行政院國家科學委員會專題研究計畫 成果報告

破裂介質中的熱傳導方程式
研究成果報告(精簡版)

計 畫 類 別 ： 個別型
計 畫 編 號 ： NSC 98-2115-M-009-011-
執 行 期 間 ： 98 年 08 月 01 日至 99 年 07 月 31 日
執 行 單 位 ： 國立交通大學應用數學系（所）

計 畫 主 持 人 ： 葉立明

報 告 附 件 ： 出席國際會議研究心得報告及發表論文

處 理 方 式 ： 本計畫可公開查詢

中 華 民 國 99 年 10 月 29 日

背景及目的

很多的化學工廠的廢水經由水井排放到地底，地面的有毒廢棄物也經由雨水進入地底，核能電廠儲存在地底下的核廢棄物由於時間的關係造成容器腐蝕、或是由於地層變動造成容器破裂，放射性物質也因此進入地下水。這些都造成日常飲用水的不安全。這些問題不是只發生在臺灣，世界其它國家也有同樣的情形。歐美一些國家都有專責機構負責這些污染源的清除工作。身體內的血液在血管中流動的問題，地底下的碳氫化合物的抽取等以上都屬於多相流問題[6,10,12,40]。因此，了解多孔介質中的多相流的變化對解決很多實際問題是重要的。

這是個一年期的計劃。我們計劃探討熱傳導在破裂多孔介質中的宏觀模式。地底下的縫隙結構與地質性質是十分複雜的，然而流體在地底下的運動卻與它們息息相關，譬如流體運動的方向、快慢、time-scale 等[17,18,19,20,21,25,26, 27,30, 31,33]。破裂多孔介質是一種特殊的地質結構，在這種介質中流體的運動往往呈現出兩種不同的 time scale 的現象。污染源擴散時平流與對流現象則常發生在污染源的附近或較遠處。熱傳導現象(一種 transport equation)則對應到地下污染源在地底擴散時的平流與對流問題。熱傳導在破裂多孔介質中自然也有流速快慢不同的區域。描述多孔介質中的可混合與不可混合流體的運動方程式都包含有 transport equation。換言之，了解熱傳導現象在破裂多孔介質中的的數學模式是要進一步了解污染源在地底擴散情形的基礎。

探討在破裂多孔介質中的多相流的問題不是一個新的問題。但到目前為止討論的方式主要仍是數值模擬加上一些平均的方法，數值模擬的缺點是需要很多的計算時間且是 case by case 的討論。至於用數學分析的討論方式則是最近才開始。數學文獻中關於破裂多孔介質中的多相流的問題大多只單討論 flow equation[51,52]的部份。之前我們的計劃也討論 flow equation（一種橢圓形方程式）在破裂多孔介質中的情形。我們希望借助之前的經驗來幫助了解 transport equation 在破裂多孔介質中的情形。底下是我們考慮的抛物線微分方程式。

$\Omega \in \Re^3$ is a domain and $\Omega = \Omega_f^\varepsilon \cup \Omega_m^\varepsilon$. Denote absolute permeability by K_ε in Ω_f^ε and k_ε in Ω_m^ε, phase pressure by P_ε in Ω_f^ε and p_ε in Ω_m^ε, and external source by $Q_\varepsilon, F_\varepsilon$ in Ω_f^ε and $q_\varepsilon, f_\varepsilon$ in Ω_m^ε. The equations are

$$\partial_t P_\varepsilon - \nabla\cdot(K_\varepsilon \nabla P_\varepsilon + Q_\varepsilon) = F_\varepsilon \qquad \text{in } \Omega_f^\varepsilon \times (0,T),$$

$$\partial_t p_\varepsilon - \varepsilon\nabla\cdot(k_\varepsilon \varepsilon \nabla p_\varepsilon + q_\varepsilon) = f_\varepsilon \qquad \text{in } \Omega_m^\varepsilon \times (0,T)$$

$$(K_\varepsilon \nabla P_\varepsilon + Q_\varepsilon)\cdot \vec{n} = \varepsilon k_\varepsilon (\varepsilon \nabla p_\varepsilon + q_\varepsilon)\cdot \vec{n} \qquad \text{on } \partial\Omega_m^\varepsilon \times (0,T),$$

$$P_\varepsilon = p_\varepsilon \qquad \text{on } \partial\Omega_m^\varepsilon \times (0,T)$$

$$P_\varepsilon = P \qquad \text{in } \Omega_f^\varepsilon$$

$p_\varepsilon = p_0$ in Ω_m^ε

with periodic boundary condition on $\partial\Omega$. 我們想考慮當 ε 變動時，P_ε 的何種 norm 與 ε 的變化無關?

除了拋物線微分方程式，我們也考慮橢圓形方程式在 perforated domain 的數值近似解的誤差估計問題。

研究方法、進行步驟及執行進度。

這是一個拋物線微分方程式，我們以底下的方式進行: 一. 利用半群理論(semigroup theory) 我們得到時間方面的解的均勻 Holder norm 的估計。 二.利用 three-step compactness method 我們得到空間方面的解的均勻 Holder norm 的估計。三.為了估計不連續截面的變化，我們用到 pseudodifferential operator [40]及 boundary integral method[14]。在函數空間方面則利用 Besov space and Holder space [2,37,41]。由於考慮的拋物線微分方程式是在破裂多孔介質中，介質基本上是不連續的，因此在空間方面的解的 Holder norm 只在局部較平滑的區域得到，在不平滑的區域則沒有均勻 Holder norm 的估計。

關於橢圓形方程式，我們則利用 three-step compactness method 與 Taylor expansion 得到 數值近似解的誤差估計。

（四）成果與自評。

一、得到不均勻拋物線微分方程式的解的 Holder norm 的均勻估計結果[58]。有了此結再利用 two-scale method 我們可以很容易的導出熱傳導在破裂多孔介質中的宏觀模式。

二、得到橢圓形方程式在 perforated domain 的數值近似解的誤差估計[59]。

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Hölder estimate for non-uniform parabolic equations in highly heterogeneous media

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Uniform bound for the solutions of non-uniform parabolic equations in highly heterogeneous media is concerned. The space domains are periodic as well as consist of a connected high permeability sub-region and a disconnected matrix block sub-region with low permeability. Let ϵ denote the size ratio of matrix blocks to the whole domain and let the permeability ratio of the matrix block sub-region to the connected high permeability sub-region be of the order ϵ^2. It is proved that the Hölder norm of the non-uniform parabolic solutions in connected sub-region is bounded uniformly in ϵ.

Keywords: Highly heterogeneous media, pseudo-differential operator, paramatrix, strict solution, infinitesimal generator, numerical range.

AMS Subject Classification: 35K10, 35K20, 35M13

1. Introduction

Uniform Hölder estimate for the solutions of non-uniform parabolic equations in highly heterogeneous media is presented. The equations have many applications in multiphase flows in porous media, the stress in composite materials, and so on (see [3, 12, 17] and references therein). The domain $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ has boundary $\partial\Omega$ as well as contains a connected high permeability sub-region and a disconnected matrix block sub-region with low permeability. Let $Y \equiv [0,1]^n$ be a cell consisting of a sub-domain Y_m completely surrounded by another connected sub-domain Y_f $(\equiv Y \setminus Y_m)$, $\epsilon \in (0,1)$, and $\Omega(2\epsilon) \equiv \{x \in \Omega : dist(x, \partial\Omega) > 2\epsilon\}$. The disconnected sub-region is $\Omega^\epsilon_m \equiv \{x : x \in \epsilon(Y_m + j) \subset \Omega(2\epsilon) \text{ for } j \in \mathbb{Z}^n\}$, the connected sub-region is $\Omega^\epsilon_f \equiv \Omega \setminus \Omega^\epsilon_m$, and the boundary of Ω^ϵ_m is represented by $\partial\Omega^\epsilon_m$. The non-uniform parabolic equations in $[0,T] \times \Omega$ are

$$\begin{cases} \partial_t U_\epsilon - \nabla \cdot (\Lambda_\epsilon \nabla U_\epsilon) = F_\epsilon & \text{in } (0,T] \times \Omega, \\ U_\epsilon = 0 & \text{on } (0,T] \times \partial\Omega, \\ U_\epsilon = U_{\epsilon,0} & \text{in } \{0\} \times \Omega, \end{cases} \tag{1.1}$$

where $\Lambda_\epsilon \equiv \begin{cases} \mathbf{K}_\epsilon & \text{in } \Omega^\epsilon_f, \\ \epsilon^2 \mathbf{k}_\epsilon & \text{in } \Omega^\epsilon_m, \end{cases}$ and $\mathbf{K}_\epsilon$ and $\mathbf{k}_\epsilon$ are positive smooth functions in Ω. Since $\epsilon \in (0,1)$, equations (1.1) are non-uniform parabolic equations with discontinuous coefficients. In [20], existence of solution in $W_p^{2,1}([0,T]\times\Omega)$ space for uniform parabolic equations with discontinuous coefficients can be found. For non-uniform parabolic equations with smooth coefficients, existence of solution in $C^{2,\alpha}([0,T]\times\Omega)$ space was studied in [13]. It is also known that if $F_\epsilon, U_{\epsilon,0}$ are smooth, a piecewise regular solution of (1.1) exists uniquely for each ϵ and, by energy method, the H^1 norm of the parabolic solution in the connected high permeability sub-region is bounded uniformly in ϵ [12, 16]. Many studies of the uniform estimate in ϵ for the elliptic equations in highly heterogeneous media had been done [4, 12, 15, 17, 19, 23], but little for parabolic equations. Existence of piecewise regular solutions for elliptic diffraction equations in Hilbert space was considered in [12, 15]. Uniform Lipschitz estimate in ϵ for Laplace equation in perforated domains was given in [23], and uniform L^p estimate in ϵ of the same problem was considered in [19]. Lipschitz estimate for uniform elliptic equations was studied in [17]. Uniform Hölder and Lipschitz estimates in ϵ for uniform elliptic equations in periodic domains were obtained in [4]. This work is to present uniform Hölder estimate in ϵ for the solutions of the non-uniform parabolic equations with discontinuous coefficients. Permeability fields (that is, Λ_ϵ) are not periodic and are allowed to have large deviation. It is proved that the Hölder norm of the non-uniform parabolic solutions in connected sub-region is bounded uniformly in ϵ.

2. Notation and main result

Let L^p (resp.$H^k, W^{k,p}$) denote complex Sobolev space with norm $\|\cdot\|_{L^p}$ (resp. $\|\cdot\|_{H^k}, \|\cdot\|_{W^{k,p}}$), C_0^∞ be the set containing all infinite differentiable functions with compact support, and C^σ (resp. $C^{1,\sigma}$) denote Hölder space with norm $\|\cdot\|_{C^\sigma}$ (resp. $\|\cdot\|_{C^{1,\sigma}}$) for $\sigma\in(0,1], k\geq -1$, and $p\in[1,\infty]$ [11]. $[\varphi]_{C^\sigma}$ (resp. $[\varphi]_{C^{1,\sigma}}$) denotes the Hölder semi-norms of φ (resp. $\nabla\varphi$). If φ is a complex function, $\overline{\varphi}$ denotes its complex conjugate. If $\mathbf{B}_1$ and $\mathbf{B}_2$ are two Banach spaces, $\mathcal{L}(\mathbf{B}_1,\mathbf{B}_2)$ is the set of all bounded linear maps from $\mathbf{B}_1$ to $\mathbf{B}_2$ with norm $\|\cdot\|_{\mathcal{L}(\mathbf{B}_1,\mathbf{B}_2)}$. For any Banach space $\mathbf{B}$, define $\|\varphi_1,\varphi_2,\cdots,\varphi_m\|_{\mathbf{B}}\equiv\|\varphi_1\|_{\mathbf{B}}+\|\varphi_2\|_{\mathbf{B}}+\cdots+\|\varphi_m\|_{\mathbf{B}}$, denote its dual space by $\mathbf{B}'$, and denote the pairing between $\mathbf{B}$ and its dual space $\mathbf{B}'$ by $\langle\cdot,\cdot\rangle_{\mathbf{B},\mathbf{B}'}$. The function spaces $L^\infty(I;\mathbf{B}), C(I;\mathbf{B}), C^\sigma(I;\mathbf{B})$ for $\sigma\in(0,1]$ and an interval $I\subset\mathbb{R}$ are defined as those in [18]. $B_r(x)$ represents a ball centered at x with radius r. For any domain $\mathbb{D}$, $\overline{\mathbb{D}}$ is the closure of $\mathbb{D}$, $\mathbb{D}/r\equiv\{x: rx\in\mathbb{D}\}$, $|\mathbb{D}|$ is the volume of $\mathbb{D}$, and $\mathcal{X}_{\mathbb{D}}$ is the characteristic function on $\mathbb{D}$. For any $\varphi\in L^1(B_r(x)\cap\Omega)$, we define

$$(\varphi)_{x,r}\equiv\fint_{B_r(x)\cap\Omega}\varphi(y)dy\equiv\frac{1}{|B_r(x)\cap\Omega|}\int_{B_r(x)\cap\Omega}\varphi(y)dy.$$

For any $p\in(1,\infty)$ and $\epsilon>0$, we define

$$\begin{cases}\mathcal{A}^\epsilon\varphi\equiv-\nabla\cdot(\Lambda_\epsilon\nabla\varphi),\\ \mathbb{B}_p(\mathcal{A}^\epsilon)\equiv\big\{\varphi\in W_0^{1,p}(\Omega):\varphi\in W^{2,p}(\Omega_f^\epsilon)\cup W^{2,p}(\Omega_m^\epsilon),\\ \qquad\qquad\mathbf{K}_\epsilon\nabla\varphi\cdot\vec{\mathbf{n}}^\epsilon|_{\partial\Omega_m^\epsilon}=\epsilon^2\mathbf{k}_\epsilon\nabla\varphi\cdot\vec{\mathbf{n}}^\epsilon|_{\partial\Omega_m^\epsilon}\big\},\end{cases}$$

where $\vec{\mathbf{n}}^\epsilon$ is a normal vector on $\partial\Omega_m^\epsilon$. $\mathbb{B}_p(\mathcal{A}^\epsilon)$ with norm $\|\varphi\|_{\mathbb{B}_p(\mathcal{A}^\epsilon)} \equiv \|\mathcal{A}^\epsilon\varphi\|_{L^p(\Omega)}$ is a normed space. $\mathcal{M}(\alpha,\beta;\mathbb{D}) \equiv \{\varphi : \mathbb{D} \to \mathbb{R} | \varphi \in L^\infty(\mathbb{D}), 0 < \alpha \le \varphi \le \beta\}$.

For any $\sigma \in (0,1)$ and $\delta, \alpha, \beta > 0$, we assume

A1. Ω and Y_m are smooth domains,

A2. $\mathbf{K}_\epsilon, \mathbf{k}_\epsilon \in \mathcal{M}(\alpha,\beta;\Omega)$ and

$$\|\mathbf{K}_\epsilon(\epsilon x) - \alpha_{\epsilon,j}\|_{W^{1,\infty}((Y_f+j)\cap\Omega/\epsilon)} + \|\mathbf{k}_\epsilon(\epsilon x) - \alpha_{\epsilon,j}\|_{W^{1,\infty}((Y_m+j)\cap\Omega/\epsilon)} \le c\alpha_{\epsilon,j}$$

where $j \in \mathbb{Z}^n$, $\alpha_{\epsilon,j}$ depend on ϵ, j, and c is small and depending on Y_f,

A3. $F_\epsilon \in C^\sigma([0,T]; L^{n+\delta}(\Omega))$, $-\mathcal{A}^\epsilon U_{\epsilon,0} + F_\epsilon(0,x) \in \overline{\mathbb{B}_{n+\delta}(-\mathcal{A}^\epsilon)}$, and $U_{\epsilon,0} \in \mathbb{B}_{n+\delta}(-\mathcal{A}^\epsilon)$.

The main result is:

Theorem 2.1. *Under A1-A3, the solution of (1.1) satisfies*

$$\|U_\epsilon\|_{C^1([0,T];L^{n+\delta}(\Omega))} + \|U_\epsilon\|_{C([0,T];C^\mu(\Omega_f^\epsilon))} + \sup_{\substack{j\in\mathbb{Z}^n \\ \epsilon(Y_m+j)\subset\Omega_m^\epsilon}} \epsilon\|U_\epsilon\|_{C([0,T];C^\mu(\epsilon(Y_m+j)))}$$
$$\le c\big(\|U_{\epsilon,0}\|_{\mathbb{B}_{n+\delta}(\mathcal{A}^\epsilon)} + \|F_\epsilon\|_{C^\sigma([0,T];L^{n+\delta}(\Omega))}\big), \qquad (2.1)$$

where $\delta > 0$, $\mu, \sigma \in (0,1)$, μ *is a constant depending on* $n, \delta, \alpha, \beta, Y_f, \Omega$, *and* c *is a constant independent of* ϵ. *Moreover, there is a* $\nu \in (0,\mu)$ *such that*

$$\|U_\epsilon\|_{C^\nu([0,T]\times\Omega_f^\epsilon)} \le c\big(\|U_{\epsilon,0}\|_{\mathbb{B}_{n+\delta}(\mathcal{A}^\epsilon)} + \|F_\epsilon\|_{C^\sigma([0,T];L^{n+\delta}(\Omega))}\big), \qquad (2.2)$$

where c *is a constant independent of* ϵ.

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Pointwise error estimate for elliptic equations in perforated domains

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Numerical approximation for the solutions of elliptic equations in perforated domains is concerned. Let ϵ denote the size ratio of the holes of some perforated domains to their whole domains. As ϵ closes to 0, the elliptic solutions in perforated domains approach a solution of some homogenized equation. So it is expected that the numerical approximation of the solution of the homogenized equation is a good approximation for the elliptic solutions in perforated domains when ϵ is small. In this work, the L^∞ estimate and the Lipschitz estimate for the difference between the elliptic solutions in perforated domains and the numerical approximation of the homogenized solution are derived. Higher order estimate in Lipschitz norm for the difference between the elliptic solutions in perforated domains and the homogenized solution is also derived.

Keywords: elliptic solution, perforated domain, homogenized solution.

AMS Subject Classification: 65N12, 65N15, 65N22

1. Introduction

Pointwise error estimate for the numerical approximation of the solutions of elliptic equations in perforated domains is presented. Let $\Omega \subset \mathbb{R}^n$ ($n = 2$ or 3) be a smooth domain with boundary $\partial\Omega$, $Y \equiv [0,1]^n$ consist of a sub-domain Y_m completely surrounded by another connected sub-domain Y_f ($\equiv Y \setminus Y_m$), $\epsilon \in (0,1)$, $\Omega(2\epsilon) \equiv \{x \in \Omega : dist(x, \partial\Omega) > 2\epsilon\}$, $\Omega_m^\epsilon \equiv \{x : x \in \epsilon(Y_m + j) \subset \Omega(2\epsilon) \text{ for } j \in \mathbb{Z}^n\}$ with boundary $\partial\Omega_m^\epsilon$, and $\Omega_f^\epsilon \equiv \Omega \setminus \Omega_m^\epsilon$ be a connected region. The equations in the perforated domain Ω_f^ϵ are

$$\begin{cases} -\nabla \cdot (\mathbf{K}_\epsilon \nabla U_\epsilon) + \lambda U_\epsilon = F & \text{in } \Omega_f^\epsilon, \\ \mathbf{K}_\epsilon \nabla U_\epsilon \cdot \vec{\mathbf{n}}^\epsilon = 0 & \text{on } \partial\Omega_m^\epsilon, \\ U_\epsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\lambda \geq 0$, $\mathbf{K}_\epsilon(x) = \mathbf{K}(\frac{x}{\epsilon})$, $\mathbf{K}$ is a positive periodic function in $\mathbb{R}^n$ with period Y, and $\vec{\mathbf{n}}^\epsilon$ is the unit normal vector on $\partial\Omega_m^\epsilon$. When ϵ is small, direct numerical simulation of the solution of (1.1) can be very expensive. It is known that if $F \in L^2(\Omega)$, the H^1 solution of (1.1) exists uniquely and satisfies

$$\|U_\epsilon\|_{H^1(\Omega_f^\epsilon)} \leq c\|F\|_{L^2(\Omega)},$$

where c is a constant independent of ϵ [9]. By compactness principle [2], there exists a function $U \in H^1(\Omega)$ such that the solution U_ϵ of (1.1) satisfies

$$\mathbf{K}_\epsilon \nabla U_\epsilon \mathcal{X}_{\Omega_f^\epsilon} \to \mathbf{K}^* \nabla U \qquad \text{in } L^2(\Omega) \text{ weakly as } \epsilon \to 0, \tag{1.2}$$

where $\mathcal{X}_{\Omega_f^\epsilon}$ is the characteristic function on Ω_f^ϵ and $\mathbf{K}^*$ is a constant positive definite matrix depending on $\mathbf{K}, Y_f$ (explicit form of $\mathbf{K}^*$ is in (2.2) below). The function U in (1.2) satisfies

$$\begin{cases} -\nabla \cdot (\mathbf{K}^* \nabla U) + \lambda |Y_f| U = |Y_f| F & \text{in } \Omega, \\ U = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $|Y_f|$ is the volume of Y_f. Therefore, it is expected that the numerical approximation for the solution of (1.3) is a good approximation for the solution of (1.1), especially when ϵ is small. The error estimate between the numerical solution and the analytic solution of (1.3) had been extensively studied (see [4, 7, 10, 15] to name a few). So we shall focus on the error estimate for the solutions of (1.1) and (1.3).

By homogenization theory, solutions of elliptic equations in periodic domains in general converge to a solution of some homogenized elliptic equation with convergence rate ϵ in L^2 norm and with convergence rate $\sqrt{\epsilon}$ in H^1 norm as ϵ closes to 0 (see [3, 11, 14] and references therein). In [5, 13], higher order asymptotic expansion for the solutions of elliptic equations in perforated domains was given. Higher order convergence rate for the solution of (1.1) for $\lambda = 0$ case was derived in Hilbert spaces [3, 6, 14]. Different from the literatures mentioned above in which L^2 space was considered, we present pointwise error estimate for the solutions of (1.1) and (1.3) for $\lambda \geq 0$. More precisely, the L^∞ error estimate with convergence rate ϵ for the solutions of (1.1) and (1.3) is proved. For equation (1.1) with $\lambda > 0$ and with periodic boundary condition case, $W^{1,\infty}$ error estimate with convergence rate ϵ is also derived. In particular, for equation (1.1) with $\lambda = 0$ and with periodic boundary condition case, higher order approximation in Lipschitz norm is obtained as well.

2. Notation and main results

Denote by $C^{k,\alpha}$ the Hölder space with norm $\|\cdot\|_{C^{k,\alpha}}$, by $[g]_{C^{k,\alpha}}$ the Hölder seminorm of g, and by L^p (resp. $H^s, W^{s,p}$) the Sobolev space with norm $\|\cdot\|_{L^p}$ (resp. $\|\cdot\|_{H^s}$, $\|\cdot\|_{W^{s,p}}$) for $k \geq 0, \alpha \in [0,1], s \geq 1$, and $p \in [1,\infty]$ (see [9]). For any Banach space $\mathbf{B}$, we define $\|g_1, g_2, \cdots, g_k\|_{\mathbf{B}} \equiv \|g_1\|_{\mathbf{B}} + \|g_2\|_{\mathbf{B}} + \cdots + \|g_k\|_{\mathbf{B}}$. $B_r(x)$ is a ball centered at x with radius r. For any domain D, $\overline{D}$ is the closure of D, $D/r \equiv \{x : rx \in D\}$, $|D|$ is the volume of D, and $\mathcal{X}_D$ is the characteristic function on D. For any $g \in L^1(\Omega)$,

$$(g)_{x,r} \equiv \fint_{B_r(x)\cap\Omega} g(y)dy \equiv \frac{1}{|B_r(x)\cap\Omega|} \int_{B_r(x)\cap\Omega} g(y)dy.$$

Define $\mathcal{Z}_m \equiv \cup_{j\in\mathbb{Z}^n}(Y_m + j)$ with boundary $\partial\mathcal{Z}_m$, $\mathcal{Z}_f \equiv \mathbb{R}^n \setminus \mathcal{Z}_m$, $\mathcal{Z}_m^\epsilon \equiv \epsilon\mathcal{Z}_m$ with boundary $\partial\mathcal{Z}_m^\epsilon$, and $\mathcal{Z}_f^\epsilon \equiv \mathbb{R}^n \setminus \mathcal{Z}_m^\epsilon$. For $D \in \{\mathbb{R}^n, \mathcal{Z}_f, \mathcal{Z}_f^\epsilon\}$, we define

$$W_{per}^{s,p}(D) \equiv \{g \in W_{loc}^{s,p}(D) : g \text{ is a periodic function in } D \text{ with period } [0,1]^n\}$$

with norm $\|g\|_{W_{per}^{s,p}(D)} \equiv \|g\|_{W^{s,p}(D\cap Y)}$ for $s \geq 1$ and $p \in [1,\infty]$. Similar definition for $L_{per}^p(D), H_{per}^s(D), C_{per}^{k,\alpha}(D)$ when $D \in \{\mathbb{R}^n, \mathcal{Z}_f, \mathcal{Z}_f^\epsilon\}$, $k \geq 0, \alpha \in [0,1], s \geq 1$, and $p \in [1,\infty]$. If $G(x) = g(\epsilon x), g \in C_{per}^{0,\alpha}(\mathcal{Z}_f^\epsilon)$ for $\alpha \in (0,1)$, define $\|g\|_{C_{per}^{0,\alpha}(\mathcal{Z}_f^\epsilon)} \equiv \|G\|_{C^{0,\alpha}(\mathcal{Z}_f)}$.

For each $i = 1, \cdots, n$, we find $\mathbb{X}^{(i)}(y) \in H_{per}^1(\mathcal{Z}_f)$ satisfying, in cell Y_f,

$$\begin{cases} -\nabla\cdot(\mathbf{K}(\nabla\mathbb{X}^{(i)} + \vec{e}_i)) = 0 & \text{in } Y_f, \\ \mathbf{K}(\nabla\mathbb{X}^{(i)} + \vec{e}_i)\cdot\vec{\mathbf{n}}_y = 0 & \text{on } \partial Y_m, \\ \int_{Y_f}\mathbb{X}^{(i)}dy = 0, \end{cases} \tag{2.1}$$

where $\vec{\mathbf{n}}_y$ denotes the unit normal vector on ∂Y_m and $\vec{e}_i$ is a unit vector in the i-th coordinate direction for $i = 1, \cdots, n$. For any $\nu > 0$, we define $\mathbb{X}_\nu^{(i)}(x) \equiv \nu\mathbb{X}^{(i)}(\frac{x}{\nu})$, $\mathbb{X} \equiv (\mathbb{X}^{(1)}, \cdots, \mathbb{X}^{(n)})$, and $\mathbb{X}_\nu \equiv (\mathbb{X}_\nu^{(1)}, \cdots, \mathbb{X}_\nu^{(n)})$. Denote by Ξ a $n\times n$ matrix function whose (i,j) component is $\partial_{y_i}\mathbb{X}^{(j)}$ and define

$$\mathbf{K}^* \equiv \int_{Y_f}\mathbf{K}(y)(I + \Xi(y))dy, \tag{2.2}$$

where I is the identity matrix. By [2] and remark in page 90 [11], $\mathbf{K}^*$ is a constant symmetric positive definite matrix. For $i_1, i_2 = 1, \cdots, n$, find $\mathbb{X}^{(i_1,i_2)}(y) \in H_{per}^1(\mathcal{Z}_f)$ satisfying, in cell Y_f,

$$\begin{cases} \nabla\cdot(\mathbf{K}\nabla\mathbb{X}^{(i_1,i_2)}) + \partial_{i_1}(\mathbf{K}\mathbb{X}^{(i_2)}) = -\mathbf{K}(\delta_{i_1,i_2} + \partial_{i_1}\mathbb{X}^{(i_2)}) + \frac{\mathbf{K}^*_{i_1,i_2}}{|Y_f|} & \text{in } Y_f, \\ \mathbf{K}(\nabla\mathbb{X}^{(i_1,i_2)}\cdot\vec{\mathbf{n}}_y + \mathbb{X}^{(i_2)}\mathbf{n}_{y_{i_1}}) = 0 & \text{on } \partial Y_m, \\ \int_{Y_f}\mathbb{X}^{(i_1,i_2)}dy = 0. \end{cases} \tag{2.3}$$

Here $\delta_{i_1,i_2} \equiv \begin{cases} 1 & \text{if } i_1 = i_2, \\ 0 & \text{if } i_1 \neq i_2, \end{cases}$ $\mathbf{K}^*_{i_1,i_2}$ is the (i_1,i_2) component of $\mathbf{K}^*$ in (2.2), and $\mathbf{n}_{y_{i_1}}$ is the i_1 component of $\vec{\mathbf{n}}_y$. Similarly define $\mathbb{X}^{(i_1,i_2,\ldots,i_\ell)} \in H_{per}^1(\mathcal{Z}_f)$ for $\ell \geq 3$, $i_j \in \{1, \cdots, n\}$, and $j \in \{1, 2, \cdots, \ell\}$ as, in cell Y_f,

$$\begin{cases} \nabla\cdot(\mathbf{K}\nabla\mathbb{X}^{(i_1,\ldots,i_\ell)}) + \partial_{i_1}(\mathbf{K}\mathbb{X}^{(i_2,\ldots,i_\ell)}) \\ \qquad = -\mathbf{K}(\delta_{i_1,i_2}\mathbb{X}^{(i_3,\ldots,i_\ell)} + \partial_{i_1}\mathbb{X}^{(i_2,\ldots,i_\ell)}) & \text{in } Y_f, \\ \mathbf{K}\left(\nabla\mathbb{X}^{(i_1,\ldots,i_\ell)}\cdot\vec{\mathbf{n}}_y + \mathbb{X}^{(i_2,\ldots,i_\ell)}\mathbf{n}_{y_{i_1}}\right) = 0 & \text{on } \partial Y_m, \\ \int_{Y_f}\mathbb{X}^{(i_1,\ldots,i_\ell)}dy = 0. \end{cases} \tag{2.4}$$

By energy method and Lax-Milgram theorem [9], $\mathbb{X}^{(i_1,\ldots,i_\ell)}$ for $\ell \geq 1$ in (2.1), (2.3), and (2.4) are solvable uniquely. By Lemma 6.29 [9], if $\mathbf{K} \in C_{per}^{1,\alpha}(\mathcal{Z}_f)$ for $\alpha \in (0,1)$, then

$$\|\mathbb{X}^{(i_1,\ldots,i_\ell)}\|_{C^{2,\alpha}(\mathcal{Z}_f)} \leq c \qquad \text{for } \ell \geq 1. \tag{2.5}$$

2.1. L^∞ *error estimate*

We shall assume

A1. Ω and Y_m are bounded $C^{1,\alpha}$ domains,
A2. $\mathbf{K} \in C^{1,\alpha}_{per}(\mathcal{Z}_f)$ is a positive function,
A3. $F \in W^{1,n+\delta}(\Omega)$,

where $\alpha \in (\mu, 1)$, $\mu \equiv \frac{\delta}{n+\delta}$, $n \in \{2, 3\}$, and $\delta \in (0, 3)$. We recall an extension result in [1].

Lemma 2.1. *For $1 \le p < \infty$, there is a constant $c(Y_f, p)$ and a linear continuous extension operator $\Pi_\epsilon : W^{1,p}(\Omega^\epsilon_f) \to W^{1,p}(\Omega)$ such that (1) If $\varphi \in W^{1,p}(\Omega^\epsilon_f)$, then*

$$\begin{cases} \Pi_\epsilon \varphi = \varphi \quad \text{in } \Omega^\epsilon_f \text{ almost everywhere}, \\ \|\Pi_\epsilon \varphi\|_{L^p(\Omega)} \le c(Y_f, p)\|\varphi\|_{L^p(\Omega^\epsilon_f)}, \\ \|\nabla \Pi_\epsilon \varphi\|_{L^p(\Omega)} \le c(Y_f, p)\|\nabla \varphi\|_{L^p(\Omega^\epsilon_f)}, \\ \|\Pi_\epsilon \varphi\|_{L^\infty(\Omega)} \le c(Y_f, p)\|\varphi\|_{L^\infty(\Omega^\epsilon_f)} \quad \text{if } \varphi \in L^\infty(\Omega^\epsilon_f), \\ \Pi_\epsilon \varphi = \zeta \quad \text{in } \Omega \text{ if } \varphi = \zeta|_{\Omega^\epsilon_f} \text{ for some linear function } \zeta \text{ in } \Omega. \end{cases}$$

(2) If $\zeta(x) \equiv \varphi(rx)$ in $B_1(z) \cap \Omega^\epsilon_f / r$ for any $z \in \mathbb{R}^n$ and any constant $r > \epsilon$, there is a $\theta \in (0, 1)$ so that $\Pi_{\epsilon/r}\zeta(x) = (\Pi_\epsilon \varphi)(rx)$ in $B_\theta(z) \cap \Omega / r$

Lemma 2.1 also holds if Ω^ϵ_f (resp. Ω) is replaced by $\mathcal{Z}^\epsilon_f$ (resp. $\mathbb{R}^n$). For the solutions of (1.1) and (1.3), we have the following error estimate:

Lemma 2.2. *Under A1–A3, there is a constant ϵ_0 such that the solutions of (1.1) and (1.3) satisfy, for any $\epsilon \in (0, 1)$ and $\lambda \in [0, \epsilon_0)$,*

$$\|U_\epsilon - U\|_{L^\infty(\Omega^\epsilon_f)} \le c\epsilon \|F\|_{W^{1,n+\delta}(\Omega)},$$

where c is a constant independent of ϵ, λ.

We now describe the families of subspaces to be used to approximate the solution of (1.3). Let $\{\mathcal{F}^h : 0 < h \le 1\}$ be a family of subdivisions of Ω into disjoint, non-empty, connected, open sets $\tau \in \mathcal{F}^h$ of diameter not greater than h (a subdivision means $\overline{\Omega} = \cup_{\tau \in \mathcal{F}^h} \overline{\tau}$). Then we assume

A4. $\{\mathcal{F}^h : 0 < h \le 1\}$ is a quasi-uniform (see page 5 [10]) family of subdivisions of Ω and $\{\mathcal{S}^h : 0 < h \le 1\}$ is a family of linear spaces of functions on Ω such that, for each $\varphi \in \mathcal{S}^h$ and $\tau \in \mathcal{F}^h$, we have $\varphi|_\tau \in \mathcal{P}$, where $\mathcal{P}$ is a fixed finite dimensional space of polynomials independent of h, τ, and φ.

A5. For some non-negative integer m not greater than n and for some integer k greater than m, there is a linear mapping $\mathcal{J}^h$ for each $h \in (0, 1]$ of

$$\mathcal{D}^h \equiv \{\zeta \in C^m(\overline{\Omega}) : \zeta|_{\partial\Omega} = 0\} \cup \{\zeta\varphi : \zeta \in C^m(\overline{\Omega}), \varphi \in \mathcal{S}^h\}$$

into $\mathcal{S}^h$ such that

(i) $\mathcal{J}^h \varphi = \varphi$ for all $\varphi \in \mathcal{S}^h$.

Furthermore, for any integer ℓ and any $p \in [1, \infty]$ such that

$$\begin{cases} \max\{1, m\} + n \le \ell \le k & \text{if } p = 1, \\ \max\{1, m\} + n/p < \ell \le k & \text{if } 1 < p < \infty, \\ m < \ell \le k & \text{if } p = \infty, \end{cases}$$

it follows that

$$\text{(ii)} \sum_{j=0}^{1} h^{j+n/p} \|D^j(\zeta - \mathcal{J}^h \zeta)\|_{L^\infty(\tau)} \le c_{\ell,p} h^\ell \|\zeta\|_{W^{\ell,p}(\tau)}$$

for all $\tau \in \mathcal{F}^h$ and $\zeta \in \mathcal{D}^h \cap W^{\ell,p}(\tau)$, where $c_{\ell,p}$ is independent of ζ, τ, h.

A6. For all $h \in (0, 1]$, $\mathcal{S}^h \subset W^{1,\infty}(\Omega)$.

Many finite element spaces satisfying *A*4–*A*6 have been constructed, see [7, 10] and references therein. Inverse inequalities (see Theorem 3.2.6 [7]) hold in these finite element spaces. Define a bilinear form $\mathcal{L}^h_\gamma$ on $\mathcal{S}^h \times \mathcal{S}^h$ as

$$\begin{aligned} \mathcal{L}^h_\gamma(\varphi, \zeta) \equiv & \int_\Omega \mathbf{K}^* \nabla\varphi \nabla\zeta \; dx + \int_\Omega \lambda |Y_f| \varphi\zeta \; dx - \int_{\partial\Omega} \mathbf{K}^* \varphi \nabla\zeta \cdot \vec{\mathbf{n}} \; d\sigma \\ & - \int_{\partial\Omega} \mathbf{K}^* \zeta \nabla\varphi \cdot \vec{\mathbf{n}} \; d\sigma + \gamma h^{-1} \int_{\partial\Omega} \varphi\zeta \; d\sigma \end{aligned}$$

where $\varphi, \zeta \in \mathcal{S}^h$, $\vec{\mathbf{n}}$ is the unit outward normal vector on $\partial\Omega$, $\mathbf{K}^*$ is that in (2.2), and γ is some positive number. Now we find $U^h \in \mathcal{S}^h$ such that

$$\mathcal{L}^h_\gamma(U^h, \zeta) = \int_\Omega |Y_f| F \zeta dx \qquad \text{for all } \zeta \in \mathcal{S}^h. \tag{2.6}$$

Here U^h is the numerical approximation of the solution of (1.3). By Lax-Milgram theorem [9], (2.6) is solvable uniquely. Let us recall Theorem 3.1 [10].

Theorem 2.1. *Suppose A*4*–A*6 *hold, U solves (1.3), U^h solves (2.6), and $n = 2, 3$. Let $U \in W^{s,\infty}(\Omega)$ with $\max\{m, 1\} < s \le k$, where m and k are the parameters in A*5 *and $k \ge 3$. Then there are constants $\gamma_1 < \infty$ and $h_1 > 0$ such that, for $\gamma \ge \gamma_1$ and $0 < h \le h_1$,*

$$\|U - U^h\|_{L^\infty(\Omega)} \le c h^s \|U\|_{W^{s,\infty}(\Omega)}, \tag{2.7}$$

where c is independent of h and U.

By *A*3 and Theorem 9.15 [9], we know $U \in W^{2,\infty}(\Omega)$. Lemma 2.2 and Theorem 2.1 imply that the maximum norm estimate of $U_\epsilon - U^h$ (that is, difference between the solution of (1.1) and the solution of (2.6)) satisfies

$$\begin{aligned} \|U_\epsilon - U^h\|_{L^\infty(\Omega^\epsilon_f)} & \le \|U_\epsilon - U\|_{L^\infty(\Omega^\epsilon_f)} + \|U - U^h\|_{L^\infty(\Omega^\epsilon_f)} \\ & \le c(\epsilon + h^2) \|F\|_{W^{1,n+\delta}(\Omega)}. \end{aligned} \tag{2.8}$$

where c is a constant independent of ϵ, h. In other words, if Lemma 2.2 holds, we have the following results:

Theorem 2.2. *Under A1–A6, there exist positive constants $\epsilon_0, h_1, \gamma_1$ such that when $\epsilon \in (0,1), \lambda \in [0,\epsilon_0), h \in (0,h_1)$, and $\gamma > \gamma_1$, the solution of (1.1) and the numerical approximation in (2.6) satisfy the L^∞ error estimate (2.8).*

2.2. *Lipschitz error estimate*

In this subsection, functions considered are periodic and have period $[0,1]^n$ for $n \in \{2,3\}$. If $\lambda > 0$ and $F \in L^2_{per}(\mathbb{R}^n)$, we find $U_\epsilon \in H^1_{per}(\mathcal{Z}^\epsilon_f)$ satisfying

$$\begin{cases} -\nabla \cdot (\mathbf{K}_\epsilon \nabla U_\epsilon) + \lambda U_\epsilon = F & \text{in } \mathcal{Z}^\epsilon_f, \\ \mathbf{K}_\epsilon \nabla U_\epsilon \cdot \vec{\mathbf{n}}^\epsilon = 0 & \text{on } \partial \mathcal{Z}^\epsilon_m. \end{cases} \tag{2.9}$$

By Lax-Milgram theorem [9], (2.9) is solvable uniquely in $H^1_{per}(\mathcal{Z}^\epsilon_f)$ and the solution satisfies $\|U_\epsilon\|_{H^1_{per}(\mathcal{Z}^\epsilon_f)} \leq c\|F\|_{L^2_{per}(\mathbb{R}^n)}$, where c is a constant independent of ϵ. By compactness principle [2] and Lemma 2.1, there is a function $U \in H^1_{per}(\mathbb{R}^n)$ such that the solution U_ϵ of (2.9) satisfies

$$\begin{cases} \Pi_\epsilon U_\epsilon \to U & \text{in } H^1_{per}(\mathbb{R}^n) \text{ weakly} \\ \mathbf{K}_\epsilon \nabla U_\epsilon \mathcal{X}_{\mathcal{Z}^\epsilon_f} \to \mathbf{K}^* \nabla U & \text{in } L^2_{per}(\mathbb{R}^n) \text{ weakly} \end{cases} \quad \text{as } \epsilon \to 0$$

and U satisfies

$$-\nabla \cdot (\mathbf{K}^* \nabla U) + \lambda |Y_f| U = |Y_f| F \qquad \text{in } \mathbb{R}^n. \tag{2.10}$$

Here $|Y_f|$ is the volume of Y_f and $\mathbf{K}^*$ is the positive definite matrix in (2.2).

Lemma 2.3. *Under the following conditions*

A1′. $\delta \in (0,3)$, $\alpha > 0$, *and Y_m is a $C^{1,\alpha}$ domain,*
A2 . $\mathbf{K} \in C^{1,\alpha}_{per}(\mathcal{Z}_f)$ *is a positive function,*
A3′. $F \in W^{2,n+\delta}_{per}(\mathbb{R}^n)$,

there is a constant $\epsilon_0 < 1$ such that the solutions of (2.9) and (2.10) satisfy, for any $\epsilon \in (0,1)$ and $\lambda \in (0,\epsilon_0)$,

$$\|\nabla U_\epsilon(x) - (I + \nabla \mathbb{X}(x/\epsilon)) \nabla U(x)\|_{L^\infty(\mathcal{Z}^\epsilon_f)} \leq c\epsilon \|F\|_{W^{2,n+\delta}_{per}(\mathbb{R}^n)}, \tag{2.11}$$

where c is a constant independent of ϵ.

Next we describe the finite element spaces to approximate the solution of (2.10). Let $\{\mathcal{F}^h : 0 < h < 1\}$ be a family of subdivisions of $\mathbb{R}^n$ into disjoint, non-empty, connected, open sets $\tau \in \mathcal{F}^h$ of diameter not greater than h (a subdivision means $\mathbb{R}^n = \cup_{\tau \in \mathcal{F}^h} \overline{\tau}$) and each subdivision is periodic with period $[0,1]^n$ (that is, $\tau + j \in \mathcal{F}^h$ for any $\tau \in \mathcal{F}^h$ and $j \in \mathbb{Z}^n$). For $r \geq 2$, $\mathcal{S}^h_r(\mathbb{R}^n)$ is a family of finite dimensional subspace of $W^{1,\infty}_{per}(\mathbb{R}^n)$. If $D \subset \mathbb{R}^n$, then $\mathcal{S}^h_r(D)$ (resp. $W^{1,\infty}_{per}(D)$) denotes the restriction of functions in $\mathcal{S}^h_r(\mathbb{R}^n)$ (resp. $W^{1,\infty}_{per}(\mathbb{R}^n)$) to D. Let $kh + d_i < d_{i+1}$ for some $k \in \mathbb{N}$ and $i \in \{-1,0,1,2,3\}$. Let us assume

A7. If $t \in \{0,1\}$, $t \le \ell \le r$, $1 \le p \le \infty$, then for each $\varphi \in W^{\ell,p}(B_{d_2})$ there exists a $\zeta \in \mathcal{S}_r^h(B_{d_2})$ such that

$$\|\varphi - \zeta\|_{W^{t,p}(B_{d_1})} \le ch^{\ell-t}\|\varphi\|_{W^{\ell,p}(B_{d_2})}.$$

If $n < p \le \infty$,

$$\|\varphi - \zeta\|_{W^{1,\infty}(B_{d_1})} \le ch^{r-1-n/p}\|\varphi\|_{W^{r,p}(B_{d_2})}.$$

Furthermore if φ vanishes outside of B_{d_0}, ζ vanishes outside of B_{d_1}. The constant c is independent of $h, \varphi, \zeta, B_{d_1}$, and B_{d_2}.

A8. If $\zeta \in \mathcal{S}_r^h(\mathbb{R}^n)$, then for $t \in \{0,1\}$ and $\ell \ge 0$ is an integer and $1 \le q \le p \le \infty$,

$$\|\zeta\|_{W^{t,p}(B_{d_1})} \le ch^{-(n/q-n/p)-t-\ell}\|\zeta\|_{W^{-\ell,q}(B_{d_2})}.$$

The constant c is independent of h, ζ, B_{d_1}, and B_{d_2}.

A9. Let $\varphi \in C_0^\infty(B_{d_1})$, then for each $\zeta \in \mathcal{S}_r^h(B_{d_3})$ there exists an $\eta \in \mathcal{S}_r^h(B_{d_3})$, vanishing outside of B_{d_2}, such that for some integer $\gamma > 0$

$$\|\varphi\zeta - \eta\|_{H^1(B_{d_3})} \le ch\|\varphi\|_{W^{\gamma,\infty}(B_{d_1})}\|\zeta\|_{H^1(B_{d_3})}.$$

Furthermore, if $\varphi = 1$ on B_{d_0}, then $\eta = \zeta$ on $B_{d_{-1}}$ and

$$\|\varphi\zeta - \eta\|_{H^1(B_{d_2})} \le ch\|\varphi\|_{W^{\gamma,\infty}(B_{d_1})}\|\zeta\|_{H^1(B_{d_2}\setminus B_{d_0})}.$$

Here c is independent of $\varphi, \zeta, \eta, h, B_{d_0}, B_{d_1}, B_{d_2}$.

A10. Let $x_0 \in \mathbb{R}^n$ and $d \ge kh$. The transformation $y = \frac{x-x_0}{d}$ takes $B_d(x_0)$ into a domain $B_1(x_0)$ and $\mathcal{S}_r^h(B_d(x_0))$ into a function space $\hat{\mathcal{S}}_r^{h/d}(B_1(x_0))$. Then $\hat{\mathcal{S}}_r^{h/d}(B_1(x_0))$ satisfies *A*7-*A*9 with h replaced by h/d. The constants in *A*7-*A*9 remain unchanged, in particular independent of d.

Finite element spaces satisfying *A*7-*A*10 can be found in [15] and reference therein. We find $U^h \in \mathcal{S}_r^h(\mathbb{R}^n)$ such that

$$\int_{[0,1]^n} \mathbf{K}^*\nabla U^h \nabla\zeta dx + \int_{[0,1]^n} \lambda|Y_f|U^h\zeta dx = \int_{[0,1]^n} |Y_f|F\zeta dx, \tag{2.12}$$

where $\zeta \in \mathcal{S}_r^h(\mathbb{R}^n)$. By $A3'$ and Theorem 9.11 [9], the solution of (2.10) satisfies $U \in W_{per}^{4,n+\delta}(\mathbb{R}^n)$. Theorem 3.1 [15] implies

Theorem 2.3. *If A7-A10 hold, $U \in W_{per}^{1,\infty}(\mathbb{R}^n)$ solves (2.10), and $U^h \in \mathcal{S}_4^h(\mathbb{R}^n)$ solves (2.12), then there exists a constant c independent of U, U^h, h such that*

$$\|U - U^h\|_{W_{per}^{1,\infty}(\mathbb{R}^n)} \le ch^{3-n/(n+\delta)}\|F\|_{W_{per}^{2,n+\delta}(\mathbb{R}^n)}. \tag{2.13}$$

Lemma 2.3, Theorem 2.3, and (2.5) imply that the Lipschitz norm estimate of $U_\epsilon - U^h$ (that is, difference between the solution of (2.9) and the solution of (2.12)) satisfies

$$\|\nabla U_\epsilon - (I + \nabla\mathbb{X}_\epsilon)\nabla U^h\|_{L^\infty(\mathcal{Z}_f^\epsilon)} \le c(\epsilon + h^{3-n/(n+\delta)})\|F\|_{W^{2,n+\delta}([0,1]^n)}, \tag{2.14}$$

where c is a constant independent of ϵ, h. So we have the following results:

Theorem 2.4. *Assume A1′, A2, A3′, A7–A10. There exist positive constants ϵ_0, h_1 such that if $\epsilon \in (0,1), \lambda \in (0,\epsilon_0)$, and $h \in (0,h_1)$, the solution of (2.9) and the approximation in (2.12) satisfy the Lipschitz error estimate (2.14).*

2.3. *Higher order Lipschitz estimate*

Besides L^∞ and Lipschitz error estimates, we also have higher order Lipschitz estimate. Functions in this subsection are periodic and have period $[0,1]^n$ for $n \in \{2,3\}$. If $F \in L^2_{per}(\mathbb{R}^n)$ satisfies $\int_{[0,1]^n\cap\mathcal{Z}^\epsilon_f} F dx = 0$, we find $U_\epsilon \in H^1_{per}(\mathcal{Z}^\epsilon_f)$ such that

$$\begin{cases} -\nabla\cdot(\mathbf{K}_\epsilon\nabla U_\epsilon) = F & \text{in } \mathcal{Z}^\epsilon_f, \\ \mathbf{K}_\epsilon\nabla U_\epsilon\cdot\vec{\mathbf{n}}^\epsilon = 0 & \text{on } \partial\mathcal{Z}^\epsilon_m, \\ \int_{[0,1]^n\cap\mathcal{Z}^\epsilon_f} U_\epsilon dx = 0. \end{cases} \tag{2.15}$$

By Lax-Milgram theorem [9], (2.15) is solvable uniquely in $H^1_{per}(\mathcal{Z}^\epsilon_f)$ and the solution satisfies $\|U_\epsilon\|_{H^1_{per}(\mathcal{Z}^\epsilon_f)} \le c\|F\|_{L^2_{per}(\mathbb{R}^n)}$, where c is a constant independent of ϵ. By compactness principle [2] and Lemma 2.1, there is a function $U \in H^1_{per}(\mathbb{R}^n)$ such that the solution U_ϵ of (2.15) satisfies

$$\begin{cases} \Pi_\epsilon U_\epsilon \to U & \text{in } H^1_{per}(\mathbb{R}^n) \text{ weakly} \\ \mathbf{K}_\epsilon\nabla U_\epsilon\mathcal{X}_{\mathcal{Z}^\epsilon_f} \to \mathbf{K}^*\nabla U & \text{in } L^2_{per}(\mathbb{R}^n) \text{ weakly} \end{cases} \quad \text{as } \epsilon\to 0$$

and U satisfies

$$\begin{cases} -\nabla\cdot(\mathbf{K}^*\nabla U) = |Y_f|F & \text{in } \mathbb{R}^n, \\ \int_{[0,1]^n} U dx = 0. \end{cases} \tag{2.16}$$

Here $|Y_f|$ is the volume of Y_f and $\mathbf{K}^*$ is the positive definite matrix in (2.2).

Theorem 2.5. *Under A1′, A2, and*

A3″. $F \in W^{k,n+\delta}_{per}(\mathbb{R}^n)$ for $k \ge 2$,

the solutions of (2.15) and (2.16) satisfy, for any $\epsilon \in (0,1)$,

$$\|\nabla\varphi_\epsilon\|_{L^\infty(\mathcal{Z}^\epsilon_f)} \le c\epsilon^{k-1}\|F\|_{W^{k,n+\delta}_{per}(\mathbb{R}^n)}.$$

where c is a constant independent of ϵ and

$$\varphi_\epsilon(x) \equiv U_\epsilon(x) - U(x) - \sum_{\ell=1}^{k}\epsilon^\ell \sum_{i_1,\dots,i_\ell=1}^{n} \mathbb{X}^{(i_1,\dots,i_\ell)}(\frac{x}{\epsilon})\partial_{i_1,\dots,i_\ell}U(x).$$

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參加會議經過

2010 年 SIAM 年會是在美國賓州的匹茲堡 (Pittsburg)市舉行。時間由九十九年七月十二日至十六日共五天。會議的主題包含了大氣物理、洋流、地球科學、電磁學、生物數學、材料學、影像處理、計算方法等。演講內容包含有問題的模式的推導、計算方法的設計、及數學性質的探討。我的報告時間是安排在會議的第二天上午。講題是``Numerical Methods for Elliptic and Parabolic Equations in Perforated Domains”。主要是介紹如何有效率的得到橢圓方程式與拋物線方程式在 Perforated Domains 的解。

與會心得

和往年一樣我都會參加一些與我的研究主題相關的 session。參加完這次會議我的最大的心得就是對 point source 的了解。它主要是描述在很小的範圍內有很大的外加進來的 source 情形。point source 在很多的應用問題上都會出現，如材料力學、地球科學、流體力學、地質學等的研究都可能會遇到 point source 這問題。在橢圓形方程式或拋物線方程式的 Green function 即為對應到 point source 的解，關於 Green function 在數學上已經有很多的研究，大家對它也有一定程度的了解。在數值計算上，無論是有限差分法、有限元素法、或是 finite volume method 也都有簡單的方法來處理 point source。一切看起來似乎都很美好，point source 在數學理論上、在數值計算上不應該給我們造成任何的困擾。不過再仔細想一想有 point source 與沒有 point source 最大的差異是甚麼?我覺得最大的差異是在 point source 附近壓力與流速趨近於無限大，而且在一般情形下它幾乎無法計算。若是我們只在乎整體的變化，則 point source 的計算倒是容易，畢竟它只佔整個區域的極少部份，即使 point source 附近的計算有很大的誤差，它對整體的影響還是很小，所以一般的有限差分法、有限元素法、或是 finite volume method 都能處理 point source 的問題。但若是我們在乎每一個小範圍的變化，則 point source 附近的計算就很重要了。譬如需要保有每一個小範圍都有質量守恆性質的數值計算方法，就在乎每一個小範圍的變化。真正從是計算它的值，即可發現它的困難度。我的最大的心得就是對如何計算 point source 附近的的壓力與流速找到了一個好的辦法。

無衍生研發成果推廣資料

98 年度專題研究計畫研究成果彙整表

計畫主持人：葉立明			計畫編號：98-2115-M-009-011-				
計畫名稱：破裂介質中的熱傳導方程式							
成果項目			量化			單位	備註（質化說明：如數個計畫共同成果、成果列為該期刊之封面故事...等）
			實際已達成數（被接受或已發表）	預期總達成數(含實際已達成數)	本計畫實際貢獻百分比		
國內	論文著作	期刊論文	0	0	100%	篇	
		研究報告/技術報告	2	2	100%		
		研討會論文	0	0	100%		
		專書	0	0	100%		
	專利	申請中件數	0	0	100%	件	
		已獲得件數	0	0	100%		
	技術移轉	件數	0	0	100%	件	
		權利金	0	0	100%	千元	
	參與計畫人力（本國籍）	碩士生	2	2	100%	人次	
		博士生	0	0	100%		
		博士後研究員	0	0	100%		
		專任助理	0	0	100%		
國外	論文著作	期刊論文	0	0	100%	篇	
		研究報告/技術報告	0	0	100%		
		研討會論文	0	0	100%		
		專書	0	0	100%	章/本	
	專利	申請中件數	0	0	100%	件	
		已獲得件數	0	0	100%		
	技術移轉	件數	0	0	100%	件	
		權利金	0	0	100%	千元	
	參與計畫人力（外國籍）	碩士生	0	0	100%	人次	
		博士生	0	0	100%		
		博士後研究員	0	0	100%		
		專任助理	0	0	100%		

其他成果 (無法以量化表達之成果如辦理學術活動、獲得獎項、重要國際合作、研究成果國際影響力及其他協助產業技術發展之具體效益事項等，請以文字敘述填列。)	無

	成果項目	量化	名稱或內容性質簡述
科教處計畫加填項目	測驗工具(含質性與量性)	0	
	課程/模組	0	
	電腦及網路系統或工具	0	
	教材	0	
	舉辦之活動/競賽	0	
	研討會/工作坊	0	
	電子報、網站	0	
	計畫成果推廣之參與（閱聽）人數	0	

國科會補助專題研究計畫成果報告自評表

請就研究內容與原計畫相符程度、達成預期目標情況、研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）、是否適合在學術期刊發表或申請專利、主要發現或其他有關價值等，作一綜合評估。

1. 請就研究內容與原計畫相符程度、達成預期目標情況作一綜合評估

■達成目標

□未達成目標（請說明，以 100 字為限）

　　□實驗失敗

　　□因故實驗中斷

　　□其他原因

說明：

2. 研究成果在學術期刊發表或申請專利等情形：

論文：□已發表 ■未發表之文稿 □撰寫中 □無

專利：□已獲得 □申請中 ■無

技轉：□已技轉 □洽談中 ■無

其他：（以 100 字為限）

3. 請依學術成就、技術創新、社會影響等方面，評估研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）（以 500 字為限）

1. H\' ' ' ' older estimate for non-uniform parabolic equations in highly heterogeneous media

此篇論文研究的方程式是 non-uniform parabolic equations with discontinuous coefficients.從文獻中我們知道 uniform parabolic equations with discontinuous coefficients 及 non-uniform parabolic equations with smooth coefficients 有人研究過.以上兩種情況下解都有可能是平滑的.但是我們考慮的方程式的解是否平滑要看是在甚麼區域.這與之前的認知很不同.在學術上,這是值得繼續深究的.在實際問題上,一如成果報告中所述是很有應用價值的研究. 此篇論文研究的成果告訴我們在各個不同區域的平滑性.

2.Pointwise error estimate for elliptic equations in perforated domains

它的重要性在很多的文獻中都可看出.這裡不再贅述. 此篇論文則告訴我們每一點的誤差估計.