a (2^{n-1})-cycle $C' = \langle \mathbf{x}, P, \mathbf{z}, \mathbf{y}, \mathbf{x} \rangle$ of Q_n^0 where $d_P(\mathbf{x}, \mathbf{z}) = 2^{n-1} - 2$. Suppose that $k - 2^{n-1} = 2$. Then $C = \langle \mathbf{x}, P, \mathbf{z}, \bar{\mathbf{z}}, \bar{\mathbf{y}}, \mathbf{y}, \mathbf{x} \rangle$ forms a $(2^{n-1} + 2)$-cycle with $d_C(\mathbf{x}, \mathbf{y}) = 1$. Suppose that $k - 2^{n-1} \geq 4$. By induction, there is a $(k - 2^{n-1})$-cycle C'' of Q_n^1 such that $d_{C''}(\bar{\mathbf{z}}, \bar{\mathbf{y}}) = 1$. We write $C'' = \langle \bar{\mathbf{z}}, R, \bar{\mathbf{y}}, \bar{\mathbf{z}} \rangle$ with $d_R(\bar{\mathbf{z}}, \bar{\mathbf{y}}) = k - 2^{n-1} - 1$. Then $C = \langle \mathbf{x}, P, \mathbf{z}, \bar{\mathbf{z}}, R, \bar{\mathbf{y}}, \mathbf{y}, \mathbf{x} \rangle$ forms a k-cycle of Q_n with $d_C(\mathbf{x}, \mathbf{y}) = l = 1$.

Case 1.2: $l \geq 3$. Suppose that $k - l - 1 \leq 2^{n-1}$. By induction, there is an $(l + 1)$-cycle C' of Q_n^0 with $d_{C'}(\mathbf{x}, \mathbf{y}) = 1$. We write $C' = \langle \mathbf{x}, P, \mathbf{y}, \mathbf{x} \rangle$ where $d_P(\mathbf{x}, \mathbf{y}) = l$. By induction, there is a $(k - l - 1)$-cycle C'' of Q_n^1 with $d_{C''}(\bar{\mathbf{x}}, \bar{\mathbf{y}}) = 1$. We then write $C'' = \langle \bar{\mathbf{y}}, R, \bar{\mathbf{x}}, \bar{\mathbf{y}} \rangle$ such that $d_R(\bar{\mathbf{y}}, \bar{\mathbf{x}}) = k - l - 2$. Then $C = \langle \mathbf{x}, P, \mathbf{y}, \bar{\mathbf{y}}, R, \bar{\mathbf{x}}, \mathbf{x} \rangle$ forms a k-cycle of Q_n with $d_C(\mathbf{x}, \mathbf{y}) = l$. Suppose that $k - l - 2 \geq 2^{n-1} + 1$. By induction, there is a $(k - 2^{n-1})$-cycle C' of Q_n^0 with $d_{C'}(\mathbf{x}, \mathbf{y}) = l$. We write $C' = \langle \mathbf{x}, P, \mathbf{y}, R, \mathbf{u}, \mathbf{x} \rangle$ with $d_P(\mathbf{x}, \mathbf{y}) = l$ and $d_R(\mathbf{y}, \mathbf{x}) = k - (2^{n-1} - 1) - l - 2$. By induction, there is a (2^{n-1})-cycle C'' of Q_n^1 with $d_{C''}(\bar{\mathbf{x}}, \bar{\mathbf{u}}) = 1$. We write $C'' = \langle \bar{\mathbf{x}}, \bar{\mathbf{u}}, S, \bar{\mathbf{x}} \rangle$ with $d_S(\bar{\mathbf{u}}, \bar{\mathbf{x}}) = 2^{n-1} - 1$. Then $C = \langle \mathbf{x}, P, \mathbf{y}, R, \mathbf{u}, \bar{\mathbf{u}}, S, \bar{\mathbf{x}}, \mathbf{x} \rangle$ forms a k-cycle of Q_n with $d_C(\mathbf{x}, \mathbf{y}) = l$.

Case 2: $d_{Q_n}(\mathbf{x}, \mathbf{y}) \geq 2$ and $l = 2$. Since $d_{Q_n}(\mathbf{x}, \mathbf{y}) \leq l$ and $l = 2$, so $d_{Q_n}(\mathbf{x}, \mathbf{y}) = 2$. Without loss of generality, we may assume that $\mathbf{x}$ is in Q_n^0 and $\mathbf{y}$ is in Q_n^1. Then $d_{Q_n}(\bar{\mathbf{x}}, \mathbf{y}) = 1$ and $d_{Q_n}(\bar{\mathbf{y}}, \mathbf{x}) = 1$.

Suppose that $k = 4$. Then $C = \langle \mathbf{x}, \bar{\mathbf{x}}, \mathbf{y}, \bar{\mathbf{y}}, \mathbf{x} \rangle$ forms a 4-cycle of Q_n with $d_{Q_n}(\mathbf{x}, \mathbf{y}) = 2$. Suppose that $6 \leq k \leq 2^{n-1} + 2$. By induction, there is a $(k-2)$-cycle $C' = \langle \mathbf{x}, P, \bar{\mathbf{y}}, \mathbf{x} \rangle$ of Q_n^0 such that $d_P(\mathbf{x}, \bar{\mathbf{y}}) = k - 3$. Then $C = \langle \mathbf{x}, P, \bar{\mathbf{y}}, \mathbf{y}, \bar{\mathbf{x}}, \mathbf{x} \rangle$ forms a k-cycle of Q_n with $d_C(\mathbf{x}, \mathbf{y}) = 2$. Suppose that $k \geq 2^{n-1} + 4$. By induction, there is a 2^{n-1}-cycle C' of Q_n^0 with $d_{C'}(\mathbf{x}, \bar{\mathbf{y}}) = 1$. We write $C' = \langle \mathbf{x}, P, \mathbf{z}, \bar{\mathbf{y}}, \mathbf{x} \rangle$ with $d_P(\mathbf{x}, \mathbf{z}) = 2^{n-1} - 2$. By induction, there is a $(k - 2^{n-1})$-cycle C'' of Q_n^1 with $d_{C''}(\mathbf{y}, \bar{\mathbf{z}}) = 1$. We write $C'' = \langle \mathbf{y}, \bar{\mathbf{z}}, R, \mathbf{y} \rangle$ with $d_R(\mathbf{y}, \bar{\mathbf{z}}) = k - 2^{n-1} - 1$. Then $C = \langle \mathbf{x}, P, \mathbf{z}, \bar{\mathbf{z}}, R, \mathbf{y}, \bar{\mathbf{y}}, \mathbf{x} \rangle$ forms a k-cycle of Q_n with $d_C(\mathbf{x}, \mathbf{y}) = 2$.

Case 3: $d_{Q_n}(\mathbf{x}, \mathbf{y}) \geq 2$ and $l \geq 3$. Without loss of generality, we may assume that $\mathbf{x}$ is in Q_n^0 and $\mathbf{y}$ is in Q_n^1. Suppose that $k - l - d_{Q_n}(\mathbf{x}, \mathbf{y}) + 2 \leq 2^{n-1}$. By induction, there is an $(l + d_{Q_n}(\mathbf{x}, \mathbf{y}) - 2)$-cycle $C' = \langle \mathbf{x}, P, \bar{\mathbf{y}}, \mathbf{u}, R, \mathbf{x} \rangle$ of Q_n^0 such that $d_P(\mathbf{x}, \bar{\mathbf{y}}) = l - 1$ and $d_R(\mathbf{u}, \mathbf{x}) = d_{Q_n}(\mathbf{x}, \mathbf{y}) - 2$. For $k - l - d_{Q_n}(\mathbf{x}, \mathbf{y}) + 2 \leq 2$, by induction, there is a $(k - l - d_{Q_n}(\mathbf{x}, \mathbf{y}) + 2)$-cycle C'' of Q_n^1 with $d_{C''}(\mathbf{y}, \bar{\mathbf{u}}) = 1$. We write $C'' = \langle \mathbf{y}, S, \bar{\mathbf{u}}, \mathbf{y} \rangle$ with $d_S(\mathbf{y}, \bar{\mathbf{u}}) = k - l - d_{Q_n}(\mathbf{x}, \mathbf{y}) + 1$. We then set $C = \langle \mathbf{x}, P, \bar{\mathbf{y}}, \mathbf{y}, \bar{\mathbf{u}}, \mathbf{u}, R, \mathbf{x} \rangle$ if $k - l - d_{Q_n}(\mathbf{x}, \mathbf{y}) + 2 = 2$ or $C = \langle \mathbf{x}, P, \bar{\mathbf{y}}, \mathbf{y}, S, \bar{\mathbf{u}}, \mathbf{u}, R, \mathbf{x} \rangle$ if $k - l - d_{Q_n}(\mathbf{x}, \mathbf{y}) + 2 \leq 4$. Then C forms a k-cycle of Q_n with $d_C(\mathbf{x}, \mathbf{y}) = l$. Suppose that $k - l - d_{Q_n}(\mathbf{x}, \mathbf{y}) + 4 \geq 2^{n-1}$. By induction, there is a $(k - 2^{n-1})$-cycle $C' = \langle \mathbf{x}, P, \bar{\mathbf{y}}, \mathbf{u}, R, \mathbf{x} \rangle$ of Q_n^0 such that $d_P(\mathbf{x}, \bar{\mathbf{y}}) = l - 1$ and $d_R(\mathbf{u}, \mathbf{x}) = k - 2^{n-1} - l$. By induction, there is a 2^{n-1}-cycle C'' of Q_n^1 with $d_{C''}(\mathbf{y}, \bar{\mathbf{u}}) = 1$. We write $C'' = \langle \mathbf{y}, S, \bar{\mathbf{u}}, \mathbf{y} \rangle$ with $d_S(\mathbf{y}, \bar{\mathbf{u}}) = 2^{n-1} - 1$. Then $C = \langle \mathbf{x}, P, \bar{\mathbf{y}}, \mathbf{y}, S, \bar{\mathbf{u}}, \mathbf{u}, R, \mathbf{x} \rangle$ forms a k-cycle of Q_n with $d_C(\mathbf{x}, \mathbf{y}) = l$.

The theorem is proved. □

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Conditional fault hamiltonian connectivity of the complete graph

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ABSTRACT

A path in G is a hamiltonian path if it contains all vertices of G. A graph G is hamiltonian connected if there exists a hamiltonian path between any two distinct vertices of G. The degree of a vertex u in G is the number of vertices of G adjacent to u. We denote by $\delta(G)$ the minimum degree of vertices of G. A graph G is conditional k edge-fault tolerant hamiltonian connected if $G - F$ is hamiltonian connected for every $F \subset E(G)$ with $|F| \leqslant k$ and $\delta(G - F) \geqslant 3$. The conditional edge-fault tolerant hamiltonian connectivity $\mathcal{HC}_e^3(G)$ is defined as the maximum integer k such that G is k edge-fault tolerant conditional hamiltonian connected if G is hamiltonian connected and is undefined otherwise. Let $n \geqslant 4$. We use K_n to denote the complete graph with n vertices. In this paper, we show that $\mathcal{HC}_e^3(K_n) = 2n - 10$ for $n \notin \{4, 5, 8, 10\}$, $\mathcal{HC}_e^3(K_4) = 0$, $\mathcal{HC}_e^3(K_5) = 2$, $\mathcal{HC}_e^3(K_8) = 5$, and $\mathcal{HC}_e^3(K_{10}) = 9$.

1. Introduction

For the graph definitions and notations, we follow [1]. Let $G = (V, E)$ be a graph if V is a finite set and E is a subset of $\{(u, v) \mid (u, v)$ is an unordered pair of $V\}$. We say that V is the *vertex set* and E is the *edge set*. Two vertices u and v are *adjacent* if $(u, v) \in E$. The *complete graph* K_n is the graph with n vertices such that any two distinct vertices are adjacent. The *degree* of a vertex u in G, denoted by $\deg_G(u)$, is the number of vertices adjacent to u. We use $\delta(G)$ to denote $\min\{\deg_G(u) \mid u \in V(G)\}$. A *path* of length $m - 1$, $\langle v_0, v_1, \ldots, v_{m-1}\rangle$, is an ordered list of distinct vertices such that v_i and v_{i+1} are adjacent for $0 \leqslant i \leqslant m - 2$. We also write the path $\langle v_0, \ldots, v_k, P, v_l, \ldots, v_m\rangle$ for $P = \langle v_k, \ldots, v_l\rangle$. A *cycle* is a path with at least three vertices such that the first vertex is the same as the last one. A *hamiltonian cycle* of G is a cycle that traverses every vertex of G exactly once. A graph is *hamiltonian* if it has a hamiltonian cycle. A *hamiltonian path* is a path of length $V(G) - 1$.

A hamiltonian graph G is k *edge-fault tolerant hamiltonian* if $G - F$ remains hamiltonian for every $F \subset E(G)$ with $|F| \leqslant k$. The *edge-fault tolerant hamiltonicity*, $\mathcal{H}_e(G)$, is defined as the maximum integer k such that G is k edge-fault tolerant hamiltonian if G is hamiltonian and is undefined otherwise. Assume that G is a hamiltonian graph, and x is a vertex such that $\deg_G(x) = \delta(G)$. We arbitrary choose $\deg_G(x) - 1$ edges from those edges incident to x to form an edge faulty set F. Obviously, $\deg_{G-F}(x) = 1$; hence, $G - F$ is not hamiltonian. Therefore, $\mathcal{H}_e(G) \leqslant \delta(G) - 2$ if $\mathcal{H}_e(G)$ is defined. Assume that n is an integer with $n \geqslant 3$. It is proved by Ore [9] that any n-vertex graph with at least $C(n, 2) - (n - 3)$ edges is hamiltonian. Moreover, there exists a non-hamiltonian n-vertex graph with $C(n, 2) - (n - 2)$ edges. In other words, $\mathcal{H}_e(K_n) = n - 3$ for $n \geqslant 3$. In [5], it is proved that $\mathcal{H}_e(Q_n) = n - 2$ for $n \geqslant 2$ where Q_n is the n-dimensional hypercube. In [6], it is proved that $\mathcal{H}_e(S_n) = n - 3$ for $n \geqslant 3$ where S_n is the n-dimensional star graph.

Chan and Lee [2] began the study of the existence of hamiltonian cycle in a graph such that each vertex is incident to at least two fault-free edges. A graph G is *conditional k edge-fault tolerant hamiltonian* if $G - F$ is hamilto-

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nian for every $F \subset E(G)$ with $|F| \leqslant k$ and $\delta(G-F) \geqslant 2$. The *conditional edge-fault tolerant hamiltonicity*, $\mathcal{H}_e^2(G)$, is defined as the maximum integer k such that G is conditional k edge-fault tolerant hamiltonian if G is hamiltonian and is undefined otherwise. Chan and Lee [2] proved that $\mathcal{H}_e^2(Q_n) = 2n-5$ for $n \geqslant 3$. Recently, Fu [3] studies the conditional edge-fault tolerant hamiltonicity of the complete graph.

Fault tolerant hamiltonian connectivity is another important parameter for graphs [4]. A graph G is *hamiltonian connected* if there exists a hamiltonian path between any two distinct vertices of G. It is easy to see that a hamiltonian connected graph with at least three vertices is hamiltonian. It is proved by Moon [7] that the degree of any vertex in a hamiltonian connected graph with at least four vertices is at least 3. A graph G is *k edge-fault tolerant hamiltonian connected* if $G-F$ remains hamiltonian connected for any $F \subset E(G)$ with $|F| \leqslant k$. The *edge-fault tolerant hamiltonian connectivity* of a graph G, $\mathcal{HC}_e(G)$, is defined as the maximum integer k such that G is k edge-fault tolerant hamiltonian connected if G is hamiltonian connected and is undefined otherwise. Assume that G is a hamiltonian connected graph with at least four vertices and x is a vertex such that $\deg_G(x) = \delta(G)$. We arbitrary choose $\deg_G(x) - 2$ edges from those edges incident to x to form an edge faulty set F. Obviously, $\deg_{G-F}(x) = 2$; hence, $G-F$ is not hamiltonian connected. Therefore, $\mathcal{HC}_e(G) \leqslant \delta(G) - 3$ if $\mathcal{HC}_e(G)$ is defined. Again, Ore [8] proved that $\mathcal{HC}_e(K_n) = n-4$ for $n \geqslant 4$.

In this paper, we study the concept of conditional edge-fault tolerant hamiltonian connectivity. Since the degree of any vertex in a hamiltonian connected graph with at least four vertices is at least 3, it is natural to assume that each vertex is incident to at least three fault-free edges. A graph G is *conditional k edge-fault tolerant hamiltonian connected* if $G-F$ is hamiltonian connected for every $F \subset E(G)$ with $|F| \leqslant k$ and $\delta(G-F) \geqslant 3$. The *conditional edge-fault tolerant hamiltonian connectivity*, $\mathcal{HC}_e^3(G)$, is defined to be the maximum integer k such that G is conditional k edge-fault tolerant hamiltonian connected if G is hamiltonian connected and is undefined otherwise.

Assume that n is an integer with $n \geqslant 4$. In this paper, we prove that $\mathcal{HC}_e^3(K_n) = 2n-10$ for $n \notin \{4, 5, 8, 10\}$, $\mathcal{HC}_e^3(K_4) = 0$, $\mathcal{HC}_e^3(K_5) = 2$, $\mathcal{HC}_e^3(K_8) = 5$, and $\mathcal{HC}_e^3(K_{10}) = 9$. To reach this goal, we present some preliminary in the following section. In Section 3, we prove our main result.

2. Preliminary

Let F be a faulty edge set. We define $K_n(F)$ be a graph with $E(K_n(F)) = F$ and $V(K_n(F)) = V(K_n)$. The following statement is proved in [3]:

Suppose $F \subset E(K_n)$ and $\delta(K_n - F) \geqslant 2$, where $n \geqslant 4$. If $n \notin \{7, 9\}$ (respectively, $n \in \{7, 9\}$) then $K_n - F$ is hamiltonian, where $|F| \leqslant 2n-8$ (respectively, $|F| \leqslant 2n-9$).

In the conclusion of [3], it is claimed that the above statement is optimal. Using our terminology, we obtain the following statement.

$\mathcal{H}_e^2(K_n) = 2n-8$ for $n \notin \{7, 9\}$ and $n \geqslant 4$, $\mathcal{H}_e^2(K_7) = 5$, and $\mathcal{H}_e^2(K_9) = 9$.

Yet, it is easy to check that $\mathcal{H}_e^2(K_3)$ is 0 and $\mathcal{H}_e^2(K_4)$ is 2 (not 0.) Thus, we have the following theorem.

Theorem 1. *$\mathcal{H}_e^2(K_n) = 2n-8$ for $n \notin \{7, 9\}$ and $n \geqslant 5$, $\mathcal{H}_e^2(K_3) = 0$, $\mathcal{H}_e^2(K_4) = 2$, $\mathcal{H}_e^2(K_7) = 5$, and $\mathcal{H}_e^2(K_9) = 9$.*

Lemma 1. *Assume that n is an integer with $n \geqslant 6$ and F is any subset of $E(K_n)$ with $|F| = 2n-10$ if $n \notin \{8, 10\}$ and $|F| = 2n-11$ if $n \in \{8, 10\}$. There exists a vertex w in $K_n(F)$ such that $1 \leqslant \deg_{K_n(F)}(w) \leqslant \lfloor \frac{n-1}{2} \rfloor - 1$.*

Proof. Suppose that the lemma is false. Then $\deg_{K_n(F)}(w) \geqslant \lfloor \frac{n-1}{2} \rfloor$ for every vertices with $\deg_{K_n(F)}(w) \neq 0$. Obviously, there are at least $\lfloor \frac{n-1}{2} \rfloor + 1$ vertices with $\deg_{K_n(F)}(w) \neq 0$. Hence, $|F| \geqslant (\lfloor \frac{n-1}{2} \rfloor (\lfloor \frac{n-1}{2} \rfloor + 1))/2$. However, $(\lfloor \frac{n-1}{2} \rfloor \times (\lfloor \frac{n-1}{2} \rfloor + 1))/2 > 2n-10$ for $n \notin \{8, 10\}$ and $(\lfloor \frac{n-1}{2} \rfloor (\lfloor \frac{n-1}{2} \rfloor + 1))/2 > 2n-11$ for $n \in \{8, 10\}$. It is a contradiction. The lemma is proved. □

The following theorem can be found in [1].

Theorem 2. *(See [1].) Let $D = (d_1, d_2, \ldots, d_n)$ be a nonincreasing sequence with $d_1 \geqslant 1$ and $d_i \geqslant 0$ for $2 \leqslant i \leqslant n$. We set $D' = (d'_1, d'_2, \ldots, d'_{n-1}) = (d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n)$. Then there exists a graph G with vertex set $\{x_1, x_2, \ldots, x_n\}$ such that $\deg_G(x_i) = d_i$ for $1 \leqslant i \leqslant n$ if and only if there exists a graph G' with vertex set $\{y_1, y_2, \ldots, y_{n-1}\}$ such that $\deg_{G'}(y_j) = d'_j$ for $1 \leqslant j \leqslant n-1$.*

By the above theorem, we know that there is a graph G with degree sequence D if and only if there is a graph G' with degree sequence D'. If $d'_i < 0$ for some i, then D' is not the degree sequence of any graph, neither is D.

Lemma 2. *Let F be a subset of $E(K_9)$ with $|F| = 8$ and $\delta(K_9 - F) \geqslant 3$. Let u and v be any two distinct vertices in K_9 such that $\deg_{K_9(F)}(u) = 0$ and $\deg_{K_9(F)}(v) = 0$. Then there exists a vertex w with $\deg_{K_9(F)}(w) \in \{2, 3\}$.*

Proof. Let $\{x_1, x_2, \ldots, x_8 = u, x_9 = v\}$ be the vertex set of K_9 such that $\deg_{K_9(F)}(x_i) = d_i$ and $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_9$. Obviously, $\sum_{i=1}^{9} d_i = 16$. Assume that the lemma is false. Then $\deg_{K_9(F)}(x_i) \in \{0, 1, 4, 5\}$ for $1 \leqslant i \leqslant 9$. By brute force, all such sequences are listed below: $(5, 5, 5, 1, 0, 0, 0, 0, 0)$, $(5, 5, 4, 1, 1, 0, 0, 0, 0)$, $(5, 4, 4, 1, 1, 1, 0, 0, 0)$, $(4, 4, 4, 4, 0, 0, 0, 0, 0)$, and $(4, 4, 4, 1, 1, 1, 1, 0, 0)$. By Theorem 2, we can check that such a graph does not exist. Hence, the lemma is proved. □

Lemma 3. *Let F be a subset of $E(K_{11})$ with $|F| = 12$ and $\delta(K_{11} - F) \geqslant 3$. Let u and v be any two distinct vertices in K_{11} such that $\deg_{K_{11}(F)}(u) = 0$ and $\deg_{K_{11}(F)}(v) = 0$. Then there exists a vertex w with $\deg_{K_{11}(F)}(w) \in \{2, 3, 4\}$.*

Proof. Let $\{x_1, x_2, \ldots, x_{10} = u, x_{11} = v\}$ be the vertex set of K_{11} such that $\deg_{K_{11}(F)}(x_i) = d_i$ and $d_1 \geqslant d_2 \geqslant \cdots \geqslant$

d_{11}. Obviously, $\sum_{i=1}^{11} d_i = 24$. Assume that the lemma is false. Then $\deg_{K_{11}(F)}(x_i) \in \{0, 1, 5, 6, 7\}$ for $1 \leqslant i \leqslant 11$. By brute force, all such sequences are listed below: $(7, 7, 7, 1, 1, 1, 0, 0, 0, 0, 0)$, $(7, 7, 6, 1, 1, 1, 1, 0, 0, 0, 0)$, $(7, 7, 5, 5, 0, 0, 0, 0, 0, 0, 0)$, $(7, 7, 5, 1, 1, 1, 1, 1, 0, 0, 0)$, $(7, 6, 6, 5, 0, 0, 0, 0, 0, 0, 0)$, $(7, 6, 6, 1, 1, 1, 1, 1, 0, 0, 0)$, $(7, 6, 5, 5, 1, 0, 0, 0, 0, 0, 0)$, $(7, 6, 5, 1, 1, 1, 1, 1, 1, 0, 0)$, $(7, 5, 5, 5, 1, 1, 0, 0, 0, 0, 0)$, $(6, 6, 6, 6, 0, 0, 0, 0, 0, 0, 0)$, $(6, 6, 6, 5, 1, 0, 0, 0, 0, 0, 0)$, $(6, 6, 6, 1, 1, 1, 1, 1, 1, 0, 0)$, $(6, 6, 5, 5, 1, 1, 0, 0, 0, 0, 0)$, $(6, 5, 5, 5, 1, 1, 1, 0, 0, 0, 0)$, and $(5, 5, 5, 5, 1, 1, 1, 1, 0, 0, 0)$. By Theorem 2, we can check that such a graph does not exist. The lemma is proved. □

We can easily obtain the following lemma.

Lemma 4. *Let $k \geqslant 2$. Let G be a hamiltonian connected graph. Then deleting any set S of k vertices from G, the resulting graph $G - S$ contains at most $k - 1$ connected components.*

By the above lemma, we have a simple observation.

Lemma 5. *Let $k \geqslant 2$. Let G be a graph. If there is a set S of k vertices such that $G - S$ contains k or more connected components, then G is not hamiltonian connected.*

3. Main result

Lemma 6. *Let $n \geqslant 4$ and $F \subset E(K_n)$ with $\delta(K_n - F) \geqslant 3$. Then $K_n - F$ is hamiltonian connected if $|F| \leqslant 2n - 10$ for $n \notin \{4, 5, 8, 10\}$, $|F| = 0$ for $n = 4$, $|F| \leqslant 2$ for $n = 5$, and $|F| \leqslant 2n - 11$ for $n \in \{8, 10\}$.*

Proof. We prove this lemma by induction on n. Yet, we should be very careful because the size of $|F|$ is depending on n. Without loss of generality, we assume that $|F| = 2n - 10$ for $n \notin \{4, 5, 8, 10\}$, $|F| = 0$ for $n = 4$, $|F| = 2$ for $n = 5$, and $|F| = 2n - 11$ for $n \in \{8, 10\}$. The induction bases are $n = 4$, $n = 5$, and $n = 6$. Suppose $n = 4$ and $|F| = 0$. It is easy to see that the complete graph K_4 is hamiltonian connected. Suppose $n = 5$ and $|F| = 2$. To keep $\delta(K_5 - F) \geqslant 3$, F forms two independent edges. By brute force, it is easy to check whether $K_5 - F$ is hamiltonian connected. Suppose that $n = 6$ and $|F| = 2$. Obviously, F is either two adjacent edges or two independent edges. Again, by brute force, we can check that $K_6 - F$ is hamiltonian connected.

Now, we assume that $n \geqslant 7$. Let u and v be any two vertices of K_n. The lemma follows if we can find a hamiltonian path of $K_n - F$ between u and v.

Case 1. $\deg_{K_n(F)}(u) \neq 0$ or $\deg_{K_n(F)}(v) \neq 0$. Without loss of generality, we assume that $\deg_{K_n(F)}(u) = k \neq 0$. Let $i_1, \ldots, i_k$ be the vertices such that $(u, i_j) \in F$ for $1 \leqslant j \leqslant k$. Let $F' = (F - \{(u, i_1), \ldots, (u, i_k)\}) \cup \{(v, i_1), \ldots, (v, i_k)\}$. Obviously, $|F'| \leqslant |F|$. Now, we consider $K_n - \{u\}$ as a complete graph of $(n-1)$ vertices with faulty edge set F'. Obviously, $|F'| \leqslant 2(n-1) - 8$ for $n \notin \{8, 10\}$ and $|F'| \leqslant 2(n-1) - 9$ for $n \in \{8, 10\}$. Moreover, $\delta(K_n - \{u\} - F') \geqslant 2$. Thus, we can apply Theorem 1 to obtain a hamiltonian cycle C in $K_n - \{u\} - F'$. Without loss of generality, we write C as $\langle v, x, \ldots, y, v\rangle$. Then, $\langle u, x, \ldots, y, v\rangle$ forms a hamiltonian path of $K_n - F$ joining u to v.

Case 2. $\deg_{K_n(F)}(u) = 0$ and $\deg_{K_n(F)}(v) = 0$. By Lemmas 1, 2, and 3, there exists a vertex w such that $2 \leqslant \deg_{K_n(F)}(w) \leqslant \lfloor \frac{n-1}{2} \rfloor - 1$ for $n \in \{9, 11\}$ and $1 \leqslant \deg_{K_n(F)}(w) \leqslant \lfloor \frac{n-1}{2} \rfloor - 1$ for $n \notin \{9, 11\}$.

Obviously, $\delta(K_n - F - \{w\}) \geqslant 2$. Suppose that $\delta(K_n - F - \{w\}) = 2$. Let x be any vertex in $K_n - \{w\}$ such that $\deg_{K_n - \{w\} - F}(x) = 2$. Obviously, $(x, w) \notin F$, $\deg_{K_n - F}(x) = 3$, and $\deg_{K_n(F)}(x) = n - 4$. We claim that x is the only vertex in $K_n - \{w\}$ with $\deg_{K_n - \{w\} - F}(x) = 2$. If otherwise, let z be another vertex in $K_n - \{w\}$ with $\deg_{K_n - \{w\} - F}(z) = 2$. Then $|F| \geqslant \deg_{K_n(F)}(x) + \deg_{K_n(F)}(z) - 1 = 2n - 9$. This is impossible because $|F| \leqslant 2n - 10$. Thus, x is the only vertex in $K_n - \{w\}$ such that $\deg_{K_n - \{w\} - F}(x) = 2$. Thus, $\delta(K_n - F - \{u, x\}) \geqslant 3$.

Let $F' = F - \{(x, i) \mid i \in V(K_n)\}$. We consider $K_n - \{u, x\}$ as a complete graph of $(n-2)$ vertices with faulty edge set F'. Obviously, $|F'| = 1 \leqslant 2$ for $n = 7$, $|F'| = n - 7 \leqslant 2(n-2) - 10$ for $n \notin \{10, 12\}$, and $|F'| = n - 7 \leqslant 2(n-2) - 11$ for $n \in \{10, 12\}$. By induction, we have a hamiltonian path P of $K_n - \{u, x\} - F'$ joining w to v. So $\langle u, x, w, P, v\rangle$ forms a hamiltonian path of $K_n - F$ joining u to v.

Now, we consider $\delta(K_n - \{w\} - F) \geqslant 3$. Since $2 \leqslant \deg_{K_n(F)}(w) \leqslant \lfloor \frac{n-1}{2} \rfloor - 1$ for $n \in \{9, 11\}$ and $1 \leqslant \deg_{K_n(F)}(w) \leqslant \lfloor \frac{n-1}{2} \rfloor - 1$ for $n \notin \{9, 11\}$, there exists $(x, y) \in F$ such that $\{(w, x), (w, y)\} \cap F = \emptyset$. We set F' as $F - \{(w, z) \mid (w, z) \in F\} - \{(x, y)\}$ and consider $K_n - \{w\}$ with faulty set F'. We have $|F'| = 2n - 10 - \deg_{K_n(F)}(w) - 1 \leqslant 2n - 13$ for $n \in \{9, 11\}$ and $|F'| = 2n - 10 - \deg_{K_n(F)}(w) - 1 \leqslant 2n - 12$ for $n \notin \{9, 11\}$. By induction, there exists a hamiltonian path $P = \langle u = x_1, x_2, \ldots, x_{n-1} = v\rangle$ of $K_n - \{w\} - F'$ joining u to v. Suppose that $(x, y) \in P$. There exists an integer i such that $\{x_i, x_{i+1}\} = \{x, y\}$ for some i. Suppose that $(x, y) \notin P$. Since $\deg_{K_n(F)}(w) \leqslant \lfloor \frac{n-1}{2} \rfloor - 1$ and $\deg_{K_n(F)}(w) + \deg_{K_n - F}(w) = n - 1$, $\deg_{K_n - F}(w) \geqslant \lfloor \frac{n}{2} \rfloor + 1$. Hence, there exists an integer i such that $(x_i, x_{i+1}) \in P$ and $\{(w, x_i), (w, x_{i+1})\} \cap F = \emptyset$. Therefore, $\langle u = x_1, x_2, \ldots, x_i, w, x_{i+1}, x_{i+2}, \ldots, v\rangle$ forms a hamiltonian path of $K_n - F$ joining u to v. □

Theorem 3. *Let $n \geqslant 4$. Then $\mathcal{HC}_e^3(K_n) = 2n - 10$ for $n \notin \{4, 5, 8, 10\}$, $\mathcal{HC}_e^3(K_4) = 0$, $\mathcal{HC}_e^3(K_5) = 2$, $\mathcal{HC}_e^3(K_8) = 5$, and $\mathcal{HC}_e^3(K_{10}) = 9$.*

Proof. Let F be any subset of $E(K_n)$ with $\delta(K_n - F) \geqslant 3$. Since $\delta(K_n - F) \geqslant 3$, $|F| = 0$ for $n = 4$ and $|F| \leqslant 2$ for $n = 5$. Thus, $\mathcal{HC}_e^3(K_4) = 0$ and $\mathcal{HC}_e^3(K_5) = 2$.

Suppose $n = 8$. Let $V(K_8) = \{x_1, x_2, \ldots, x_8\}$. We set $R = \{x_1, \ldots, x_4\}$, $S = \{x_5, \ldots, x_8\}$, and $F = \{(u, v) \mid u, v \in R\}$. We can check that $\delta(K_8 - F) \geqslant 3$, $|F| = 6$ and $(K_8 - F) - S$ has four connected components. By Lemma 5, $K_8 - F$ is not hamiltonian connected. See Fig. 1(a) for illustration. Thus, $\mathcal{HC}_e^3(K_8) < 6$. By Lemma 6, $\mathcal{HC}_e^3(K_8) = 5$.

Suppose $n = 10$. Let $V(K_{10}) = \{x_1, x_2, \ldots, x_{10}\}$. We set $R = \{x_1, \ldots, x_5\}$, $S = \{x_6, \ldots, x_{10}\}$, and $F = \{(u, v) \mid u, v \in$

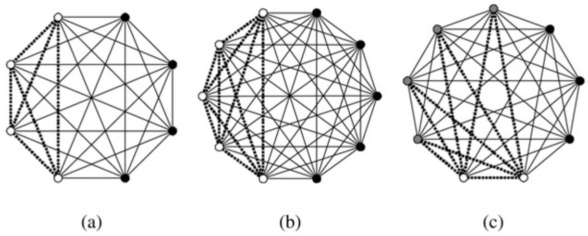

Fig. 1. All white vertices are in R, all black vertices are in S, and all gray vertices are in T. All dashed lines are in F.

$R\}$. Then, $\delta(K_{10} - F) \geqslant 3$, $|F| = 10$, and $(K_{10} - F) - S$ has five connected components. By Lemma 5, $K_{10} - F$ is not hamiltonian connected. See Fig. 1(b) for illustration. Thus, $\mathcal{HC}_e^3(K_{10}) < 10$. By Lemma 6, $\mathcal{HC}_e^3(K_{10}) = 9$.

Suppose that $n \in \{6, 7, 9\} \cup \{i \mid i \geqslant 11\}$. Let $V(K_n) = \{x_1, x_2, \ldots, x_n\}$. We set $R = \{x_1, x_2\}$, $S = \{x_3, x_4, x_5\}$, $T = \{x_6, \ldots, x_n\}$, and $F = \{(u, v) \mid u \in R, v \in R \cup T\}$. Obviously, $\delta(K_n - F) \geqslant 3$, $|F| = 2(n-5)+1 = 2n-9$, and $(K_n - F) - S$ has three connected components. See Fig. 1(c) for illustration for case $n = 9$. By Lemma 5, $K_n - F$ is not hamiltonian connected. Thus, $\mathcal{HC}_e^3(K_n) < 2n - 9$. By Lemma 6, $\mathcal{HC}_e^3(K_n) = 2n - 10$.

The theorem is proved. □

References

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附件五

國科會補助專題研究計畫項下赴國外(或大陸地區)出差或研習心得報告

日期：99年7月20日

計畫編號	NSC 96－2221－E－009－137－MY3		
計畫名稱	連結網路上的連通性相關之研究（第 3 年）		
出國人員姓名	譚建民	服務機構及職稱	國立交通大學資訊工程學系教授
出國時間	99 年 7 月 12 日至 99 年 7 月 16 日	出國地點	美國

一、 國外(大陸)研究過程

這次出國是參加在美國舉辦的 The 2010 International Conference on Parallel and Distributed Processing Techniques and Applications。會議日期 2010 七月 12-16。整個會議共四天，每天都有 keynote lecture 及 tutorials。其中 Keynote lecture U.C. Berkeley 的教授 Lotfi A. Zadeh，講題 Computing with Words and Perceptions。Zadeh 教授是 FuzzyTheory 的開創者，我的系上同事孫春在教授在 Berkeley 的論文指導教授。聽他的演講，增廣見識，並獲得啟發。

二、 研究成果

本人與博士生林政寬發表了一篇論文。

Cheng-Kuan Lin, Tzu-Liang Kung, Shao-Lun Peng, Jimmy J.M. Tan and Lih-Hsing Hsu “The Diagnosability of g-good-neighbor Conditional-Faulty Hypercube under PMC Model”, Proceedings of the 2101 International Conference on Parallel and Distributed Processing Techniques and Applications, Volume 2 pp. 494-499.

三、 建議

無。

四、 其他

會議中有機會能夠與來自各國的學者交流，擴展自己的視野，增加研究動力。

無研發成果推廣資料

96 年度專題研究計畫研究成果彙整表

<table>
<tr><td colspan="3">計畫主持人：譚建民</td><td colspan="5">計畫編號：96-2221-E-009-137-MY3</td></tr>
<tr><td colspan="8">計畫名稱：連結網路上的連通性相關之研究</td></tr>
<tr><td colspan="3" rowspan="2">成果項目</td><td colspan="3">量化</td><td rowspan="2">單位</td><td rowspan="2">備註（質化說明：如數個計畫共同成果、成果列為該期刊之封面故事...等）</td></tr>
<tr><td>實際已達成數（被接受或已發表）</td><td>預期總達成數(含實際已達成數)</td><td>本計畫實際貢獻百分比</td></tr>
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<tr><td>專任助理</td><td>0</td><td>0</td><td>100%</td><td></td></tr>
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（外國籍）</td><td>碩士生</td><td>0</td><td>0</td><td>100%</td><td rowspan="4">人次</td><td></td></tr>
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<tr><td>專任助理</td><td>0</td><td>0</td><td>100%</td><td></td></tr>
</table>

其他成果 (無法以量化表達之成果如辦理學術活動、獲得獎項、重要國際合作、研究成果國際影響力及其他協助產業技術發展之具體效益事項等，請以文字敘述填列。)	無

	成果項目	量化	名稱或內容性質簡述
科教處計畫加填項目	測驗工具(含質性與量性)	0	
	課程/模組	0	
	電腦及網路系統或工具	0	
	教材	0	
	舉辦之活動/競賽	0	
	研討會/工作坊	0	
	電子報、網站	0	
	計畫成果推廣之參與（閱聽）人數	0	

國科會補助專題研究計畫成果報告自評表

請就研究內容與原計畫相符程度、達成預期目標情況、研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）、是否適合在學術期刊發表或申請專利、主要發現或其他有關價值等，作一綜合評估。

1. 請就研究內容與原計畫相符程度、達成預期目標情況作一綜合評估 ■達成目標 □未達成目標（請說明，以 100 字為限） 　　□實驗失敗 　　□因故實驗中斷 　　□其他原因 說明：
2. 研究成果在學術期刊發表或申請專利等情形： 論文：■已發表 □未發表之文稿 □撰寫中 □無 專利：□已獲得 □申請中 ■無 技轉：□已技轉 □洽談中 ■無 其他：（以 100 字為限）
3. 請依學術成就、技術創新、社會影響等方面，評估研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）（以 500 字為限） 這次的計劃執行可以說是非常的成功，目前已經發表且刊登出來的論文已經有三篇，還有其它的論文也已經被接受。總括來說，這次三年計劃可以說完全達到我們預期的目標，更可說是已經超越我們預設的目標。