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Synchronization of mutual coupled chaotic systems via partial stability theory

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Abstract

A scheme is proposed to achieve chaos synchronization for mutual coupled systems via partial stability theory. Under this scheme, three criteria are given to ensure chaos synchronization. The first criterion applies to the case without system perturbation and the other two apply to systems possessing vanishing and nonvanishing perturbations, respectively. Finally, coupled Lorenz systems are simulated to illustrate the theoretical analysis.

1. Introduction

Chaotic systems are thought difficult to be synchronized or controlled in the past since they exhibit sensitive dependence on initial conditions. From the work of Pecora and Carroll [1], the researchers have realized that the synchronism of chaotic motions is possible. Hence chaos synchronization is of great interest in these years. In particular, it is pointed out that chaos synchronization has the potential in secure communication. Many engineers and scientists are attracted by this discipline.

Synchronization means that the state variables of a response system approach eventually to that of a driving system. Zero crossing of a Lyapunov exponent is used as a criterion of chaos synchronization widely. There is a drawback that we can only calculate finite evolution time in computer simulation but infinite evolution time is needed by definition of the Lyapunov exponent. On the other hand, it may be difficult to use the traditional Lyapunov direct method since the equation of state errors is not a pure function of state errors in general. In the paper of Ge and Chen [9], a general scheme is proposed to achieve chaos synchronization of unidirectional coupled systems via the partial stability theory. Preceding two obstacles can be overcome by this scheme. Furthermore, it not only applies for unidirectional coupled systems but also works for mutual coupled systems. The objective of this paper is to accomplish the theoretical analysis

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of chaos synchronization for mutual coupled systems via the partial stability theory. Some other achievement about synchronization of mutual coupled systems can be found in [2–8].

In this paper, three criteria are given to ensure synchronization for mutual coupled systems. The first criterion suits for systems without perturbation and the other two suit for systems under vanishing and nonvanishing perturbations, respectively. The only assumption is that system equations meet the Lipschitz condition. Since there is no further restriction on the type of systems, all criteria derived work for nonlinear nonautonomous systems. When these criteria are used, a matrix should be negative definite and an estimation of a Lipschitz constant is needed in advance.

Theoretical analyses are arranged in Section 2 and coupled Lorenz systems are simulated to demonstrate analytical results in Section 3. Conclusions follow sequentially in Section 4.

2. Theoretical analyses

Consider the following mutual coupled system

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(t,\mathbf{x}) + \mathbf{G}_1(t,\mathbf{x},\mathbf{y}),\\ \dot{\mathbf{y}} &= \mathbf{f}(t,\mathbf{y}) + \mathbf{G}_2(t,\mathbf{x},\mathbf{y}), \end{aligned} \tag{1}$$

where $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$ and $\mathbf{f} : \Omega \subset \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ satisfy the Lipschitz condition $\|\mathbf{f}(t,\mathbf{x}_1) - \mathbf{f}(t,\mathbf{x}_2)\| \leqslant L\|\mathbf{x}_1 - \mathbf{x}_2\|$ in $\mathbf{x}$ for all $(t,\mathbf{x}_1)$ and $(t,\mathbf{x}_2)$ in Ω with a Lipschitz constant L. This constant L is not unique since any number larger than L is also a Lipschitz constant. $\mathbf{G}_1$ and $\mathbf{G}_2$ are coupling functions which satisfy $\mathbf{G}_1(t,\mathbf{x},\mathbf{y}) = \mathbf{0}$ and $\mathbf{G}_2(t,\mathbf{x},\mathbf{y}) = \mathbf{0}$ for $\mathbf{x}(t) = \mathbf{y}(t)$, $\forall t \geqslant t_0$.

Define $\mathbf{e} = \mathbf{y} - \mathbf{x}$ to be the state error. Then the error dynamic equation can be written as

$$\dot{\mathbf{e}} = \mathbf{f}(t,\mathbf{e}+\mathbf{x}) - \mathbf{f}(t,\mathbf{x}) + \mathbf{G}_2(t,\mathbf{x},\mathbf{e}+\mathbf{x}) - \mathbf{G}_1(t,\mathbf{x},\mathbf{e}+\mathbf{x}). \tag{2}$$

In general the right hand side of Eq. (2) is not a function of the state error $\mathbf{e}$ only. As a result the traditional Lyapunov method might hardly be used. Herein, we take the first equation of Eqs. (1) and (2) together with $\mathbf{y} = \mathbf{e} + \mathbf{x}$ to form an extended system of states $\mathbf{x}$ and $\mathbf{e}$ as follows

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(t,\mathbf{x}) + \mathbf{G}_1(t,\mathbf{x},\mathbf{e}+\mathbf{x}),\\ \dot{\mathbf{e}} &= \mathbf{f}(t,\mathbf{e}+\mathbf{x}) - \mathbf{f}(t,\mathbf{x}) + \mathbf{G}_2(t,\mathbf{x},\mathbf{e}+\mathbf{x}) - \mathbf{G}_1(t,\mathbf{x},\mathbf{e}+\mathbf{x}). \end{aligned} \tag{3}$$

If the partial state variable $\mathbf{e}$ in Eq. (3) is asymptotically stable about $\mathbf{e} = \mathbf{0}$, then $\mathbf{x}$ and $\mathbf{y}$ in Eq. (1) are synchronized. The stability of partial state variables can be verified via the partial stability theory. A brief review of the partial stability theory can be found in the appendix of paper [9] or in [10]. Although the acquirement of the extended system Eq. (3) doubles the order of the original error dynamic equation Eq. (2), only partial variables $\mathbf{e}$ are handled. This scheme does not increase any difficulty due to the increase of the order. Furthermore, the usage of the partial stability theory is similar to the traditional Lyapunov method.

The proposed scheme not only applies to mutual coupled systems but also applies to unidirectional cases. Actually, it reduces to unidirectional cases if $\mathbf{G}_1 = \mathbf{0}$ is satisfied [9]. The rest mission is to choose appropriate controllers $\mathbf{G}_1$ and $\mathbf{G}_2$ to guarantee the occurrence of synchronization. There are many forms of $\mathbf{G}_1$ and $\mathbf{G}_2$ for choice. We choose $\mathbf{G}_1 = \Gamma_1(\mathbf{y}-\mathbf{x})$ and $\mathbf{G}_2 = \Gamma_2(\mathbf{x}-\mathbf{y})$ and Eq. (1) can be rewritten as

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(t,\mathbf{x}) + \Gamma_1(\mathbf{y}-\mathbf{x}),\\ \dot{\mathbf{y}} &= \mathbf{f}(t,\mathbf{y}) + \Gamma_2(\mathbf{x}-\mathbf{y}), \end{aligned} \tag{4}$$

where $\Gamma_1, \Gamma_2 \in M_{n\times n}$ are two constant matrices whose entries represent the coupling strength. An extended system can be obtained as

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(t,\mathbf{x}) + \Gamma_1\mathbf{e},\\ \dot{\mathbf{e}} &= \mathbf{f}(t,\mathbf{e}+\mathbf{x}) - \mathbf{f}(t,\mathbf{x}) - (\Gamma_1+\Gamma_2)\mathbf{e}. \end{aligned} \tag{5}$$

A synchronization criterion of Eq. (5) is derived as follows.

Theorem 1. The partial state $\mathbf{e}$ in Eq. (5) uniformly asymptotically approaches $\mathbf{0}$ if $L\mathbf{I}_n - (\Gamma_1+\Gamma_2)$ is negative definite. This means that two subsystems in Eq. (4) are synchronized if $L\mathbf{I}_n - (\Gamma_1+\Gamma_2)$ is negative definite.

Proof. Choose a function $V(\mathbf{x},\mathbf{e}) = \frac{1}{2}\mathbf{e}^{\mathrm{T}}\mathbf{e}$ which is positive definite with respect to $\mathbf{e}$ and possesses an infinitesimal upper bound. Using the Cauchy–Schwarz inequality and the Lipschitz condition, we get

$$\dot{V}(\mathbf{x},\mathbf{e}) = \mathbf{e}^{\mathrm{T}}\dot{\mathbf{e}} \leqslant L\|\mathbf{e}\|^2 - \mathbf{e}^{\mathrm{T}}(\Gamma_1+\Gamma_2)\mathbf{e} = \mathbf{e}^T[L\mathbf{I}_n-(\Gamma_1+\Gamma_2)]\mathbf{e}.$$

The state error **e** approaches **0** uniformly asymptotically if $L\mathbf{I}_n-(\Gamma_1+\Gamma_2)$ is negative definite by Theorem A2 in the appendix of [9]. □

A special case $\Gamma_1=\Gamma_2=\mathrm{diag}(\gamma_1,\gamma_2,\ldots,\gamma_n)$ with $\gamma_i>0$, $i=1,\ldots,n$ is commonly used. Since the time derivative of $V(\mathbf{x},\mathbf{e})$ along $\mathbf{x}(t)$ and $\mathbf{e}(t)$ satisfies $\dot{V}(\mathbf{x},\mathbf{e})\leqslant(L-2\gamma_{\min})\|\mathbf{e}\|^2$, the synchronization criterion reduces to $\gamma_{\min}>L/2$, $\gamma_{\min}\leqslant\gamma_i$, $i=1,\ldots,n$. When $\gamma=\gamma_1,\ldots,\gamma_n$, synchronization occurs if $\gamma>L/2$. This means that the synchronization of mutual coupled chaotic systems is guaranteed by the large coupling strength γ. If the system is autonomous, the threshold value of γ for the occurrence of synchronization is one half of the largest Lyapunov exponent of the chaotic system [2].

If perturbations exist in the system, similar criterion can also be obtained. Consider a mutual coupled nonautonomous system with the perturbations in the form of

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}(t,\mathbf{x})+\Delta\mathbf{f}_1(t,\mathbf{x},\mathbf{y})+\Gamma_1(\mathbf{y}-\mathbf{x}),\\ \dot{\mathbf{y}} &= \mathbf{f}(t,\mathbf{y})+\Delta\mathbf{f}_2(t,\mathbf{x},\mathbf{y})+\Gamma_2(\mathbf{x}-\mathbf{y}),\end{aligned} \tag{6}$$

where $\Delta\mathbf{f}_1(t,\mathbf{x},\mathbf{y})$ and $\Delta\mathbf{f}_2(t,\mathbf{x},\mathbf{y})$ are the vanishing perturbation. Vanishing perturbation means that $\Delta\mathbf{f}_j(t,\mathbf{x},\mathbf{y})=\mathbf{0}$ whenever $\mathbf{x}(t)=\mathbf{y}(t)$, $\forall t$ for $j=1,2$. $\Delta\mathbf{f}_j(t,\mathbf{x},\mathbf{y})$ can be rephrased as $\Delta\mathbf{f}_j(t,\mathbf{x},\mathbf{e})$ for $j=1,2$. Then an extended system can be obtained as

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}(t,\mathbf{x})+\Delta\mathbf{f}_1(t,\mathbf{x},\mathbf{e})+\Gamma_1\mathbf{e},\\ \dot{\mathbf{e}} &= \mathbf{f}(t,\mathbf{e}+\mathbf{x})-\mathbf{f}(\mathbf{x})+\Delta\mathbf{f}_2(t,\mathbf{x},\mathbf{e})-\Delta\mathbf{f}_1(t,\mathbf{x},\mathbf{e})-(\Gamma_1+\Gamma_2)\mathbf{e}.\end{aligned} \tag{7}$$

Theorem 2. Assume that $\exists K_j>0\Rightarrow\|\Delta\mathbf{f}_j\|<K_j\|\mathbf{e}\|$, $j=1,2$. Then null solution of the partial state **e** of Eq. (7) is uniformly asymptotically stable if $(L+K_1+K_2)\mathbf{I}_n-(\Gamma_1+\Gamma_2)$ is negative definite, i.e., the two subsystems in Eq. (6) are synchronized if $(L+K_1+K_2)\mathbf{I}_n-(\Gamma_1+\Gamma_2)$ is negative definite.

Proof. Choose a function $V(\mathbf{x},\mathbf{e})=\frac{1}{2}\mathbf{e}^{\mathrm{T}}\mathbf{e}$ which is positive definite with respect to **e** and possesses an infinitesimal upper bound. By the Cauchy-Schwarz inequality and the Lipschitz condition, $\dot{V}(\mathbf{x},\mathbf{e})$ satisfies

$$\dot{V}(\mathbf{x},\mathbf{e})\leqslant\mathbf{e}^{\mathrm{T}}[(L+K_1+K_2)\mathbf{I}_n-(\Gamma_1+\Gamma_2)]\mathbf{e}.$$

Hence the null solution of Eq. (7) is uniformly asymptotically **e**-stable if $(L+K_1+K_2)\mathbf{I}_n-(\Gamma_1+\Gamma_2)$ is negative definite. □

When $\Gamma_1=\Gamma_2=\mathrm{diag}\ (\gamma_1,\gamma_2,\ldots,\gamma_n)$ with $\gamma_i>0$ for $i=1,\ldots,n$, synchronization occurs if $\gamma_{\min}>(L+K_1+K_2)/2$, where $\gamma_{\min}$ is the minimum of γ_i. If $\gamma=\gamma_1,\ldots,\gamma_n$, the synchronization criterion reduces to $\gamma>(L+K_1+K_2)/2$. Moreover, by Theorem A4 [11], the synchronization in Theorem 1 and 2 are global if **f** is globally Lipschitzian.

If perturbations $\Delta\mathbf{f}_1(t,\mathbf{x}_1,\mathbf{x}_2)$ and $\Delta\mathbf{f}_2(t,\mathbf{x}_1,\mathbf{x}_2)$ are not vanishing, it is difficult to design a controller to guarantee the occurrence of asymptotically partial stability as that in Theorem 2. The reason is that the origin is not an equilibrium point anymore. The stability under constantly acting perturbation small on the average [11] must be studied instead.

Theorem 3. Assume that the functions **f** and $D\mathbf{f}(\mathbf{x})$ are continuous and bounded. The null solution of Eq. (6) is uniformly e-stable under constantly acting perturbation small on the average if $L\mathbf{I}_n-2\Gamma$ is negative definite.

Proof. From Theorem 1, the partial state **e** uniformly asymptotically approaches **0** in Eq. (5) if $L\mathbf{I}_n-2\Gamma$ is negative definite. By corollary in [11], the null solution of Eq. (7) is uniformly **e**-stable under constantly acting perturbation small on the average if $L\mathbf{I}_n-2\Gamma$ is negative definite with the assumption that **f** and $D\mathbf{f}(\mathbf{x})$ are continuous and bounded. This completes the proof. □

If $\Gamma=\mathrm{diag}(\gamma_1,\gamma_2,\ldots,\gamma_n)$ with $\gamma_i>0$ for $i=1,\ldots,n$, practical synchronization occurs if $\gamma_{min}>L$, where $\gamma_{\min}\leqslant\gamma_i$, $i=1,\ldots,n$. Moreover, the larger $\gamma_{\min}$ is, the smaller bounds of the state errors are. This criterion is global if **f** is globally Lipschitzian.

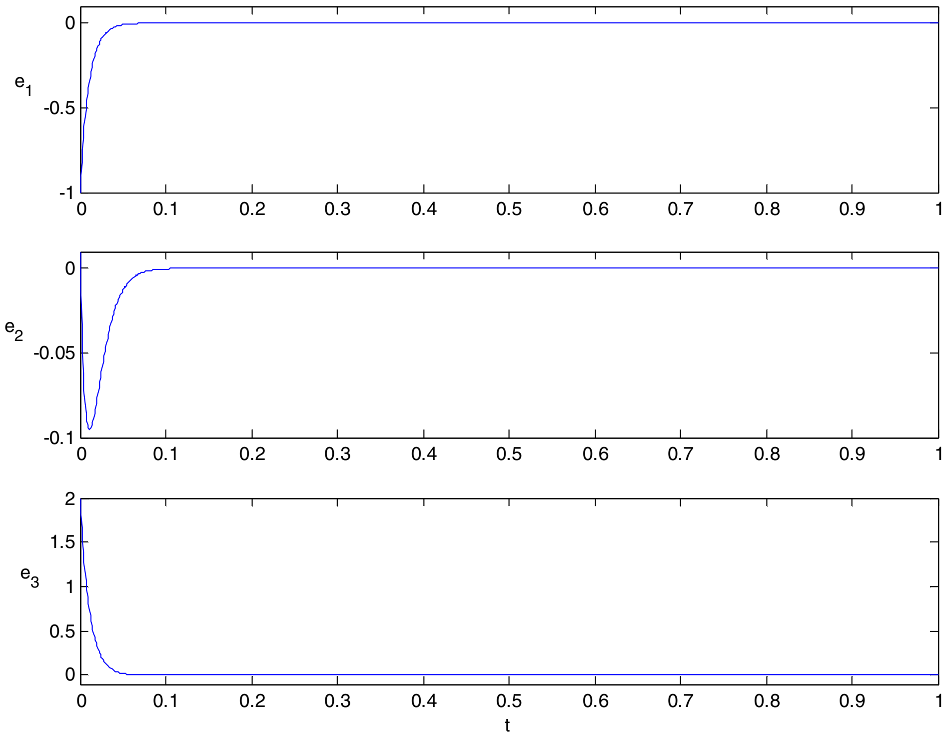

Fig. 1. State errors versus time for mutual coupled Lorenz systems without perturbation while $\gamma = 0.6$.

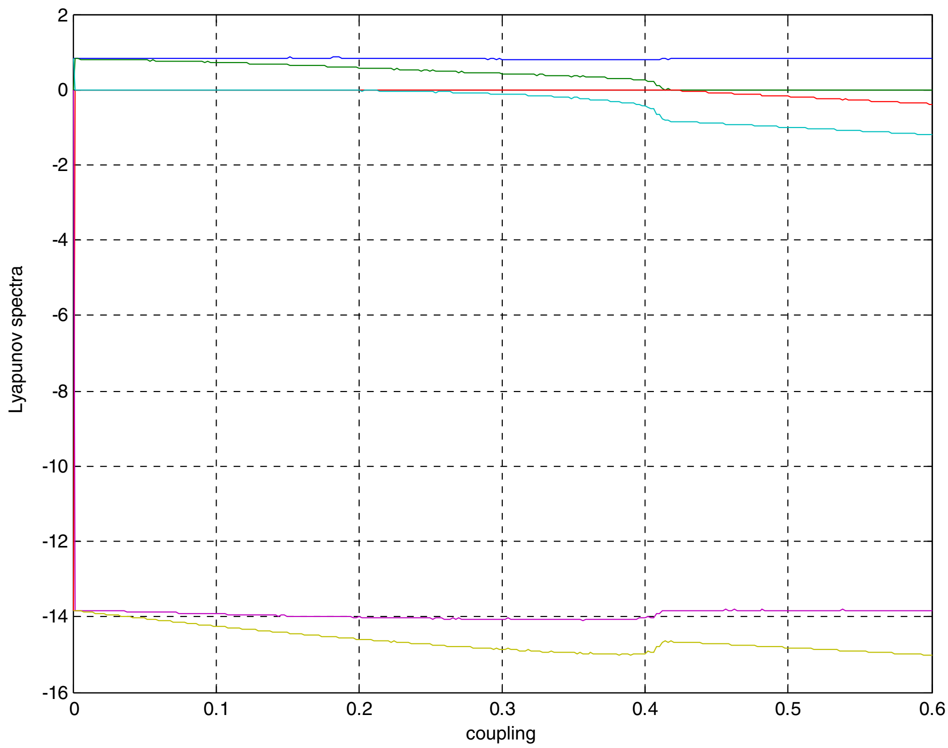

Fig. 2. Lyapunov spectra for mutual coupled Lorenz systems without perturbation.

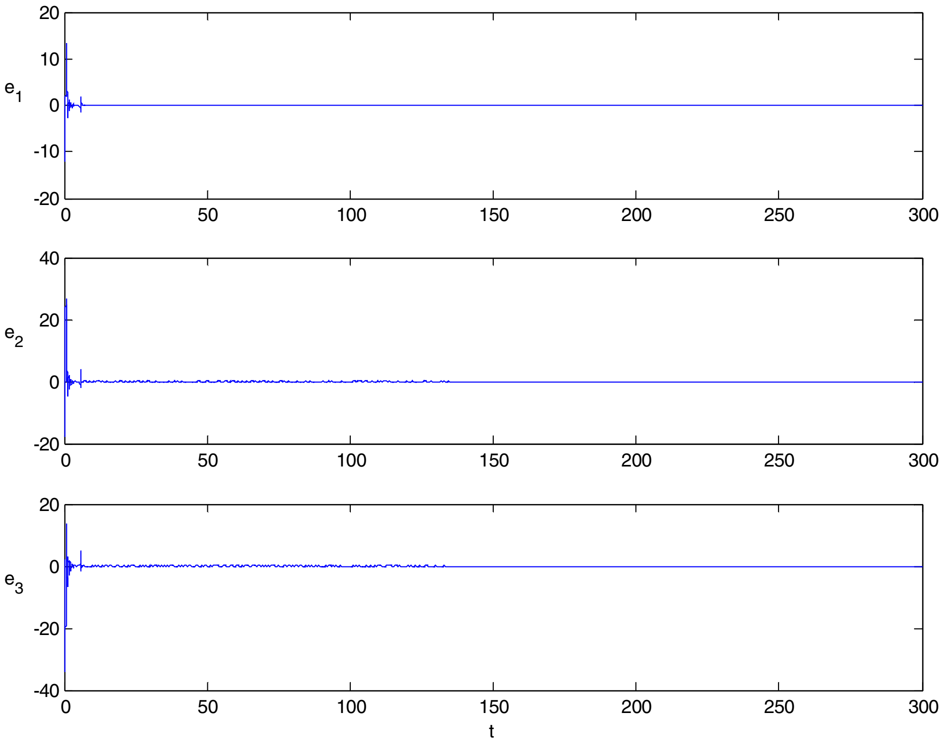

Fig. 3. State errors versus time for mutual coupled Lorenz systems without perturbation while $\gamma = 0.6$.

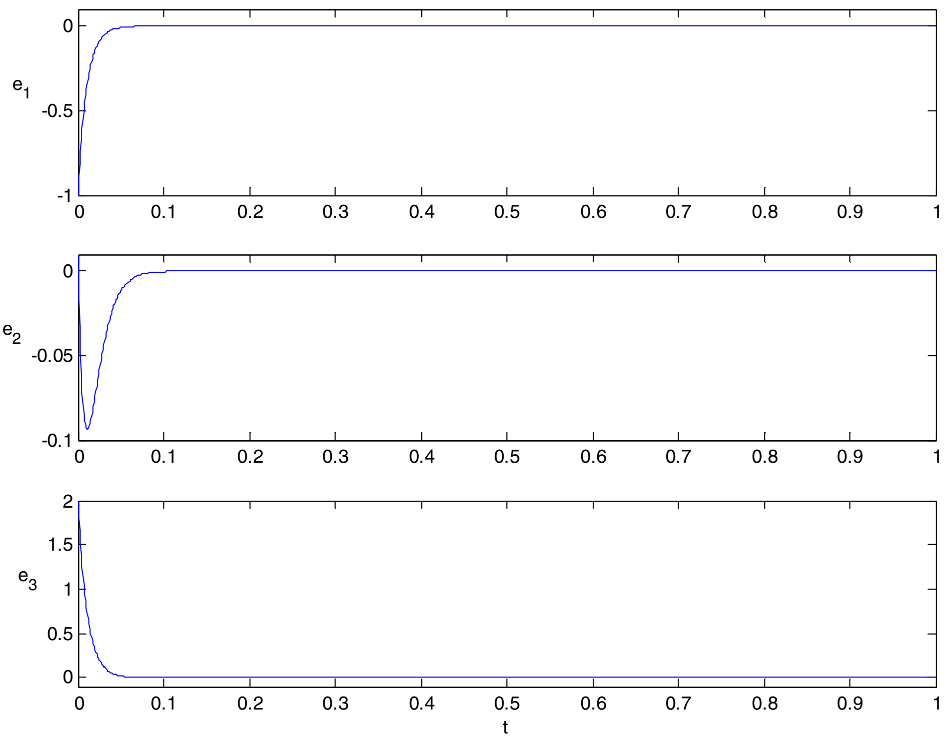

Fig. 4. State errors versus time for mutual coupled Lorenz systems with vanishing perturbations $\Delta f_1 = \cos t \cdot (y_1 - y_2)$ and $\Delta f_6 = x_1 - x_2$.

3. Numerical illustrations

Consider a Lorenz system

$$\dot{x} = -\sigma(x - y),$$
$$\dot{y} = rx - y - xz,$$
$$\dot{z} = xy - bz,$$

where $\sigma = 10$, $r = 28$ and $b = 8/3$ ensure that there exists chaotic behavior. When Theorem 1 is applied, an estimation of a Lipschitz constant is needed. By Cauchy–Schwarz inequality, we have

$$|f_1(\mathbf{x}_2) - f_1(\mathbf{x}_1)| = |-\sigma e_1 + \sigma e_2| \leqslant \|[-\sigma \quad \sigma \quad 0]\|\|\mathbf{x}_2 - \mathbf{x}_1\|,$$
$$|f_2(\mathbf{x}_2) - f_2(\mathbf{x}_1)| = |re_1 - e_2 - x_2 e_3 - z_1 e_1|\|\mathbf{x}_2 - \mathbf{x}_1\| \leqslant \|[r + B_3 \quad -1 \quad B_1]\|\|\mathbf{x}_2 - \mathbf{x}_1\|,$$
$$|f_3(\mathbf{x}_2) - f_3(\mathbf{x}_1)| = |x_2 y_2 - x_1 y_1 - be_3| \leqslant \|[B_2 \quad B_1 \quad -b]\|\|\mathbf{x}_2 - \mathbf{x}_1\|,$$

for any $\mathbf{x}_2 = [x_2 \quad y_2 \quad z_2]^{\mathrm{T}}, \mathbf{x}_1 = [x_1 \quad y_1 \quad z_1]^{\mathrm{T}}$, where $|x(t)| \leqslant B_1$, $|y(t)| \leqslant B_2$, $|z(t)| \leqslant B_3$ $\forall t > t_0$. Hence a Lipschitz constant is obtained as

$$L = \sqrt{\|[-\sigma \quad \sigma \quad 0]\|^2 + \|[r + B_3 \quad -1 \quad B_1]\|^2 + \|[B_2 \quad B_1 \quad -b]\|^2}.$$

From numerical simulation, $B_1 = 20$, $B_2 = 28$, $B_3 = 49$, then $L = 87.87$. The mutual coupled systems are in the form of Eq. (4) with $\Gamma_1 = \Gamma_2 = \mathrm{diag}(\gamma, \ldots, \gamma)$ and $\gamma = 44 > L/2$. The initial value is $\mathbf{x_0} = [1, -0.01, 3, 0, 0, 5]^{\mathrm{T}}$. The simulated results are shown in Figs. 1–6. In Fig. 1, three state errors approach zero as time evolves. Lyapunov exponents versus coupling strength γ are shown in Fig. 2. There is a zero-crossing of one Lyapunov spectrum while $\gamma \approx 0.41$. This value of γ is a threshold value where synchronization occurs. Fujisaka and Yamada [2] proved that synchronization of linear mutual coupled autonomous systems occurs if the coupling strength larger than one half of the largest Lyapunov exponent. The largest Lyapunov exponent of the Lorenz system is 0.82 and its half is 0.41. This coincides with the value of γ

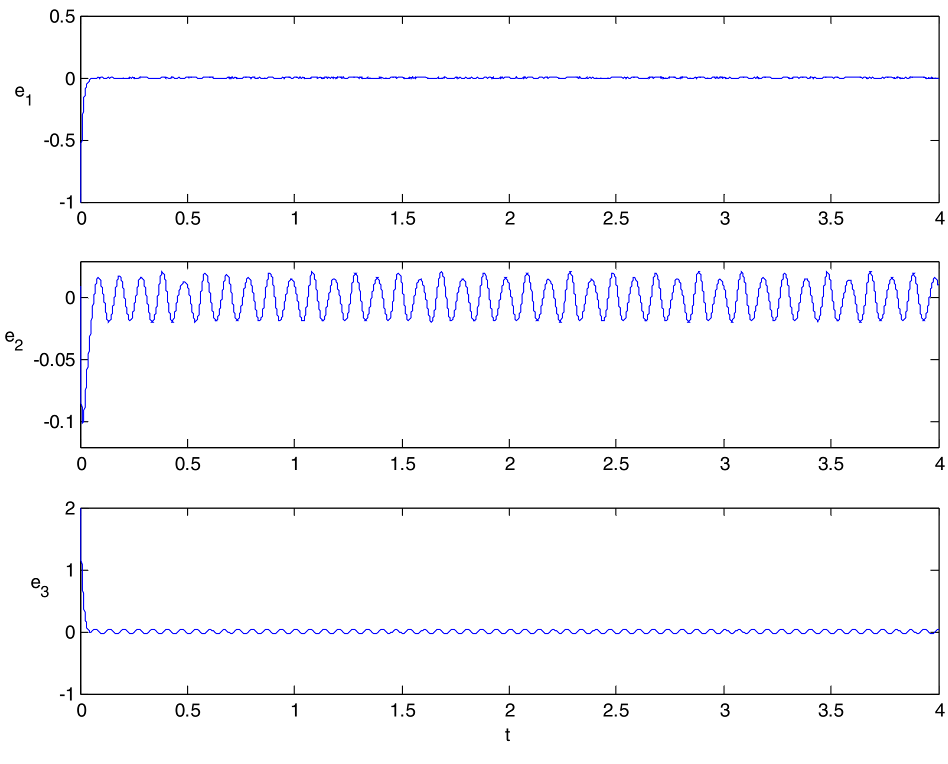

Fig. 5. State errors versus time for mutual coupled Lorenz systems with nonvanishing perturbations $\Delta f_2 = 2\sin(20\pi t)$, $\Delta f_4 = r(t)$, $\Delta f_6 = 5\cos(30\pi t)$ and $\gamma = 44$.

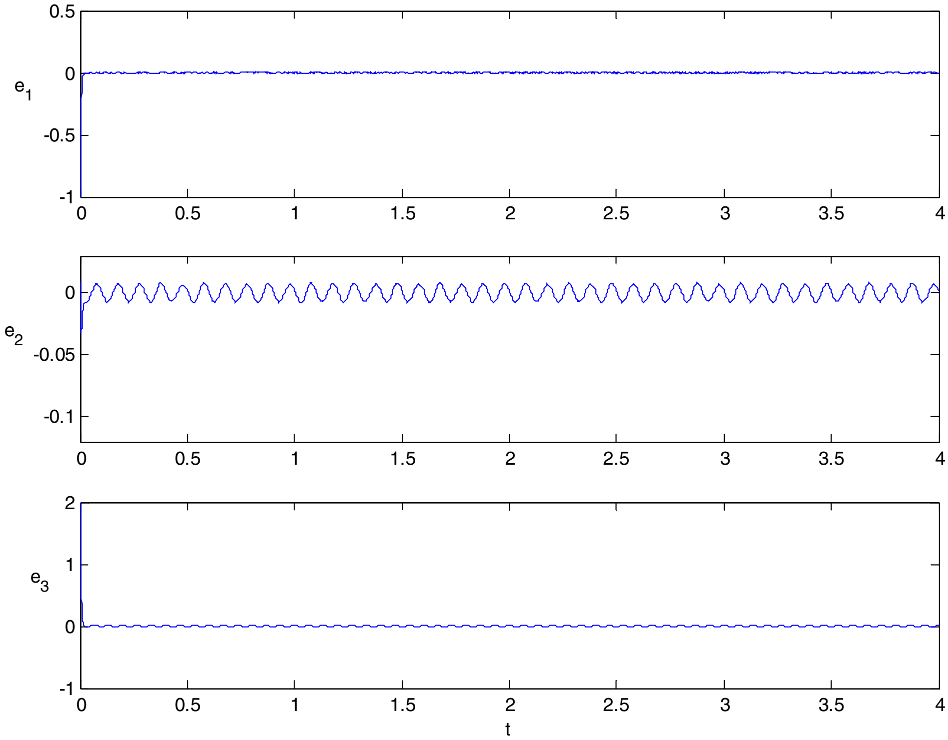

Fig. 6. State errors versus time for mutual coupled Lorenz systems with nonvanishing perturbations $\Delta f_2 = 2\ \sin(20\pi t)$, $\Delta f_4 = r(t)$, $\Delta f_6 = 5\ \cos(30\pi t)$ and $\gamma = 130$.

at the zero-crossing of the Lyapunov spectrum. Choose $\gamma = 0.6$, the simulated result in Fig. 3 shows that the state errors still converge to zero but the transient time of convergence is long. This fact agrees with our intuition.

If there exist vanishing perturbations in mutual coupled Lorenz systems as

$$\begin{aligned}
\dot{x}_1 &= -\sigma(x_1 - y_1) + \Delta f_1 + \gamma(x_2 - x_1),\\
\dot{y}_1 &= rx_1 - y_1 - x_1 z_1 + \gamma(y_2 - y_1),\\
\dot{z}_1 &= x_1 y_1 - bz_1 + \gamma(z_2 - z_1),\\
\dot{x}_2 &= -\sigma(x_2 - y_2) + \gamma(x_1 - x_2),\\
\dot{y}_2 &= rx_2 - y_2 - x_2 z_2 + \gamma(y_1 - y_2),\\
\dot{z}_2 &= x_2 y_2 - bz_2 + \Delta f_6 + \gamma(z_1 - z_2).
\end{aligned}$$

System perturbations are bounded since $|\Delta f_1| = |\cos t \cdot (y_1 - y_2)| \leqslant \|\mathbf{e}\|$ and $|\Delta f_6| = |x_1 - x_2| \leqslant \|\mathbf{e}\|$. Choose $\gamma = 45$ to satisfy $\gamma > (L + K_1 + K_2)/2$. In Fig. 4, state errors approach zero as time goes to infinity although there are persistent acting perturbations.

If not all perturbations are vanishing as $\Delta f_1 = \Delta f_3 = \Delta f_5 = 0$, $\Delta f_2 = 2\ \sin(20\pi t)$, $\Delta f_4 = r(t)$ and $\Delta f_6 = 5\ \cos(30\pi t)$, where $r(t)$ is the unit normal random variable. These perturbations are bounded on the average since $\int_t^{t+T} \sup\{|\Delta f_2|\}\mathrm{d}\tau \leqslant 2T$, $\int_t^{t+T} \sup\{|\Delta f_4|\}\mathrm{d}\tau \leqslant T$ and $\int_t^{t+T} \sup\{|\Delta f_6|\}\mathrm{d}\tau \leqslant 5T$, $\forall t \in [0, \infty), T > 0$. The initial condition is the same and $\gamma = 44$. State errors versus time are shown in Fig. 5 and they are bounded as time evolves. If $\gamma = 130$, results are shown in Fig. 6. As coupling strength γ increases, the error bounds decrease.

When these criteria are used, a matrix should be negative definite and an estimation of a Lipschitz constant is needed in advance. Moreover, this estimation is often conservative. An adaptive method can improve these two shortcomings [12].

4. Conclusions

A general scheme to achieve the chaos synchronization of mutual coupled systems via the partial stability theory is proposed in this paper. By the procedure of the proposed scheme, three criteria are proven to ensure the chaos synchronization for a general kind of mutual coupled systems. The first theorem applies for the system without perturbation. The other two theorems suit for systems possessing vanishing and nonvanishing perturbations, respectively. All these criteria work for nonlinear nonautonomous systems. Numerical simulations show that these criteria are effective.

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