The Fading Number of a Multiple-Access Rician Fading Channel

Intermediate Report of NSC Project

"Capacity Analysis of Various Multiple-Antenna Multiple-Users Communication Channels with Joint Estimation and Detection"

Date:	31 May 2009
Project-Number:	NSC 96-2221-E-009-012-MY3
Project Duration:	1 August 2007 – 31 July 2010
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Abstract

The fading number of a noncoherent two-user Rician fading channel is derived. The fading number is the second term in the high-SNR expansion of the sum-rate capacity of this multiple-access channel. It is shown that the fading number is identical to the fading number of the single-user Rician fading channel that is obtained when the user seeing the worse channel is switched off.

Keywords: Channel capacity, fading number, flat fading, high signal-to-noise ratio (SNR), multiple-access.

Contents

1 Introduction

In a noncoherent fading channel where neither transmitter nor receiver know the fading realization, it has been shown in [1] that the capacity at high SNR behaves fundamentally different from the usual asymptotics seen in AWGN channels or in coherent fading channels: instead of a logarithmic growth in the SNR, the capacity only grows double-logarithmically. In detail, we have

$$\mathsf{C}(\text{SNR}) = \log(1 + \log(1 + \text{SNR})) + \chi + o(1) \tag{1}$$

where $o(1)$ denote terms that tend to zero as the SNR tends to infinity; and where χ is a constant independent of the SNR that is denoted *fading number*. The value of χ depends on the exact specifications of the fading law. Note that even though the fading number is defined only in the limit when the available SNR tends to infinity, it has practical relevance also for finite SNR because it is a good estimator for the threshold where the capacity changes from the normal logarithmic growth to the highly inefficient double-logarithmic growth.

In the situation of single-user fading channels with multiple antennas both at transmitter and receiver and with a fading process that may contain memory, a formula for the fading number has been derived in [2]. The present paper is a first step towards extending the setup to a multiple-user situation.

2 Channel Model and Previous Results

We consider a multiple-access channel with two independently transmitting users and one receiver. All terminals are assumed to have only one antenna, and the two fading channels from each user to the receiver are assumed to suffer from independent Rician fading. The channel output $Y \in \mathbb{C}$ can then be written as

$$\begin{aligned} Y &= H_1 x_1 + H_2 x_2 + Z && (2)\\ &= d_1 x_1 + \tilde{H}_1 x_1 + d_2 x_2 + \tilde{H}_2 x_2 + Z, && (3) \end{aligned}$$

where $x_i \in \mathbb{C}$ denotes the input of user i, $i = 1, 2$; where the random variables H_i describe Rician fading

$$\tilde{H}_i + d_i = H_i \sim \mathcal{N}_{\mathbb{C}}(d_i, 1), \qquad i = 1, 2, \tag{4}$$

(hence, $\tilde{H}_i$ are zero-mean, circularly symmetric, complex Gaussian random variables with variance 1) and are assumed to be independent

$$H_1 \perp\!\!\!\perp H_2; \tag{5}$$

and where $Z \sim \mathcal{N}_{\mathbb{C}}\bigl(0, \sigma^2\bigr)$ denotes additive, zero-mean, circularly symmetric Gaussian noise, independent from the fading vector $(H_1, H_2)^{\mathsf{T}}$.

We assume a noncoherent situation, *i.e.*, neither transmitter nor receiver have knowledge of the current fading realization, they only know the fading distributions. Moreover we assume two different types of input constraints: a peak-power constraint and an average-power constraint. We use $\mathcal{E}$ to denote the maximum allowed total instantaneous power in the former case, and to denote the allowed total average power in the latter case. This total power then still must be split and distributed among the users.

The sum-rate capacity $\mathsf{C}(\mathcal{E})$ of the channel (3) is given by

$$\mathsf{C}(\mathcal{E}) = \sup I(X_1, X_2; Y) \tag{6}$$

where the supremum is over the set of all probability distributions on $\mathbf{X} = (X_1, X_2)^{\mathsf{T}}$ for which both components are independent and satisfy the input constraint, *i.e.*,

$$|X_1|^2 + |X_2|^2 \leq \mathcal{E}, \quad \text{almost surely} \tag{7}$$

for a peak-power constraint, or

$$\mathsf{E}\bigl[|X_1|^2 + |X_2|^2\bigr] \leq \mathcal{E} \tag{8}$$

for an average-power constraint.

Note that the difference of this multiple-access channel (MAC) to the multiple-input single-output (MISO) fading channel with two transmit antennas and one receive antenna is that in the latter both transmit antennas can cooperate, while in the former they are assumed to be independent. Hence, it immediately follows from this that the MAC sum-rate capacity can be upper-bounded by the MISO capacity:

$$\mathsf{C}_{\text{MAC}}(\mathcal{E}) \leq \mathsf{C}_{\text{MISO}}(\mathcal{E}). \tag{9}$$

On the other hand, obviously the sum rate cannot be smaller than the single-user rate that can be achieved if the weaker of the two users is switched off, *i.e.*,

$$\mathsf{C}_{\text{MAC}}(\mathcal{E}) \geq \max_{i=1,2} \mathsf{C}_{\text{SISO},i}(\mathcal{E}). \tag{10}$$

Using this and specializing [1, Theorem 4.2] and [4, Theorem 6.10] to memoryless MISO and SISO fading, respectively, we get

$$\varlimsup_{\mathcal{E}\uparrow\infty} \left\{ \mathsf{C}_{\text{MAC}}(\mathcal{E}) - \log\log\frac{\mathcal{E}}{\sigma^2} \right\} < \infty. \tag{11}$$

We now define the *MAC fading number* accordingly (see [1, Definition 4.6], [4, Definition 6.13]) as

$$\chi_{\text{MAC}} \triangleq \varlimsup_{\mathcal{E}\uparrow\infty} \left\{ \mathsf{C}_{\text{MAC}}(\mathcal{E}) - \log\log\frac{\mathcal{E}}{\sigma^2} \right\}. \tag{12}$$

Prima facie the MAC fading number depends on whether a peak-power constraint (7) or an average-power constraint (8) is imposed on the input. However, it will turn out that the MAC fading number is identical for both cases.

From [1, Corollary 4.28] we know that

$$\chi_{\text{MISO}} = \log\left(|d_1|^2 + |d_2|^2\right) - \text{Ei}\left(-|d_1|^2 + |d_2|^2\right) - 1 \tag{13}$$

were $\text{Ei}(\cdot)$ is the exponential integral function defined as

$$\text{Ei}(-x) \triangleq -\int_x^\infty \frac{e^{-t}}{t}\,\text{d}t, \qquad x > 0. \tag{14}$$

Hence, from (9) we have

$$\chi_{\text{MAC}} \leq \chi_{\text{MISO}} = \log\left(|d_1|^2 + |d_2|^2\right) - \text{Ei}\left(-|d_1|^2 + |d_2|^2\right) - 1. \tag{15}$$

On the other hand from (10)

$$\chi_{\text{MAC}} \geq \max_i \chi_{i,\text{SISO}} = \max_i \left\{\log\left(|d_i|^2\right) - \text{Ei}\left(-|d_i|^2\right) - 1\right\}. \tag{16}$$

Using the monotonicity of $\xi \mapsto \log(\xi) - \text{Ei}(-\xi)$ we therefore see that

$$\chi_{\text{MAC}} = \log\left(d_{\text{MAC}}^2\right) - \text{Ei}\left(-d_{\text{MAC}}^2\right) - 1 \tag{17}$$

where we have introduced d_{MAC} to be a nonnegative real number satisfying

$$\max\{|d_1|, |d_2|\} \leq d_{\text{MAC}} \leq \sqrt{|d_1|^2 + |d_2|^2}. \tag{18}$$

In the following we will evaluate the exact value of d_{MAC}.

Note that from [5] we actually know that

$$d_{\text{MAC}} < \sqrt{|d_1|^2 + |d_2|^2} \tag{19}$$

with strict inequality.

3 Main Result

Theorem 1. *Assume a Rician fading channel as defined in Section 2. Then the sum-rate fading number is given by*

$$\chi_{\text{MAC}} = \log\left(d_{\text{MAC}}^2\right) - \text{Ei}\left(-d_{\text{MAC}}^2\right) - 1, \tag{20}$$

where

$$d_{\text{MAC}} = \max\{|d_1|, |d_2|\}. \tag{21}$$

This shows that the lower bound in (18) *is tight.*

Note that if the magnitude of the line-of-sight component of one user is strictly smaller than of the other user, then this sum rate can only be achieved if the user with the weaker $|d_i|$ is switched off. If both line-of-sight components have identical magnitudes, then the sum rate can be achieved by time-sharing.

Remark 2. Note that Theorem 1 continues to hold even if we do not allow power optimization over the users, *i.e.*, if we constrain the inputs to satisfy

$$|X_i|^2 \leq \frac{\mathcal{E}}{2}, \quad \text{almost surely}, \ \forall\, i. \tag{22}$$

4 Proof of Main Result

The proof of Theorem 1 consists of two parts. The first part is given already in (18). There it is shown that $\max\{|d_1|, |d_2|\}$ is a lower bound to d_{MAC}. Note that this lower bound can be achieved using an input that satisfies the peak-power constraint.

The second part will be to prove that $\max\{|d_1|, |d_2|\}$ is also an upper bound. We will prove this under the assumption of an average-power constraint. Since a peak-power constraint is more stringent than an average-power constraint, the result follows.

The proof of this upper bound relies strongly on a generalization of the concept of input distributions that escape to infinity as introduced in [1] and in [6].

Proposition 3. *Let $\{Q_{\mathcal{E}}\}_{\mathcal{E}>0}$ be a family of joint input distributions of the two-user MAC given in* (3) *parameterized by the available power $\mathcal{E}$, $\mathcal{E} > 0$, satisfying the average-power constraint* (8) *and*

$$\lim_{\mathcal{E}\uparrow\infty} \frac{I(Q_{\mathcal{E}}, W)}{\log\log\mathcal{E}} = 1. \tag{23}$$

Then at least one user's input distribution must escape to infinity, i.e., *for any $\mathcal{E}_0 > 0$,*

$$\lim_{\mathcal{E}\uparrow\infty} Q_{\mathcal{E}}\left(\left\{|X_1|^2 \geq \frac{\mathcal{E}_0}{2}\right\} \cup \left\{|X_2|^2 \geq \frac{\mathcal{E}_0}{2}\right\}\right) = 1. \tag{24}$$

Proof. See the appendix. □

Using [1, Eq. (25)] for our channel model we get after some steps the bound

$$\begin{aligned} I(X_1, X_2; Y) \leq{}& \frac{\nu}{\beta} - 1 + \mathsf{E}\left[\log\frac{|d_1X_1 + d_2X_2|^2}{|X_1|^2 + |X_2|^2}\right] - \mathsf{E}\left[\mathrm{Ei}\left(-\frac{|d_1X_1 + d_2X_2|^2}{|X_1|^2 + |X_2|^2}\right)\right] \\ &+ \epsilon_\nu + \log\Gamma\left(\alpha, \frac{\nu}{\beta}\right) + \alpha(\log\beta - \log\sigma^2 + \gamma) \\ &+ \frac{1}{\beta}\Big((1 + d_{\max}^2)\mathcal{E} + \sigma^2\Big) \end{aligned} \tag{25}$$

which with the right choice of the free parameters α, β, and ν leads to the upper bound

$$\begin{aligned} \chi_{\mathrm{MAC}} \leq \varlimsup_{\mathcal{E}\uparrow\infty} \sup_{Q_{\mathcal{E}}\in\mathcal{A}} \Bigg\{ & \log\mathsf{E}\left[\frac{|d_1X_1 + d_2X_2|^2}{|X_1|^2 + |X_2|^2}\right] \\ & - \mathrm{Ei}\left(-\mathsf{E}\left[\frac{|d_1X_1 + d_2X_2|^2}{|X_1|^2 + |X_2|^2}\right]\right) - 1\Bigg\}. \end{aligned} \tag{26}$$

Here we define $\mathcal{A}$ be the set of joint input distributions such that $X_1 \perp\!\!\!\perp X_2$ and the input distribution of at least one user escapes to infinity when the available power $\mathcal{E}$ tends to infinity, *i.e.*,

$$\begin{aligned} \mathcal{A} \triangleq \Bigg\{X_1, X_2 \,\Bigg|\, & X_1 \perp\!\!\!\perp X_2, \lim_{\mathcal{E}\uparrow\infty} Q_{\mathcal{E}}\left(\{|X_1|^2 \geq \mathcal{E}_0/2\} \cup \{|X_2|^2 \geq \mathcal{E}_0/2\}\right) = 1 \\ & \text{for any fixed } \mathcal{E}_0 > 0\Bigg\}. \end{aligned} \tag{27}$$

In the following we will only prove the special case where $d_1 = d_2 = d$. The proof of the case of general d_1, d_2 is omitted.

Noting that $\xi \mapsto \log(\xi) - \mathrm{Ei}\,(-\xi)$ is monotonically increasing and writing $X_i = R_i e^{\mathrm{i}\Phi_i}$, we see that our problem becomes

$$\sup_{Q_{\mathcal{E}} \in \mathcal{A}} \mathsf{E}\left[\frac{|dX_1 + dX_2|^2}{|X_1|^2 + |X_2|^2}\right] = \sup_{Q_{\mathcal{E}} \in \mathcal{A}} \mathsf{E}\left[\frac{|d|^2 |X_1 + X_2|^2}{|X_1|^2 + |X_2|^2}\right] \tag{28}$$

$$= \sup_{Q_{\mathcal{E}} \in \mathcal{A}} \mathsf{E}\left[|d|^2 \left(1 + \frac{2|X_1||X_2|\cos(\Phi_1 - \Phi_2)}{|X_1|^2 + |X_2|^2}\right)\right] \tag{29}$$

$$= |d|^2 \left(1 + \sup_{Q_{\mathcal{E}} \in \mathcal{A}} \mathsf{E}\left[\frac{2|X_1||X_2|\cos(\Phi_1 - \Phi_2)}{|X_1|^2 + |X_2|^2}\right]\right) \tag{30}$$

$$\leq |d|^2 \left(1 + \sup_{Q_{\mathcal{E}} \in \mathcal{A}} \mathsf{E}\left[\frac{2R_1R_2}{R_1^2 + R_2^2}\right]\right), \tag{31}$$

where in the last inequality we upper-bounded $\cos(\Phi_1 - \Phi_2) \leq 1$. The result now follows once we can show that

$$\varlimsup_{\mathcal{E}\uparrow\infty} \sup_{Q_{\mathcal{E}} \in \mathcal{A}} \mathsf{E}\left[\frac{2R_1R_2}{R_1^2 + R_2^2}\right] = 0. \tag{32}$$

To that goal let

$$\mathsf{E}\left[|X_1|^2\right] \leq \mathcal{E}_1 \tag{33}$$

$$\mathsf{E}\left[|X_2|^2\right] \leq \mathcal{E}_2 \tag{34}$$

where

$$\mathcal{E}_1 + \mathcal{E}_2 = \mathcal{E}. \tag{35}$$

Note that from Proposition 3 we know that if $\mathcal{E} \uparrow \infty$ then $\mathcal{E}_1 \uparrow \infty$ or $\mathcal{E}_2 \uparrow \infty$ or both. Without loss of generality assume that $\mathcal{E}_1 \uparrow \infty$. Note further that

$$\frac{2r_1r_2}{r_1^2 + r_2^2} \leq 1 \tag{36}$$

and that $r_1 \mapsto \frac{2r_1r_2}{r_1^2+r_2^2}$ is monotonically decreasing in r_1 if $r_1 > r_2$. Hence, for an arbitrary choice of $a > 1$,

$$\begin{aligned}
&\varlimsup_{\mathcal{E}\uparrow\infty} \sup_{Q_{\mathcal{E}} \in \mathcal{A}} \mathsf{E}\left[\frac{2R_1R_2}{R_1^2 + R_2^2}\right] \\
&\quad \leq \sup_{Q_{R_2}} \varlimsup_{\mathcal{E}_1\uparrow\infty} \sup_{Q_{R_1} \in \mathcal{A}_1} \mathsf{E}\left[\frac{2R_1R_2}{R_1^2 + R_2^2}\right] && (37) \\
&\quad = \sup_{Q_{R_2}} \varlimsup_{\mathcal{E}_1\uparrow\infty} \sup_{Q_{R_1} \in \mathcal{A}_1} \int_0^\infty \int_0^\infty \frac{2r_1r_2}{r_1^2 + r_2^2} \,\mathrm{d}Q_{R_1}(r_1) \,\mathrm{d}Q_{R_2}(r_2) && (38) \\
&\quad \leq \sup_{Q_{R_2}} \varlimsup_{\mathcal{E}_1\uparrow\infty} \sup_{Q_{R_1} \in \mathcal{A}_1} \int_0^\infty \int_0^{ar_2} \frac{2r_1r_2}{r_1^2 + r_2^2} \,\mathrm{d}Q_{R_1}(r_1) \,\mathrm{d}Q_{R_2}(r_2) \\
&\qquad + \sup_{Q_{R_2}} \varlimsup_{\mathcal{E}_1\uparrow\infty} \sup_{Q_{R_1} \in \mathcal{A}_1} \int_0^\infty \int_{ar_2}^\infty \frac{2r_1r_2}{r_1^2 + r_2^2} \,\mathrm{d}Q_{R_1}(r_1) \,\mathrm{d}Q_{R_2}(r_2). && (39)
\end{aligned}$$

Here in the first inequality we define $\mathcal{A}_1$ as the set of all input distributions of the first user that escape to infinity, we use that from $\mathcal{E} \uparrow \infty$ we know that $\mathcal{E}_1 \uparrow \infty$ and take the supremum over all Q_{R_2} without any constraint on the average power and no dependence on Q_{R_1}. The last inequality then follows from splitting the integration into two parts and from the property that the supremum of a sum is always upper-bounded by the sum of the suprema.

Next, let's look at the first term in (39):

$$
\begin{aligned}
&\varlimsup_{\mathcal{E}_1\uparrow\infty} \sup_{Q_{R_1}\in\mathcal{A}_1} \int_0^\infty \int_0^{ar_2} \underbrace{\frac{2r_1r_2}{r_1^2+r_2^2}}_{\leq 1} \,\mathrm{d}Q_{R_1}(r_1)\,\mathrm{d}Q_{R_2}(r_2) \\
&\leq \varlimsup_{\mathcal{E}_1\uparrow\infty} \sup_{Q_{R_1}\in\mathcal{A}_1} \int_0^\infty \int_0^{ar_2} \mathrm{d}Q_{R_1}(r_1)\,\mathrm{d}Q_{R_2}(r_2) && (40)\\
&\leq \varlimsup_{\mathcal{E}_1\uparrow\infty} \int_0^\infty \sup_{Q_{R_1}\in\mathcal{A}_1} \int_0^{ar_2} \mathrm{d}Q_{R_1}(r_1)\,\mathrm{d}Q_{R_2}(r_2) && (41)\\
&= \int_0^\infty \varlimsup_{\mathcal{E}_1\uparrow\infty} \sup_{Q_{R_1}\in\mathcal{A}_1} \int_0^{ar_2} \mathrm{d}Q_{R_1}(r_1)\,\mathrm{d}Q_{R_2}(r_2) && (42)\\
&= \int_0^\infty 0\,\mathrm{d}Q_{R_2}(r_2) && (43)\\
&= 0. && (44)
\end{aligned}
$$

Here, (43) follows because Q_{R_1} escapes to infinity; and in (42) we exchange the limit and the integration which can be justified as follows: let

$$
\begin{aligned}
g_{\mathcal{E}_1}(r_2) &\triangleq \sup_{Q_{R_1}\in\mathcal{A}_1} \int_0^{ar_2} \mathrm{d}Q_{R_1}(r_1) && (45)\\
&\leq \sup_{Q_{R_1}\in\mathcal{A}_1} \int_0^{\infty} \mathrm{d}Q_{R_1}(r_1) && (46)\\
&= 1 \triangleq g_{\text{upper}}(r_2). && (47)
\end{aligned}
$$

Then note that

$$
\int_0^\infty g_{\text{upper}}(r_2)\,\mathrm{d}Q_{R_2}(r_2) = \int_0^\infty \mathrm{d}Q_{R_2}(r_2) = 1, \qquad (48)
$$

i.e., $g_{\text{upper}}(\cdot)$ is independent of $\mathcal{E}_1$ and integrable. Hence by the Dominated Convergence Theorem (DCT) [7, Ch. 14] we are allowed to swap limit and integration.

Hence, we continue with (39) as follows:

$$
\begin{aligned}
&\varlimsup_{\mathcal{E}\uparrow\infty} \sup_{Q_{\mathcal{E}}\in\mathcal{A}} \mathsf{E}\left[\frac{2R_1R_2}{R_1^2+R_2^2}\right] \\
&\leq \sup_{Q_{R_2}} \varlimsup_{\mathcal{E}_1\uparrow\infty} \sup_{Q_{R_1}\in\mathcal{A}_1} \int_0^\infty \int_{ar_2}^\infty \frac{2r_1r_2}{r_1^2+r_2^2}\,\mathrm{d}Q_{R_1}(r_1)\,\mathrm{d}Q_{R_2}(r_2) && (49)\\
&\leq \sup_{Q_{R_2}} \varlimsup_{\mathcal{E}_1\uparrow\infty} \sup_{Q_{R_1}\in\mathcal{A}_1} \int_0^\infty \int_{ar_2}^\infty \frac{2(ar_2)r_2}{(ar_2)^2+r_2^2}\,\mathrm{d}Q_{R_1}(r_1)\,\mathrm{d}Q_{R_2}(r_2) && (50)\\
&= \sup_{Q_{R_2}} \varlimsup_{\mathcal{E}_1\uparrow\infty} \sup_{Q_{R_1}\in\mathcal{A}_1} \int_0^\infty \int_{ar_2}^\infty \frac{2a}{a^2+1}\,\mathrm{d}Q_{R_1}(r_1)\,\mathrm{d}Q_{R_2}(r_2) && (51)\\
&\leq \sup_{Q_{R_2}} \varlimsup_{\mathcal{E}_1\uparrow\infty} \sup_{Q_{R_1}\in\mathcal{A}_1} \int_0^\infty \int_0^\infty \frac{2a}{a^2+1}\,\mathrm{d}Q_{R_1}(r_1)\,\mathrm{d}Q_{R_2}(r_2) && (52)
\end{aligned}
$$

$$= \frac{2a}{a^2+1} \sup_{Q_{R_2}} \varlimsup_{\mathcal{E}_1 \uparrow \infty} \sup_{Q_{R_1} \in \mathcal{A}_1} \int_0^\infty \int_0^\infty \mathrm{d}Q_{R_1}(r_1)\, \mathrm{d}Q_{R_2}(r_2) \tag{53}$$

$$= \frac{2a}{a^2+1} < \epsilon \qquad \text{if } a \text{ large enough.} \tag{54}$$

Here (50) follows because $r_1 \mapsto 2r_1r_2/(r_1^2+r_2^2)$ is monotonically decreasing in r_1 if $r_1 > r_2$. Since a is arbitrary, the result follows.

5 Conclusions

We have shown that at very high SNR, the sum-rate capacity of a noncoherent multiple-access Rician fading channel is limited by the capacity of the better of the two users. This rather pessimistic result fits to the very pessimistic behavior of noncoherent fading channels at high SNR.

Currently we are working on an extension of this result to more than two users where each user is allowed to use several antennas. We believe that the stated theorem generalizes accordingly, *i.e.*, the MAC fading number corresponds to the MISO fading number if only the user seeing the best channel is allowed to transmit.

A Proof of Proposition 3

In the following we will prove Proposition 3. From (9) we know that

$$\lim_{\mathcal{E} \uparrow \infty} \frac{\mathsf{C}_{\mathrm{MAC}}(\mathcal{E})}{\log\log \mathcal{E}} = 1. \tag{55}$$

Moreover note that

$$\varliminf_{\mathcal{E} \uparrow \infty} \left\{ \sup_{\mu \in (0,\mu_0]} \frac{\mu \log\log \frac{\mathcal{E}}{\mu}}{\log\log \mathcal{E}} \right\} < 1, \quad \forall\, 0 < \mu_0 < 1. \tag{56}$$

Fix some $\mathcal{E}_0 > 0$ and let

$$E \triangleq \begin{cases} 1 & \text{if } |X_1|^2 \geq \mathcal{E}_0/2 \text{ or } |X_1|^2 \geq \mathcal{E}_0/2, \\ 0 & \text{if } |X_1|^2 < \mathcal{E}_0/2 \text{ and } |X_1|^2 < \mathcal{E}_0/2. \end{cases} \tag{57}$$

and

$$\mu \triangleq \Pr[E=1]. \tag{58}$$

Then

$$\begin{aligned}
I(Q_{X_1}, Q_{X_2}, W) &= I(X_1, X_2; Y) && (59)\\
&= I(X_1, X_2, E; Y) && (60)\\
&= I(E;Y) + I(X_1, X_2; Y \mid E) && (61)\\
&= I(E;Y) + I(X_1, X_2; Y \mid E=0)\, \Pr[E=0] \\
&\quad + I(X_1, X_2; Y \mid E=1)\, \Pr[E=1] && (62)\\
&\leq \log 2 + I(X_1, X_2; Y \mid E=0) + \mu I(X_1, X_2; Y \mid E=1) && (63)\\
&\leq \log 2 + \mathsf{C}(\mathcal{E}_0) + \mu \mathsf{C}\left(\frac{\mathcal{E}}{\mu}\right), && (64)
\end{aligned}$$

where the first inequality follows because E is a binary random variable and because $\Pr[E=0] \leq 1$; the subsequent inequality follows because conditional on $E=0$, $\mathsf{E}\left[|X_1|^2+|X_2|^2\right] < \mathcal{E}_0$ and

$$\begin{aligned}\mathsf{E}\left[|X_1|^2+|X_2|^2\right] &= \mu\mathsf{E}\left[|X_1|^2+|X_2|^2 \,\middle|\, E=1\right] \\ &\quad + \underbrace{(1-\mu)\mathsf{E}\left[|X_1|^2+|X_2|^2 \,\middle|\, E=0\right]}_{\geq 0} \qquad (65)\\ &\geq \mu\mathsf{E}\left[|X_1|^2+|X_2|^2 \,\middle|\, E=1\right] \qquad (66)\end{aligned}$$

from which follows that

$$\mathsf{E}\left[|X_1|^2+|X_2|^2 \,\middle|\, E=1\right] \leq \frac{\mathsf{E}\left[|X_1|^2+|X_2|^2\right]}{\mu} \leq \frac{\mathcal{E}}{\mu}. \qquad (67)$$

To show $\mu \uparrow 1$, let $\mathcal{E}_n$ be a sequence with $\mathcal{E}_n \uparrow \infty$. Let $Q_{\mathcal{E}_n}$ be a family of joint input distributions on the MAC channel (3) such that

$$\lim_{n\uparrow\infty} \frac{I(Q_{\mathcal{E}_n}, W)}{\log\log\mathcal{E}_n} = 1 \qquad (68)$$

and define

$$\mu_n \triangleq Q_{\mathcal{E}_n}\left(\left\{|X_1|^2 \geq \frac{\mathcal{E}_0}{2}\right\} \cup \left\{|X_2|^2 \geq \frac{\mathcal{E}_0}{2}\right\}\right). \qquad (69)$$

By contradiction, assume $\mu_n \to \mu^* < 1$. Then $\exists\, \mu_0 < 1$ such that

$$\mu_n < \mu_0, \quad n \text{ sufficiently large.} \qquad (70)$$

From (64) we have

$$\underbrace{\frac{I(Q_{\mathcal{E}_n}, W)}{\log\log\mathcal{E}_n}}_{\to 1} \leq \underbrace{\frac{\log 2 + \mathsf{C}(\mathcal{E}_0)}{\log\log\mathcal{E}_n}}_{\to 0} + \underbrace{\frac{\mathsf{C}\left(\frac{\mathcal{E}_n}{\mu_n}\right)}{\log\log\frac{\mathcal{E}_n}{\mu_n}}}_{\to 1} \cdot \frac{\mu_n \log\log\frac{\mathcal{E}_n}{\mu_n}}{\log\log\mathcal{E}_n}. \qquad (71)$$

Here the limiting behavior of the LHS follows from (68); the limiting behavior of the first term on the RHS is because $\mathsf{C}(\mathcal{E}_0) < \infty$; the second term on the RHS tends to one because $\mathcal{E}_n \uparrow \infty$ implies $\mathcal{E}_n/\mu_n \uparrow \infty$ and because of (55). Hence, when $n \uparrow \infty$ we obtain the contradiction

$$\begin{aligned} 1 &\leq \varliminf_{n\uparrow\infty} \frac{\mu_n \log\log\frac{\mathcal{E}_n}{\mu_n}}{\log\log\mathcal{E}_n} \qquad (72)\\ &\leq \varliminf_{\mathcal{E}\uparrow\infty} \left\{ \sup_{\mu\in(0,\mu_0]} \frac{\mu\log\log\frac{\mathcal{E}}{\mu}}{\log\log\mathcal{E}} \right\} \qquad (73)\\ &< 1, \qquad (74)\end{aligned}$$

where the first inequality follows from (71); the second inequality follows from (70); and the last inequality follows from (56).

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