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TWO-SIDED EXIT FOR PHASE-TYPE LÉVY MODELS AND PERPETUAL CALLABLE BOND

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ABSTRACT. We consider a firm whose asset value follows a jump diffusion and both the upward and downward jumps are distributed as mixtures of exponential distributions. Parallel to Leland(1994) and Goldstein et al.(2001), we consider corporate debt for which default, tax effect and bankruptcy are all considered. As in Black and Cox(1976), Brennan and Schwartz(1978), Fischer et al.(1989), Duffie and Singleton(2001) and Leland(1998), we allow the possibility of redemption of debt. We give a closed form solution for such bond. For related work on pricing securities in jump diffusion model, see Asmussen et al.(2004) and Chen et al.(2006).

1. Introduction

It is commonly stipulated in a bond covenant that the bond under consideration is callable or redeemed under some circumstances. And such redemption can be seen as recapitalization of firm. Therefore, in addition to the common practice of modeling corporate bond by the discounted recovery at default, one should also consider in the case of callable debt the discounted recovery of redemption prior to default. Indeed, such *two-sided exit problem* for bond pricing is well recognized and discussed in, for example, Black and Cox(1976), Brennan and Schwartz(1978), Fischer et al.(1989), Duffie and Lando(2001)and Leland(1998). Bodie and Taggart(1978) also provided a simplified model to explain the phenomenon why firms prefer callable bonds.

In this paper, we assume that the firm asset value is a jump diffusion for which both upward and downward jumps are controlled by mixtures of exponential distributions. We will give a closed form solution of the risk neutral price of a perpetual corporate bond for which both the default and redemption of bond are possible for two exogenously determined boundaries.

2. Pricing Perpetual Callable Coupon Bond

As in Black and Cox(1976), Leland(1994), Goldstein et al.(2001) and many others, we assume the existence of a constant risk free rate $r > 0$ for all maturities. Let $\mathbb{P}$ be an equivalent martingale measure such that a given firm has its asset value following the dynamics

$$dV_t = V_{t-}\left(\mu dt + dM_t\right)$$

up to the time of default, where $\mu \in \mathbb{R}$ and $M = (M_t; t \geq 0)$ is a martingale. Then the asset value process $V = (V_t; t \geq 0)$ takes the form

$$V_t = V_0 e^{X_t}, \quad t \geq 0,$$

for some process $X = (X_t; t \geq 0)$. In this paper, we assume X is given by a jump diffusion

$$X_t = ct + \sigma W_t - \sum_{n=1}^{N_t} Y_n, \quad t \geq 0. \tag{2.1}$$

Here $c \in \mathbb{R}$, $\sigma > 0$, $W = (W_t; t \geq 0)$ is a standard Brownian motion, $N = (N_t; t \geq 0)$ is a compound Poisson process with rate $\lambda > 0$, and the jump sizes $(Y_n, n \geq 1)$ are independent and identically distributed; all the aforementioned objects being mutually independent. We assume the distribution F of Y_1 has probability density function

$$f(y) = \begin{cases} \sum_{j=1}^{m_{(+)}} p_j \eta_j^+ e^{-\eta_j^+ y}, & y > 0, \\ 0, & y = 0, \\ \sum_{j=1}^{m_{(-)}} q_j \eta_j^- e^{\eta_j^- y}, & y < 0. \end{cases} \tag{2.2}$$

where η_j^+'s are distinct, η_j^-'s are distinct, $\sum_{j=1}^{m_{(+)}} p_j + \sum_{j=1}^{m_{(-)}} q_j = 1$, and $p_j, q_j, \eta_j^\pm > 0$. We will denote by $\mathbb{P}_x$ the law of $X + x$ under $\mathbb{P}$ and hence by the definition of $\mathbb{P}$, $\mathbb{P}_{\log V_0} = \mathbb{P}$.

Assume the management decides to follow the **upward capital structure strategy** throughout time (see Goldstein et al.(2001) Section III). That is, at time 0, the firm chooses two thresholds V_L^0 and V_U^0 satisfying $V_L^0 < V_0 < V_U^0$ and issues a perpetual callable coupon bond whose covenant specifies

- **(1):** The life time of the bond ends if either of the following events occurs:
  - **(a):** The firm asset value first crosses V_U^0. Then recapitalization takes place, and the bond is called.
  - **(b):** The firm asset value first crosses V_L^0. Then the firm declares bankruptcy, and liquidation occurs.
- **(2):** The bond pays a constant coupon rate $C > 0$ up to the life time of the bond.
- **(3):** In case (a), the firm promises a time-inhomogeneous callable price K. In case (b), the bondholder takes over the firm and receives the remaining value of the firm. However, a fraction α of the remaining firm value is lost due to bankruptcy costs.

Period 0 ends whenever (a) or (b) occurs. Set $\gamma_R = V_U^0/V_0 > 1$ and $\gamma_D = V_0/V_L^0 \in (0,1)$. Then in general, period n begins at the time R_n the firm has not declared bankruptcy and the firm asset value first rises above $\gamma_R^{n-1}V_{R_{n-1}}$ after R_n. Throughout period n, a perpetual callable coupon bond whose covenant is the same as the one issued in period 0 except that the bankruptcy level and the recapitalization level are set respectively as $\gamma_D^n V_{D_n}$ and $\gamma_R^n V_{R_n}$. Period n ends when the firm has not declared bankruptcy and accumulates sufficiently large asset value above $\gamma_R^n V_{R_n}$. We also assume the bondholder has a personal tax rate τ_{p} and the corporate tax rate is τ_{c}.

Such a capital structure leads us some natural pricing problems:

(1) What are the no arbitrage values of the perpetual callable coupon bonds issued at the beginning of each period under $\mathbb{P}$ (and hence the perpetual bond issued at the beginning of period n)?
(2) What is the value of the firm?
(3) What are the optimal parameters V_L^0, V_U^0, γ_R and γ_D that can maximize the shareholders' value? Then what are the maximized equity value and maximized firm value?

We will answer these questions in the subsequent section.

According to bond covenant, the **first default time** is given by

$$D_1 = \inf\left\{t \geq 0; \sup_{s\leq t} V_s < V_U^0, V_t \leq V_L^0\right\},$$

the **first recapitalization time** is given by

$$R_1 = \inf\left\{t \geq 0; \inf_{s\leq t} V_s > V_L^0, V_t \geq V_U^0\right\}.$$

and the **first contract ceasing time** is given by:

$$\tau_1 = R_1 \wedge D_1.$$

Then under the risk neutral probability measure, a no-arbitrage price of the corporate bond is given by

$$D(V_0) = \mathbb{E}\left[\int_0^{\tau_1} C(1-\tau_{\mathrm{p}})e^{-rt}dt\right] + \mathbb{E}\left[\widehat{g}(V_{\tau_1})e^{-r\tau_1}\right], \tag{2.3}$$

where

$$\widehat{g}(y) = \begin{cases} (1-\alpha)y, & \text{if } y \leq V_L, \\ K, & \text{if } y \geq V_U. \end{cases}$$

The right hand side of equation (2.3) has the following explanation. The first term comes from the discounted after-tax coupon payment up to the first contract ceasing time. And the second term can be written as the sum of

$$\mathbb{E}\left[(1-\alpha)V_{\tau_1}e^{-r\tau_1}\mathbf{1}(R_1 > D_1)\right] \tag{2.4}$$

and

$$\mathbb{E}\left[Ke^{-r\tau_1}\mathbf{1}(R_1 < D_1)\right]. \tag{2.5}$$

So, we see that (2.4) and (2.5) are the discounted payoffs upon default and recapitalization, respectively. To facilitate our study of bond price, we write $V_t = e^{X_t}$ and straightforward computations give the following lemma.

Lemma 2.1. *Under $\mathbb{P}_x$ with $x = \log V_0$, $V_t = e^{X_t}$ for all $t \geq 0$ and the components of the bond price (2.3) can be written as*

$$\begin{aligned}\mathbb{E}\left[\int_0^{\tau_1} C(1-\tau_{\mathrm{p}})e^{-rt}dt\right] =& \frac{C(1-\tau_{\mathrm{p}})}{r}\left(1-\mathbb{E}_x\left[e^{-r\tau_B}\right]\right),\\ \mathbb{E}\left[(1-\alpha)V_{\tau_1}e^{-r\tau_1}\mathbf{1}(R_1>D_1)\right] =& (1-\alpha)\mathbb{E}_x\left[e^{-r\tau_B}\mathbf{1}_{e^{X_{\tau_B}}\leq V_L}e^{X_{\tau_B}}\right]\\ \mathbb{E}\left[Ke^{-r\tau_1}\mathbf{1}(R_1<D_1)\right] =& K\mathbb{E}_x\left[e^{-r\tau_B}\mathbf{1}_{e^{X_{\tau_B}}\geq V_U}\right].\end{aligned}$$

Theorem 2.1. *Let $\psi(\zeta)$ be the characteristic exponent of the process X. That is, $\mathbb{E}_0\left[e^{\zeta X_1}\right] = e^{\psi(\zeta)}$, for $\zeta \in i\mathbb{R}$. Then ψ is an analytic function on $\mathbb{C}$ except at a finite number of poles. Suppose the equation $\psi(\zeta) - r = 0$ admits distinct zeros, then the bond price is given by*

$$D(V_0) = \frac{C(1-\tau_{\mathrm{p}})}{r}\left[1-\boldsymbol{Q}(g_1)^\top \boldsymbol{e}^{\boldsymbol{\rho}}(\log V_0)\right] + (1-\alpha)\boldsymbol{Q}(g_2)^\top\boldsymbol{e}^{\boldsymbol{\rho}}(\log V_0) + K\boldsymbol{Q}(g_3)^\top\boldsymbol{e}^{\boldsymbol{\rho}}(\log V_0).$$

Here $g_1(y) \equiv 1$, $g_2(y) = e^y\mathbf{1}_{y\leq \log V_L}$, $g_3(y) = \mathbf{1}_{y\geq\log V_U}$ and $\boldsymbol{Q}(g_k)$ is a vector of constants that solves the following system of linear equations

$$(2.6)\qquad \begin{cases} \boldsymbol{Q}(g_k)^\top\boldsymbol{e}^{\boldsymbol{\rho}}(\log V_L) & = g(\log V_L),\\ \boldsymbol{Q}(g_k)^\top\boldsymbol{e}^{\boldsymbol{\rho}}(\log V_U) & = g(\log V_U),\\ \sum_{i=1}^{m+2}\frac{\boldsymbol{Q}(g_k)_i\eta_j^+e^{(\rho_i+\eta_j^+)\log V_L}}{\rho_i+\eta_j^+} & = \int_{-\infty}^{\log V_L} g(y)\eta_j^+e^{\eta_j^+y}dy, 1\leq j\leq m_{(+)},\\ \sum_{i=1}^{m+2}\frac{\boldsymbol{Q}_i(g_k)\eta_j^-e^{-(\eta_j^--\rho_i)\log V_U}}{\rho_i+\eta_j^-} & = \int_{\log V_U}^{\infty} g(y)\eta_j^-e^{-\eta_j^-y}dy, 1\leq j\leq m_{(-)}.\end{cases}$$

Proof. See Theorem A.1 in Appendix. □

3. A Two-Sided Exit Problem

In this appendix, we solve the valuation problem of (2.3). We begin with the observation that we can alternatively write the debt value as

$$\begin{aligned}D(V_0) &= C(1-\tau_{\mathrm{p}})\mathbb{E}\left[\int_0^{\tau_B} e^{-rt}dt\right] + (1-\alpha)\mathbb{E}\left[e^{-r\tau_B}\mathbf{1}_{V_{\tau_B}\leq V_L}V_{\tau_B}\right] + K\mathbb{E}\left[e^{-r\tau_B}\mathbf{1}_{V_{\tau_B}\geq V_U}\right]\\ &= \frac{C(1-\tau_{\mathrm{p}})}{r}\left(1-\mathbb{E}\left[e^{-r\tau_B}\right]\right) + (1-\alpha)\mathbb{E}\left[e^{-r\tau_B}\mathbf{1}_{e^{X_{\tau_B}}\leq V_L}e^{X_{\tau_B}}\right] + K\mathbb{E}\left[e^{-r\tau_B}\mathbf{1}_{e^{X_{\tau_B}}\geq V_U}\right].\end{aligned}$$

So, to give solution for the valuation problem of debt, it suffices to compute the first passage time functional

$$\Phi(x) = \mathbb{E}_x\left[e^{-r\tau_B}g(X_{\tau_B})\right],$$

where g is a nonnegative bounded Borel measurable function and $X_0 = x$ a.s. under $\mathbb{P}_x$. On the other hand, by Dynkin's formula and Theorem of Feynman and Kac, one needs to solve the following boundary value problem which admits at most one solution: find $\Phi \in \mathcal{C}([L,U]) \cap \mathcal{C}^2((L,U))$ such that

$$(3.1)\qquad \begin{cases} (L-r)\Phi = 0, & \text{in } (L,U)\\ \Phi = g, & \text{on } (-\infty,L]\cup[U,\infty).\end{cases}$$

Here $U = \log V_U$, $L = \log V_L$ and L is the infinitesimal generator of X acting on $h \in \mathcal{C}_0^2(\mathbb{R})$ by

$$(3.2)\qquad Lh(x) = \frac{\sigma^2}{2}h''(x) + ch'(x) + \lambda\int h(x-y)dF(y) - \lambda h(x).$$

For details, see Bertoin(1996) or Sato(1999).

By the assumption that both tails of f are mixtures of exponential distributions, the Fourier multiplier $\psi(\zeta) - r$ of $L - r$ is a rational function, where ψ is the characteristic exponent of X. Let $\mathcal{P}_0(\zeta)$ be the **minimal polynomial** such that $\mathcal{P}_1(\zeta) = \mathcal{P}_0(\zeta)(\psi(\zeta) - r)$ is a polynomial whose zeros coincide with those of $\psi(\zeta) - r$. If we denote by D the differential operator such that its characteristic polynomial is $\mathcal{P}_1(\zeta)$, then, as we will show below in Lemma A.1,

$$(3.3)\qquad D\Phi \equiv 0, \text{ on } (L,U),$$

which is a homogeneous ODE of higher order.

Lemma 3.1. *Suppose there exists a bounded solution Φ to the boundary value problem (3.1) and the jump distribution F has a density f given by (2.2). Then on (L,U), Φ is infinitely differentiable on (L,U) and satisfies (3.3).*

Proof. We now prove this lemma by direct computation. And we will first show that Φ is infinitely differentiable and then transform the integro-differential equation $(L-r)\Phi\equiv 0$ into an ODE.

Plugging the density function f given by (2.2) into (3.2), we deduce that the generator L acting on Φ is given by

$$\begin{aligned} L\Phi(x) =& \frac{\sigma^2}{2}\Phi''(x)+c\Phi'(x)+\lambda\left(\sum_{j=1}^{m_{(+)}} p_j\eta_j^+\int_0^\infty \Phi(x-y)e^{-\eta_j^+ y}dy+\sum_{j=1}^{m_{(-)}} q_j\eta_j^-\int_{-\infty}^0 \Phi(x-y)e^{\eta^- y}dy\right)-\lambda\Phi(x)\\ =&\frac{\sigma^2}{2}\Phi''(x)+c\Phi'(x)+\lambda\left(\sum_{j=1}^{m_{(+)}} p_j\eta_j^+e^{-\eta_j^+x}\int_{-\infty}^x \Phi(y)e^{\eta_j^+ y}dy+\sum_{j=1}^{m_-} q_j\eta_j^-e^{\eta_j^- x}\int_x^\infty \Phi(y)e^{-\eta_j^- y}dy\right)-\lambda\Phi(x). \end{aligned}$$

From the last equation and by the fact that $\sigma>0$ and $(L-r)\Phi\equiv 0$, Φ is infinitely differentiable on (L,U) by an induction argument.

Next, we show that Φ satisfies an ODE. Observe the following differentiation rule:

$$\begin{aligned} &\left(\frac{d}{dx}+\eta_j^+\right)p_j\eta_j^+e^{-\eta_j^+x}\int_{-\infty}^x\Phi(y)e^{\eta_j^+y}dy\\ =&p_j\eta_j^+\left[\left(-\eta_j^+e^{-\eta_j^+x}\int_{-\infty}^x\Phi(y)e^{\eta_j^+y}dy+\Phi(x)\right)+\eta_j^+e^{-\eta_j^+x}\int_{-\infty}^x\Phi(y)e^{\eta_j^+y}dy\right]=p_j\eta_j^+\Phi(x), \end{aligned}$$

and similarly

$$\left(\frac{d}{dx}-\eta_j^-\right)q_j\eta_j^-e^{\eta_j^-x}\int_x^\infty\Phi(y)e^{-\eta_j^-y}dy=-q_j\eta_j^-\Phi(x).$$

So, by the fact that Φ is infinitely differentiable on (L,U) and $(L-r)\Phi\equiv 0$ on (L,U), we get that

$$\begin{aligned} 0=&\left(\frac{d}{dx}+\eta_1^+\right)\cdots\left(\frac{d}{dx}+\eta_{m_{(+)}}^+\right)\left(\frac{d}{dx}-\eta_1^-\right)\cdots\left(\frac{d}{dx}-\eta_{m_{(-)}}^-\right)(L-r)\Phi(x)\\ =&\left(\frac{d}{dx}+\eta_1^+\right)\cdots\left(\frac{d}{dx}+\eta_{m_{(+)}}^+\right)\left(\frac{d}{dx}-\eta_1^-\right)\cdots\left(\frac{d}{dx}-\eta_{m_{(-)}}^-\right)\left(\frac{\sigma^2}{2}\frac{d^2}{dx^2}+c\frac{d}{dx}-\lambda-r\right)\Phi(x)\\ &+\sum_{j=1}^{m_{(+)}}\prod_{k=1,k\neq j}^{m_{(+)}}\left(\frac{d}{dx}+\eta_k^+\right)p_j\eta_j^+\Phi(x)-\sum_{j=1}^{m_{(-)}}\prod_{k=1,k\neq j}^{m_{(-)}}\left(\frac{d}{dx}-\eta_k^-\right)q_j\eta_j^-\Phi(x). \end{aligned}\tag{3.4}$$

Note in the last equation, we have used the fact that the order of differentiation for an infinitely differentiable function is irrelevant. In addition, we have adopted the notation that

$$\prod_{k=1}^N\left(\frac{d}{dx}-a_k\right)\Phi(x)=\left(\frac{d}{dx}-a_1\right)\cdots\left(\frac{d}{dx}-a_N\right)\Phi(x);\quad a_k\in\mathbb{R},1\le k\le N.$$

So, (3.4) shows us the transformation of the integro-differential equation $(L-r)\Phi\equiv 0$ into an ODE: $D'\Phi\equiv 0$, where D' is a higher order differential operator.

To complete the proof, we need to show that D' coincides with D(see the definition of D in the paragraph above (3.3)). First, we note that the Laplace exponent $\psi(\zeta)$ of X is given by

$$\begin{aligned} \psi(\zeta)=&\frac{\sigma^2}{2}\zeta^2+c\zeta+\lambda\int e^{-\zeta y}dF(y)-\lambda\\ =&\frac{\sigma^2}{2}\zeta^2+c\zeta+\lambda\left(\int_0^\infty e^{-\zeta y}f(y)dy+\int_{-\infty}^0 e^{-\zeta y}f(y)dy\right)-\lambda\\ =&\frac{\sigma^2}{2}\zeta^2+c\zeta+\lambda\left(\sum_{j=1}^{m_{(+)}}\frac{p_j\eta_j^+}{\zeta+\eta_j^+}+\sum_{j=1}^{m_{(-)}}\frac{-q_j\eta_j^-}{\zeta-\eta_j^-}\right)-\lambda,\qquad\zeta\in i\mathbb{R}. \end{aligned}\tag{3.5}$$

Therefore, by the definition of the minimal polynomial $\mathcal{P}_0(\zeta)$, we get

$$\mathcal{P}_0(\zeta)=\prod_{j=1}^{m_{(+)}}(\zeta+\eta_j^+)\prod_{j=1}^{m_{(-)}}(\zeta-\eta_j^-),\tag{3.6}$$

Now, we are in the position to show $D = D'$. And it suffices to show that the characteristic polynomials of D and D' coincide. Write $\mathcal{P}'(\zeta)$ as the characteristic polynomial of D'. Then by (3.4), $\mathcal{P}'$ is given by

$$\begin{aligned}\mathcal{P}'(\zeta) = &\prod_{j=1}^{m_{(+)}}(\zeta+\eta_j^+)\prod_{j=1}^{m_{(-)}}(\zeta-\eta_j^-)\left[\frac{\sigma^2}{2}\zeta^2 + c\zeta + \lambda\left(\sum_{j=1}^{m_{(+)}}\frac{p_j\eta_j^+}{\zeta+\eta_j^+} + \sum_{j=1}^{m_{(-)}}\frac{-q_j\eta_j^-}{\zeta-\eta_j^-}\right) - (\lambda+r)\right]\\ =&\mathcal{P}_0(\zeta)(\psi(\zeta)-r),\end{aligned}$$

by (3.5) and (3.6). This shows the characteristic polynomial $\mathcal{P}_1(\zeta)$ of D is equal to that $\mathcal{P}'(\zeta)$ of D'. We have completed the proof. $\square$

If we assume the zeros of $\mathcal{P}_1(\zeta)$ are distinct and are given by $\{-\infty < \rho_1 < \rho_2 < \cdots < \rho_S < \infty\}$, a general solution of the last equation is given by

$$\Phi(x) = \sum_{i=1}^{S} \boldsymbol{Q}_i e^{\rho_i x}, \tag{3.7}$$

for some constants $\boldsymbol{Q}_i$. Note that that $S = m_{(+)} + m_{(-)} + 2$. For details of these arguments, see Chen et al.(2006).

Proposition 3.1. *The constant $\boldsymbol{Q}$ satisfies (2.6) with $\boldsymbol{Q}(g_i)$ replaced by $\boldsymbol{Q}$.*

Proof. Let $m = m_{(+)} + m_{(-)}$. Since $(L - r)\Phi = 0$ on (L, U), we have for $x \in (L, U)$,

$$0 = D\Phi''(x) + c\Phi'(x) + \lambda\int\Phi(x-y)f(y)dy - (\lambda+r)\Phi(x) \tag{3.8}$$

$$= \sum_{i=1}^{m+2}\boldsymbol{Q}_i e^{\rho_i x}(D\rho_i^2 + c\rho_i - (\lambda+r)) + \lambda\int\Phi(x-y)f(y)dy. \tag{3.9}$$

Furthermore, we have

$$\begin{aligned}\int\Phi(x-y)f(y)dy =& \left(\int_{-\infty}^{L} + \int_{U}^{\infty}\right)g(y)f(x-y)dy + \int_{x-U}^{x-L}\Phi(x-y)f(y)dy\\ =&\sum_{j=1}^{m_{(+)}}p_j e^{-\eta_j^+ x}\int_{-\infty}^{L}g(y)\eta_j^+ e^{\eta_j^+ y}dy + \sum_{j=1}^{m_{(-)}}q_j e^{\eta_j^- x}\int_{U}^{\infty}g(y)\eta_j^- e^{-\eta_j^- y}dy\\ &+\sum_{i=1}^{m+2}\boldsymbol{Q}_i e^{\rho_i x}\sum_{j=1}^{m_{(-)}}q_j\eta_j^-\int_{x-U}^{0}e^{-\rho_i y}e^{\eta_j^- y}dy + \sum_{i=1}^{m+2}\boldsymbol{Q}_i e^{\rho_i x}\sum_{j=1}^{m_{(+)}}p_j\eta_j^+\int_{0}^{x-L}e^{-\rho_i y}e^{-\eta_j^+ y}dy\\ =&\sum_{j=1}^{m_{(+)}}p_j e^{-\eta_j^+ x}\int_{-\infty}^{L}g(y)\eta_j^+ e^{\eta_j^+ y}dy + \sum_{j=1}^{m_{(-)}}q_j e^{\eta_j^- x}\int_{U}^{\infty}g(y)\eta_j^- e^{-\eta_j^- y}dy\\ &+\sum_{i=1}^{m+2}\boldsymbol{Q}_i e^{\rho_i x}\sum_{j=1}^{m_{(-)}}\frac{q_j\eta_j^-}{\eta_j^- - \rho_i}\left(1 - e^{-(\eta_j^- - \rho_i)(U-x)}\right)\\ &+\sum_{i=1}^{m+2}\boldsymbol{Q}_i e^{\rho_i x}\sum_{j=1}^{m_{(+)}}\frac{p_j\eta_j^+}{\rho_i + \eta_j^+}\left(1 - e^{-(\rho_i + \eta_j^+)(x-L)}\right)\end{aligned} \tag{3.10}$$

Now, by (3.9), (3.10) and the fact $\psi(\rho_i) - r = 0$ for all i, we deduce that

$$\begin{aligned}0 =& \sum_{j=1}^{m_{(+)}}p_j e^{-\eta_j^+ x}\int_{-\infty}^{L}g(y)\eta_j^+ e^{-\eta_j^+ y}dy + \sum_{j=1}^{m_{(-)}}q_j e^{\eta_j^- x}\int_{U}^{\infty}g(y)\eta_j^- e^{-\eta_j^- y}dy\\ &+\sum_{i=1}^{m+2}\boldsymbol{Q}_i e^{\rho_i x}\sum_{j=1}^{m_{(-)}}\frac{\eta_j^-}{\eta_j^- - \rho_i}\left(-e^{-(\eta_j^- - \rho_i)(U-x)}\right)\\ &+\sum_{i=1}^{m+2}\boldsymbol{Q}_i e^{\rho_i x}\sum_{j=1}^{m_{(+)}}\frac{\eta_j^+}{\rho_i + \eta_j^+}\left(-e^{-(\rho_i + \eta_j^+)(x-L)}\right).\end{aligned}$$

By comparing $e^{-\eta^{+}jx}$ and $e^{\eta_j^{-}x}$, we get (2.6). This completes the proof. $\square$

Write $\boldsymbol{Q} = [\boldsymbol{Q}_1, \cdots, \boldsymbol{Q}_S]^\top$ and $\boldsymbol{e}^{\boldsymbol{\rho}}(x) = [e^{\rho_1 x}, \cdots, e^{\rho_S x}]^\top$. By (3.7) and Proposition A.2, we conclude that

Theorem 3.1. *Suppose there exists a solution Φ to the boundary value problem (3.1). Then on $[U, L]$, $\Phi(x) = \boldsymbol{Q}^\top \boldsymbol{e}^{\boldsymbol{\rho}}(x)$, where $\boldsymbol{Q}$ is a constant vector that solves (2.6) with $\boldsymbol{Q}(g_k)$ replaced by $\boldsymbol{Q}$. Conversely, if (2.6) admits a solution, then the solution must be unique and the function $\Phi(x)$ which is equal to $\boldsymbol{Q}\boldsymbol{e}^{\boldsymbol{\rho}}(x)$ on $[L, U]$ and $g(x)$ on $[L, U]^c$ solves the boundary value problem (3.1).*

Proof. We have shown the first statement in the above. The second statement follows directly from the Theorem of Feynman and Kac. For details, see Chen et al.(2006). $\square$

Appendix A. Approximation of Bond with Finite Maturity

We now consider a bond whose covenant has the same term as the perpetual bond that we consider in the previous section, except that it has a finite maturity and a par value P upon maturity date. More precisely, the bond price under consideration has its risk neutral price as the following:

$$D(V_0, T) = \mathbb{E}\left[\int_0^{\tau_B \wedge T} C(1-\tau_{\mathrm{p}}) e^{-rt} dt\right] + \mathbb{E}\left[e^{-r\tau_B} \widehat{g}(V_{\tau_B}) \mathbf{1}_{\tau_B \le T}\right] + P\mathbb{P}[\tau_B > T].$$

In this case, closed form solutions of $D(V_0, T)$ is not available. Indeed, solving $D(V_0, T)$ in the probabilistic way will require the density of the contract ceasing time, which is not available. And solving $D(V_0, T)$ in the analytic way will require us solving a partial integro-differential equation, which is not an easy work. Instead, we will show in the following that we can approximate the bond price $D(V_0, T)$ via *a sequence of linear combinations of the perpetual bond prices considered in the previous section and some zero coupon defaultable bonds.*

To see this, we first rewrite $D(V_0, T)$ as

$$\begin{aligned}
D(V_0, T) =& \frac{C(1-\tau_{\mathrm{p}})}{r}\left(1 - \mathbb{E}\left[e^{-r\tau_B \wedge T}\right]\right) + (1-\alpha)\mathbb{E}\left[e^{-r\tau_B} \mathbf{1}_{V_{\tau_B} \le V_L} V_{\tau_B} \mathbf{1}_{\tau_B \le T}\right] \\
&+ K\mathbb{E}\left[e^{-r\tau_B} \mathbf{1}_{V_{\tau_B} \ge V_U} \mathbf{1}_{\tau_B \le T}\right] + P(1 - \mathbb{P}[\tau_B \le T]) \\
=& \frac{C(1-\tau_{\mathrm{p}})}{r}\left\{1 - \mathbb{E}\left[e^{-r\tau_B} \mathbf{1}_{\tau_B \le T}\right] - e^{-rT} + e^{-rT}\mathbb{P}[\tau_B \le T]\right\} + (1-\alpha)\mathbb{E}\left[e^{-r\tau_B} \mathbf{1}_{V_{\tau_B} \le V_L} V_{\tau_B} \mathbf{1}_{\tau_B \le T}\right] \\
&+ K\mathbb{E}\left[e^{-r\tau_B} \mathbf{1}_{V_{\tau_B} \ge V_U} \mathbf{1}_{\tau_B \le T}\right] + P(1 - \mathbb{P}[\tau_B \le T]).
\end{aligned}$$

We show how to approximate the functional $\mathbb{E}_x\left[e^{-r\tau_B} \widehat{g}(X_{\tau_B}) \mathbf{1}_{\tau_B \le T}\right]$ by linear combinations of functions of the form $\mathbb{E}_x\left[e^{-r\tau_B} h(X_{\tau_B})\right]$. Moreover, the error bound of such approximation will be provided.

First, for each $n \ge 1$, define a piecewise linear function f_n on $\mathbb{R}_+$ by

$$f_n(t) = \begin{cases} 0, t \in \left[0, \frac{1}{2n}\right] \cup [T, \infty), \\ 1, t \in \left[\frac{1}{n}, T - \frac{1}{2n}\right], \\ \text{linear, otherwise.} \end{cases} \tag{A.1}$$

Also, take $\phi(t) = \mathbf{1}_{[0,T]}(t)$. Then

$$\left|\mathbb{E}_x\left[e^{-r\tau} g(X_\tau)\phi(\tau)\right] - \mathbb{E}_x\left[e^{-r\tau} g(X_\tau) f_n(\tau)\right]\right| \le \|g\|_\infty \left(\mathbb{P}_x[\tau \in [0, 1/n] + e^{-r(T-1/n)}\mathbb{P}_x\left[\tau \in [T - 1/n, T]\right]\right).$$

By the absolute continuity of $\mathbb{P}_x\left[\tau_B \in dt, \tau_B < \infty\right]$, we have $\mathbb{P}_x\left[\tau_B \in (0, 1/n]\right], \mathbb{P}_x\left[\tau_B \in (T - 1/n, T]\right] \to 0$, as $n \to \infty$. On the other hand, take the Bernstein polynomial

$$B_n(x) = \sum_{k=0}^{n^7} f_n\left(-\log \frac{k}{n^7}\right) \binom{n^7}{k} e^{-kx}\left(1 - e^{-x}\right)^{n^7 - k}, 0 \le x < \infty. \tag{A.2}$$

Here, we define $f_n(-\log 0) = 0$. Then $f_n(-\log x)$ is continuous on $[0, 1]$ and we have

$$\|B_n - f_n\|_\infty \le \sup\left\{|f_n(-\log t) - f_n(-\log s)|; |t - s| \le \frac{1}{n^3}, 0 \le t, s \le 1\right\} + \frac{1}{2n};$$

see Resnick(1997) page 177 for details. We estimate the supremum term. First, note that for $s, t \geq e^{-T}$ and $|s-t| \leq \frac{1}{n^3}$, by Mean Value Theorem, we have

$$|\log s - \log t| \leq e^T |s-t| \leq \frac{e^T}{n^3}.$$

Hence,

$$\sup\{\cdots\} \leq \frac{e^T}{n^2}.$$

We now conclude that

Theorem A.1. *We can approximate the function* $\mathbb{E}_x\left[e^{-r\tau_B} g(X_{\tau_B})\mathbf{1}_{\tau_B \leq T}\right]$ *by* $\mathbb{E}_x\left[e^{-r\tau_B} g(X_{\tau_B}) B_n(\tau_B)\right]$*, where* B_n *is given by (A.2) and* $\mathbb{E}_x\left[e^{-r\tau_B} g(X_{\tau_B}) B_n(\tau)\right]$ *has a closed form given by Theorem A.1. Moreover, the approximation error bound is given by*

$$\begin{aligned}&\left|\mathbb{E}_x\left[e^{-r\tau_B} g(X_{\tau_B})\mathbf{1}_{[\tau_B \leq T]}\right] - \mathbb{E}_x\left[e^{-r\tau_B} g(X_{\tau_B}) B_n(\tau_B)\right]\right| \\ &\leq \|g\|_\infty \Big(\mathbb{P}_x[\tau \in [0, 1/n] + e^{-r(T-1/n)}\mathbb{P}_x\left[\tau \in [T-1/n, T]\right]\Big) + \frac{e^T}{n^2} + \frac{1}{2n}. \qquad \text{(A.3)}\end{aligned}$$

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無研發成果推廣資料

98 年度專題研究計畫研究成果彙整表

計畫主持人：許元春			計畫編號：98-2115-M-009-005-				
計畫名稱：馬可夫過程離開問題及應用							
成果項目			量化			單位	備註（質化說明：如數個計畫共同成果、成果列為該期刊之封面故事...等）
			實際已達成數（被接受或已發表）	預期總達成數(含實際已達成數)	本計畫實際貢獻百分比		
國內	論文著作	期刊論文	0	0	100%	篇	
		研究報告/技術報告	3	3	100%		
		研討會論文	0	0	100%		
		專書	0	0	100%		
	專利	申請中件數	0	0	100%	件	
		已獲得件數	0	0	100%		
	技術移轉	件數	0	0	100%	件	
		權利金	0	0	100%	千元	
	參與計畫人力（本國籍）	碩士生	2	2	100%	人次	
		博士生	2	2	100%		
		博士後研究員	0	0	100%		
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國外	論文著作	期刊論文	3	3	100%	篇	
		研究報告/技術報告	0	0	100%		
		研討會論文	0	0	100%		
		專書	0	0	100%	章/本	
	專利	申請中件數	0	0	100%	件	
		已獲得件數	0	0	100%		
	技術移轉	件數	0	0	100%	件	
		權利金	0	0	100%	千元	
	參與計畫人力（外國籍）	碩士生	0	0	100%	人次	
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其他成果 (無法以量化表達之成果如辦理學術活動、獲得獎項、重要國際合作、研究成果國際影響力及其他協助產業技術發展之具體效益事項等，請以文字敘述填列。)	無。

	成果項目	量化	名稱或內容性質簡述
科教處計畫加填項目	測驗工具(含質性與量性)	0	
	課程/模組	0	
	電腦及網路系統或工具	0	
	教材	0	
	舉辦之活動/競賽	0	
	研討會/工作坊	0	
	電子報、網站	0	
	計畫成果推廣之參與（閱聽）人數	0	

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請就研究內容與原計畫相符程度、達成預期目標情況、研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）、是否適合在學術期刊發表或申請專利、主要發現或其他有關價值等，作一綜合評估。

1. 請就研究內容與原計畫相符程度、達成預期目標情況作一綜合評估 ■達成目標 □未達成目標（請說明，以 100 字為限） 　　□實驗失敗 　　□因故實驗中斷 　　□其他原因 說明：
2. 研究成果在學術期刊發表或申請專利等情形： 論文：□已發表 ■未發表之文稿 □撰寫中 □無 專利：□已獲得 □申請中 ■無 技轉：□已技轉 □洽談中 ■無 其他：（以 100 字為限）
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