Zero-Dispersion Limit of the Klein-Gordon Equation in Electromagnetic Fields

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Abstract

This paper deals with the zero-dispersion limit of the modulated nonlinear Klein-Gordon equation in electromagnetic fields. First, we derive the hydrodynamical structure of the modulated nonlinear Klein-Gordon equation with divergence free magnetic potential and prove the convergence of the modulated nonlinear Klein-Gordon equation with divergence free magnetic potential to the anelastic system. Second, we investigate the singular limit, indeed the nonrelativistic-semiclassical limit, of the modulated nonlinear Klein-Gordon equation with Ginzburg-Landau type potential directly; the wave map equation (with or without magnetic potential) is recovered as a nonrelativistic-semiclassical limit. The magnetic effect depends on the relation between the scaled Planck's constant ε and the scaled light speed ν.

Keywords and phrases: Nonlinear Klein-Gordon equation, electromagnetic potential, nonrelativistic-semiclassical limit, anelastic system, wave map.

2000 Mathematical Subject Classification: 35Q40; 35Q55; 76Y05.

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1 Introduction

The nonlinear Klein-Gordon equation for a relativistic particle of spin zero moving in an electromagnetic potential (ϕ, A), where $\phi : \mathbb{R}^n \mapsto \mathbb{R}$ the electric (scalar) potential and $A : \mathbb{R}^n \mapsto \mathbb{R}^n$ the magnetic (vector) potential, is given by

$$-\frac{\hbar^2}{2mc^2}\partial_t^2\Psi + \frac{1}{2m}\Big(\hbar\nabla - \frac{ie}{c}A\Big)^2\Psi - \frac{1}{2}mc^2\Psi - \big(|\Psi|^{2(\gamma-1)} - e\phi\big)\Psi = 0 \quad (1.1)$$

in which e and m indicate the charge and the rest mass of a particle, $\hbar$ and c are Planck's constant and the speed of light respectively and $|\Psi|^{2(\gamma-1)}$ is the external potential [25, 27]. The wave function $\Psi(x,t)$ is a complex-valued field over a spatial domain $\Omega \subset \mathbb{R}^n$. Based on the dimensional balance principle we can consider the modulated wave function [16, 19, 33]

$$\psi(x,t) = \Psi(x,t)\exp(imc^2t/\hbar) \quad (1.2)$$

then ψ will satisfy the modulated nonlinear Klein-Gordon equation

$$i\hbar\partial_t\psi - \frac{\hbar^2}{2mc^2}\partial_t^2\psi + \frac{1}{2m}\Big(\hbar\nabla - \frac{ie}{c}A\Big)^2\psi - \big(|\psi|^{2(\gamma-1)} - e\phi\big)\,\psi = 0 \quad (1.3)$$

which is a combination of the nonlinear Schrödinger and the nonlinear wave equations. This equation is not only interesting by itself but also important in physics. It involves the speed of light c and Planck's constant $\hbar$ which are referred to as relativity and quantum respectively. Besides the nonrelativistic limit $(c \to \infty)$ and semiclassical limit $(\hbar \to 0)$, the massless limit $(m \to 0)$ is also interesting. However, we will not discuss this limit in this paper. In fact after proper rescaling, we may assume $m = e = 1$ and rewrite (1.3) in the dimensionless form

$$i\varepsilon\partial_t\psi + \frac{1}{2}\big(\varepsilon\nabla - i\nu A\big)^2\psi - \big(|\psi|^{2(\gamma-1)} - \phi\big)\psi = \frac{\nu^2\varepsilon^2}{2}\partial_t^2\psi\,, \quad (1.4)$$

where ε is the dimensionless scaled Planck's constant and another dimensionless parameter ν is the ratio of the reference velocity and speed of light. The relativistic effects not only occur on the right hand side of (1.4) but also in the magnetic potential as a part of the covariant gradient. Assuming the electromagnetic potentials A and ϕ are of order $O(1)$ with respect to ν, then keeping ε fixed and formally letting $\nu \to 0$ (i.e. $c \to \infty$) in (1.4), the so-called nonrelativistic limit, we obtain the nonlinear Schrödinger equation with scalar potential ϕ

$$i\varepsilon\partial_t\psi + \frac{\varepsilon^2}{2}\Delta\psi - \big(|\psi|^{2(\gamma-1)} - \phi\big)\psi = 0\,, \tag{1.5}$$

though one has to be extremely careful with heuristics due to the double time derivative on the right side of (1.4). The reader is referred to [20, 31] for the up-to-date account of results and methods in the well-posedness theory for initial-value problems of nonlinear Schrödinger and more general nonlinear dispersive equations. For the general introduction and physical background to the nonlinear Schrödinger equations we will refer to [30].

The above discussion motivates that we may think of the Klein-Gordon equation (even with magnetic potential) as the relativistic generalization of the nonlinear Schrödinger equation (see [22] and references therein). Note that the magnetic potential A vanishes in this limit. When $\phi = 1$ the hydrodynamic and singular limits of the nonlinear Schrödinger equation (1.5) is very well studied in the past two decades ([6, 7, 10, 14]). Hence it is interesting and important to investigate similar problems for (1.4) from the point of view of the Schrödinger equation. In particular, we would like to see what the role played by the electromagnetic potential in the limiting process is and therefore this paper will be devoted to the singular and hydrodynamic limits of the nonlinear Klein-Gordon equation with electromagnetic potentials.

Rigorous studies for the singular and hydrodynamic limits of the nonlinear Klein-Gordon equation have been carried out recently. When there is no magnetic potential and the scalar potential is constant, say $(\phi, A) = (1, 0)$ for example, the singular limits, including semiclassical, nonrelativistic and nonrelativistic-semiclassical limits, i.e., $\hbar \to 0$, $c \to \infty$ and $c = \hbar^{-\beta}$ for $\beta > 0, \hbar \to 0$ respectively, of the Cauchy problem for the modulated defocusing nonlinear Klein-Gordon equation (1.3) were studied in [16] (see also [19] for the fluid dynamical approximation and [17] for the review). In particular, the authors established the connections between the solution of the Klein-Gordon equation and the solution of the wave map equation. The nonrelativistic-semiclassical limit of the modulated cubic nonlinear Klein-Gordon equation with magnetic potential was also discussed by one of the authors in [32]. When Planck's constant $\hbar$ and the speed of light c are related by $c = \hbar^{-\alpha}$ for some $\alpha \geq 1$, then as $\hbar \to 0$ it is shown that for $\alpha = 1$ the limit wave function satisfies the wave map with one extra term due to the effect of the magnetic potential, however, for $\alpha > 1$, the effect of the magnetic potential disappears and the limit equation is the typical wave map equation. The reader is also referred to [22] for a very complete answer of the nonrelativistic limit of the Klein-Gordon equation.

The hydrodynamic limit is also considered by the authors in [19]. In fact, before the formation of singularities in the limit hydrodynamic system, the nonrelativistic-semiclassical limit is shown to be the compressible Euler equation. If we further rescale the time variable and keep the light speed fixed, then in the semiclassical limit, the incompressible Euler equations is recovered. The main idea to prove the convergence of the hydrodynamic limit is the modulated energy method introduced by Brenier [1], following an ideal due to P.-L. Lions in [21]. We successfully extend this method to the nonlinear Klein-Gordon equation by introducing a correction term to control the propagation of the relativistic charge and current and prove the convergence of the charge and current defined by the modulated nonlinear Klein-Gordon equation towards the solution of the compressible Euler equations. It is well-known that incompressible approximation may be derived by filtering out or averaging over the fast acoustic motion; the incompressible limit formalizes and makes rigorous these approximation. Thus, for the incompressible limit, we have to introduce one more correction term which describes the propagation of the density fluctuation, i.e., the acoustic wave and the detail is referred to [19].

The paper is organized as follows. In section 2, we apply the Madelung transformation to obtain the relativistic quantum hydrodynamics equations of the modulated nonlinear Klein-Gordon equation (1.4). The formal different hydrodynamics limits from the relativistic quantum hydrodynamical equations are also derived. When the vector potential A is stationary, $\partial_t A = 0$, the nonrelativistic limit, $\nu \to 0$, will be the quantum hydrodynamics equations which are the same as derived from the defocusing nonlinear Schrödinger equations, then letting $\varepsilon \to 0$, the semiclassical limit will be the compressible Euler equation. In other word, if we let ε and ν tend to 0 simultaneously then the nonrelativistic-semiclassical limit of (1.4) will be the compressible Euler equations. In this case (hyperbolic scaling) the electromagnetic potentials do not effect the singular limit. However, if we consider the divergence free magnetic potential $\nabla \cdot A = 0$ and rescale the time variable $t \to \varepsilon^\alpha t$ and assume $\nu = \varepsilon^\beta$ then, depending on the choice of the exponent β, two different asymptotic limits appear as the effect equations. When $\beta = \alpha$, it is the rotating anelastic approximation (2.27)–(2.28) and for $\beta > \alpha$ the limit is the anelastic approximation (2.30).

Section 3 is devoted to the rigorous proof of the hydrodynamic limits. When $\beta = \alpha$, we show that the nonrelativistic-semiclassical limit of the modulated nonlinear Klein-Gordon equation (1.4) with divergence free magnetic potential is the anelastic approximation (3.8) with rotating $\mathbb{G}_A$ and nonconstant density ρ_0. However, when $\beta > \alpha$, the rotating effect vanishes and the limit is the typical anelastic system (3.9) with nonconstant density ρ_0.

In section 4, we study the nonrelativistic-semiclassical limit of the electromagnetic Klein-Gordon equation with Ginzburg-Landau type potential. The scaled Planck's constant $\hbar$ and the dimensionless light speed c are related by $\hbar = \varepsilon$ and $c = \varepsilon^{\beta}$, $\beta \geq 1$. When $\beta = 1$, the limit wave function ψ satisfies the wave map equation with electromagnetic potentials (ϕ, A), but for $\beta > 1$ the magnetic potential A does not effect the singular limit and the limiting equation is the wave map equation with scalar potential only. The main reason is that other than the linear momentum W, there is an extra term ϕA appearing in the limiting density fluctuation w (see (4.31)–(4.32)). We can interpret ϕA as the background momentum occurring from the electromagnetic potentials. The existence of the weak solution of the wave map equation with electromagnetic potential (4.14)–(4.15) is given in the appendix.

Notation. In this paper, $L^p(\Omega), (p \geq 1)$ denotes the classical Lebesgue space with norm $\|f\|_p = (\int_\Omega |f|^p dx)^{1/p}$, the Sobolev space of functions with all its k-th partial derivatives in $L^2(\Omega)$ will be denoted by $H^k(\Omega)$, and its dual space is $H^{-k}(\Omega)$. We use $\langle f, g\rangle = \int_\Omega fg dx$ to denote the standard inner product on the Hilbert space $L^2(\Omega)$. Given two vectors A and F, we define $\mathbb{G}_A(F) = (\operatorname{curl} A) \times F$. Let $u = (u_1, u_2)$ be a two dimensional vector, we define its orthogonal vector as $u^\perp = (-u_2, u_1)$. It is everywhere orthogonal to u and of the same length. Finally, we abbreviate " $\leq C$ " to " $\lesssim$ ", where C is a positive constant depending only on fixed parameters.

2 Hydrodynamics Structure

This section is devoted to the hydrodynamical structure of the n dimensional ($n = 2, 3$) modulated nonlinear Klein-Gordon equation (1.4). Following Madelung's idea [7, 9, 10, 15, 30, 33], we introduce the complex-valued wave function

$$\psi = R\exp(iS/\varepsilon), \tag{2.1}$$

in which both R, the amplitude, and S, the action function, are real-valued functions. The amplitude R is positive, $R(x,t) > 0$ for all x and t. Plugging (2.1) into (1.4) and separating the real and imaginary parts, we obtain

$$\partial_t R + \frac{R}{2}\Box_\nu S + (\nabla S - \nu A)\cdot\nabla R - \nu^2\partial_t S\partial_t R = 0\,, \tag{2.2}$$

and

$$\partial_t S + \frac{1}{2}\Big(\,|\nabla S - \nu A|^2 - \nu^2\left(\partial_t S\right)^2\Big) + \left(R^{2(\gamma-1)} - \phi\right) = \frac{\varepsilon^2}{2}\frac{\Box_\nu R}{R}\,, \tag{2.3}$$

where $\Box_\nu \equiv \Delta - \nu^2\partial_t^2$ is the d'Alembertian. Equations (2.2) and (2.3) are equivalent to the modulated nonlinear Klein-Gordon equation (1.4) for smooth functions R and S. Furthermore, (2.2) turns out to be the continuity equation for the relativistic quantum fluid and (2.3) is the relativistic quantum Hamilton-Jacobi equation. Introducing the hydrodynamical variables ρ, u and ρ_K defined respectively by

$$\rho = R^2 = |\psi|^2 = \psi\overline{\psi}\,, \tag{2.4}$$

$$u = \nabla S = \frac{i\varepsilon}{2}\frac{1}{|\psi|^2}(\psi\nabla\overline{\psi} - \overline{\psi}\nabla\psi)\,, \tag{2.5}$$

and

$$\rho_K = \nu^2 R^2\partial_t S = \frac{i\varepsilon\nu^2}{2}(\psi\partial_t\overline{\psi} - \overline{\psi}\partial_t\psi)\,, \tag{2.6}$$

we can rewrite (2.2)–(2.3) as the relative quantum hydrodynamics equations

$$\partial_t\big(\rho - \rho_K\big) + \nabla\cdot\big(\rho(u - \nu A)\big) = 0\,, \tag{2.7}$$

$$\begin{aligned}&\Big(1 - \frac{\rho_K}{\rho}\Big)\partial_t u + (u - \nu A)\cdot\nabla(u - \nu A) + \nabla V'(\rho)\\ &+\nu\mathbb{G}_A(u - \nu A) = \frac{\varepsilon^2}{2}\nabla\bigg(\frac{\Box_\nu\sqrt{\rho}}{\sqrt{\rho}}\bigg),\end{aligned} \tag{2.8}$$

where $V'(\rho) = \rho^{\gamma-1} - \phi$ and $\mathbb{G}_A(u - \nu A) = (\operatorname{curl} A)\times(u - \nu A)$. From (2.7) and (2.8) we can derive the momentum equation

$$\begin{aligned}&\partial_t\Big(\rho(u - \nu A) - \rho_K u\Big) + \nabla\cdot\Big(\rho(u - \nu A)\otimes(u - \nu A)\Big) + \nabla P(\rho)\\ &+\nu\mathbb{G}_A(\rho u - \nu\rho A) = \frac{\varepsilon^2}{4}\nabla\cdot\Big(\rho\nabla^2\log\rho\Big) - \frac{\varepsilon^2\nu^2}{4}\partial_t\Big(\rho\nabla\partial_t\log\rho\Big),\end{aligned} \tag{2.9}$$

where $P(\rho) = \rho V'(\rho) - V(\rho)$ is the pressure and ∇^2 denotes the Hessian. The above derivation shows that the magnetic potential A affects both the equation of continuity and the momentum equation, but the electric potential ϕ only appears in the momentum equation. Next, we define the Schrödinger part energy density E_S and relativistic part energy density E_K respectively by

$$E_S = \frac{1}{2}\rho|u-\nu A|^2 + \frac{\varepsilon^2}{8}\frac{|\nabla\rho|^2}{\rho} + V(\rho)\,, \tag{2.10}$$

$$E_K = \frac{1}{2\nu^2}\frac{|\rho_K|^2}{\rho} + \frac{\varepsilon^2\nu^2}{8}\frac{|\partial_t\rho|^2}{\rho}\,, \tag{2.11}$$

then the associated energy equation of (1.4) is

$$\frac{d}{dt}\int (E_S + E_K)dx = -\nu \int \partial_t A \cdot (\rho(u-\nu A))dx\,. \tag{2.12}$$

It is obvious that the total energy $E = E_S + E_K$ is conservative when the magnetic potential A is stationary, $\partial_t A = 0$. Formally, in the nonrelativistic limit $\nu \to 0$, the d'Alembertian $\Box_\nu$ becomes the Laplacian Δ and $\rho_K \to 0$, one neglects all the $O(\nu)$ and $O(\nu^2)$ terms in (2.7) and (2.9) and the limit densities satisfy the quantum hydrodynamic equations [10, 11, 30]

$$\partial_t\rho + \nabla\cdot(\rho u) = 0\,, \tag{2.13}$$

$$\partial_t(\rho u) + \nabla\cdot\big(\rho u\otimes u\big) + \nabla P(\rho) = \frac{\varepsilon^2}{4}\nabla\cdot\Big[\rho\nabla^2\log\rho\Big]\,, \tag{2.14}$$

which are the same as derived from the defocusing nonlinear Schrödinger equation. Furthermore, letting $\varepsilon \to 0$ the above quantum hydrodynamics equations will be reduced to the compressible Euler equations

$$\partial_t\rho + \nabla\cdot(\rho u) = 0\,, \tag{2.15}$$

$$\partial_t(\rho u) + \nabla\cdot\big(\rho u\otimes u\big) + \nabla P(\rho) = 0\,. \tag{2.16}$$

The above formal analysis shows that the electromagnetic potential (A, ϕ) does not effect the singular limits. However, there is rarely good reason to suppose that electromagnetic potential effects are totally absent in the singular limit process. If we go back to (2.7) and (2.8) the real situation won't behave like this when the electromagnetic potentials are more involved. To this end we introduce the scaling $\widetilde{t} = \varepsilon^\alpha t$, $\widetilde{x} = x$, $0 < \alpha < 1$, and modulated nonlinear Klein-Gordon equation (1.4) becomes (after dropping the tilde)

$$i\varepsilon^{1-\alpha}\partial_t\psi - \frac{\nu^2\varepsilon^2}{2}\partial_t^2\psi + \frac{1}{2}\big(\varepsilon^{1-\alpha}\nabla - i\nu\varepsilon^{-\alpha}A\big)^2\psi - \varepsilon^{-2\alpha}\big(|\psi|^{2(\gamma-1)} - \phi\big)\psi = 0\,. \tag{2.17}$$

The magnetic potential A is assumed to be divergence free, $\nabla \cdot A = 0$. Now, the Madelung transformation (2.1) becomes

$$\psi = R\exp(iS/\varepsilon^{1-\alpha}), \tag{2.18}$$

and the hydrodynamics equations (2.7) and (2.8) will be

$$\partial_t\big(\rho - \rho_K\big) + \nabla \cdot \big(\rho(u - \varepsilon^{-\alpha}\nu A)\big) = 0\,, \tag{2.19}$$

$$\begin{aligned}&\Big(1 - \frac{\rho_K}{\rho}\Big)\partial_t u + (u - \varepsilon^{-\alpha}\nu A)\cdot\nabla(u - \varepsilon^{-\alpha}\nu A) + \nu\varepsilon^{-\alpha}\mathbb{G}_A(u - \varepsilon^{-\alpha}\nu A)\\ &+\frac{1}{\varepsilon^{2\alpha}}\nabla V'(\rho) = \frac{\varepsilon^{2-2\alpha}}{2}\nabla\Big(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\Big) - \frac{\varepsilon^2\nu^2}{2}\nabla\Big(\frac{\partial_t^2\sqrt{\rho}}{\sqrt{\rho}}\Big)\end{aligned} \tag{2.20}$$

where the hydrodynamical variables ρ, u and ρ_K are now given by

$$\rho = R^2 = |\psi|^2 = \psi\overline{\psi}\,, \tag{2.21}$$

$$u = \nabla S = \frac{i\varepsilon^{1-\alpha}}{2}\frac{1}{|\psi|^2}\big(\psi\nabla\overline{\psi} - \overline{\psi}\nabla\psi\big)\,, \tag{2.22}$$

$$\rho_K = \nu^2\varepsilon^{2\alpha}R^2\partial_t S = \frac{i\nu^2\varepsilon^{1+\alpha}}{2}\big(\psi\partial_t\overline{\psi} - \overline{\psi}\partial_t\psi\big)\,. \tag{2.23}$$

Similarly, the Schrödinger part energy density E_S and relativistic part energy density E_K are now given respectively by

$$E_S = \frac{1}{2}\rho|u - \varepsilon^{-\alpha}\nu A|^2 + \frac{\varepsilon^{2-2\alpha}}{8}\frac{|\nabla\rho|^2}{\rho} + \frac{1}{\varepsilon^{2\alpha}}V(\rho)\,, \tag{2.24}$$

$$E_K = \frac{1}{2\varepsilon^{2\alpha}\nu^2}\frac{|\rho_K|^2}{\rho} + \frac{\varepsilon^2\nu^2}{8}\frac{|\partial_t\rho|^2}{\rho}\,, \tag{2.25}$$

then the associated energy equation of the modulated nonlinear Klein-Gordon equation (2.17) is

$$\frac{d}{dt}\int(E_S + E_K)dx = -\varepsilon^{-\alpha}\nu\int\partial_t A\cdot(\rho u - \varepsilon^{-\alpha}\nu\rho A)dx\,. \tag{2.26}$$

For the sake of simplicity, the scaled light speed ν and the scaled Planck's constant ε are chosen to satisfy the relation $\nu = \varepsilon^\beta$, for some $\beta \geq \alpha$. Since $V(\rho)$ is a convex function with minimum occuring at $\rho = \rho_0$, where $\rho_0 = \phi^{\frac{1}{\gamma-1}}$, if we have the proper apriori estimate of the energy equation, then the uniform boundedness of the Schrödinger part energy density E_S will imply $\rho \to \rho_0$ as $\varepsilon \to 0$.

To derive the limiting equations, we will discus two different situations. First, $\beta = \alpha$, we expect that the continuity equation (2.19) yields the limit:

$$\nabla \cdot \big[\rho_0(u - A)\big] = 0\,. \tag{2.27}$$

Since $V'(\rho_0) = 0$ then writing $\nabla V'(\rho) = \nabla(V'(\rho) - V'(\rho_0))$, we deduce from (2.20) that

$$\partial_t u + \big[(u - A)\cdot\nabla\big](u - A) + \mathbb{G}_A(u - A) + \nabla\pi = 0\,. \tag{2.28}$$

If A is stationary, $\partial_t A = 0$, then we can further rewrite (2.27)–(2.28) as

$$\partial_t u_A + (u_A \cdot \nabla)u_A + \mathbb{G}_A(u_A) + \nabla\pi = 0\,, \qquad \nabla\cdot(\rho_0 u_A) = 0 \tag{2.29}$$

where $u_A = u - A$ is the relative velocity. When $A = 0$ the system of equations (2.27)–(2.28) or (2.29) is usually termed anelastic approximation. An anelastic approximation is a filtering approximation for the equations of motion that eliminates sound waves by assuming that the flow has velocities and phase speeds much smaller than the speed of sound. This approximation has been used to model astrophysical and geophysical fluids [23, 26]. Next, when $\beta > \alpha$, the effect of the magnetic potential vanishes, $\mathbb{G}_A(u_A) = 0$ and the limiting equations of (2.19)–(2.20) will be the typical anelastic approximation [2, 3, 8, 24]

$$\partial_t u + u\cdot\nabla u + \nabla\pi = 0\,, \qquad \nabla\cdot(\rho_0 u) = 0 \tag{2.30}$$

where the pressure π is the limit of $\frac{1}{\varepsilon^{2\alpha}}(V'(\rho) - V'(\rho_0))$.

3 Hydrodynamic Limit

3.1 Main results

The first result we shall prove rigorously in this paper is the convergence of the n dimensional ($n = 2, 3$) modulated nonlinear Klein-Gordon equation with electromagnetic potential to the anelastic system with rotation

(3.8) and without rotation (3.9) respectively. In fact, we will consider the nonrelativistic-semiclassical limit, i.e., $\nu \to 0$ and $\varepsilon \to 0$ simultaneously. In order to avoid carrying out a double limit, the two parameters ν and ε must be related, $\nu = \varepsilon^\beta, \beta \geq \alpha, 0 < \alpha < 1$. By the energy estimate discussed in the previous section, we have $\rho_0 = \phi^{\frac{1}{\gamma-1}}$ or $\phi = \rho_0^{\gamma-1}$, where ρ_0 is the limiting initial density. Thus, instead of (2.17) we will investigate the time-scaled modulated nonlinear Klein-Gordon equation with divergence free magnetic vector potential A,

$$\begin{aligned} i\varepsilon^{1-\alpha}\partial_t\psi^\varepsilon - \frac{1}{2}\varepsilon^{2+2\beta}\partial_t^2\psi^\varepsilon + \frac{1}{2}(\varepsilon^{1-\alpha}\nabla - i\varepsilon^{\beta-\alpha}A)^2\psi^\varepsilon \\ -\frac{1}{\varepsilon^{2\alpha}}\left(|\psi^\varepsilon|^{2(\gamma-1)} - \rho_0^{\gamma-1}\right)\psi^\varepsilon = 0\,. \end{aligned} \tag{3.1}$$

The initial conditions are supplemented by

$$\psi^\varepsilon(x,0) = \psi_0^\varepsilon(x)\,, \qquad \partial_t\psi^\varepsilon(x,0) = \psi_1^\varepsilon(x)\,. \tag{3.2}$$

To avoid the complications at the boundary, we concentrate below on the case when $x \in \mathbb{T}^n$, the n-dimensional torus. We define the hydrodynamical variables: Schrödinger part charge ρ^ε, relativistic part charge ρ_K^ε, Schrödinger part momentum (current) J^ε, Schrödinger part velocity u^ε, relativistic part momentum J_K^ε and the energy e^ε as follows:

$$\begin{aligned} &\rho^\varepsilon = |\psi^\varepsilon|^2, \qquad \rho_K^\varepsilon = \frac{i}{2}\varepsilon^{1+\alpha+2\beta}\left(\psi^\varepsilon\partial_t\overline{\psi^\varepsilon} - \overline{\psi^\varepsilon}\partial_t\psi^\varepsilon\right), \\ &J^\varepsilon = \frac{i}{2}\varepsilon^{1-\alpha}\left(\psi^\varepsilon\nabla\overline{\psi^\varepsilon} - \overline{\psi^\varepsilon}\nabla\psi^\varepsilon\right), \\ &J_K^\varepsilon = \frac{\rho_K^\varepsilon}{\rho^\varepsilon}J^\varepsilon + \varepsilon^{2+2\beta}\partial_t\sqrt{\rho^\varepsilon}\nabla\sqrt{\rho^\varepsilon}\,, \qquad u^\varepsilon = \frac{J^\varepsilon}{\rho^\varepsilon}\,, \\ &e^\varepsilon = \frac{1}{2}\rho^\varepsilon\left|u^\varepsilon - \varepsilon^{\beta-\alpha}A\right|^2 + \frac{\varepsilon^{2-2\alpha}}{2}\left|\nabla\sqrt{\rho^\varepsilon}\,\right|^2 + \frac{1}{2\varepsilon^{2\alpha+2\beta}}\left|\frac{\rho_K^\varepsilon}{\sqrt{\rho^\varepsilon}}\right|^2 \\ &\quad + \frac{\varepsilon^{2+2\beta}}{2}\left|\partial_t\sqrt{\rho^\varepsilon}\,\right|^2 + \frac{1}{\varepsilon^{2\alpha}}\Theta(\rho^\varepsilon)\,, \end{aligned} \tag{3.3}$$

where

$$\Theta(\rho) = \frac{1}{\gamma}\Big(\rho^{\gamma} - \rho_0^{\gamma} - \gamma\rho_0^{\gamma-1}(\rho-\rho_0)\Big), \qquad \rho > 0 \tag{3.4}$$

is a convex function, where minimum occurs at $\rho = \rho_0$ and satisfies $\Theta(\rho) \geq 0$. We also define the relative Schrödinger part momentum and velocity by $J_A^\varepsilon = J^\varepsilon - \varepsilon^{\beta-\alpha}\rho^\varepsilon A$, $u_A^\varepsilon = u^\varepsilon - \varepsilon^{\beta-\alpha}A$, respectively. The most important hydrodynamic equations associated with the time-scaled modulated nonlinear Klein-Gordon equation (3.1) are the charge, momentum and energy equations given respectively by:

(A) Charge equation

$$\frac{\partial}{\partial t}\big(\rho^\varepsilon - \rho_K^\varepsilon\big) + \nabla\cdot J_A^\varepsilon = 0\,, \tag{3.5}$$

(B) Momentum equation

$$\begin{aligned}
&\frac{\partial}{\partial t}\big(J_A^\varepsilon - J_K^\varepsilon\big) + \nabla\cdot(\rho^\varepsilon u_A^\varepsilon \otimes u_A^\varepsilon) + \frac{1}{4}\varepsilon^{2+2\beta}\nabla\partial_t^2\rho^\varepsilon \\
&+\varepsilon^{2-2\alpha}\nabla\cdot\big(\nabla\sqrt{\rho^\varepsilon}\otimes\nabla\sqrt{\rho^\varepsilon}\big) - \frac{1}{4}\varepsilon^{2-2\alpha}\nabla\Delta\rho^\varepsilon \\
&+\varepsilon^{\beta-\alpha}\rho^\varepsilon\partial_t A + \varepsilon^{\beta-\alpha}\mathbb{G}_A(J_A^\varepsilon) + \frac{1}{\varepsilon^{2\alpha}}\rho^\varepsilon\nabla\Big((\rho^\varepsilon)^{\gamma-1} - \rho_0^{\gamma-1}\Big) = 0\,,
\end{aligned} \tag{3.6}$$

(C) Energy equation

$$\frac{d}{dt}\int e^\varepsilon(\cdot,t)dx = -\varepsilon^{\beta-\alpha}\int \partial_t A\cdot J_A^\varepsilon dx\,. \tag{3.7}$$

The nonrelativistic-semiclassical limit of (3.1) depends on the relation of α and β. If $\beta = \alpha$, then the limit is the anelastic system with rotating effect $\mathbb{G}_A$ and nonconstant density ρ_0

$$\begin{cases}
\partial_t(\rho_0 u) + \big[\rho_0(u-A)\cdot\nabla\big](u-A) + \rho_0\mathbb{G}_A(u-A) + \rho_0\nabla\pi = 0\,, \\
\qquad \nabla\cdot\big[\rho_0(u-A)\big] = 0\,. \\
u(x,0) = u_0(x)\,, \qquad \nabla\cdot(\rho_0 u_0) = 0,
\end{cases} \tag{3.8}$$

However, when $\beta > \alpha$, the limit is the typical anelastic system

$$\begin{cases} \partial_t(\rho_0 u) + \nabla \cdot (\rho_0 u \otimes u) + \rho_0 \nabla \pi = 0\,, \\ \qquad \nabla \cdot (\rho_0 u) = 0\,, \\ u(x,0) = u_0(x)\,, \qquad \nabla \cdot (\rho_0 u)(x,0) = 0\,. \end{cases} \tag{3.9}$$

The well-posedness results of anelastic systems (3.8) and (3.9) can be found in [4, 12, 13]. Before the presentation of the main result of this paper, let us make the following assumptions of the initial conditions:
(A1) The initial data is assumed to be with Sobolev regularity, $(\psi_0^\varepsilon, \psi_1^\varepsilon) \in H^{s+1}(\mathbb{T}^n) \oplus H^s(\mathbb{T}^n)$, $s > \frac{n}{2} + 2$. It will guarantee the local existence and uniqueness of classical solution of the time-scaled modulated nonlinear Klein-Gordon equation (3.1).
(A2) The initial potential energy converges to 0 as ε goes to zero:

$$\varepsilon^{-2\alpha} \int_{\mathbb{T}^n} \Theta(\rho_0^\varepsilon) dx \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

(A3) The quantum and relativistic part of the initial energy converges to 0 as ε tends to zero:

$$\int_{\mathbb{T}^n} \frac{\varepsilon^{2-2\alpha}}{2} \big|\nabla\sqrt{\rho_0^\varepsilon}\,\big|^2 + \frac{1}{2\varepsilon^{2\alpha+2\beta}} \Big|\frac{\rho_{0K}^\varepsilon}{\sqrt{\rho_0^\varepsilon}}\Big|^2 + \frac{\varepsilon^{2+2\beta}}{2} \big|\partial_t \sqrt{\rho_0^\varepsilon}\,\big|^2 dx \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

(A4) The initial kinetic energy is well prepared:

$$\sqrt{\rho_0^\varepsilon} u_0^\varepsilon \to \sqrt{\rho_0} u_0 \quad \text{in} \quad L^2(\mathbb{T}^n) \quad \text{as} \quad \varepsilon \to 0.$$

(A5) The initial density is bounded away from zero, $\rho_0 \geq c > 0$, $u_0 \in (H^s(\mathbb{T}^n))^2$, $s > \frac{n}{2}+1$, and satisfies $\nabla\cdot(\rho_0 u_{0A}) = 0$ for $\beta = \alpha$ and $\nabla\cdot(\rho_0 u_0) = 0$ for $\beta > \alpha$. This condition will guarantee the local existence and uniqueness of classical solution of the rotating anelastic system (3.8) and typical anelastic system (3.9) respectively for the well prepared initial condition.

Theorem 3.1 *Let $n = 2$ or 3, $0 < \alpha < 1$, $\beta \geq \alpha$, $\gamma \geq 2$ and ψ^ε be the solution of the time scaled modulated nonlinear Klein-Gordon equation* (3.1) *with divergence free magnetic potential, $\nabla \cdot A = 0$, and the initial condition*

$(\psi_0^\varepsilon, \psi_1^\varepsilon)$ *satisfying the assumptions* (A1)–(A5)*. Then there exists* $T > 0$ *such that*

$$\|(\rho^\varepsilon - \rho_0)(\cdot,t)\|_{L^\gamma(\mathbb{T}^n)} \to 0\,, \qquad \|\rho_K^\varepsilon(\cdot,t)\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^n)} \to 0\,, \tag{3.10}$$

$$\|(J^\varepsilon - \rho_0 u)(\cdot,t)\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^n)} \to 0\,, \quad \|J_K^\varepsilon(\cdot,t)\|_{L^1(\mathbb{T}^n)} \to 0\,, \tag{3.11}$$

for all $t \in [0,T]$ *as* $\varepsilon \to 0$*, where* u *is the unique local smooth solution of the rotating anelastic approximation system* (3.8) *for* $\beta = \alpha$ *and anelastic system* (3.9) *for* $\beta > \alpha$ *rspectively.*

3.2 Proof of Theorem 3.1

The proof of Theorem 3.1 is based on Lemma 3.2 and Lemma 3.3 given below.

Lemma 3.2 *Under the same hypothesis of Theorem* 3.1*, we have*

$$\|\rho_K^\varepsilon(\cdot,t)\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^n)} = O(\varepsilon^{\alpha+\beta})\,, \quad \|J_K^\varepsilon(\cdot,t)\|_{L^1(\mathbb{T}^n)} = O(\varepsilon^{\alpha+\beta})\,, \tag{3.12}$$

and

$$\|(\rho^\varepsilon - \rho_0)(\cdot,t)\|_{L^\gamma(\mathbb{T}^n)} = O\big(\varepsilon^{\frac{2\alpha}{\gamma}}\big)\,, \qquad t \in [0,T]\,. \tag{3.13}$$

Proof. Applying the Cauchy-Schwarz and Young's inequalities, we obtain from the charge equation (3.5) that

$$\int_{\mathbb{T}^n} \rho^\varepsilon dx \le C + \int_{\mathbb{T}^n} \left(\varepsilon^{2\alpha+2\beta}\rho^\varepsilon + \frac{1}{\varepsilon^{2\alpha+2\beta}} \Big| \frac{\rho_K^\varepsilon}{\sqrt{\rho^\varepsilon}} \Big|^2 \right) dx\,, \tag{3.14}$$

for all $t \in [0,T]$, i.e.,

$$\int_{\mathbb{T}^n} \rho^\varepsilon dx \le C + \int_{\mathbb{T}^n} \frac{1}{\varepsilon^{2\alpha+2\beta}} \Big| \frac{\rho_K^\varepsilon}{\sqrt{\rho^\varepsilon}} \Big|^2 dx\,. \tag{3.15}$$

We also derive from the assumptions (A1)–(A4) and the equation of energy (3.7) the following estimate

$$\int_{\mathbb{T}^n} e^\varepsilon(x,t)dx \le C + \int_0^t \int_{\mathbb{T}^n} \varepsilon^{\beta-\alpha} |\partial_t A| \Big(|\sqrt{\rho^\varepsilon} u_A^\varepsilon|^2 + \rho^\varepsilon \Big) dx d\tau\,. \tag{3.16}$$

Combining (3.14) and (3.16) together yields

$$\int_{\mathbb{T}^n} E^\varepsilon(x,t)dx \le C_1 + C_2 \int_0^t \int_{\mathbb{T}^n} E^\varepsilon(x,s)dxds\,, \tag{3.17}$$

where

$$\begin{aligned} E^\varepsilon(x,t) = \rho^\varepsilon + \rho^\varepsilon\big|u_A^\varepsilon\big|^2 + \frac{\varepsilon^{2-2\alpha}}{2}\big|\nabla\sqrt{\rho^\varepsilon}\,\big|^2 + \frac{1}{2\varepsilon^{2\alpha+2\beta}}\Big|\frac{\rho_K^\varepsilon}{\sqrt{\rho^\varepsilon}}\Big|^2 \\ + \frac{\varepsilon^{2+2\beta}}{2}\big|\partial_t\sqrt{\rho^\varepsilon}\,\big|^2 + \frac{1}{\varepsilon^{2\alpha}}\Theta(\rho^\varepsilon)\,, \end{aligned} \tag{3.18}$$

then the Gronwall inequality gives the uniform bound

$$\int_{\mathbb{T}^n} E^\varepsilon(x,t)dx \le C\,. \tag{3.19}$$

In particular, we have

$$\int_{\mathbb{T}^n} \Theta(\rho^\varepsilon)dx = O(\varepsilon^{2\alpha})\,. \tag{3.20}$$

It is easy to see that (3.20) will imply (3.13) by the following elementary inequality

$$\frac{1}{\gamma}\big|\rho^\varepsilon - \rho_0\big|^\gamma \le \Theta(\rho^\varepsilon) \quad \text{if} \quad \gamma \ge 2\,. \tag{3.21}$$

To prove the estimates (3.12), we deduce from (3.19) and the Hölder inequality that

$$\|\rho_K^\varepsilon\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^n)} \le \big\|\sqrt{\rho^\varepsilon}\big\|_{L^{2\gamma}(\mathbb{T}^n)}\Big\|\frac{1}{\sqrt{\rho^\varepsilon}}\rho_K^\varepsilon\Big\|_{L^2(\mathbb{T}^n)} \tag{3.22}$$

and

$$\begin{aligned} \|J_K^\varepsilon\|_{L^1(\mathbb{T}^n)} \le \big\|\sqrt{\rho^\varepsilon}u^\varepsilon\big\|_{L^2(\mathbb{T}^n)}\Big\|\frac{1}{\sqrt{\rho^\varepsilon}}\rho_K^\varepsilon\Big\|_{L^2(\mathbb{T}^n)} \\ + \varepsilon^{\alpha+\beta}\big\|\varepsilon^{1+\beta}\partial_t\sqrt{\rho^\varepsilon}\big\|_{L^2(\mathbb{T}^n)}\big\|\varepsilon^{1-\alpha}\nabla\sqrt{\rho^\varepsilon}\big\|_{L^2(\mathbb{T}^n)}\,. \end{aligned} \tag{3.23}$$

Therefore we have proved the estimates (3.12). ∎

Lemma 3.3 *Let $v = u - A$ for $\beta = \alpha$ and $v = u$ for $\beta > \alpha$. Under the same hypothesis of Theorem 3.1, the modulated energy defined by*

$$
\begin{aligned}
H^\varepsilon(t) &= \frac{1}{2}\int_{\mathbb{T}^n} \rho^\varepsilon \left|u_A^\varepsilon - v\right|^2 dx + \frac{\varepsilon^{2-2\alpha}}{2}\int_{\mathbb{T}^n} \left|\nabla\sqrt{\rho^\varepsilon}\,\right|^2 dx \\
&+ \frac{1}{2\varepsilon^{2\alpha+2\beta}}\int_{\mathbb{T}^n} \left|\frac{\rho_K^\varepsilon}{\sqrt{\rho^\varepsilon}}\right|^2 dx + \frac{\varepsilon^{2+2\beta}}{2}\int_{\mathbb{T}^n} \left|\partial_t\sqrt{\rho^\varepsilon}\,\right|^2 dx + \frac{1}{\varepsilon^{2\alpha}}\int_{\mathbb{T}^n} \Theta(\rho^\varepsilon) dx\,,
\end{aligned}
\tag{3.24}
$$

satisfies the decay property

$$
H^\varepsilon(t) \to 0 \quad as \quad \varepsilon \to 0\,, \quad t \in [0,T]\,.
$$

Proof. The modulated energy $H^\varepsilon(t)$ can be further rewritten in terms of the hydrodynamic variables as

$$
H^\varepsilon(t) = \int_{\mathbb{T}^n} e^\varepsilon dx - \int_{\mathbb{T}^n} v \cdot J_A^\varepsilon dx + \frac{1}{2}\int_{\mathbb{T}^n} \rho^\varepsilon |v|^2 dx\,. \tag{3.25}
$$

Differentiating the modulated energy (3.25) with respect to t and using energy equation (3.7), we obtain

$$
\begin{aligned}
\frac{d}{dt}H^\varepsilon(t) &= -\int_{\mathbb{T}^n} \varepsilon^{\beta-\alpha}\partial_t A \cdot J_A^\varepsilon dx - \frac{d}{dt}\int_{\mathbb{T}^n} v \cdot J_A^\varepsilon dx + \frac{d}{dt}\int_{\mathbb{T}^n} \frac{1}{2}\rho^\varepsilon |v|^2 dx \\
&\equiv I_1 + I_2 + I_3\,.
\end{aligned}
\tag{3.26}
$$

Employing the momentum equation (3.6) and integration by part, we can write the integral I_2 as

$$
\begin{aligned}
I_2 = & -\frac{d}{dt}\int_{\mathbb{T}^n} v\cdot J_K^\varepsilon + \frac{1}{4}\varepsilon^{2+2\beta}\nabla\cdot v\partial_t\rho^\varepsilon dx + \int_{\mathbb{T}^n}\varepsilon^{\beta-\alpha}\rho^\varepsilon v\cdot\partial_t A dx \\
& -\int_{\mathbb{T}^n}\partial_t v\cdot(J_A^\varepsilon - J_K^\varepsilon)dx + \int_{\mathbb{T}^n}\varepsilon^{\beta-\alpha} v\cdot\mathbb{G}_A(J_A^\varepsilon)dx \\
& -\int_{\mathbb{T}^n}\Big[\big(\rho^\varepsilon u_A^\varepsilon\otimes u_A^\varepsilon\big) + \varepsilon^{2-2\alpha}\big(\nabla\sqrt{\rho^\varepsilon}\otimes\nabla\sqrt{\rho^\varepsilon}\big)\Big] : \nabla v dx \\
& -\frac{\varepsilon^{2-2\alpha}}{4}\int_{\mathbb{T}^n}\nabla\rho^\varepsilon\cdot(\nabla\nabla\cdot v)dx - \frac{1}{\varepsilon^{2\alpha}}\int_{\mathbb{T}^n}\rho^\varepsilon v\cdot\nabla\Big((\rho^\varepsilon)^{\gamma-1} - \rho_0^{\gamma-1}\Big)dx \\
& +\frac{1}{4}\varepsilon^{2+2\beta}\int_{\mathbb{T}^n}\nabla\cdot\partial_t v\partial_t\rho^\varepsilon dx\,.
\end{aligned}
\tag{3.27}
$$

Applying the same idea as above to the charge equation (3.5), we have

$$
\begin{aligned}
I_3 = & \frac{d}{dt}\int_{\mathbb{T}^n}\frac{1}{2}\rho_K^\varepsilon|v|^2 dx + \int_{\mathbb{T}^n}\rho^\varepsilon v\cdot\partial_t v dx \\
& + \int_{\mathbb{T}^n}\frac{1}{2}\nabla|v|^2\cdot J_A^\varepsilon dx - \int_{\mathbb{T}^n}\frac{1}{2}\rho_K^\varepsilon\partial_t|v|^2 dx\,.
\end{aligned}
\tag{3.28}
$$

To discuss the hydrodynamic limit, we introduce the relativistic correction term of the modulation energy defined by

$$
G^\varepsilon(t) = -\frac{1}{2}\int_{\mathbb{T}^n}\rho_K^\varepsilon|v|^2 dx + \int_{\mathbb{T}^n} v\cdot J_K^\varepsilon + \frac{1}{4}\varepsilon^{2+2\beta}\nabla\cdot v\partial_t\rho^\varepsilon dx. \tag{3.29}
$$

Combining (3.26)–(3.29), we have the evolution of the modified modulated energy $H^\varepsilon(t) + G^\varepsilon(t)$

$$
\begin{aligned}
& \frac{d}{dt}(H^\varepsilon(t) + G^\varepsilon(t)) \\
= & \int_{\mathbb{T}^n}\frac{1}{2}\nabla|v|^2\cdot J_A^\varepsilon dx + \int_{\mathbb{T}^n}\partial_t(v + \varepsilon^{\beta-\alpha})\cdot(\rho^\varepsilon v - J_A^\varepsilon)dx \\
& -\int_{\mathbb{T}^n}\varepsilon^{2-2\alpha}\big(\nabla\sqrt{\rho^\varepsilon}\otimes\nabla\sqrt{\rho^\varepsilon}\big) : \nabla v dx + R_1 + R_2 + R_3 + R_4
\end{aligned}
\tag{3.30}
$$

where R_1, R_2, R_3 and R_4 are defined respectively by

$$R_1 = -\int_{\mathbb{T}^n} \big(\rho^\varepsilon u_A^\varepsilon \otimes u_A^\varepsilon\big) : \nabla v dx \,,$$

$$R_2 = \frac{1}{\varepsilon^{2\alpha}} \int_{\mathbb{T}^n} \rho^\varepsilon v \cdot \nabla\Big((\rho^\varepsilon)^{\gamma-1} - \rho_0^{\gamma-1}\Big) dx \,,$$

$$R_3 = \int_{\mathbb{T}^n} \varepsilon^{\beta-\alpha} v \cdot \mathbb{G}_A(J_A^\varepsilon) dx \,,$$

$$R_4 = \int_{\mathbb{T}^n} \partial_t v \cdot J_K^\varepsilon - \frac{1}{2}\rho_K^\varepsilon \partial_t |v|^2 - \frac{\varepsilon^{2-2\alpha}}{4} \nabla\rho^\varepsilon \cdot (\nabla\nabla \cdot v)$$
$$+\frac{1}{4}\varepsilon^{2+2\beta} \nabla \cdot \partial_t v \partial_t \rho^\varepsilon dx \,.$$

We can rewrite the kinetic part R_1 as

$$\begin{aligned} R_1 = &-\int_{\mathbb{T}^n} \big(\rho^\varepsilon (u_A^\varepsilon - v) \otimes (u_A^\varepsilon - v)\big) : \nabla v dx \\ &-\int_{\mathbb{T}^n} \big(\rho^\varepsilon v \otimes u_A^\varepsilon\big) : \nabla v dx \\ &+\int_{\mathbb{T}^n} \big(\rho^\varepsilon v \otimes v\big) : \nabla v dx - \int_{\mathbb{T}^n} \big(\rho^\varepsilon u_A^\varepsilon \otimes v\big) : \nabla v dx, \end{aligned} \tag{3.31}$$

where $A : B$ denotes the trace of the product of the tensors, such as $A : B \equiv \mathrm{tr}(AB)$. Simple computation gives the following two equalities

$$-\int_{\mathbb{T}^n} \big(\rho^\varepsilon u_A^\varepsilon \otimes v\big) : \nabla v dx = \int_{\mathbb{T}^n} \frac{1}{2} |v|^2 \nabla \cdot J_A^\varepsilon dx \,, \tag{3.32}$$

and

$$\begin{aligned} &\int_{\mathbb{T}^n} \big(\rho^\varepsilon v \otimes v\big) : \nabla v dx - \int_{\mathbb{T}^n} \big(\rho^\varepsilon v \otimes u_A^\varepsilon\big) : \nabla v dx \\ &= \int_{\mathbb{T}^n} \Big[\big(v \cdot \nabla\big) v\Big] \cdot (\rho^\varepsilon v - J_A^\varepsilon) dx \,. \end{aligned} \tag{3.33}$$

Thus R_1 given by (3.31) becomes

$$\begin{aligned} R_1 = &-\int_{\mathbb{T}^n} \big(\rho^\varepsilon(u_A^\varepsilon - v)\otimes(u_A^\varepsilon - v)\big) : \nabla v dx \\ &+ \int_{\mathbb{T}^n} \frac{1}{2}|v|^2 \nabla\cdot J_A^\varepsilon dx + \int_{\mathbb{T}^n} \Big[\big(v\cdot\nabla\big)v\Big]\cdot(\rho^\varepsilon v - J_A^\varepsilon)dx\,. \end{aligned} \tag{3.34}$$

To deal with the potential part R_2, we employ the relation

$$\rho^\varepsilon v\cdot\nabla(\rho^\varepsilon)^{\gamma-1} = \tfrac{\gamma-1}{\gamma} v\cdot\nabla(\rho^\varepsilon)^\gamma\,, \tag{3.35}$$

and use the divergence free of $\rho_0 v$ to obtain

$$-\rho^\varepsilon v\cdot\nabla\rho_0^{\gamma-1} = (\gamma-1)\rho^\varepsilon\rho_0^{\gamma-1}\nabla\cdot v \tag{3.36}$$

then combining (3.35) and (3.36) together we obtain

$$\rho^\varepsilon v\cdot\nabla\Big((\rho^\varepsilon)^{\gamma-1} - \rho_0^{\gamma-1}\Big) = \tfrac{\gamma-1}{\gamma}\Big[v\cdot\nabla(\rho^\varepsilon)^\gamma + \gamma\rho^\varepsilon\rho_0^{\gamma-1}\nabla\cdot v\Big]\,. \tag{3.37}$$

Moreover, using integration by parts and divergence free of $\rho_0 v$ again, we have

$$\int_{\mathbb{T}^n}\rho_0^\gamma\nabla\cdot v dx = -\int_{\mathbb{T}^n} v\cdot\nabla\rho_0^\gamma dx = -\frac{\gamma}{\gamma-1}\int_{\mathbb{T}^n}\rho_0 v\cdot\nabla\rho_0^{\gamma-1}dx = 0\,. \tag{3.38}$$

Consequently, by (3.37) and (3.38) we can represent R_2 as

$$R_2 = -\frac{1}{\varepsilon^{2\alpha}}\int_{\mathbb{T}^n}\frac{\gamma-1}{\gamma}\nabla\cdot v\Big[(\rho^\varepsilon)^\gamma - \gamma\rho^\varepsilon\rho_0^{\gamma-1} + (\gamma-1)\rho_0^\gamma\Big]dx\,. \tag{3.39}$$

For the rotating part R_3, we have

$$\begin{aligned} v\cdot\mathbb{G}_A(J_A^\varepsilon) &= v\cdot\big(\operatorname{curl} A\times J_A^\varepsilon\big) = -J_A^\varepsilon\cdot\big(\operatorname{curl} A\times v\big) \\ &= (\rho^\varepsilon v - J_A^\varepsilon - \rho^\varepsilon v)\cdot\big(\operatorname{curl} A\times v\big) = \mathbb{G}_A(v)\cdot(\rho^\varepsilon v - J_A^\varepsilon)\,, \end{aligned}$$

and hence

$$R_3 = \int_{\mathbb{T}^n}\varepsilon^{\beta-\alpha}v\cdot\mathbb{G}_A(J_A^\varepsilon)d = \int_{\mathbb{T}^n}\varepsilon^{\beta-\alpha}\mathbb{G}_A(v)\cdot(\rho^\varepsilon v - J_A^\varepsilon)dx\,.$$

To treat the quantum-relativistic potential part R_4, we need the following inequalities

$$\int_{\mathbb{T}^n} \partial_t v \cdot J_K^\varepsilon dx \lesssim \varepsilon^{\alpha+\beta} \|\partial_t v\|_{L^\infty(\mathbb{T}^n)}\,, \tag{3.40}$$

$$\int_{\mathbb{T}^n} \rho_K^\varepsilon \partial_t |v|^2 dx \quad \lesssim \quad \varepsilon^{\alpha+\beta} \|\partial_t |v|^2\|_{L^{\frac{2\gamma}{\gamma-1}}(\mathbb{T}^n)}\,, \tag{3.41}$$

and

$$\int_{\mathbb{T}^n} \varepsilon^{\beta-\alpha} \rho_K^\varepsilon v \cdot \partial_t A dx \lesssim \varepsilon^{2\beta} \|v\|_{L^\infty(\mathbb{T}^n)}\,. \tag{3.42}$$

Furthermore, by the Hölder inequality we have the estimates

$$\begin{aligned} &\varepsilon^{2-2\alpha} \int_{\mathbb{T}^n} \nabla \rho^\varepsilon \cdot (\nabla\nabla \cdot v) dx \\ &\leq \varepsilon^{1-\alpha} \|\varepsilon^{1-\alpha} \nabla \sqrt{\rho^\varepsilon}\|_{L^2(\mathbb{T}^n)} \|\sqrt{\rho^\varepsilon}\|_{L^{2\gamma}(\mathbb{T}^n)} \|\nabla\nabla \cdot v\|_{L^{\frac{2\gamma}{\gamma-1}}(\mathbb{T}^n)} \\ &\lesssim \varepsilon^{1-\alpha} \|v\|_{H^s(\mathbb{T}^n)}\,, \end{aligned} \tag{3.43}$$

and

$$\begin{aligned} &\varepsilon^{2+2\beta} \int_{\mathbb{T}^n} \nabla \cdot \partial_t v \partial_t \rho^\varepsilon dx \\ &\leq \varepsilon^{1+\beta} \|\varepsilon^{1+\alpha} \partial_t \sqrt{\rho^\varepsilon}\|_{L^2(\mathbb{T}^n)} \|\sqrt{\rho^\varepsilon}\|_{L^{2\gamma}(\mathbb{T}^n)} \|\nabla \cdot \partial_t v\|_{L^{\frac{2\gamma}{\gamma-1}}(\mathbb{T}^n)} \\ &\lesssim \varepsilon^{1+\beta} \|v\|_{H^s(\mathbb{T}^n)}\,. \end{aligned} \tag{3.44}$$

Thus $R_4 \to 0$ as $\varepsilon \to 0$. Combining the above estimates, the evolution of the modified modulated energy (3.30) becomes

$$
\begin{aligned}
\frac{d}{dt}\Big(H^\varepsilon(t)+G^\varepsilon(t)\Big) &= -\int_{\mathbb{T}^n}\big(\rho^\varepsilon(u_A^\varepsilon-v)\otimes(u_A^\varepsilon-v)\big):\nabla v dx\\
&-\int_{\mathbb{T}^n}\varepsilon^{2-2\alpha}\big(\nabla\sqrt{\rho^\varepsilon}\otimes\nabla\sqrt{\rho^\varepsilon}\big):\nabla v dx\\
&-\frac{1}{\varepsilon^{2\alpha}}\int_{\mathbb{T}^n}\frac{\gamma-1}{\gamma}\Big[(\rho^\varepsilon)^\gamma-\gamma\rho^\varepsilon\rho_0^{\gamma-1}+(\gamma-1)\rho_0^\gamma\Big]\nabla\cdot v dx\\
&+\int_{\mathbb{T}^n}\Big[\partial_t(v+\varepsilon^{\beta-\alpha}A)+\big(v\cdot\nabla\big)v+\varepsilon^{\beta-\alpha}\mathbb{G}_A(v)\Big]\cdot(\rho^\varepsilon v-J_A^\varepsilon)dx+o(1)\,.
\end{aligned}
\tag{3.45}
$$

We will estimate the first three integrals of the right side of (3.45), and show that they can be bounded by $\|\nabla v\|_{L^\infty(\mathbb{T}^n)}H^\varepsilon(t)$. Indeed, if $\beta>\alpha$, we have (by uniform bound of the total energy and the Hölder inequality)

$$
\int_\Omega\big[\partial_t A+\mathbb{G}_A(v)\big]\cdot(\rho^\varepsilon v-J_A^\varepsilon)dx\lesssim\|\rho^\varepsilon\|_{L^1(\mathbb{T}^n)}+\|\rho^\varepsilon\|^2_{L^1(\mathbb{T}^n)}\|\sqrt{\rho^\varepsilon}u_A^\varepsilon\|_{L^2(\mathbb{T}^n)}\,.
\tag{3.46}
$$

Note that (3.45) can be transformed into

$$
\frac{d}{dt}\Big(H^\varepsilon(t)+G^\varepsilon(t)\Big)\le C_1H^\varepsilon(t)-\int_{\mathbb{T}^n}\nabla\pi\cdot(\rho^\varepsilon v-J_A^\varepsilon)dx+o(1)\,.
\tag{3.47}
$$

Now we will estimate the second term of the right side of (3.47). By (3.12), (3.13) and the Hölder inequality, we arrive at the inequality

$$
\begin{aligned}
\int_{\mathbb{T}^n}\rho^\varepsilon v\cdot\nabla\pi dx &= \int_{\mathbb{T}^n}(\rho^\varepsilon-\rho_0)v\cdot\nabla\pi dx\\
&\lesssim\varepsilon^{\frac{2\alpha}{\gamma}}\|v\|_{L^\infty(\mathbb{T}^n)}\|\nabla\pi\|_{L^{\frac{\gamma}{\gamma-1}}(\mathbb{T}^n)}\,.
\end{aligned}
\tag{3.48}
$$

To proceed, we need the relation

$$
\begin{aligned}
&\int_{\mathbb{T}^n}J_A^\varepsilon\cdot\nabla\pi dx=\int_{\mathbb{T}^n}\pi\partial_t\big[(\rho^\varepsilon-\rho_0)-\rho_K^\varepsilon\big]dx\\
&=\frac{d}{dt}\int_{\mathbb{T}^n}\pi[(\rho^\varepsilon-\rho_0)-\rho_K^\varepsilon]dx-\int_{\mathbb{T}^n}\partial_t\pi[(\rho^\varepsilon-\rho_0)-\rho_K^\varepsilon]dx\,,
\end{aligned}
\tag{3.49}
$$

and the last integral of (3.49) can be estimated by the Hölder inequality

$$\int_{\mathbb{T}^n} \partial_t\pi[(\rho^\varepsilon - \rho_0) - \rho_K^\varepsilon]dx \lesssim \varepsilon^{\frac{2\alpha}{\gamma}} \|\partial_t\pi\|_{L^{\frac{\gamma}{\gamma-1}}(\mathbb{T}^n)} + \varepsilon^{\alpha+\beta}\|\partial_t\pi\|_{L^{\frac{2\gamma}{\gamma-1}}(\mathbb{T}^n)} . \quad (3.50)$$

In addition to $G^\varepsilon(t)$, we have to introduce one more correction term W^ε of the modulated energy $H^\varepsilon(t)$ defined by

$$W^\varepsilon(t) = \int_{\mathbb{T}^n} \big[\rho_K^\varepsilon - (\rho^\varepsilon - \rho_0)\big]\pi dx\,. \quad (3.51)$$

Note that $W^\varepsilon(t)$ can be served as the acoustic part (density fluctuation) of the modulated energy $H^\varepsilon(t)$. It is designed to control the propagation of the acoustic wave. For $t \in [0, T]$ we have

$$\frac{d}{dt}\Big(H^\varepsilon(t) + G^\varepsilon(t) + W^\varepsilon(t)\Big) \lesssim C_1 H^\varepsilon(t) + o(1)\,, \quad (3.52)$$

then after integrating it becomes

$$\begin{aligned} H^\varepsilon(t) &\leq H^\varepsilon(0) + G^\varepsilon(0) + W^\varepsilon(0) - G^\varepsilon(t) - W^\varepsilon(t) \\ &\quad + C_1 \int_0^t H^\varepsilon(\tau)d\tau + o(1)\,. \end{aligned} \quad (3.53)$$

One can show that $G^\varepsilon(0) + W^\varepsilon(0) - G^\varepsilon(t) - W^\varepsilon(t) = o(1)$, and hence

$$H^\varepsilon(t) \leq C_1 \int_0^t H^\varepsilon(\tau)d\tau + H^\varepsilon(0) + o(1). \quad (3.54)$$

In order to obtain the convergent results, we have to estimate the initial modulated energy functional $H^\varepsilon(0)$. For $\beta = \alpha$, it is easy to see that

$$\begin{aligned} &\|\sqrt{\rho_0^\varepsilon}u_0^\varepsilon - \sqrt{\rho_0^\varepsilon}u_0\|_{L^2(\mathbb{T}^n)} \\ \leq\ &\|\sqrt{\rho_0^\varepsilon}u_0^\varepsilon - \sqrt{\rho_0}u_0\|_{L^2(\mathbb{T}^n)} + \|(\sqrt{\rho_0} - \sqrt{\rho_0^\varepsilon})u_0\|_{L^2(\mathbb{T}^n)}\,, \end{aligned} \quad (3.55)$$

and the first term of the right side of (3.55) converges to 0 by assumption (A4). For the second term, using the boundedness of $|\mathbb{T}^n|$, assumption (A1) and an elementary inequality

$$|\sqrt{x} - \sqrt{a}|^2 \leq a^{-1}|x - a|^2,$$

for $x \geq 0$ and $a \geq c > 0$, we have

$$\begin{aligned}\|(\sqrt{\rho_0} - \sqrt{\rho_0^\varepsilon})u_0\|_{L^2(\mathbb{T}^n)} &\leq \|u_0\|_{L^\infty(\mathbb{T}^n)}\|\sqrt{\rho_0} - \sqrt{\rho_0^\varepsilon}\|_{L^2(\mathbb{T}^n)} \\ &\leq \frac{1}{\sqrt{\rho_0}}\|u_0\|_{L^\infty(\mathbb{T}^n)}\|\rho_0 - \rho_0^\varepsilon\|_{L^2(\mathbb{T}^n)} \\ &\leq C\|u_0\|_{L^\infty(\mathbb{T}^n)}\|\rho_0 - \rho_0^\varepsilon\|_{L^\gamma(\mathbb{T}^n)}\,,\end{aligned} \tag{3.56}$$

which converges to 0 as ε tends to 0 by assumption (A2). Thus $H^\varepsilon(0) \to 0$ as $\varepsilon \to 0$. Then applying the Gronwall's inequality we can show that $H^\varepsilon(t) \to 0$ for all $t \in [0,T]$ as ε tends to 0. The case $\beta > \alpha$ is similar, and we omit the detail. ∎

Proof of Theorem 3.1. We rewrite the modulated energy $H^\varepsilon(t)$ defined by (3.24) as

$$\begin{aligned}H^\varepsilon(t) = &\frac{\varepsilon^{2-2\alpha}}{2}\int_{\mathbb{T}^n}\left|\nabla\sqrt{\rho^\varepsilon}\right|^2 dx + \frac{1}{2}\int_{\mathbb{T}^n}\left|\frac{1}{\sqrt{\rho^\varepsilon}}(J_A^\varepsilon - \rho^\varepsilon v)\right|^2 dx \\ &+\frac{1}{2\varepsilon^{2\alpha+2\beta}}\int_{\mathbb{T}^n}\left|\frac{\rho_K^\varepsilon}{\sqrt{\rho^\varepsilon}}\right|^2 dx + \frac{\varepsilon^{2+2\beta}}{2}\int_{\mathbb{T}^n}\left|\partial_t\sqrt{\rho^\varepsilon}\right|^2 dx + \frac{1}{\varepsilon^{2\alpha}}\int_{\mathbb{T}^n}\Theta(\rho^\varepsilon)dx\,,\end{aligned} \tag{3.57}$$

then from Lemma 3.3 and (3.57) we have

$$\int_{\mathbb{T}^n}\left|\frac{1}{\sqrt{\rho^\varepsilon}}(J^\varepsilon - \rho^\varepsilon u)\right|^2 dx \to 0 \tag{3.58}$$

for $\beta = \alpha$ and

$$\int_{\mathbb{T}^n}\left|\frac{1}{\sqrt{\rho^\varepsilon}}(J_A^\varepsilon - \rho^\varepsilon u)\right|^2 dx \to 0 \tag{3.59}$$

for $\beta > \alpha$, as $\varepsilon \to 0$. Firstly, if $\beta = \alpha$, we deduce from (3.58) and the Hölder inequality that

$$\begin{aligned}\|(J^\varepsilon - \rho_0 u)\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^n)} \leq &\left\|\sqrt{\rho^\varepsilon}\right\|_{L^{2\gamma}(\mathbb{T}^n)}\left\|\frac{1}{\sqrt{\rho^\varepsilon}}(J^\varepsilon - \rho^\varepsilon u)\right\|_{L^2(\mathbb{T}^n)} \\ &+\|\rho^\varepsilon - \rho_0\|_{L^\gamma(\mathbb{T}^n)}\|u\|_{L^2(\mathbb{T}^n)}\,,\end{aligned} \tag{3.60}$$

which converges to zero as $\varepsilon \to 0$. This resolves the case $\beta = \alpha$. The case for $\beta > \alpha$ is similar and we omit the details. This completes the proof of Theorem 3.1. ∎

4 Singular Limits

4.1 Main results

In this section, we will consider the singular limit of the modulated nonlinear Klein-Gordon equation with Ginzburg-Landau type potential directly without transforming into the hydrodynamic equations. We limit our discussion to the nonrelativistic-semiclassical limit to avoid carrying out double limits. In this case Planck's constant $\hbar$ and the speed of light c are chosen such that $\hbar = \varepsilon$ and $c = \varepsilon^{-\beta}$, $0 < \varepsilon \ll 1, \beta \geq 1$. After proper rescaling or nondimensionlization, we may assume the unit mass $m = 1$ and unit charge $e = 1$, then (1.3) is rewritten as

$$i\partial_t \psi^\varepsilon - \frac{\varepsilon^{1+2\beta}}{2}\partial_t^2 \psi^\varepsilon + \frac{\varepsilon}{2}(\nabla - i\varepsilon^{\beta-1}A)^2\psi^\varepsilon - \Big(\frac{|\psi^\varepsilon|^2 - \phi}{\varepsilon}\Big)\psi^\varepsilon = 0\,, \qquad (4.1)$$

where $\nabla - i\varepsilon^{\beta-1}A$ is the covariant derivative and the initial conditions are supplemented by

$$\psi^\varepsilon(x,0) = \psi_0^\varepsilon(x)\,, \qquad \partial_t\psi^\varepsilon(x,0) = \psi_1^\varepsilon(x)\,. \qquad (4.2)$$

Similar to the hydrodynamic limits discussed in the previous section, we concentrate below on the n-dimensional torus $\mathbb{T}^n$ and state the existence theorem of (4.1)–(4.2) for finite initial energy.

Theorem 4.1 *Let* $0 < \varepsilon \ll 1$, $\beta \geq 1$, $\phi(x) \in C^1(\mathbb{T}^n)$ *satisfying* $\phi(x) \geq c > 0$ *and* $A(x,t) \in C^1([0,\infty) \times \mathbb{T}^n)$ *for all* $x \in \mathbb{T}^n$ *and* $t > 0$. *Given* $(\psi_0^\varepsilon, \psi_1^\varepsilon) \in H^1(\mathbb{T}^n) \oplus L^2(\mathbb{T}^n)$ *and* $\frac{|\psi_0^\varepsilon|^2 - \phi}{\varepsilon} \in L^2(\mathbb{T}^n)$, *there exists a function* ψ^ε *such that*

$$\psi^\varepsilon \in L^\infty([0,T]; H^1(\mathbb{T}^n)) \cap C([0,T]; L^2(\mathbb{T}^n))\,,$$
$$\partial_t\psi^\varepsilon \in L^\infty([0,T]; L^2(\mathbb{T}^n)) \cap C([0,T]; H^{-1}(\mathbb{T}^n))\,,$$
$$\frac{|\psi^\varepsilon|^2 - \phi}{\varepsilon} \in L^\infty([0,T]; L^2(\mathbb{T}^n))\,,$$

and solves the weak formulation of (4.1) *given by*

$$i\Big\langle \psi^\varepsilon(\cdot,t), \varphi\Big\rangle\Big|_{t=t_1}^{t_2} + \frac{\varepsilon}{2}\int_{t_1}^{t_2} \Big\langle (\nabla - i\varepsilon^{\beta-1}A)\psi^\varepsilon(\cdot,\tau), (\nabla + i\varepsilon^{\beta-1}A)\varphi\Big\rangle d\tau$$
$$-\frac{\varepsilon^{2\beta}}{2}\Big\langle \varepsilon\partial_t\psi^\varepsilon(\cdot,t), \varphi\Big\rangle\Big|_{t=t_1}^{t_2} - \int_{t_1}^{t_2} \Big\langle \Big(\frac{|\psi^\varepsilon|^2 - \phi}{\varepsilon}\Big)\psi^\varepsilon(\cdot,\tau), \varphi\Big\rangle\, d\tau = 0\,, \qquad (4.3)$$

for every $[t_1, t_2] \subset [0, T]$ *and for all* $\varphi \in C_0^\infty(\mathbb{T}^n)$. *Moreover, it satisfies the charge-energy inequality*

$$\sup_{t\in[0,T]} \int_{\mathbb{T}^n} |\psi^\varepsilon|^2 + |\varepsilon^\beta \partial_t \psi^\varepsilon|^2 + |\nabla\psi^\varepsilon|^2 + \Big(\frac{|\psi^\varepsilon|^2 - \phi}{\varepsilon}\Big)^2 dx \le C\,, \tag{4.4}$$

where $C = C(\psi_0^\varepsilon, \psi_1^\varepsilon, T, \|(\phi, A, \partial_t A)\|_{L^\infty([0,T]\times\mathbb{T}^n)})$ *is a constant.*

We make a formal analysis on the model (4.1)–(4.2) and begin with a derivation of the apriori estimates through the standard energy estimate. For simplicity, we define some quantities associated with the modulated cubic nonlinear Klein-Gordon equation (4.1) as follows:

$$W(\psi^\varepsilon) = \frac{i}{2}\Big(\psi^\varepsilon \nabla\overline{\psi^\varepsilon} - \overline{\psi^\varepsilon}\nabla\psi^\varepsilon\Big), \quad Z(\psi^\varepsilon) = \frac{i}{2}\varepsilon^{2\beta}\Big(\overline{\psi^\varepsilon}\partial_t\psi^\varepsilon - \psi^\varepsilon\partial_t\overline{\psi^\varepsilon}\Big),$$

$$e^\varepsilon = \frac{1}{2}|\varepsilon^\beta\partial_t\psi^\varepsilon|^2 + \frac{1}{2}|(\nabla - i\varepsilon^{\beta-1}A)\psi^\varepsilon|^2 + \frac{1}{2}\Big(\frac{|\psi^\varepsilon|^2 - \phi}{\varepsilon}\Big)^2.$$

The equations of charge and energy associated with (4.1) are given below:
(A) Equation of charge

$$\frac{\partial}{\partial t}\Big[|\psi^\varepsilon|^2 + \varepsilon Z(\psi^\varepsilon)\Big] + \varepsilon\nabla\cdot\Big[W(\psi^\varepsilon) - \varepsilon^{\beta-1}|\psi^\varepsilon|^2 A\Big] = 0\,, \tag{4.5}$$

(B) Equation of energy

$$\frac{d}{dt}\int_{\mathbb{T}^n} e^\varepsilon(x,t)dx = -\int_{\mathbb{T}^n} \varepsilon^{\beta-1}\partial_t A\cdot\Big[W(\psi^\varepsilon) - \varepsilon^{\beta-1}|\psi^\varepsilon|^2 A\Big]dx\,. \tag{4.6}$$

Proof of the charge-energy inequality (4.4). The charge-energy inequality (4.4) playing the role of apriori estimate follows easily from (4.5)–(4.6). In fact, by the Cauchy-Schwarz and Young's inequalities, we deduce from the charge equation (4.5) that

$$\int_{\mathbb{T}^n} |\psi^\varepsilon|^2 dx \le C + \varepsilon^{1+\beta}\int_{\mathbb{T}^n}\big(|\psi^\varepsilon|^2 + |\varepsilon^\beta\partial_t\psi^\varepsilon|^2\big)dx\,, \tag{4.7}$$

for all $t \in [0, T]$, i.e.,

$$\int_{\mathbb{T}^n} |\psi^\varepsilon|^2 dx \le C + \varepsilon^{1+\beta}\int_{\mathbb{T}^n} |\varepsilon^\beta\partial_t\psi^\varepsilon|^2 dx\,. \tag{4.8}$$

We also derive from the equation of energy (4.6) that

$$\int_{\mathbb{T}^n} e^\varepsilon(x,t)dx \le C + \int_0^t \int_{\mathbb{T}^n} \varepsilon^{\beta-1}|\partial_t A| \Big(|(\nabla - i\varepsilon^{\beta-1}A)\psi^\varepsilon|^2 + |\psi^\varepsilon|^2 \Big) dx d\tau \,. \quad (4.9)$$

Combining (4.7) and (4.9) together yields

$$\int_{\mathbb{T}^n} E^\varepsilon(x,t)dx \le C_1 + C_2 \int_0^t \int_{\mathbb{T}^n} E^\varepsilon(x,s)dxds \,, \quad (4.10)$$

where

$$E^\varepsilon(x,t) = |\psi^\varepsilon|^2 + |\varepsilon^\beta \partial_t \psi^\varepsilon|^2 + |(\nabla - i\varepsilon^{\beta-1}A)\psi^\varepsilon|^2 + \Big(\frac{|\psi^\varepsilon|^2 - \phi}{\varepsilon} \Big)^2 ,$$

then the Gronwall inequality gives the uniform bound of the energy

$$\int_{\mathbb{T}^n} E^\varepsilon(x,t)dx \le C \,, \quad (4.11)$$

From (4.8) and (4.11), we have the charge-energy inequality (4.4). ■

The idea of the proof of Theorem 4.1 is to obtain a family of approximation solutions $\{\psi_n\}$ constructed by any method that yields a consistent weak formulation (4.3) and charge-energy inequality (4.4), for example, the Fourier-Galerkin method, then apply the compactness arguments to prove the existence of weak solutions, which is similar to the defocusing cubic nonlinear Klein-Gordon equation without electromagnetic potential as given in the appendix of [16] with modification. Therefore, we omit the details. The main result concerning the nonrelativistic-semiclassical limit of (4.1)–(4.2) is stated as follows:

Theorem 4.2 *Let $\phi(x) \ge c > 0$, $\phi(x) \in C^1(\mathbb{T}^n)$, $A(x,t) \in C^1([0,\infty) \times \mathbb{T}^n)$, $(\psi_0^\varepsilon, \psi_1^\varepsilon) \in H^1(\mathbb{T}^n) \oplus L^2(\mathbb{T}^n)$, $|\psi_0^\varepsilon|^2 = \phi$ almost everywhere and satisfy*

$$(\psi_0^\varepsilon, \psi_1^\varepsilon) \to (\psi_0, 0) \quad in \quad H^1(\mathbb{T}^n) \oplus L^2(\mathbb{T}^n),$$

such that $|\psi_0|^2 = \phi$ almost everywhere and ψ^ε be the corresponding weak solution of the Cauchy problem (4.1)–(4.2). For $\beta = 1$, the weak limit ψ of $\{\psi^\varepsilon\}_\varepsilon$, satisfying $|\psi|^2 = \phi$ almost everywhere, solves the initial value problem

$$\partial_t^2 \psi - \phi \Delta \psi = \Big[|\nabla \psi|^2 - \frac{1}{\phi} |\partial_t \psi|^2 - \frac{1}{2} \Delta \phi - i \nabla \cdot (\phi A) \Big] \psi \,, \quad (4.12)$$

$$\psi(x,0) = \psi_0(x), \qquad \partial_t\psi(x,0) = 0. \tag{4.13}$$

Using the fact $|\psi|^2 = \phi$ *a.e., we can write* $\psi = \sqrt{\phi}h$ *then* h *solves the initial value problem of the wave map equation with electromagnetic potential*

$$\partial_t^2 h - \nabla\cdot(\phi\nabla h) = \Big[\phi|\nabla h|^2 - |\partial_t h|^2 - i\nabla\cdot(\phi A)\Big]h\,, \quad |h| = 1\,, \tag{4.14}$$

$$h(x,0) = \frac{\psi_0}{|\psi_0|}, \qquad \partial_t h(x,0) = 0\,. \tag{4.15}$$

Moreover, let $h = e^{i\theta}$*; then the phase function* θ *solves the initial value problem of the linear wave equation with electromagnetic potential*

$$\partial_t^2\theta - \nabla\cdot(\phi\nabla\theta) = -\nabla\cdot(\phi A)\,, \tag{4.16}$$

$$\theta(x,0) = \arg\,\psi_0, \qquad \partial_t\theta(x,0) = 0\,. \tag{4.17}$$

For $\beta > 1$*, the effect of the magnetic potential* A *vanishes,* (4.12) *becomes*

$$\partial_t^2\psi - \phi\Delta\psi = \Big(|\nabla\psi|^2 - \frac{1}{\phi}|\partial_t\psi|^2 - \frac{1}{2}\Delta\phi\Big)\psi\,, \tag{4.18}$$

and equations (4.14) *and* (4.16) *are replaced by*

$$\partial_t^2 h - \nabla\cdot(\phi\nabla h) = \Big(\phi|\nabla h|^2 - |\partial_t h|^2\Big)h\,, \qquad |h| = 1\,, \tag{4.19}$$

and

$$\partial_t^2\theta - \nabla\cdot(\phi\nabla\theta) = 0\,, \tag{4.20}$$

respectively. The initial conditions (4.13)*,* (4.15) *and* (4.17) *remain unchanged.*

Remarks: For $\beta = 1$, when $\phi = 1$ and vector potential A satisfies $\nabla\cdot A = 0$, the electromagnetic wave map equation (4.12) will be reduced to the typical wave map equation

$$\partial_t^2\psi - \Delta\psi = (|\nabla\psi|^2 - |\partial_t\psi|^2)\psi\,. \tag{4.21}$$

However, when $\beta > 1$ even without the assumption $\nabla\cdot A = 0$ we still obtain the wave map (4.21) as the limit equation. The existence of weak solution of (4.14)–(4.15) will be given in the appendix.

4.2 Proof of the Theorem 4.2

We deduce from charge-energy inequality (4.4) that

$$\{\psi^\varepsilon\}_\varepsilon \quad \text{is bounded in} \quad L^\infty\big([0,T];H^1(\mathbb{T}^n)\big)\,, \tag{4.22}$$

$$\{\varepsilon^\beta \partial_t \psi^\varepsilon\}_\varepsilon \quad \text{is bounded in} \quad L^\infty\big([0,T];L^2(\mathbb{T}^n)\big)\,, \tag{4.23}$$

$$\left\{\frac{|\psi^\varepsilon|^2-\phi}{\varepsilon}\right\}_\varepsilon \quad \text{is bounded in} \quad L^\infty\big([0,T];L^2(\mathbb{T}^n)\big)\,. \tag{4.24}$$

It follows from (4.22) that there exists a subsequence still denoted by $\{\psi^\varepsilon\}_\varepsilon$ and a function $\psi \in L^\infty\big([0,T];H^1(\mathbb{T}^n)\big)$ such that

$$\psi^\varepsilon \quad \rightharpoonup \quad \psi \quad \text{weakly} * \text{in} \quad L^\infty\big([0,T];H^1(\mathbb{T}^n)\big)\,. \tag{4.25}$$

Next, from (4.24), we have

$$|\psi^\varepsilon|^2 \to \phi \quad \text{a.e. and strongly in } L^2(\mathbb{T}^n). \tag{4.26}$$

Note that (4.24) only shows that $\{\frac{|\psi^\varepsilon|^2-\phi}{\varepsilon}\}_\varepsilon$ converges weakly $*$ to some function $\eta \in L^\infty\big([0,T];L^2(\mathbb{T}^n)\big)$. Thus to overcome the difficulty caused by nonlinearity, i.e., the fourth term on the left hand side of (4.1), we can use (4.22)–(4.23) via the Arzela-Ascoli theorem to prove

$$\psi^\varepsilon \to \psi \quad \text{in} \quad C\big([0,T];L^2(\mathbb{T}^n)\big) \quad \text{as} \quad \varepsilon \to 0\,. \tag{4.27}$$

To find the explicit form of η, we rewrite the equation of charge (4.5) as

$$\frac{|\psi^\varepsilon|^2-\phi}{\varepsilon} = -Z(\psi^\varepsilon)\Big|_0^t - \int_0^t \nabla\cdot\Big[W(\psi^\varepsilon) - \varepsilon^{\beta-1}|\psi^\varepsilon|^2 A\Big]d\tau\,. \tag{4.28}$$

Thus to obtain the compactness of the sequence $\left\{\frac{|\psi^\varepsilon(x,t)|^2-\phi}{\varepsilon}\right\}_\varepsilon$, we have to treat the integral term of the right hand side of (4.28). By (4.22)–(4.23), we have $Z(\psi^\varepsilon) \rightharpoonup 0$ in $\mathcal{D}'((0,T)\times\mathbb{T}^n)$. Therefore we can apply integration by parts, Fubini's theorem and Lebesgue's dominated convergence theorem to show that

$$\int_0^t \nabla\cdot\big(|\psi^\varepsilon|^2 A\big)d\tau \rightharpoonup \int_0^t \nabla\cdot(\phi A)d\tau \tag{4.29}$$

and

$$\int_0^t \nabla\cdot\big(\psi^\varepsilon\nabla\overline{\psi^\varepsilon}\big)d\tau \rightharpoonup \int_0^t \nabla\cdot\big(\psi\nabla\overline{\psi}\big)d\tau \tag{4.30}$$

in $\mathcal{D}'((0,T)\times\mathbb{T}^n)$. Therefore we have the convergences

$$\frac{|\psi^\varepsilon|^2-\phi}{\varepsilon} \rightharpoonup \eta \equiv -\int_0^t \nabla\cdot\Big[W(\psi)-\phi A\Big]d\tau\,. \tag{4.31}$$

for $\beta=1$ and

$$\frac{|\psi^\varepsilon|^2-\phi}{\varepsilon} \rightharpoonup \eta \equiv -\int_0^t \nabla\cdot W(\psi)d\tau\,. \tag{4.32}$$

for $\beta>1$. By (4.22), (4.23) and (4.31), passing to the limit of the weak formulation (4.3), we can show that if $\beta=1$, the limit wave function ψ satisfies

$$i\partial_t\psi+\Big[\int_0^t\nabla\cdot\big(W(\psi)-\phi A\big)d\tau\Big]\psi=0 \tag{4.33}$$

in the sense of distribution. Since $|\psi(x,t)|^2=\phi(x)$ is t-independent, then we have $\overline{\psi}\partial_t\psi+\psi\partial_t\overline{\psi}=0$ and $\overline{\psi}\nabla\psi+\psi\nabla\overline{\psi}=\nabla\phi$, hence

$$\frac{1}{2}\Big(\overline{\psi}\partial_t\psi-\psi\partial_t\overline{\psi}\Big)=\overline{\psi}\partial_t\psi=-\psi\partial_t\overline{\psi}\,,\quad \frac{1}{2}\Big(\overline{\psi}\nabla\psi-\psi\nabla\overline{\psi}\Big)=\overline{\psi}\nabla\psi-\frac{1}{2}\nabla\phi\,.$$

Differentiating (4.33) with respect to t, we have

$$\partial_t^2\psi-\nabla\cdot\Big(\overline{\psi}\nabla\psi-\frac{1}{2}\nabla\phi-i\phi A\Big)\psi-\frac{\partial_t\psi}{\psi}\partial_t\psi=0\,.$$

Therefore ψ satisfies the wave map equation with electromagnetic potential

$$\partial_t^2\psi-\phi\Delta\psi=\Big[|\nabla\psi|^2-\frac{1}{\phi}|\partial_t\psi|^2-\frac{1}{2}\Delta\phi-i\nabla\cdot(\phi A)\Big]\psi\,,\quad |\psi|^2=\phi\quad \text{a.e.}$$

Furthermore, using the fact $|\psi|^2=\phi$ and writing $\psi=\sqrt{\phi}h$, then h solves the wave map equation

$$\partial_t^2h-\nabla\cdot(\phi\nabla h)=\left[\phi|\nabla h|^2-|\partial_t h|^2-i\nabla\cdot(\phi A)\right]h\,,\qquad |h|=1\,.$$

Moreover, using the fact $|h|=1$ and writing $h=e^{i\theta}$, we have

$$\partial_t^2\theta-\nabla\cdot(\phi\nabla\theta)=-\nabla\cdot(\phi A)\,,$$

i.e., θ is a distribution solution of wave equation with electromagnetic potential. The case for $\beta>1$ is similar, and we omit the details. This completes the proof of Theorem 4.2. ∎

5 Appendix

Let $\mathcal{R}e\,h$ and $\mathcal{I}m\,h$ denote the real and imaginary parts of the complex-valued function h, i.e., $h = \mathcal{R}e\,h + i\mathcal{I}m\,h$. We define the 2-dimensional vector function ω by $\omega = (\omega_1, \omega_2) \equiv (\mathcal{R}e\,h, \mathcal{I}m\,h)^t$, then $\omega^\perp = (-\omega_2, \omega_1)$ and (4.14)–(4.15) can be rewritten as

$$\partial_t^2 \omega - \nabla \cdot (\phi \nabla \omega) = \big(\phi |\nabla \omega|^2 - |\partial_t \omega|^2\big)\omega - \nabla \cdot (\phi A)\omega^\perp\,, \quad |\omega| = 1\,, \tag{5.1}$$

$$\omega(x,0) = \omega_0(x), \qquad \partial_t \omega(x,0) = 0\,. \tag{5.2}$$

Lemma 5.1 *If $|\omega| = 1$ almost everywhere and satisfies*

$$\nabla \omega \in L^\infty\big([0,T]; L^2(\mathbb{T}^n)\big), \qquad \partial_t \omega \in L^\infty\big([0,T]; L^2(\mathbb{T}^n)\big)$$

then ω is a weak solution of the Cauchy problem (5.1)–(5.2) *if and only if*

$$\partial_t(\partial_t \omega \times \omega) - \nabla \cdot (\phi \nabla \omega \times \omega) = \nabla \cdot (\phi A)\,.$$

Proof. Suppose that ω is a weak solution of (5.1)–(5.2), then taking the cross product of (5.1), we have $\partial_t(\partial_t \omega \times \omega) - \nabla \cdot (\phi \nabla \omega \times \omega) = \nabla \cdot (\phi A)$. Conversely, assuming $\partial_t(\partial_t \omega \times \omega) - \nabla \cdot (\phi \nabla \omega \times \omega) = \nabla \cdot (\phi A)$ or equivalently

$$\big[\partial_t^2 \omega - \nabla \cdot (\phi \nabla \omega) + \nabla \cdot (\phi A)\omega^\perp\big] \times \omega = 0\,,$$

which implies $\partial_t^2 \omega - \nabla \cdot (\phi \nabla \omega) + \nabla \cdot (\phi A)\omega^\perp = \lambda \omega$ for some scalar function λ. Taking the inner product with ω and employing the property $|\omega| = 1$, we have

$$\lambda = \phi |\nabla \omega|^2 - |\partial_t \omega|^2\,.$$

This completes the proof of the lemma. ∎

Theorem 5.2 *Given $\omega_0 \in H^1(\mathbb{T}^n)$ and $|\omega_0| = 1$ a.e., there exists a function ω satisfying $|\omega| = 1$ a.e., such that*

$$\nabla \omega \in L^\infty\big([0,T]; L^2(\mathbb{T}^n)\big)\,, \quad \partial_t \omega \in L^\infty\big([0,T]; L^2(\mathbb{T}^n)\big) \tag{5.3}$$

and solves the wave map equation with electromagnetic potential (ϕ, A) :

$$\partial_t^2 \omega - \nabla \cdot (\phi \nabla \omega) = \big(\phi |\nabla \omega|^2 - |\partial_t \omega|^2\big)\omega - \nabla \cdot (\phi A)\omega^\perp \tag{5.4}$$

in $\mathcal{D}'((0,T) \times \mathbb{T}^n)$.

Proof. As discussed in [29] for the wave map equation, we need to construct an approximate equation ω_n which preserves the energy $|\partial_t\omega_n|^2 + \phi|\nabla\omega_n|^2$ and satisfies $|\omega_n| \to 1$, so we can approximate the equation (5.4) by

$$\partial_t^2\omega_n - \nabla\cdot(\phi\nabla\omega_n) + n^2(|\omega_n|^2 - 1)\omega_n = -\nabla\cdot(\phi A)\omega_n^{\perp}\,, \tag{5.5}$$

$$\omega_n(x,0) = \omega_0(x), \qquad \partial_t\omega_n(x,0) = 0\,. \tag{5.6}$$

For each n the initial value problem (5.5)–(5.6) has a strong solution ω_n such that $\omega_n \in C([0,T];H^1(\mathbb{T}^n))$, $\partial_t\omega_n \in C([0,T];L^2(\mathbb{T}^n))$ and satisfies the energy relation

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{T}^n} |\partial_t\omega_n|^2 + \phi|\nabla\omega_n|^2 + \frac{1}{2}n^2(|\omega_n|^2-1)^2 dx \\
&\qquad \le C(\mathbb{T}^n) + C\int_{\mathbb{T}^n} |\partial_t\omega_n|^2 + (\omega_n^2 - 1)dx \qquad (5.7)\\
&\qquad \le C(\mathbb{T}^n) + C\int_{\mathbb{T}^n} |\partial_t\omega_n|^2 + n^2(\omega_n^2-1)^2 dx\,,
\end{aligned}$$

this shows

$$\int_{\mathbb{T}^n} |\partial_t\omega_n|^2 + \phi|\nabla\omega_n|^2 + n^2(|\omega_n|^2-1)^2 dx \le C\,. \tag{5.8}$$

It follows from (5.8) that there exists a subsequence still denoted by $\{\omega_n\}_n$ and a function $\omega \in L^\infty\big([0,T];H^1(\mathbb{T}^n)\big)$ such that

$$\omega_n \rightharpoonup \omega \quad \text{weakly} * \text{ in} \quad L^\infty\big([0,T];H^1(\mathbb{T}^n)\big)\,, \tag{5.9}$$

$$\partial_t\omega_n \rightharpoonup \partial_t\omega \quad \text{weakly} * \text{ in} \quad L^\infty\big([0,T];L^2(\mathbb{T}^n)\big)\,, \tag{5.10}$$

and

$$|\omega_n|^2 \to 1 \quad \text{a.e. and strongly in } L^2(\mathbb{T}^n). \tag{5.11}$$

Moreover, after applying the Lions-Aubin's lemma (5.9)–(5.10) shows that $\omega_n \to \omega$ strongly in $L^\infty([0,T];L^2(\mathbb{T}^n))$.

Now, in order to show that ω satisfies the equation (5.4) by Lemma 5.1, we take the cross product of (5.5) with ω_n to obtain

$$\partial_t(\partial_t\omega_n \times \omega_n) - \nabla\cdot(\phi\nabla\omega_n \times \omega_n) = \nabla\cdot(\phi A)\,. \tag{5.12}$$

Since $\omega_n \to \omega$ strongly in $L^\infty\big([0,T];L^2(\mathbb{T}^n)\big)$, $\partial_t\omega_n$ and $\nabla\omega_n$ are weakly $*$ convergent in $L^\infty\big([0,T];L^2(\mathbb{T}^n)\big)$, we can pass to the limit in (5.12) and show that ω satisfies

$$\partial_t(\partial_t\omega \times \omega) - \nabla\cdot(\phi\nabla\omega\times\omega) = \nabla\cdot(\phi A)\,. \tag{5.13}$$

By Lemma 5.1 we have completed the proof of Theorem 5.2. ■

ACKNOWLEDGMENTS

C.K. Lin is supported by National Science Council of Taiwan under the grant NSC98-2115-M-009-004-MY3. K.C. Wu is supported by the Tsz-Tza Foundation in Institute of Mathematics, Academia Sinica, Taipei, Taiwan. K.C. Wu would like to thank Dr. Clément Mouhot for his kind invitation to visit Cambridge during the 2011-2013 academic year.

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