行政院國家科學委員會專題研究計畫 成果報告

卡拉比—邱流型與齊苟模型式的關聯(3/3)
研究成果報告(完整版)

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中 華 民 國 99 年 10 月 28 日

REPORT

YIFAN YANG

1. SUMMARY

During the period of 2007 to 2010, I worked in the following areas of number theory. See the references for a list of the published and submitted manuscripts.

1. *Modular curves*, [8, 11, 7, 13].
2. *Picard-Fuchs differential equations of Calabi-Yau manifolds*, [3, 9, 14].
3. *Modular forms*, [4, 5, 2].
4. *L-functions of algebraic curves*, [1, 6].
5. *Riemann zeta function*, [10].
6. *Partition function*, [12].

In the following sections, I will give a brief overview of the results obtained.

2. MODULAR CURVES

2.1. **Reference** [8]**.** In this note we obtain defining equations of modular curves $X_0(2^{2n})$. The key ingredient is a recursive formula for certain generators of the function fields on $X_0(2^{2n})$. This is a joint work with my Ph.D. student Miss Fang-Ting Tu.

2.2. **Reference** [11]**.** In this article, we consider the group $\mathscr{F}_1^\infty(N)$ of modular units on $X_1(N)$ that have divisors supported on the cusps lying over ∞ of $X_0(N)$, called the ∞-cusps. For each positive integer N, we will give an explicit basis for the group $\mathscr{F}_1^\infty(N)$. This enables us to compute the group structure of the rational torsion subgroup $\mathscr{C}_1^\infty(N)$ of the Jacobian $J_1(N)$ of $X_1(N)$ generated by the differences of the ∞-cusps. In addition, based on our numerical computation, we make a conjecture on the structure of the p-primary part of $\mathscr{C}_1^\infty(p^n)$ for a regular prime p.

2.3. **Reference** [13]**.** In this article, we determine the structure of the p-primary subgroup of the cuspidal rational torsion subgroup of the Jacobian $J_1(p^n)$ of the modular curve $X_1(p^n)$ for a regular prime p. This is a joint work with Professor Jeng-Daw Yu of the National Taiwan University.

2.4. **Reference** [7]**.** In this paper, we study the automorphism groups of the reduction $X_0(N) \times \bar{\mathbb{F}}_p$ of a modular curve $X_0(N)$ over primes $p \nmid N$. This is a joint work with Aristides Kontogeorgis of the Aegean University, Greece.

3. PICARD-FUCHS DIFFERENTIAL EQUATIONS OF CALABI-YAU MANIFOLDS

3.1. **Reference** [3]**.** In this paper we are concerned with the monodromy of Picard-Fuchs differential equations associated with one-parameter families of Calabi-Yau threefolds. Our results show that in the hypergeometric cases the matrix representations of monodromy relative to the Frobenius bases can be expressed in terms of the geometric invariants of the underlying Calabi-Yau threefolds. This phenomenon is also verified numerically for other families of Calabi-Yau threefolds in the paper. Furthermore, we discover that under a suitable change of bases the monodromy groups are contained in certain congruence subgroups of $\mathrm{Sp}(4,\mathbb{Z})$ of finite index and whose levels are related to the geometric invariants of the Calabi-Yau threefolds. This is a joint work with Noriko Yui of the Queen's University, Canada and my Ph.D. student Mr. Yao-Han Chen.

3.2. **Reference** [14]**.** Motivated by the relationship of classical modular functions and Picard–Fuchs linear differential equations of order 2 and 3, we present an analogous concept for equations of order 4 and 5. This is a joint work with Wadim Zudilin, now at the University of Newcastle, Australia.

3.3. **Reference** [9]**.** We describe a general method to determine the Apéry limits of a differential equation that has a modular-function origin. As a by-product of our analysis, we discover a family of identities involving the special values of L-functions associated with modular forms. The proof of these identities is independent of differential equations and Apéry limits.

4. MODULAR FORMS

4.1. **Reference** [4]**.** Let $S_{w+2}(\Gamma_0(N))$ be the vector space of cusp forms of weight $w+2$ on the congruence subgroup $\Gamma_0(N)$. We first determine explicit formulas for period polynomials of elements in $S_{w+2}(\Gamma_0(N))$ by means of Bernoulli polynomials. When $N=2$, from these explicit formulas we obtain new bases for $S_{w+2}(\Gamma_0(2))$, and extend the Eichler-Shimura-Manin isomorphism theorem to $\Gamma_0(2)$. This implies that there are natural correspondences between the spaces of cusp forms on $\Gamma_0(2)$ and the spaces of period polynomials. Based on these results, we will find explicit form of Hecke operators on $S_{w+2}(\Gamma_0(2))$. As an application of our main theorems, we will also give an affirmative answer to a speculation of Imamoḡlu and Kohnen on a basis of $S_{w+2}(\Gamma_0(2))$. This is a joint work with Shinji Fukuhara of the Tsuda College, Japan.

4.2. **Reference** [5]. Let $f_1, \ldots, f_d$ be an orthogonal basis for the space of cusp forms of even weight $2k$ on $\Gamma_0(N)$. Let $L(f_i, s)$ and $L(f_i, \chi, s)$ denote the L-function of f_i and its twist by a Dirichlet character χ, respectively. In this note, we obtain a "trace formula" for the values $L(f_i, \chi, m)\overline{L(f_i, n)}$ at integers m and n with $0 < m, n < 2k$ and proper parity. In the case $N = 1$ or $N = 2$, the formula gives us a convenient way to evaluate precisly the value of the ratio $L(f, \chi, m)/L(f, n)$ for a Hecke eigenform f. This is a joint work with Shinji Fukuhara of the Tsuda College, Japan.

4.3. **Reference** [2]. Around 1828, T. Clausen discovered that the square of certain hypergeometric ${}_2F_1$ function can be expressed as a hypergeometric ${}_3F_2$ function. Special cases of Clausen's identities were later used by S. Ramanujan in his derivation of 17 series for $1/\pi$. Since then, there were several attempts to find new analogues of Clausen's identities with the hope to derive new classes of series for $1/\pi$. Unfortunately, none were successful. In this article, we will present three new analogues of Clausen's identities. Their discovery is motivated by the study of relations between modular forms of weight 2 and modular functions associated with modular groups of genus 0. This is a joint work with Heng Huat Chan of the National University of Singapore, Yoshio Tanigawa of the Nagoya University, Japan, and Wadim Zudilin of the University of Newcastle, Australia.

5. L-FUNCTIONS OF ALGEBRAIC CURVES

5.1. **Reference** [1]. It is well known that if p is a prime such that $p \equiv 1 \pmod 4$, then p can be expressed as a sum of two squares. Several proofs of this fact are known and one of them, due to E. Jacobsthal, involves the identity $p = x^2+y^2$, with x and y expressed explicitly in terms of sums involving the Legendre symbol. These sums are now known as the Jacobsthal sums. In this short note, we prove that if $p \equiv 1 \pmod 6$, then $3p = u^2 + uv + v^2$ for some integers u and v using an analogue of Jacobsthal's identity. This is a joint work with Heng Huat Chan of the National University of Singapore and Ling Long of the Iowa State Universtiy, USA.

5.2. **Reference** [6]. Let p be a prime congruent to 1 or 3 modulo 8 so that the equation $p = a^2 + 2b^2$ is solvable in integers. In this paper, we obtain closed-form expressions for a and b in terms of Jacobsthal sums. This is analogous to a classical identity of Jacobsthal. This is a joint work with

Ki-Ichiro Hashimoto of the Waseda University and Ling Long of the Iowa State University, USA.

6. Riemann zeta function

6.1. **Reference** [10]. Let $\Delta(T)$ and $E(T)$ be the error terms in the classical Dirichlet divisor problem and in the asymptotic formula for the mean square of the Riemann zeta function on the critical strip, respectively. We show that $\Delta(T)$ and $E(T)$ are asymptotic integral transforms of each other. We then use this integral representation of $\Delta(T)$ to give a new proof of a result of M. Jutila.

7. Partition function

7.1. **Reference** [12]. Let $p(n)$ denote the partition function. In this article, we will show that congruences of the form

$$p(m\ell^k n + B) \equiv 0 \mod m \text{ for all } n \geq 0$$

exist for all primes m and ℓ satisfying $m \geq 13$ and $\ell \neq 2, 3, m$, where B is a suitably chosen integer depending on m and ℓ. Here the integer k depends on the Hecke eigenvalues of a certain invariant subspace of $S_{m/2-1}(\Gamma_0(576), \chi_{12})$ and can be explicitly computed.

More generally, we will show that for each integer $i > 0$ there exists an integer k such that with a properly chosen B the congruence

$$p(m^i \ell^k n + B) \equiv 0 \mod m^i$$

holds for all integers n not divisible by ℓ.

References

[1] Heng Huat Chan, Ling Long, and Yifan Yang. A cubic analogue of the Jacobsthal identity. *Amer. Math. Monthly*, to appear.

[2] Heng Huat Chan, Yoshio Tanigawa, Yifan Yang, and Wadim Zudilin. New analogues of clausen's identities arising from the theory of modular forms. *submitted*.

[3] Yao-Han Chen, Yifan Yang, and Noriko Yui. Monodromy of Picard-Fuchs differential equations for Calabi-Yau threefolds. *J. Reine Angew. Math.*, 616:167–203, 2008. With an appendix by Cord Erdenberger.

[4] Shinji Fukuhara and Yifan Yang. Period polynomials and explicit formulas for Hecke operators on $\Gamma_0(2)$. *Math. Proc. Cambridge Philos. Soc.*, 146(2):321–350, 2009.

[5] Shinji Fukuhara and Yifan Yang. Twisted Hecke L-values and period polynomials. *J. Number Theory*, 130(4):976–999, 2010.

[6] Ki-Ichiro Hashimoto, Ling Long, and Yifan Yang. Jacobsthal identity for $\mathbb{Q}(\sqrt{-2})$. *submitted*.

[7] Aristides Kontogeorgis and Yifan Yang. Automorphisms of hyperelliptic modular curves $X_0(N)$ in positive characteristic. *LMS J. Comput. Math.*, 13:144–163, 2010.

[8] Fang-Ting Tu and Yifan Yang. Defining equations of $X_0(2^{2n})$. *Osaka J. Math.*, 46(1):105–113, 2009.

[9] Yifan Yang. Apéry limits and special values of L-functions. *J. Math. Anal. Appl.*, 343(1):492–513, 2008.
[10] Yifan Yang. On the mean square of the Riemann zeta function and the divisor problem. *Publ. Inst. Math. (Beograd) (N.S.)*, 83(97):71–86, 2008.
[11] Yifan Yang. Modular units and cuspidal divisor class groups of $X_1(N)$. *J. Algebra*, 322(2):514–553, 2009.
[12] Yifan Yang. Congruences of the partition function. *Int. Math. Res. Not.*, in press.
[13] Yifan Yang and Jeng-Daw Yu. Structure of the cuspidal rational torsion subgroups of $J_1(p^n)$. *J. London Math. Soc.*, 82:203–228, 2010.
[14] Yifan Yang and Wadim Zudilin. An Sp_4-modularity of picard-fuchs differential equations for calabi-yau threefolds. *Contemp. Math.*, 517:381–413.

REPORT ON NSC-SPONSORED VISITS, AUGUST 2007 TO JULY 2010

YIFAN YANG

1. INVITED TALKS

1. *Structure of the cuspidal rational torsion subgroups of* $J_1(p^n)$, 10th Canadian number theory meeting, Waterloo, Canada, July 1318, 2008.
2. Sp_4*-modularity of Picard-Fuchs differential equations for Calabi-Yau threefolds*, Workshop on number theory and physics at the crossroads, Tsuda College, Japan, August 46, 2008.
3. *Construction and applications of a class of modular functions*, Joint workshop on number theory between Japan and Taiwan, Waseda University, Japan, March 911, 2009.
4. *Congruences of the partition function*, East Asia number theory conference, Tsing Hua University, China, August 1922, 2009.
5. *Congruences of the partition function*, Nankai number theory conference, Nankai University, China, August 2428, 2009.
6. *Congruence of the partition function*, KMS-AMS joint meeting, Seoul, Korea, December 1520, 2009.
7. *Monodromy of Picard-Fuchs differential equations*, Postech-NCTS Workshop on number theory, Pohang, Korea, December 1920, 2009.

2. VISIT TO THE TSUDA COLLEGE, JAPAN

On August 1–13, 2008, I was invited by Professor Shinji Fukuhara to visit the Tsuda college, Japan, to work together on mathematics of common interest. Specifically, we took on the project to study the relation between the twisted periods of cusp forms and the values of the Hecke L-functions, twisted by Dirichlet characters. These values are important objects in number theory. For example, according to the Birch and Swinnterton-Dyer conjecture, the infinitude of the Mordell-Weil group of the quadratic twist of an elliptic curve over rational numbers depends on whether the twisted L-function vanishes or not at the center point. A manuscript on these twisted Hecke L-values has appeared on the Journal of Number Theory recently.

3. VISIT TO THE EAST CHINA NORMAL UNIVERSITY

On August 25–September 7, 2008, I was invited by Professor Liu Zhiguo to visit the East China Normal University, China, to work together on mathematics of common interest. Specifically, we have had intensive discussion on the following two topics.

1. *Monodromy of hypergeometric differential equations*: The monodromy of the Picard-Fuchs differential equation associated to a family of Calabi-Yau manifolds often contain important informations about the underlying geometric objects. In

the hypergeometric cases, because a hypergeometric function is the limit of a q-hypergeometric function, one may wonder whether it is possible to study the monodromy of a hypergeometric differential equation using the q-hypergeometric functions, and furthermore, whether it is possible to study other non-hypergeometric cases using other q-series.

2. *Defining equations of modular curves*: In an earlier work, we find that the modular curve $X(7)$ has a defining equation $x^3y + y^3z + z^3x = 0$, and an explicit modular-function parameterization is given in terms of the generalized Dedekind eta functions, which are essentially the values of the Jacobi theta function at 7-torsion points of a period lattice. In a recent work of Professor Liu, he rediscovered this paramterization via the theory of elliptic functions. It is natural to ask whether his method will also work for modular curves of higher level, or how far one can go using his method.

4. Visit to the Waseda University, Japan

On July 25–August 16, 2009 and January 20–February 10, 2010, I was invited by Professor Ki-Ichiro Hashimoto to visit the Waseda University, Japan. During the visits, we worked on problems related to modular curves, Jacobsthal sums, quaternion algebras, and Shimura curves. One particular beautiful identity we found is the following. For all primes p congruent to 3 modulo 8, let

$$A = \frac{1}{2}\sum_{x=0}^{p-1}\left(\frac{x^3+4x^2+2x}{p}\right).$$

and

$$B = \frac{1}{4}\left(1+\sum_{x=0}^{p-1}\left(\frac{x^6+4x^5+10x^4-20x^2-16x-8}{p}\right)\right).$$

Then A and B are integers and satisfy $A^2+2B^2=p$. The proof of this result uses the theory of Hecke characters, CM modular forms, Galois representations, and algebraic curves. A manuscript on this result has submitted for publication.

Moreover, in two projects currently underway, we are working on arithmetic and applications of quaternion algebras. One is about ideal lattices of quaternion algebras. We have been successful in constructing some well-known lattices, such as the Coxeter-Todd lattice K_{12}, the laminated lattice Λ_{16}, the Leech lattice, and so on using the ideals of definite quaternion algebras over totally real fields. The other project we are currently working on is about automorphic forms on Shimura curves. Our approach leads us to some fascinating algebraic transformation formula for hypergeometric function. One such example is

$$\begin{aligned}(1-(1-i)t)^{3/5}{}_2F_1&\left(\frac{1}{20},\frac{1}{4};\frac{4}{5};\frac{64t(1-t)(1-3t+t^2)^5}{(1-2t)(1+2t-4t^2)^5}\right)\\ &=(1-2t)^{1/20}(1+2t-4t^2)^{1/4}{}_2F_1\left(\frac{3}{20},\frac{2}{5};\frac{4}{5};\frac{8it(1-t)(1-2t)}{(1-(1-i)t)^4}\right).\end{aligned}$$

Such an identity arises from the fact that the intersection of two arithmetic triangle groups $(2,5,5)$ and $(5,10,10)$ is a Fuchsian subgroup of $\mathrm{SL}(2,\mathbb{R})$ of genus 0.

無衍生研發成果推廣資料

98 年度專題研究計畫研究成果彙整表

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<tr><td colspan="3" rowspan="2">成果項目</td><td colspan="3">量化</td><td rowspan="2">單位</td><td rowspan="2">備註（質化說明：如數個計畫共同成果、成果列為該期刊之封面故事...等）</td></tr>
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其他成果 (無法以量化表達之成果如辦理學術活動、獲得獎項、重要國際合作、研究成果國際影響力及其他協助產業技術發展之具體效益事項等，請以文字敘述填列。)	無

	成果項目	量化	名稱或內容性質簡述
科教處計畫加填項目	測驗工具(含質性與量性)	0	
	課程/模組	0	
	電腦及網路系統或工具	0	
	教材	0	
	舉辦之活動/競賽	0	
	研討會/工作坊	0	
	電子報、網站	0	
	計畫成果推廣之參與（閱聽）人數	0	

國科會補助專題研究計畫成果報告自評表

請就研究內容與原計畫相符程度、達成預期目標情況、研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）、是否適合在學術期刊發表或申請專利、主要發現或其他有關價值等，作一綜合評估。

1. 請就研究內容與原計畫相符程度、達成預期目標情況作一綜合評估

■達成目標

□未達成目標（請說明，以 100 字為限）

　　□實驗失敗

　　□因故實驗中斷

　　□其他原因

說明：

2. 研究成果在學術期刊發表或申請專利等情形：

論文：■已發表 □未發表之文稿 □撰寫中 □無

專利：□已獲得 □申請中 ■無

技轉：□已技轉 □洽談中 ■無

其他：（以 100 字為限）

3. 請依學術成就、技術創新、社會影響等方面，評估研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）（以 500 字為限）

The two papers on cuspidal divisor class groups studied the arithmetic of the modular curves X_1(N) and their Jacobians. The paper ’’ an Sp_4-modularity of Picard-Fuchs differential equations for Calabi-Yau threefolds’’ presents the first attempt in literature to give an automorphic explanation of such differential equations. The paper ’’ congruences of the partition function’’ shows that Ono’s result on the congruences of the partition function can be much more improved.