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Project: Metrical results for Diophantine approximation in positive characteristic

by

Michael Fuchs

1 General

This is the final report on the National Science Council project "Metrical results for Diophantine approximation in positive characteristic" with grant number NSC-98-2115-M-009-009 and term from August 1st, 2009 to July 31st, 2010.

Before presenting the outcomes of this project, we shortly summarize them.

- All conjectures (as well as generalizations) of the project proposal have been established.
- The two papers [2] and [3] contain the main findings of this project (preprints of the papers are attached to this report). The first one appeared this year in *Acta Arithmetica* and the second one was accepted by the same journal.

2 Results

In order to describe our results, we need some notations. First, let $\mathbb{F}_q$ denote a finite field with q elements. Moreover, denote by $\mathbb{F}_q[T]$ the polynomial ring over $\mathbb{F}_q$ and by

$$\mathbb{F}_q((T^{-1})) = \left\{ f = \sum_{i \le n} a_i T^i \ : \ n \in \mathbb{Z}, a_i \in \mathbb{F}_q, a_n \neq 0 \right\} \cup \{0\}$$

the field of formal Laurent series over $\mathbb{F}_q$. We define a norm in the usual way as $|f| = q^n$ if $f \neq 0$ and $|0| = 0$. By restricting this norm to the set

$$\mathbb{L} = \{ f \in \mathbb{F}_q((T^{-1})) \ : \ |f| < 1 \}$$

one obtains a compact Abelian group. Hence, there exists a unique, translation-invariant probability measure which we are going to denote by m.

Several recent papers have studied the following Diophantine approximation problem

$$\left| f - \frac{P}{Q} \right| < \frac{1}{q^{2n+l_n}}, \ Q \text{ monic}, \ \deg Q = n, \ \gcd(P, Q) = 1, \tag{1}$$

where $f \in \mathbb{L}$ is random (with respect to m) and l_n is a sequence of non-negative integers.

In order to put our results into context, we highlight some recent results. First, in [4] the following strong law of large number with error term was established.

Theorem 1 (K. Inoue and H. Nakada)**.** *The number of solutions of (1) with* $\deg Q \le N$ *satisfies*

$$\frac{q-1}{q} \Psi(N) + \mathcal{O}\left((\Psi(N))^{1/2} (\log \Psi(N))^{3/2+\epsilon} \right) \qquad a.s.$$

with an arbitrary $\epsilon > 0$ *and* $\Psi(N) := \sum_{n \ge N} q^{-l_n}$.

Moreover, in [5] the authors studied (1) with the condition $\gcd(P, Q) = 1$ dropped. They proved the following result.

Theorem 2 (H. Nakada and R. Natsui)**.** *Let l_n be non-decreasing. Then, under some further technical conditions on l_n, the number of solutions of (1) without the condition $\gcd(P, Q) = 1$ and $\deg Q \leq N$ is a.s. asymptotic to $\Psi(N)$.*

Note that compared to the previous result, the conditions in Theorem 2 are more restrictive and the result is less precise. Our starting point of this project was to improve this result and extend it to a more general setting.

Inhomogeneous Diophantine Approximation. Consider the inhomogeneous Diophantine approximation problem

$$\left| f - \frac{g + P}{Q} \right| < \frac{1}{q^{2n+l_n}}, \ Q \text{ monic}, \ \deg Q = n, \tag{2}$$

where $f \in \mathbb{L}$ is random, $g \in \mathbb{L}$ and l_n is a sequence of non-negative integers.

Using an ingenious method of W. M. Schmidt [6], we proved the following result in [2].

Theorem 3. *For any fixed $g \in \mathbb{L}$, the number of solutions of (2) with $\deg Q \leq N$ satisfies*

$$\Psi(N) + \mathcal{O}\left((\Psi(N))^{1/2}(\log \Psi(N))^{2+\epsilon}\right) \qquad a.s.$$

with an arbitrary $\epsilon > 0$.

This result is remarkable because of the following reasons.

- For $g = 0$, it improves upon Theorem 2 by removing ALL restrictions on l_n and providing an error term.
- For $g = 0$, it completes the result in [1] where Diophantine approximation of linear forms with at least two terms was studied (our result covers the missing case of only one term).
- The error term is better and the conditions are less restrictive as in the corresponding result in the real case; see [7].

Then, we also considered (2) with several restrictions on Q.

Restricted Diophantine Approximation. Here, we proved a variety of results in [2]. We just state some consequences of our results; for more consequences and general results the reader is referred to [2].

Theorem 4. *(i) Let $C, D \in \mathbb{F}_q[T]$ with $\deg C < \deg D$. The number of solutions of (2) with $Q \equiv C$ mod D and $\deg Q \leq N$ satisfies*

$$\frac{1}{|D|}\Psi(N) + \mathcal{O}\left((\Psi(N))^{1/2}(\log \Psi(N))^{2+\epsilon}\right) \qquad a.s.$$

with an arbitrary $\epsilon > 0$.

(ii) The number of solutions of (2) with Q square-free and $\deg Q \leq N$ satisfies

$$\frac{q-1}{q}\Psi(N) + \mathcal{O}\left((\Psi(N))^{1/2}(\log \Psi(N))^{2+\epsilon}\right) \qquad \textit{a.s.}$$

with an arbitrary $\epsilon > 0$.

Moreover, if restricting Q to the set of irreducible polynomials, we even have a better error term.

Theorem 5. *Let*

$$\Psi_1(N) := \sum_{n \leq N} \frac{1}{nq^{l_n}}.$$

Then, the number of solutions of (2) with Q irreducible and $\deg Q \leq N$ satisfies

$$\Psi_1(N) + \mathcal{O}\left((\Psi_1(N))^{1/2}(\log \Psi_1(N))^{3/2+\epsilon}\right) \qquad \textit{a.s.}$$

with an arbitrary $\epsilon > 0$.

Simultaneous Diophantine Approximation. Now, consider the simultaneous Diophantine approximation problem

$$\left| f_j - \frac{P_j}{Q} \right| < \frac{1}{q^{n+l_n^{(j)}}}, \ Q \text{ monic}, \ \deg Q = n, \ j = 1, \dots, d, \tag{3}$$

where $(f_1, \dots, f_d) \in \mathbb{L} \times \cdots \times \mathbb{L}$ is random (with respect to the m-fold product measure of m) and $l_n^{(j)}$ are sequences of non-negative integers. Moreover, set $l_n := \sum_{j=1}^{d} l_n^{(j)}$.

Using Schmidt's method once more, we proved the following result which generalizes Theorem 3 above (for $g = 0$).

Theorem 6. *Let $l_n \geq n$. Then, the number of solutions of (3) with $\deg Q \leq N$ satisfies*

$$\Psi(N) + \mathcal{O}\left((\Psi(N))^{1/2}(\log \Psi(N))^{2+\epsilon}\right) \qquad \textit{a.s.}$$

with an arbitrary $\epsilon > 0$.

Moreover, in [3], we considered (3) with the additional condition $\gcd(P_j, Q) = 1$.

Theorem 7. *Let $l_n \geq n$. Then, the number of solutions of (3) with $\gcd(P_j, Q) = 1$ and $\deg Q \leq N$ satisfies*

$$c_0\Psi(N) + \mathcal{O}\left((\Psi(N))^{1/2+\epsilon}\right) \qquad \textit{a.s.}$$

with an arbitrary $\epsilon > 0$. Here,

$$c_0 := \sum_{Q_1 \text{ monic}} \cdots \sum_{Q_d \text{ monic}} \frac{\mu(Q_1)}{|Q_1|} \cdots \frac{\mu(Q_d)}{|Q_d|} \frac{1}{|\text{lcm}(Q_1, \dots, Q_d)|} > 0,$$

where $\mu(\cdot)$ is the Moebius μ function.

This result generalizes Theorem 1 to the multi-dimensional setting. Note, however, that the error term in our result for $d = 1$ is weaker than the error term in Theorem 1. This is due to the fact that we use a completely different (and more involved) method of proof (the method of proof of Theorem 1 relied on continued fraction theory which is not available in higher dimensions).

3 Summary

In this project, we established several new results concerning inhomogeneous Diophantine approximation, restricted Diophantine approximation and simultaneous Diophantine approximation in the field of formal Laurent series over a finite base field. In particular, we were able to verify all conjectures from the project proposal. Moreover, our results improve and generalize several previous results in this area. Finally, our results hold under less restrictive assumptions and are more precise compared to the corresponding results over the real number field.

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Metrical Theorems for Inhomogeneous Diophantine Approximation in Positive Characteristic

Michael FUCHS

Dedicated to Prof. Harald Niederreiter
on the occasion of his 65th birthday

Abstract

We consider inhomogeneous Diophantine approximation for formal Laurent series over a finite base field. We establish an analogue of a strong law of large numbers due to W. M. Schmidt with a better error term than in the real case. A special case of our result improves upon a recent result by H. Nakada and R. Natsui and completes a result of M. M. Dodson, S. Kristensen, and J. Levesley. Moreover, we prove various results for inhomogeneous Diophantine approximation with restricted denominators.

1 Introduction

Several recent studies have been concerned with the metric theory of Diophantine approximation in the field of formal Laurent series; for some references see below. The aim of this paper is to make some further progress on the inhomogeneous Diophantine approximation problem. More precisely, we will establish some analogues of results from the real number case (which in the sequel will be referred to as the ”classical case”) with some improvements which are arising from the more simple nature of the metric structure of the formal Laurent series field.

First, let us fix some notation. Subsequently, we will denote by $\mathbb{F}_q$ a finite field with q elements; the polynomial ring over $\mathbb{F}_q$, the field of rational functions over $\mathbb{F}_q$, and the field of formal Laurent series over $\mathbb{F}_q$ will be denoted by $\mathbb{F}_q[T]$, $\mathbb{F}_q(T)$, and $\mathbb{F}_q((T^{-1}))$, respectively. For $f \in \mathbb{F}_q((T^{-1}))$ with

$$f = a_nT^n + a_{n-1}T^{n-1} + \cdots, \quad a_k \in \mathbb{F}_q,\ a_n \neq 0,\ n \in \mathbb{Z},$$

we define $|f| := q^n$ and $|0| := 0$. It is easily checked that $|\cdot|$ is a norm which satisfies the ultra-metric property, i.e.,

$$|f - g| \leq \max\{|f|, |g|\}$$

with equality if $|f| \neq |g|$. This property in particular implies that two balls (defined in the standard way) are either disjoint or they are contained in each other. Finally, we set

$$\mathbb{L} = \{f \in \mathbb{F}_q((T^{-1})) : |f| < 1\}.$$

Note that $\mathbb{L}$ equipped with the restriction of the norm to $\mathbb{L}$ is a compact abelian group. Consequently, there exist a unique, translation-invariant probability measure which will be denoted by m.

Key words: formal Laurent series, inhomogeneous Diophantine approximation, Diophantine approximation with restricted denominators, strong laws of large numbers, Schmidt's method.

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In the following, we will be concerned with the inhomogeneous Diophantine approximation problem: for $f, g \in \mathbb{L}$ consider the Diophantine inequality

$$|Qf - g - P| < \frac{1}{q^{n+l_n}},\ Q \text{ is monic},\ \deg Q = n, \tag{1}$$

whose solutions are pairs of polynomials $\langle P, Q\rangle \in \mathbb{F}_q[T] \times \mathbb{F}_q[T]$ with $Q \neq 0$ (throughout this work we will use $\langle \cdot, \cdot \rangle$ to denote pairs, whereas $(\cdot, \cdot)$ is reserved for the $\gcd$). Here, l_n is a sequence of non-negative integers. In particular, note that l_n just depends on $\deg Q$.

In a recent paper, C. Ma and W.-Y. Su [8] investigated the above problem and proved a Khintchine type 0-1 law for the number of solutions if both f and g are chosen randomly (with respect to m) from $\mathbb{L}$. Their result is an analogue of a result of J. W. S. Cassels [3] from the classical case, where this situation is sometimes called the "double-metric" case. Moreover, the following two "single-metric" cases were considered over the real number field as well (e.g., see [11] and [12]): (S1) fix f and choose a random $g \in \mathbb{L}$; (S2) fix g and choose a random $f \in \mathbb{L}$.

In this paper, we are interested in stochastic properties of the solution set of (1) for f, g such that the number of solutions is infinite. More precisely, we will derive strong laws of large numbers with error terms for the number of solutions $\langle P, Q\rangle$ of (1) with $\deg Q \leq N$. Such results have so far only been established for (S2) with $g = 0$; see [6] and H. Nakada and R. Natsui [9]. Here, we will further improve these results and extend them to general g. So, the main part of the paper will focus on the case (S2). The other "single-metric" case and the "double metric" case exhibit a somehow different behavior and will be only briefly discussed in the final section.

From now on, let $g \in \mathbb{L}$ be fixed. Moreover, define

$$\Psi(N) := \sum_{n \leq N} \frac{1}{q^{l_n}}.$$

Our first result reads as follows.

Theorem 1. *The number of solutions of (1) with $0 \leq \deg Q \leq N$ satisfies*

$$\Psi(N) + \mathcal{O}\left(\Psi(N)^{1/2} (\log \Psi(N))^{2+\epsilon}\right), \qquad \textit{a.s.},$$

where $\epsilon > 0$ is an arbitrary constant.

This result is an analogue of a result of W. M. Schmidt [11] from the classical case. In fact, we will use a variant of Schmidt's method to prove it. Note, however, that the error term is better than the one from the classical case. Moreover, no monotonicity assumption on l_n is required.

For $g = 0$ the improved error term was also achieved in the classical case; see G. Harman [7]. The result in this special case improves upon Theorem 3 in [9] by removing some further technical conditions on l_n and providing an error term. Moreover, our result completes the main result in [4] which was concerned with Diophantine approximation of linear forms with at least two terms. Here, the missing case of only one term is considered. As in the real case, the current situation turns out to be more complex, a claim which is further supported by the fact that the result in [4] has a better error term; for a discussion of this phenomena in the real case see [10].

In fact, our method of proof can be used to obtain even more general results. More precisely, the method will allow us to investigate inhomogeneous Diophantine approximation with restricted denominators as well. Therefore, replace (1) by

$$|F(Q)f - g - P| < \frac{1}{q^{n+l_n}},\ Q \text{ is monic},\ \deg Q = n, \tag{2}$$

where l_n is as above and F is a function from $\mathbb{F}_q[T]$ into $\mathbb{F}_q[T]$.

First, we will fix some further notation. Let

$$\mathcal{F} := \{Q : Q \text{ monic and } F(Q) \neq 0\}$$

and denote by $\mathcal{F}_n$ the subset of all polynomials $Q \in \mathcal{F}$ with $\deg Q = n$. Subsequently, we will only consider F that satisfy the following property: for $Q, Q' \in \mathcal{F}$ with $\deg Q \leq \deg Q'$, we have $\deg F(Q) \leq \deg F(Q')$. Finally, set

$$\Psi(N, \mathcal{F}) := \sum_{n \leq N} \frac{\#\mathcal{F}_n}{q^{n+l_n}}.$$

Then, the following generalization of the above result holds.

Theorem 2. *Assume that $F(Q)$ is either Q or 0. Then, the number of solutions of (2) with $Q \in \mathcal{F}$ and $0 \leq \deg Q \leq N$ satisfies*

$$\Psi(N, \mathcal{F}) + \mathcal{O}\left((\Psi(N))^{1/2} (\log \Psi(N))^{2+\epsilon}\right), \qquad \text{a.s.,} \tag{3}$$

where $\epsilon > 0$ is an arbitrary constant.

In particular, the latter result gives a meaningful asymptotic formula whenever

$$\liminf_{n \to \infty} \frac{\#\mathcal{F}_n}{q^n} > 0. \tag{4}$$

Two important special cases are collected in the following corollary, the first of which has to be compared with the results in [6].

Corollary 1. *(i) Let $C, D \in \mathbb{F}_q[T]$ with $\deg C < \deg D$. Then, the number of solutions of (1) with $Q \equiv C\ (D)$ and $0 \leq \deg Q \leq N$ satisfies*

$$\frac{1}{|D|}\Psi(N) + \mathcal{O}\left((\Psi(N))^{1/2} (\log \Psi(N))^{2+\epsilon}\right), \qquad \text{a.s.,} \tag{5}$$

where $\epsilon > 0$ is an arbitrary constant.

(ii) The number of solutions of (1) with Q monic, square-free and $0 \leq \deg Q \leq N$ satisfies

$$\frac{q-1}{q}\Psi(N) + \mathcal{O}\left((\Psi(N))^{1/2} (\log \Psi(N))^{2+\epsilon}\right), \qquad \text{a.s.,} \tag{6}$$

where $\epsilon > 0$ is an arbitrary constant.

Note that condition (4) is not satisfied for some interesting $\mathcal{F}$ such as the set of monic, irreducible polynomials. This situation, however, turns out to be more simpler and we can obtain a strong law of large numbers with an even better error term. Therefore, we first prove an analogue of Theorem 3.1 in [7] which holds for general F.

Theorem 3. *The number of solutions of (2) with $Q \in \mathcal{F}$ and $0 \leq \deg Q \leq N$ satisfies*

$$\Psi(N, \mathcal{F}) + \mathcal{O}\left((\Psi_0(N))^{1/2} (\log \Psi_0(N))^{3/2+\epsilon}\right), \qquad \text{a.s.,}$$

where $\epsilon > 0$ is an arbitrary constant and

$$\Psi_0(N) = \sum_{n \leq N} \frac{1}{q^{n+l_n}} \sum_{m \leq n} \sum_{Q \in \mathcal{F}_n} \sum_{Q' \in \mathcal{F}_m} \frac{|(F(Q), F(Q'))|}{|F(Q)|}.$$

This result entails the following corollary.

Corollary 2. *(i) Let*

$$\Psi_1(N) := \sum_{n\leq N} \frac{1}{nq^{l_n}}.$$

Then, the number of solutions of (1) with Q monic, irreducible and $0 \leq \deg Q \leq N$ satisfies

$$\Psi_1(N) + \mathcal{O}\left((\Psi_1(N))^{1/2} (\log \Psi_1(N))^{3/2+\epsilon}\right), \qquad \text{a.s.},$$

where $\epsilon > 0$ is an arbitrary constant.

(ii) Let $F(Q) = Q^t$ with $t \geq 2$. Then, the number of solutions of (2) with $0 \leq \deg Q \leq N$ satisfies

$$\Psi(N) + \mathcal{O}\left((\Psi(N))^{1/2} (\log \Psi(N))^{3/2+\epsilon}\right), \qquad \text{a.s.},$$

where $\epsilon > 0$ is an arbitrary constant.

It is worth mentioning that Theorem 3 does not give a meaningful result in the situations discussed in Theorem 1 and Corollary 1. Consequently, part (ii) of Corollary 2 shows that the complexity of $t = 1$ and $t \geq 2$ are rather different.

We conclude the introduction by giving a short plan of the paper. In the next section, we will prove a weak independence result which will form the crucial step in deriving all results above. In particular, Theorem 3 will follow rather quickly from this result and this will be demonstrated in the next section as well. Then, in Section 3, we will show how to amend Schmidt's method to the current situation to obtain a proof of Theorem 1 and Theorem 2. In the final section, we will then briefly discuss the other "single-metric" case and the "double-metric" case.

Notation. All logarithms appearing throughout this work will only attain values ≥ 1, i.e., $\log_a x$ should be interpreted as $\max\{\log_a x, 1\}$. We will use Landau's notation $f(x) = \mathcal{O}(g(x))$ as well as Vinogradov's notation $f(x) \ll g(x)$ to indicate that there exist a constant $C \geq 0$ such that $|f(x)| \leq C|g(x)|$ for all x sufficiently large.

2 A weak independence result with applications

We start by proving a technical lemma that constitutes a refinement of Lemma 2.3 in [2].

Lemma 1. *Let Q, Q' be two non-zero polynomials with $n = \deg Q, m = \deg Q'$ and $d = \deg(Q, Q')$. Let l be a non-negative integer. Then, the number N of pairs $\langle P, P'\rangle$ with $\deg P < n, \deg P' < m$ and*

$$\left|\frac{g+P}{Q} - \frac{g+P'}{Q'}\right| < \frac{1}{q^{m+l}} \tag{7}$$

is given by

$$N \begin{cases} = q^{n-l}, & \text{if } n \geq l + d; \\ \leq q^d, & \text{if } n < l + d. \end{cases}$$

Proof. First, (7) can be reformulated to

$$|g(Q' - Q) + PQ' - P'Q| < q^{n-l}.$$

Next, set $Q = (Q, Q') \cdot \bar{Q}$ and $Q' = (Q, Q') \cdot \bar{Q}'$. Then,

$$|g(\bar{Q}' - \bar{Q}) + P\bar{Q}' - P'\bar{Q}| < q^{n-l-d}.$$

Let $-C$ denote the polynomial part of $g(\bar{Q}' - \bar{Q})$. Now, we will consider two cases.

First, assume that $n < l + d$. Then, a necessary condition for $\langle P, P' \rangle$ being a solution of the above inequality is $P\bar{Q}' - P'\bar{Q} = C$. Observe that for P with $\deg P < n$ and

$$P\bar{Q}' \equiv C \bmod \bar{Q}, \tag{8}$$

we have $P\bar{Q}' = C + P'\bar{Q}$ with some polynomial P' and

$$\deg P' + \deg \bar{Q} = \deg(P\bar{Q}' - C) \leq \deg P + \deg \bar{Q}' < n + \deg \bar{Q}'.$$

Consequently, $\deg P' < m$. So, either $N = 0$ or N equals the number of solutions of (8) which is q^d.

Next, we consider $n \geq l + d$. Here, we can argue similar as above, the only difference being that N equals the number of solutions of (8) with C replaced by $C + D$ for all polynomials D with $\deg D < n - l - d$. Consequently, $N = q^{n-l}$. ∎

Next, we define for $Q \in \mathcal{F}_n$ the set

$$F_Q := \{f \in \mathbb{L} : f \text{ satisfies (2) with some } P \in \mathbb{F}_q[T]\}.$$

Obviously, F_Q is the union of $|F(Q)|$ disjoint balls. Consequently,

$$m(F_Q) = \frac{1}{q^{n+l_n}}.$$

Moreover, we have the following weak independence result.

Proposition 1. *Let $Q \in \mathcal{F}_n, Q' \in \mathcal{F}_m$, and $d = \deg(F(Q), F(Q'))$. Then,*

$$m(F_Q \cap F_{Q'}) \leq m(F_Q)m(F_{Q'}) + q^{d-\deg F(Q)-n-l_n}.$$

Proof. First assume that $n + l_n + \deg F(Q) \geq m + l_m + \deg F(Q')$. Then, all balls which make up F_Q have radius at most as large as the radius of the balls which make up $F_{Q'}$. So, by the ultra-metric property of the norm, we have to count how many of the $(g + P)/F(Q)$ are contained in balls with center $(g + P')/F(Q')$ and radius $q^{-\deg F(Q')-m-l_m}$, i.e., we have to count the number of solutions of

$$\left| \frac{g+P}{F(Q)} - \frac{g+P'}{F(Q')} \right| < \frac{1}{q^{\deg F(Q')+m+l_m}}.$$

The latter number is given by the above lemma. We first consider the case with $\deg F(Q) \geq m+l_m+d$. Here, the number of solutions equals $q^{\deg F(Q)-m-l_m}$. So, we obtain

$$m(F_Q \cap F_{Q'}) = \frac{|F(Q)|q^{-m-l_m}}{|F(Q)|q^{n+l_n}} = \frac{1}{q^{n+l_n}} \cdot \frac{1}{q^{m+l_m}} = m(F_Q)m(F_{Q'}).$$

Hence, the assertion holds in this case. Now, consider the second case where $\deg F(Q) < m + l_m + d$. Then, again by the above lemma,

$$m(F_Q \cap F_{Q'}) \leq \frac{q^d}{q^{\deg F(Q)+n+l_n}}.$$

Hence, the claim is proved in this case as well.

Next, if $n+l_n \deg F(Q) < m+l_m+\deg F(Q')$, we obtain from the arguments above the claim with the second term replaced by $q^{d-\deg F(Q')-m-l_m}$. This term is trivially bounded by $q^{d-\deg F(Q)-n-l_n}$. Hence, the proof of the proposition is finished. ∎

The above proposition will turn out to be one of the key ingredients in the prove of our results. The other key ingredient is the following important lemma which is a standard tool in metric number theory.

Lemma 2 (Lemma 1.5 in [7]). *Let $\xi_n(\omega)$ be a sequence of non-negative random variables defined on a probability space $(\Omega, \mathcal{B}, P)$. Let ψ_n and φ_n be sequences of real numbers with*

$$0 \leq \psi_n \leq \varphi_n.$$

Define

$$\Phi(N) = \sum_{n \leq N} \varphi_n$$

and assume that $\Phi(N) \to \infty$ as $N \to \infty$. Finally, assume that

$$\mathbb{E}\left(\sum_{M \leq n \leq N} \xi_n - \psi_n\right)^2 \ll \sum_{M \leq n \leq N} \varphi_n.$$

for all non-negative $M < N$. Then,

$$\sum_{n \leq N} \xi_n(\omega) = \sum_{n \leq N} \psi_n + \mathcal{O}\left((\Phi(N))^{1/2} (\log \Phi(N))^{3/2+\epsilon} + \max_{n \leq N} \psi_n\right), \qquad \text{a.s.},$$

where $\epsilon > 0$ is an arbitrary constant.

As a first application of this lemma, we show how to deduce Theorem 3 from it. Therefore, set

$$\xi_n := \#\{\langle P, Q\rangle : \langle P, Q\rangle \text{ is a solution of } (2)\}.$$

This sequence of random variables satisfies the following properties.

Proposition 2. *(i) We have,*

$$\mathbb{E}\left(\sum_{n \leq N} \xi_n\right) = \Psi(N, \mathcal{F}).$$

(ii) We have,

$$\mathbb{E}\left(\sum_{M \leq n \leq N} \xi_n - \frac{\#\mathcal{F}_n}{q^{n+l_n}}\right)^2 \ll \sum_{M \leq n \leq N} \frac{1}{q^{n+l_n}} \sum_{m \leq n} \sum_{Q \in \mathcal{F}_n} \sum_{Q' \in \mathcal{F}_m} \frac{|(F(Q), F(Q'))|}{|F(Q)|}$$

for all non-negative integers $M < N$.

Proof. Part (i) follows from

$$\xi_n = \sum_{Q \in \mathcal{F}_n} \mathbf{1}_{F_Q}$$

and basic properties of the mean value.

For part (ii), we also use the above representation which yields

$$\mathbb{E}\left(\sum_{M \leq n \leq N} \xi_n - \frac{\#\mathcal{F}_n}{q^{n+l_n}}\right)^2 = 2 \sum_{M \leq n \leq N} \sum_{M \leq m \leq n-1} \sum_{Q \in \mathcal{F}_n, Q' \in \mathcal{F}_m} m(F_Q \cap F_{Q'}) - m(F_Q) m(F_{Q'})$$
$$+ \sum_{M \leq n \leq N} \sum_{Q \in \mathcal{F}_n, Q' \in \mathcal{F}_m} m(F_Q \cap F_{Q'}) - m(F_Q) m(F_{Q'}).$$

Applying Proposition 1 immediately yields the claimed result. ∎

Now, we can prove Theorem 3.

Proof of Theorem 3. If $\Psi(N,\mathcal{F}) \to c \geq 0$ as $N \to \infty$, the result follows by a standard application of the Lemma of Borel-Cantelli. Hence, we can assume that $\Psi(N,\mathcal{F}) \to \infty$ as $N \to \infty$. But then the claim follows from the Proposition above together with Lemma 2. ∎

Corollary 2 follows from the last result as follows.

Proof of Corollary 2. For part (i), we use the well-known result (see Chapter 3 in [1])

$$\#\mathcal{F}_n = \frac{q^n}{n} + \mathcal{O}\left(q^{\epsilon n}\right), \tag{9}$$

where $\epsilon < 1$ is a suitable constant. Hence,

$$\Psi(N,\mathcal{F}) = \Psi_1(N) + \mathcal{O}(1).$$

Moreover,

$$\Psi_0(N) = \sum_{n\leq N} \frac{1}{q^{2n+l_n}} \sum_{m\leq n} \sum_{\substack{\deg Q=n \\ Q \text{ monic, irreducible}}} \sum_{\substack{\deg Q'=m \\ Q' \text{ monic, irreducible}}} |(Q,Q')| \ll \Psi_1(N),$$

where the last line again follows by (9). This proves the claim.

As for part (ii), first observe that $\#\mathcal{F}_n = q^n$ and hence $\Psi(N,\mathcal{F}) = \Psi(N)$. The bound for $\Psi_0(N)$ is slightly more tricky. First,

$$\begin{aligned}\Psi_0(N) &= \sum_{n\leq N} \frac{1}{q^{(t+1)n+l_n}} \sum_{m\leq n} \sum_{\substack{\deg Q=n \\ Q \text{ monic}}} \sum_{\substack{\deg Q'=m \\ Q' \text{ monic}}} |(Q^t,(Q')^t)| \\ &\ll \sum_{n\leq N} \frac{1}{q^{(t+1)n+l_n}} \sum_{\substack{\deg Q=n \\ Q \text{ monic}}} \sum_{\substack{D|Q \\ D \text{ monic}}} \frac{q^n}{|D|}|D|^t.\end{aligned}$$

Next, we have

$$\sum_{\substack{\deg Q=n \\ Q \text{ monic}}} \sum_{\substack{D|Q \\ D \text{ monic}}} |D|^{t-1} = \sum_{d\leq n} \sum_{\substack{\deg D=d \\ D \text{ monic}}} \frac{q^n}{|D|}|D|^{t-1} = q^n \sum_{d\leq n} q^{(t-1)d} \ll q^{tn}.$$

Plugging this into the estimate above yields $\Psi_0(N) \ll \Psi(N)$. Hence, the result is established. ∎

3 Schmidt's method in positive characteristic

Note that the method from the last section does not yield a meaningful result for the case $F(Q) = Q$. More specifically, it is easily checked that the error term from the proof of part (ii) of Corollary 2 for $t = 1$ would be larger than the main term. The same phenomena also occurs in the real case, where this problem was overcome by an ingenious method introduced by W. M. Schmidt in [10] and [11]. In this section, Schmidt's method will be amended to the current situation.

We start with a couple of (easy) lemmas.

Lemma 3 (Dirichlet's principle in positive characteristic)**.** *For all non-zero polynomials Q there exist polynomials A, B with $0 < |A| \leq |Q|$ and $(A,B) = 1$ such that*

$$\left|g - \frac{B}{A}\right| < \frac{1}{|A||Q|}.$$

Proof. This is proved as in the classical case. ∎

Observe that A and B in the previous lemma just depend on $\deg Q$. Subsequently, for any given non-zero polynomial Q, we will choose a fixed pair $\langle A, B\rangle$ satisfying the assumption of the previous lemma for a polynomial Q' with $\deg Q' = \lfloor \deg Q/2 \rfloor$.

Next, we define the following two sets

$$S(Q;k) = \{P : \deg P < \deg Q \text{ and } \deg(P,Q) \leq k\},$$
$$S^*(Q;k) = \{P : \deg P < \deg Q \text{ and } \deg(AP+B,Q) \leq k\},$$

whose cardinalities will be denote by $\varphi(Q;k)$ and $\varphi^*(Q;k)$, respectively.

Lemma 4. *We have,*

$$\varphi^*(Q;k) \geq \varphi(Q;k).$$

Proof. First, let $Q = Q_1Q_2$, where every prime factor of Q_1 is also a prime factor of A and $(Q_2, A) = 1$. Then, we have

$$\varphi(Q;k) \leq \varphi(Q_1;k)\varphi(Q_2;k) \leq |Q_1|\varphi(Q_2;k).$$

Now, note that $AP + B$ with $\deg P < \deg Q_2$ are all different module Q_2. Hence, $\varphi(Q_2;k) = \#\{P : \deg P < \deg Q_2 \text{ and } \deg(AP+B,Q_2) \leq k\}$. Finally notice that

$$(AP+B,Q_2) = (AP+B,Q_1Q_2) = (AP+B,Q).$$

Consequently,

$$\varphi^*(Q;k) = |Q_1| \cdot \#\{P : \deg P < \deg Q_2 \text{ and } \deg(AP+B,Q_2) \leq k\}.$$

Combining everything yields the claimed result. ∎

Next, we fix $F(Q) = Q$. Moreover, as in the last section, it suffices to consider the case where $\Psi(N) \to \infty$ as $N \to \infty$. The method of the last section did not work when directly applied to the sequence ξ_n. Therefore, we will approximate this sequence by the following one

$$\xi_n^* := \#\{\langle P,Q\rangle : \ P \in S^*(Q;\Gamma(n)) \text{ and } \langle P,Q\rangle \text{ is a solution of (1)}\},$$

where $\Gamma(n) = \lfloor \log_q \Psi(n)^2 \rfloor$. Moreover, similar as in the last section, we define

$$F_Q^* := \{f \in \mathbb{L} : f \text{ satisfies (1) with some } P \in S^*(Q;\Gamma(n))\}.$$

Then,

$$\xi_n^* = \sum_{\substack{\deg Q = n \\ Q \text{ monic}}} \mathbf{1}_{F_Q^*}$$

and consequently

$$\mathbb{E}\xi_n^* = \sum_{\substack{\deg Q = n \\ Q \text{ monic}}} \frac{\varphi^*(Q;\Gamma(n))}{q^{2n+l_n}}.$$

The next result shows that the mean values of the partial sums of ξ_n and ξ_n^* are very close to each other.

Proposition 3. *We have,*

$$\mathbb{E}\left(\sum_{M \leq n \leq N} \xi_n^*\right) = \sum_{M \leq n \leq N} \frac{1}{q^{l_n}} + \mathcal{O}(1)$$

for all non-negative integers $M < N$.

Proof. First, observe that

$$0 \leq \sum_{M \leq n \leq N} \frac{1}{q^{l_n}} - \mathbb{E}\left(\sum_{M \leq n \leq N} \xi_n^*\right) = \sum_{M \leq n \leq N} \sum_{\substack{\deg Q = n \\ Q \text{ monic}}} \frac{q^n - \varphi^*(Q; \Gamma(n))}{q^{2n+l_n}}$$
$$\leq \sum_{M \leq n \leq N} \sum_{\substack{\deg Q = n \\ Q \text{ monic}}} \frac{q^n - \varphi(Q; \Gamma(n))}{q^{2n+l_n}},$$

where we have used the above lemma in the last step. Next, it is well-known (see [5]) that the number of pairs $\langle P, Q \rangle$ with $\deg P = l < \deg Q = n$, P, Q monic and $\deg(P, Q) = k < l$ is given by

$$q^{n+l-k}\left(1 - \frac{1}{q}\right).$$

Consequently,

$$\sum_{\substack{\deg Q = n \\ Q \text{ monic}}} \varphi(Q, \Gamma(n)) = \frac{(q-1)^2}{q} \sum_{l=\Gamma(n)+1}^{n-1} \sum_{k=0}^{\Gamma(n)} q^{n+l-k} + \mathcal{O}\left(\sum_{l=0}^{\Gamma(n)} \sum_{k=0}^{l} q^{n+l-k}\right) = q^{2n} + \mathcal{O}\left(q^{2n-\Gamma(n)}\right).$$

Plugging this into the above expression, we obtain

$$0 \leq \sum_{M \leq n \leq N} \frac{1}{q^{l_n}} - \mathbb{E}\left(\sum_{M \leq n \leq N} \xi_n^*\right) \ll \sum_{N \leq n \leq M} \frac{1}{q^{l_n} \Psi(n)^2}.$$

Since the latter series is convergent by the Abel-Dini theorem, the claim is proved. ∎

Finally, we need the following property.

Proposition 4. *We have,*

$$\mathbb{E}\left(\sum_{M \leq n \leq N} \xi_n^* - \frac{1}{q^{l_n}}\right)^2 \ll \sum_{M \leq n \leq N} \frac{\Gamma(n)}{q^{l_n}}$$

for all non-negative integers $M < N$.

Proof. We start with an observation that is needed below. By a close inspection of the proof of Proposition 1, we have

$$m(F_Q^* \cap F_{Q'}^*) \leq \frac{1}{q^{n+l_n}} \cdot \frac{1}{q^{m+l_m}} + \frac{1}{q^{2n+l_n}} A(Q, Q'), \tag{10}$$

where $A(Q, Q)$ is the number of all pairs P, P' with $P \in S^*(Q; \Gamma(n)), P' \in S^*(Q'; \Gamma(m))$ and

$$|g(Q - Q') + P'Q - PQ'| < \min\left\{|(Q, Q')|, q^{\max\{n-m-l_m, m-n-l_n\}}\right\}. \tag{11}$$

Moreover, observe that $A(Q, Q) \leq |(Q, Q')|$.

We will use this to bound the expected value from the claim. First,

$$\mathbb{E}\left(\sum_{M \leq n \leq N} \xi_n^* - \frac{1}{q^{l_n}}\right)^2 =$$

$$= \sum_{M\le n\le N}\sum_{M\le m\le N}\mathbb{E}\xi_n^*\cdot\xi_m^* - 2\sum_{M\le n\le N}\frac{1}{q^{l_n}}\mathbb{E}\left(\sum_{N\le n\le M}\xi_n^*\right) + \sum_{M\le n\le N}\sum_{M\le m\le N}\frac{1}{q^{l_n}}\cdot\frac{1}{q^{l_m}}$$

$$= \sum_{M\le n\le N}\sum_{M\le m\le N}\left(\mathbb{E}\xi_n^*\cdot\xi_m^* - \frac{1}{q^{l_n}}\cdot\frac{1}{q^{l_m}}\right) + \mathcal{O}\left(\sum_{M\le n\le N}\frac{1}{q^{l_n}}\right)$$

$$= 2\sum_{M\le n\le N}\sum_{M\le m\le n-1}\left(\mathbb{E}\xi_n^*\cdot\xi_m^* - \frac{1}{q^{l_n}}\cdot\frac{1}{q^{l_m}}\right) + \sum_{M\le n\le N}\left(\mathbb{E}(\xi_n^*)^2 - \frac{1}{q^{2l_n}}\right) + \mathcal{O}\left(\sum_{M\le n\le N}\frac{1}{q^{l_n}}\right),$$

where the third step follows from Proposition 3. Now, applying (10) gives

$$\sum_{M\le m\le n}\mathbb{E}\xi_n^*\cdot\xi_m^* = \sum_{M\le m\le n}\sum_{\substack{\deg Q=n\\ Q\text{ monic}}}\sum_{\substack{\deg Q=m\\ Q\text{ monic}}} m(F_Q^*\cap F_{Q'}^*)$$
$$\le \frac{1}{q^{l_n}}\cdot\sum_{M\le m\le n}\frac{1}{q^{l_m}} + \frac{1}{q^{2n+l_n}}\sum_{M\le m\le n}\sum_{\substack{\deg Q=n\\ Q\text{ monic}}}\sum_{\substack{\deg Q'=m\\ Q'\text{ monic}}} A(Q,Q')$$

Using this to bound the first and second term in the expression above yields

$$\mathbb{E}\left(\sum_{M\le n\le N}\xi_n^* - \frac{1}{q^{l_n}}\right)^2 \ll \sum_{M\le n\le N}\frac{1}{q^{2n+l_n}}\sum_{M\le m\le n}\sum_{\substack{\deg Q=n\\ Q\text{ monic}}}\sum_{\substack{\deg Q'=m\\ Q'\text{ monic}}} A(Q,Q') + \sum_{M\le n\le N}\frac{1}{q^{l_n}}. \tag{12}$$

Next, we will estimate

$$\Sigma := \sum_{M\le n\le N}\frac{1}{q^{2n+l_n}}\sum_{M\le m\le n}\sum_{\substack{\deg Q=n\\ Q\text{ monic}}}\sum_{\substack{\deg Q'=m\\ Q'\text{ monic}}} A(Q,Q').$$

Therefore, we fix an arbitrary small δ and break Σ into two parts Σ' and Σ'', where the first part runs over all pairs $\langle Q,Q'\rangle$ with $\deg Q' \le \lceil n-\delta\deg(Q,Q')\rceil$ and the second part runs over the remaining pairs. In order to bound Σ', we change the order of summation as follows: first we sum over Q, then over $D|Q$ and finally over Q' with $D=(Q,Q')$. Note that for fixed Q and D the number of Q''s is bounded by $q^n/|D|^{1+\delta}$. This together with $A(Q,Q')\le |D|$ then yields

$$\Sigma' = \sum_{M\le n\le N}\frac{1}{q^{2n+l_n}}\sum_{\substack{\deg Q=n\\ Q\text{ monic}}}\sum_{\substack{D|Q\\ D\text{ monic}}}\frac{q^n}{|D|^{1+\delta}}|D| \ll \sum_{M\le n\le N}\frac{1}{q^{l_n}}\sum_{\substack{\deg D\le n\\ D\text{ monic}}}\frac{1}{|D|^{1+\delta}} \ll \sum_{M\le n\le N}\frac{1}{q^{l_n}}.$$

As for Σ'' observe that $\deg Q' > \lceil n-\delta\deg(Q,Q')\rceil$ implies

$$\min\left\{|(Q,Q')|, q^{\max\{n-m-l_m,m-n-l_n\}}\right\} < |(Q,Q')|^\delta.$$

Hence, for all $\langle Q,Q'\rangle$ involved in the range of Σ'' the relation (11) can be replaced by

$$|g(Q-Q') + P'Q - PQ'| < |(Q,Q')|^\delta. \tag{13}$$

This yields

$$\Sigma'' \ll \sum_{M\le n\le N}\frac{1}{q^{2n+l_n}}\sum_{M\le m\le n}\sum_{\substack{\deg Q=n\\ Q\text{ monic}}}\sum_{\substack{\deg Q'=m\\ Q'\text{ monic}}} B(Q,Q'),$$

where $B(Q,Q')$ denotes the number of all P,P' with $P \in S^*(Q;\Gamma(n))$ and $P' \in S^*(Q';\Gamma(m))$ that satisfy (13). Again note that $B(Q,Q') \le |(Q,Q')|$.

Collecting all bounds so far, we see that the right hand side of (12) can be replaced by

$$\sum_{M \le n \le N} \frac{1}{q^{2n+l_n}} \sum_{M \le m \le n} \sum_{\substack{\deg Q = n \\ Q \text{ monic}}} \sum_{\substack{\deg Q' = m \\ Q' \text{ monic}}} B(Q,Q') + \sum_{M \le n \le N} \frac{1}{q^{l_n}}. \tag{14}$$

Next, we will estimate the first term

$$\Sigma_0 := \sum_{M \le n \le N} \frac{1}{q^{2n+l_n}} \sum_{M \le m \le n} \sum_{\substack{\deg Q = n \\ Q \text{ monic}}} \sum_{\substack{\deg Q' = m \\ Q' \text{ monic}}} B(Q,Q')$$

which we will break into three parts $\Sigma_0', \Sigma_0'', \Sigma_0'''$, where the ranges will be given below. For every part we will proceed similar as for Σ' above. More precisely, we will change the order of summation as follows: as for Σ' the first two sums will run over Q and $D|Q$. The final sum will run over $\bar{Q}'$ with $(\bar{Q}', Q/D) = 1$. Here, we introduce the notation $Q' = D\bar{Q}'$ and $Q = D\bar{Q}$. Using this notation, we can rewrite (13) to

$$|g(\bar{Q} - \bar{Q}') + P'\bar{Q} - P\bar{Q}'| < |D|^{-1+\delta}. \tag{15}$$

Finally, we need the notation $R = g - B/A$, where $\langle A, B\rangle$ is the pair belonging to Q. Now, we will separately estimate the three parts $\Sigma_0', \Sigma_0'', \Sigma_0'''$.

As for Σ_0', the first two sums of this part run over all $\langle Q, D\rangle$ with $D|Q$ and $|A| \ge |D|^{\delta_1}$, where δ_1 will be chosen later. The last sum runs over $\bar{Q}'$ and our goal is to count the number of $\bar{Q}'$ such that (15) has solutions in P, P' (whose number will then be bounded by $|D|$). First, we consider $\bar{Q}'$ of the form $\bar{Q}' = C_1 + C_2$, where C_1 is fixed and C_2 is an arbitrary polynomial with $\deg C_2 < \deg A$. Plugging this into (15) and doing some simplifications yields

$$|gC_2 + L + \bar{g}| < |D|^{-1+\delta},$$

where $\bar{g} \in \mathbb{L}$ does not depend on $C_2 \in \mathbb{F}_q[T]$ might depend on C_2. From the ultra-metric property of the norm, we obtain

$$\left|\frac{B}{A}C_2 + L + \bar{g}\right| \le \max\{|gC_2 + L + \bar{g}|, |RC_2|\} < \max\{|D|^{-1+\delta}, |RA|\}.$$

Observe that since C_2 runs through a complete set of residues modulo A and $(A,B) = 1$, BC_2 also runs through a complete set of residues modulo A. Consequently,

$$\left|\frac{C}{A} + \bar{L} + \bar{g}\right| < \max\{|D|^{-1+\delta}, |RA|\},$$

where we now have to count the number of C's satisfying this inequality with $\deg C < \deg A$. Here, $\bar{L}$ is another polynomial that might depend on C. However, since the right hand side of the above inequality is smaller than 1, $\bar{L}$ must be equal to 0. Thus,

$$|C + A\bar{g}| < \max\{|A||D|^{-1+\delta}, |RA^2|\} \le \max\{|A||D|^{-1+\delta}, 1\}$$

and the number of such C's is clearly bounded by $|A||D|^{-1+\delta} + 1$. Next, observe that the number of C_1's above is bounded by $|Q||DA|^{-1} + 1$. Therefore, the number of $\bar{Q}'$ such that (15) has a solution in P, P' is bounded by

$$\begin{aligned}(|A||D|^{-1+\delta} + 1)(|Q||DA|^{-1} + 1) &\le |Q||D|^{-2+\delta} + |Q||D|^{-1-\delta_1} + \sqrt{|Q|}|D|^{-1+\delta} + 1 \\ &\ll |Q||D|^{-1-\delta_1} + 1,\end{aligned}$$

where δ_1, δ are chosen such that $\delta + \delta_1 \le 1/2$. Overall, this yields the following bound for Σ_0'

$$\Sigma_0' \ll \sum_{M\le n\le N} \frac{1}{q^{2n+l_n}} \sum_{\substack{\deg Q=n\\ Q \text{ monic}}} \sum_{\substack{D|Q\\ D \text{ monic}}} \left(\frac{q^n}{|D|^{1+\delta_1}} + 1\right)|D|$$
$$\ll \sum_{M\le n\le N} \frac{1}{q^{l_n}} + \sum_{M\le n\le N} \frac{1}{q^{n+l_n}} \sum_{\substack{\deg D\le n\\ D \text{ monic}}} 1 \ll \sum_{M\le n\le N} \frac{1}{q^{l_n}}. \tag{16}$$

Next, we turn to Σ_0'' whose first two sums run over all pairs $\langle Q, D\rangle$ with $D|Q, |A| < |D|^{\delta_1}$, and $|R| \ge |D|/|QA|$. Again, we will estimate the number of solutions of (15) in $\bar{Q}', P, P'$. Therefore, first observe that (15) can be rewritten as

$$\left|RC + \frac{L}{A}\right| < |D|^{-1+\delta} \tag{17}$$

for some polynomials C and L. If L is fixed, then the number of solutions in C of the above inequality is bounded by $|R|^{-1}|D|^{-1+\delta} + 1$. On the other hand, we have

$$|L| \le \max\{|A||D|^{-1+\delta}, |RCA| \le \max\{|A||D|^{-1+\delta}, |RQA|/|D|\}.$$

So, overall, we obtain for the number of C's such that there exist L satisfying (17)

$$(|R|^{-1}|D|^{-1+\delta} + 1)(|A||D|^{-1+\delta} + |RQA|/|D| + 1)$$
$$\ll |QA^2||D|^{-3+2\delta} + |QA||D|^{-2+\delta} + \sqrt{|Q|}|D|^{-1} + 1$$
$$\ll |Q||D|^{-2+\delta+\delta_1} + \sqrt{|Q|}|D|^{-1} + 1.$$

Note that the above number also equals the number of $\bar{Q}'$'s such that (14) has solutions in P, P'. Hence, Σ_0'' is bounded as follows

$$\Sigma_0'' \ll \sum_{M\le n\le N} \frac{1}{q^{2n+l_n}} \sum_{\substack{\deg Q=n\\ Q \text{ monic}}} \sum_{\substack{D|Q\\ D \text{ monic}}} \left(\frac{q^n}{|D|^{2-\delta-\delta_1}} + \frac{q^{n/2}}{|D|} + 1\right)|D|$$
$$\ll \sum_{M\le n\le N} \frac{1}{q^{l_n}} + \sum_{M\le n\le N} \frac{1}{q^{n/2+l_n}} \sum_{\substack{\deg D\le n\\ D \text{ monic}}} \frac{1}{|D|}$$
$$\ll \sum_{M\le n\le N} \frac{1}{q^{l_n}} + \sum_{M\le n\le N} \frac{n}{q^{n/2+l_n}} \ll \sum_{M\le n\le N} \frac{1}{q^{l_n}}. \tag{18}$$

So, what is left is to bound Σ_0'''. Here, the first two sums run over all pairs $\langle Q, D\rangle$ with $D|Q, |A| < |D|^{\delta_1}$, and $|R| < |D|/|QA|$. Then, (15) together with the ultra-metric property of the norm yields

$$|\bar{Q}(AP' + B) - \bar{Q}'(AP + B)| \le \max\{|R(\bar{Q} - \bar{Q}')A|, |A||g(\bar{Q} - \bar{Q}') + P'\bar{Q} - P\bar{Q}'|\} < 1.$$

Consequently,

$$\bar{Q}(AP' + B) = \bar{Q}'(AP + B).$$

Thus $AP + B \equiv 0\ (\bar{Q})$ and this implies $\deg \bar{Q} \le \Gamma(n)$. The latter in turn yields $\deg D \ge n - \Gamma(n)$. So, in this case, we obtain the bound

$$\Sigma_0''' \ll \sum_{M\le n\le N} \frac{1}{q^{2n+l_n}} \sum_{\substack{\deg Q=n\\ Q \text{ monic}}} \sum_{\substack{D|Q, Q \text{ monic}\\ \deg D\ge n-\Gamma(n)}} \frac{q^n}{|D|}|D|$$
$$= \sum_{M\le n\le N} \frac{1}{q^{n+l_n}} \sum_{\substack{\deg Q=n\\ Q \text{ monic}}} \sum_{\substack{D|Q, Q \text{ monic}\\ \deg D\le\Gamma(n)}} 1 \ll \sum_{M\le n\le N} \frac{\Gamma(n)}{q^{l_n}}. \tag{19}$$

Finally, combining (16), (18), and (19) gives the bound

$$\Sigma_0 \ll \sum_{M \le n \le N} \frac{\Gamma(n)}{q^{l_n}}.$$

Plugging this into (14) then proves the claimed result. ∎

Now, we can start with the proof of Theorem 1.

Proof of Theorem 1. First, from Proposition 4 together with Lemma 2, we obtain

$$\sum_{n \le N} \xi_n^* = \Psi(N) + \mathcal{O}\left((\Psi^*(N))^{1/2}(\log \Psi^*(N))^{3/2+\epsilon}\right), \qquad \text{a.s.,}$$

where $\epsilon > 0$ is an arbitrary constant. Next, observe

$$\Psi^*(N) = \sum_{n \le N} \frac{\Gamma(n)}{q^{l_n}} \ll \Psi(N) \log \Psi(N).$$

Hence, the claimed result holds for the sequence ξ_n^*.

In order to show that the claimed result holds for ξ_n as well, observe that from Proposition 3

$$P\left(\sum_{n \le N} (\xi_n - \xi_n^*) > \log \Psi(N)\right) \ll (\log \Psi(N))^{-1}.$$

Next, choose N_k to be the minimal positive integer with $\log \Psi(N_k) \ge 2^k$. Then, the Borel-Cantelli lemma implies that

$$\sum_{n \le N_k} (\xi_n - \xi_n^*) \le \log \Psi(N_k)$$

for almost all f and k large enough. Now, let N be a large enough integer with $N_k \le N < N_{k+1}$. Then,

$$\sum_{n \le N} (\xi_n - \xi_n^*) \le \sum_{n \le N_{k+1}} (\xi_n - \xi_n^*) \le \log \Psi(N_{k+1}) \ll \log \Psi(N_k) \ll \log \Psi(N).$$

Overall, we have shown that for almost all f

$$\sum_{n \le N} \xi_n = \sum_{n \le N} \xi_n^* + \mathcal{O}(\log \Psi(N)).$$

Combining with the above result yields the claim. ∎

We note that Theorem 2 also follows from the method above with only minor modifications. So, what is left is the proof of Corollary 1.

Proof of Corollary 1. For part (i), choose F such that

$$\mathcal{F} = \{C + LD : \text{ monic and } L \in \mathbb{F}_q[T]\}.$$

Then, $\#\mathcal{F}_n = q^n/|D|$ for all $n \ge \deg D$. Consequently,

$$\Psi(N, \mathcal{F}) = \frac{1}{|D|}\Psi(N) + \mathcal{O}(1).$$

For part (ii), it suffices to point out that it is well-known (see Chapter 3 in [1]) that the number of monic, square-free polynomials of degree $n \ge 2$ is given by $q^n - q^{n-1}$. Hence,

$$\Psi(N, \mathcal{F}) = \frac{q-1}{q}\Psi(N) + \mathcal{O}(1).$$

From this the result follows. ∎

4 The ”double-metric” and the other ”single-metric” case

We first turn our attention to the ”double-metric” case. So, in the following, we consider (1) with both f, g random. As before, we define the set

$$F_Q := \{\langle f,g\rangle \in \mathbb{L}\times\mathbb{L} : \langle f,g\rangle \text{ is a solution of (1) with some } P\in\mathbb{F}_q[T]\},$$

where Q is a non-zero polynomial.

As already mentioned in the introduction, this case is much easier than the ”single-metric” case discussed in the previous sections. The reason for this is the second property of the following lemma which was proved in [8].

Lemma 5. *(i) We have,*

$$(m\times m)(F_Q) = \frac{1}{q^{n+l_n}}.$$

(ii) For $Q \neq Q'$, we have

$$(m\times m)(F_Q\cap F_{Q'}) = (m\times m)(F_Q)(m\times m)(F_{Q'}).$$

So, if we define

$$\xi_n := \#\{\langle P,Q\rangle : \langle P,Q\rangle \text{ is a solution of (1)}\},$$

then we again have

$$\xi_n = \sum_{\substack{\deg Q=n\\ Q \text{ monic}}} \mathbf{1}_{F_Q}.$$

However, the above lemma shows that ξ_n considered as a sequence of random variables on the product probability space is pairwise independent. This yields

$$\mathbb{E}\left(\sum_{M\le n\le N}\xi_n - \frac{1}{q^{l_n}}\right)^2 = \sum_{M\le n\le N}\mathrm{Var}(\xi_n) = \sum_{M\le n\le N}\frac{1}{q^{l_n}}\left(1-\frac{1}{q^{n+l_n}}\right) = \sum_{M\le n\le N}\frac{1}{q^{l_n}} + \mathcal{O}(1).$$

Hence, if we assume that

$$\Psi(N) := \sum_{n\le N}\frac{1}{q^{l_n}} \to \infty, \qquad \text{as } N\to\infty,$$

then Lemma 2 directly applies and yields the following result (whose proof in case the above assumption does not hold is trivial).

Theorem 4. *The number of solutions of (1) with $0\le \deg Q\le N$ satisfies*

$$\Psi(N) + \mathcal{O}\left((\Psi(N))^{1/2}(\log\Psi(N))^{3/2+\epsilon}\right), \qquad \textit{a.s.},$$

where $\epsilon > 0$ is an arbitrary constant.

Note that a.s. here means with respect to the product measure $m\times m$.

Finally, we briefly discuss the other ”single-metric” case where the roles of f and g are interchanged. Therefore, assume now that f is fixed and g is random. Here, without proof, we state the following result: for any sequence l_n tending to infinity arbitrarily slowly, there exists an $f\in\mathbb{L}$ such that for almost all g the number of solutions of (1) is finite (see P. Szüsz [12] for the corresponding result in the real number case). Consequently, results of a similar type as in the cases above are impossible in this case.

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A Note on Simultaneous Diophantine Approximation in Positive Characteristic

Michael Fuchs

Abstract

In a recent paper, Inoue and Nakada proved a 0-1 law and a strong law of large numbers with error term for the number of coprime solutions of the one-dimensional Diophantine approximation problem in the field of formal Laurent series over a finite base field. In this note, we generalize their results to higher dimensions.

1 Introduction

Let $\mathbb{F}_q$ be a finite field with q elements and denote by $\mathbb{F}_q((T^{-1}))$ the field of formal Laurent series. For $f \in \mathbb{F}_q((T^{-1}))$ let $|f| = q^{\deg f}$ be the valuation induced by the generalized degree function. Set

$$\mathbb{L} = \{f \in \mathbb{F}_q((T^{-1})) : |f| < 1\}.$$

Then, with the restriction of $|\cdot|$ to $\mathbb{L}$, $\mathbb{L}$ is a compact topological group. Hence, there exists a (unique) translation-invariant probability measure which will be denoted by m.

We are interested in the Diophantine approximation problem

$$\left| f - \frac{P}{Q} \right| < \frac{1}{q^{n+l_n}}, \ \deg Q = n, \ Q \text{ monic}, \ (P, Q) = 1, \tag{1}$$

where $f \in \mathbb{L}$, $P, Q \in \mathbb{F}_q[T]$ with $Q \neq 0$, and l_n is a sequence of non-negative integers (subsequently, we will use $(\cdot, \cdot)$ to denote the $\gcd$, whereas $\langle \cdot, \cdot \rangle$ will be used for pairs).

Concerning the number of solutions of (1), Inoue and Nakada [5] proved the following 0-1 law: the number of solutions is either finite or infinite for almost all $f \in \mathbb{L}$, the latter holding if and only if

$$\sum_{n=0}^{\infty} q^{n-l_n} = \infty.$$

Moreover, the method of proof in [5] also gives a quantitative result under one additional assumption on l_n: if $l_n \geq n$, then the number of solutions of (1) with $\deg Q \leq N$ is given by

$$\left(1 - q^{-1}\right) \Psi(N) + \mathcal{O}\left(\Psi(N)^{1/2} \left(\log \Psi(N)\right)^{3/2+\epsilon}\right),$$

where $\epsilon > 0$ is an arbitrary small constant and $\Psi(N) := \sum_{n \leq N} q^{n-l_n}$.

Key words: formal Laurent series, simultaneous Diophantine approximation, 0-1 law, strong law of large numbers.
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The purpose of this note is to prove generalizations of the above two results to multidimensional Diophantine approximation. Therefore, consider

$$\left|f_j - \frac{P_j}{Q}\right| < \frac{1}{q^{n+l_n^{(j)}}},\ \deg Q = n,\ Q \text{ monic},\ (P_j, Q) = 1,\ j = 1, \dots, d, \tag{2}$$

where $(f_1, \dots, f_d) \in \mathbb{L} \times \cdots \times \mathbb{L}$, $P_j, j = 1, \dots, d, Q \in \mathbb{F}_q[T]$ with $Q \neq 0$, and $l_n^{(j)}, j = 1, \dots, d$ are sequences of non-negative integers. Moreover, set $l_n := \sum_{j=1}^{d} l_n^{(j)}$.

Then, the first result above has the following extension to the multidimensional setting.

Theorem 1. *The number of solutions of (2) is either finite or infinite for almost all $(f_1, \dots, f_d) \in \mathbb{L} \times \cdots \times \mathbb{L}$, the latter holding if and only if*

$$\sum_{n=0}^{\infty} q^{n-l_n} = \infty. \tag{3}$$

Moreover, also the second result admits an extension to higher dimensions.

Theorem 2. *Assume that $l_n \geq n$. Then, for almost all $(f_1, \dots, f_d)$, the number of solutions of (2) with $\deg Q \leq N$ is given by*

$$c_0 \Psi(N) + \mathcal{O}\left(\Psi(N)^{1/2+\epsilon}\right),$$

where $\epsilon > 0$ is an arbitrary small constant, $\Psi(N) := \sum_{n \leq N} q^{n-l_n}$, and

$$c_0 := \sum_{Q_1 \text{ monic}} \cdots \sum_{Q_d \text{ monic}} \frac{\mu(Q_1)}{|Q_1|} \cdots \frac{\mu(Q_d)}{|Q_d|} \frac{1}{|\mathrm{lcm}(Q_1, \dots, Q_d)|} > 0$$

where $\mu(\cdot)$ is the Moebius μ function.

Remark 1. Observe that the error term in the above result for $d = 1$ is weaker than the corresponding one in the result of Inoue and Nakada. The reason for this is that our method is completely different from the approach used by the latter two authors (it is not obvious how to generalize their approach to higher dimensions).

Notation. We will use $[D_1, \dots, D_d]$ to denote the lcm of the polynomials $D_1, \dots, D_d$. All sums will be over monic polynomials. Logarithms in this paper just take on values ≥ 1, i.e. $\log_a x$ should be interpreted as $\max\{\log_a x, 1\}$. We will use both Landau's notation $f(x) = \mathcal{O}(g(x))$ as well as Vinogradov's notation $f(x) \ll g(x)$. Finally, ϵ will denote an arbitrary small positive number whose value might change from one appearance to the next.

2 Proof of Theorem 1

First, note that the necessity of (3) for the number of solutions of (2) being infinite follows from a standard application of the Borel-Cantelli lemma. Hence, we only have to focus on the sufficiency part. For this purpose, we use a slight extension of the d-dimensional Duffin-Schaeffer theorem for formal Laurent series due to Inoue [4].

Theorem 3 (Inoue)**.** *Consider*

$$\left|f_j - \frac{P_j}{Q}\right| < \frac{1}{q^{n+l_Q^{(j)}}},\ \deg Q = n,\ Q \text{ monic},\ (P_j, Q) = 1,\ j = 1, \dots, d, \tag{4}$$

where $(f_1,\dots,f_d)\in\mathbb{L}\times\cdots\times\mathbb{L}$*,* $P_j, j=1,\dots,d, Q$ *with* $Q\neq 0$*, and* $l_Q^{(j)}, j=1,\dots,d$ *are sequences of non-negative integers. Assume that*

$$\sum_Q q^{-l_Q^{(1)}-\cdots-l_Q^{(j)}}=\infty$$

and that for infinitely many N

$$\sum_{\deg Q\le N} q^{-l_Q^{(1)}-\cdots-l_Q^{(j)}}<C\sum_{\deg Q\le N} q^{-l_Q^{(1)}-\cdots-l_Q^{(j)}}\varphi(Q)^d/|Q|^d,$$

where C *is some positive constant. Then, (4) has infinitely many solutions for almost all* $(f_1,\dots,f_d)\in\mathbb{L}\times\cdots\times\mathbb{L}$*.*

Remark 2. Note that the result in [4] is just stated for the special case $l_Q^{(1)}=\ldots=l_Q^{(d)}$. An inspection of the proof, however, shows that the result continues to hold for different approximation functions in every coordinate.

Before we can apply this result, we need a technical lemma.

Lemma 1. *We have,*

$$\sum_{\deg Q=n}\varphi(Q)^d=c_0q^{n(d+1)}+\mathcal{O}\left(q^{n(d+\epsilon)}\right),$$

where c_0 *is as in Theorem 2 and* $\varphi(\cdot)$ *is Euler's totient function.*

Proof. Note that

$$\begin{aligned}\sum_{\deg Q=n}\varphi(Q)^d&=q^{nd}\sum_{\deg Q=n}\left(\sum_{D|Q}\frac{\mu(D)}{|D|}\right)^d\\&=q^{nd}\sum_{\deg Q=n}\sum_{D_1|Q}\cdots\sum_{D_d|Q}\frac{\mu(D_1)}{|D_1|}\cdots\frac{\mu(D_d)}{|D_d|}\\&=q^{nd}\sum_{\deg D_1\le n}\cdots\sum_{\deg D_d\le n}\frac{\mu(D_1)}{|D_1|}\cdots\frac{\mu(D_d)}{|D_d|}\sum_{[D_1,\dots,D_d]|Q,\deg Q=n}1.\end{aligned}$$

The latter sum becomes

$$\sum_{[D_1,\dots,D_d]|Q,\deg Q=n}1=\begin{cases}q^n/|[D_1,\dots,D_d]|, & \text{if } \deg[D_1,\dots,D_d]\le n;\\ 0, & \text{otherwise.}\end{cases}$$

Consequently,

$$\begin{aligned}\sum_{\deg Q=n}\varphi(Q)^d&=q^{n(d+1)}\sum_{\deg D_1\le n}\cdots\sum_{\deg D_d\le n}\frac{\mu(D_1)}{|D_1|}\cdots\frac{\mu(D_d)}{|D_d|}\frac{1}{|[D_1,\dots,D_d]|}\\&\qquad+\mathcal{O}\left(q^{nd}\left(\sum_{\deg D\le n}\frac{1}{|D|}\right)^d\right)\\&=q^{n(d+1)}\sum_{\deg D_1\le n}\cdots\sum_{\deg D_d\le n}\frac{\mu(D_1)}{|D_1|}\cdots\frac{\mu(D_d)}{|D_d|}\frac{1}{|[D_1,\dots,D_d]|}+\mathcal{O}\left(n^dq^{nd}\right).\end{aligned}\tag{5}$$

Next, observe that

$$\Big|\sum_{\deg D_1\le n}\cdots\sum_{\deg D_d\le n}\frac{\mu(D_1)}{|D_1|}\cdots\frac{\mu(D_d)}{|D_d|}\frac{1}{|[D_1,\dots,D_d]|}-c_0\Big|$$
$$\le\sum_{\deg D>n}\sum_{[D_1,\dots,D_d]=D}\frac{1}{|D_1\cdots D_d|\cdot|D|}\le\sum_{\deg D>n}\frac{\omega(D)^d}{|D|^2},$$

where $\omega(D)$ denotes the number of monic divisors of D. Since $\omega(D)=\mathcal{O}(|D|^\epsilon)$ for arbitrary small $\epsilon>0$ (this is proved as for integers; see page 296 in [1]), we obtain

$$\sum_{\deg D>n}\frac{\omega(D)^d}{|D|^2}\ll\sum_{l=n+1}^{\infty}q^{l(\epsilon d-1)}\ll q^{n(\epsilon d-1)}.$$

So, we have

$$\sum_{\deg D_1\le n}\cdots\sum_{\deg D_d\le n}\frac{\mu(D_1)}{|D_1|}\cdots\frac{\mu(D_d)}{|D_d|}\frac{1}{|[D_1,\dots,D_d]|}=c_0+\mathcal{O}\left(q^{n(-1+\epsilon)}\right).$$

Plugging this into (5) yields the claimed expansion.

What is left to show is that $c_0>0$. Therefore, observe that

$$\sum_{\deg Q=n}\varphi(Q)^d\ge\sum_{\deg I=n}\varphi(I)^d=(q^n-1)^d\sum_{\deg I=n}1\gg(q^n-1)^dq^n/n,$$

where the second and third sum runs over all irreducible polynomials and the last bound is well-known. Hence, $c_0>0$ as claimed. ∎

Remark 3. For $d=1$, note that

$$c_0=\sum_Q\frac{\mu(Q)}{|Q|^2}=\prod_I\left(1-\frac{1}{|I|^2}\right)=\left(\sum_Q\frac{1}{|Q|^2}\right)^{-1}=1-\frac{1}{q}.$$

In this situation even more is known, namely,

$$\sum_{\deg Q=n}\varphi(Q)=\left(1-\frac{1}{q}\right)q^{2n}.$$

For a proof of the latter claim e.g. see [5].

Now, we can proof our first main result.

Proof of Theorem 1. As already mentioned before, we only have to show that (3) is sufficient for the number of solutions of (2) being infinity. For this purpose, we just have to check the two conditions in Inoue's result. First, note that since $l_Q^{(1)}+\cdots+l_Q^{(d)}=l_{\deg Q}$, we have

$$\sum_{\deg Q\le N}q^{-l_{\deg Q}}=\sum_{n\le N}q^{n-l_n}$$

and

$$\sum_{\deg Q\le N}q^{-l_{\deg Q}}\varphi(Q)^d/|Q|^d=\sum_{n\le N}q^{-nd-l_n}\sum_{\deg Q=n}\varphi(Q)^d=c_0\sum_{n\le N}q^{n-l_n}+\mathcal{O}\left(\sum_{n\le N}q^{\epsilon n-l_n}\right).$$

Moreover, by Cauchy's inequality

$$\sum_{n\le N}q^{\epsilon n-l_n}\ll\left(\sum_{n\le N}q^{n-l_n}\right)^{1/2}.$$

Hence, both conditions are satisfied and our result follows from Inoue's result. ∎

3 Proof of Theorem 2

We start with a technical lemma.

Lemma 2. *We have,*

$$\sum_{\deg(D_1),\dots,\deg(D_d)\le n} \frac{1}{|[D_1,\dots,D_d]|} \ll q^{n\epsilon}.$$

Proof. First note that

$$\begin{aligned}\sum_{\deg(D_1),\dots,\deg(D_d)\le n} \frac{1}{|[D_1,\dots,D_d]|} &\le \sum_{\deg(D_1),\dots,\deg(D_d)\le n} \frac{1}{|[D_1,\dots,D_d]|^{1-\epsilon}} \\ &\le \sum_{\deg(D_1),\dots,\deg(D_d)\le n} \frac{|([D_1,\dots,D_{d-1}],D_d)|^{1-\epsilon}}{|[D_1,\dots,D_{d-1}]|^{1-\epsilon}\cdot|D_d|^{1-\epsilon}}.\end{aligned}$$

Next we change the order of summation and obtain

$$\begin{aligned}&\sum_{\deg(D_1),\dots,\deg(D_d)\le n} \frac{1}{|[D_1,\dots,D_d]|^{1-\epsilon}} \\ &\le \sum_{\deg D\le n}\ \sum_{D|[D_1,\dots,D_{d-1}],\deg D_i\le n} \frac{1}{|[D_1,\dots,D_{d-1}]|^{1-\epsilon}} \sum_{D|D_d,\deg D_d\le n}\left(\frac{|D|}{|D_d|}\right)^{1-\epsilon} \\ &\le \sum_{\deg D\le n}\ \sum_{D|[D_1,\dots,D_{d-1}],\deg D_i\le n} \frac{1}{|[D_1,\dots,D_{d-1}]|^{1-\epsilon}} \sum_{\deg Q\le n}\frac{1}{|Q|^{1-\epsilon}} \\ &\ll q^{n\epsilon} \sum_{\deg(D_1),\dots,\deg(D_{d-1})\le n} \frac{1}{|[D_1,\dots,D_{d-1}]|^{1-\epsilon}} \sum_{D|[D_1,\dots,D_{d-1}]} 1 \\ &= q^{n\epsilon} \sum_{\deg(D_1),\dots,\deg(D_{d-1})\le n} \frac{\omega([D_1,\dots,D_{d-1}])}{|[D_1,\dots,D_{d-1}]|^{1-\epsilon}}.\end{aligned}$$

Now, as before, we use the estimate $\omega(D)=\mathcal{O}(|D|^{\epsilon})$ for all sufficiently small ϵ. Hence,

$$\sum_{\deg(D_1),\dots,\deg(D_d)\le n} \frac{1}{|[D_1,\dots,D_d]|^{1-\epsilon}} \ll q^{n\epsilon} \sum_{\deg(D_1),\dots,\deg(D_{d-1})\le n} \frac{1}{|[D_1,\dots,D_d]|^{1-2\epsilon}}.$$

Iterating this result proves the claim. ∎

Now, we turn to the proof of Theorem 2. For this purpose, we extend an approach due to Harman (see proof of Theorem 4.4 starting on page 109 in [3]) to higher dimensions.

We first need some notation. Let $\Gamma_1(N)=\lfloor\log_q \Psi(N)^2\rfloor$ and $\Gamma_2(N)=\lfloor\log_q \Psi(N)^4\rfloor$. Moreover, consider the following approximation problem

$$\left|f_j-\frac{P_j}{Q}\right|<\frac{1}{q^{n+l_n^{(j)}}},\ \deg Q=n,\ Q \text{ monic},\ D_j|(P_j,Q),\ \deg(P_j,Q)\le\Gamma_2(N),\ j=1,\dots,d, \quad (6)$$

where $D_1,\dots,D_d$ are fixed monic polynomials. For fixed $(f_1,\dots,f_d)$ and Q denote by $s(Q;D_1,\dots,D_d)$ the number of solutions of (6).

We gather some properties of $s(Q;D_1,\dots,D_d)$ needed below.

Lemma 3. *We have,*

$$\mathbb{E}\left(\sum_{M_1<n\le M_2}\ \sum_{\deg Q=n,[D_1,\dots,D_d]|Q} s(Q;D_1,\dots,D_d)\right) \ll \frac{1}{|D_1\cdots D_d|\cdot|[D_1,\dots,D_d]|} \sum_{M_1<n\le M_2} q^{n-l_n}$$

and

$$\mathbb{E}\left(\sum_{M_1<n\le M_2}\sum_{\deg Q=n,\ [D_1,\dots,D_d]|Q}\left(s(Q;D_1,\dots,D_d)-\frac{1}{|D_1\cdots D_d|}\cdot\frac{1}{q^{l_n}}\right)\right)^2 \ll \frac{\Gamma_2(N)}{|D_1\cdots D_d|\cdot|[D_1,\dots,D_d]|}\sum_{M_1<n\le M_2} q^{n-l_n}$$

for all $M_1 \le M_2$.

Proof. Both properties are easy extensions of the corresponding properties from the case $d = 1$ (see Proposition 3 and Proposition 4 in [2]). For the reader's convenience, we recall the proof of the first property.

Therefore, observe that $s(Q;D_1,\dots,D_d) \le s^*(Q;D_1,\dots,D_d)$ where the latter denotes the number of solutions of (6) with the upper bound on the gcd removed. Of course, $s^*(Q;D_1,\dots,D_d) = 0$ if $[D_1,\dots,D_d] \nmid Q$.

Now, for $[D_1,\dots,D_d]|Q$, note that $s^*(Q;D_1,\dots,D_d) = \mathbf{1}_A$ ($\mathbf{1}_A$ denotes an indicator random variable) with

$$A = \bigcup_{P_j|D_j,\deg P_j<n,1\le j\le d} B\left(P_1/Q; q^{-n-l_n^{(1)}}\right)\times\cdots\times B\left(P_d/Q; q^{-n-l_n^{(d)}}\right),$$

where $B(f;q^{-n})$ denotes the open ball with center f and radius q^{-n} and the above union is disjoint. Since

$$(m\times\cdots\times m)\left(B\left(P_1/Q; q^{-n-l_n^{(1)}}\right)\times\cdots\times B\left(P_d/Q; q^{-n-l_n^{(d)}}\right)\right) = q^{-dn-l_n}$$

and consequently

$$m(A) = \frac{1}{|D_1\cdots D_d|}q^{-l_n},$$

the result follows from elementary properties of the mean. ∎

Next, we prove the following proposition for the number of solutions of (6).

Proposition 1. *For almost all* $(f_1,\dots,f_d)$*, the number of solutions of (6) with* $\deg Q \le N$ *is given by*

$$\frac{1}{|D_1\cdots D_d|\cdot|[D_1,\dots,D_d]|}\Psi(N) + E(N;D_1,\dots,D_d),$$

where the second term satisfies

$$\sum_{\deg(D_1),\dots,\deg(D_d)\le\Gamma_1(N)} E(N;D_1,\dots,D_d) = \mathcal{O}\left(\Psi(N)^{1/2+\epsilon}\right)$$

with $\epsilon > 0$ *an arbitrary small constant.*

Proof. First note that it suffices to prove our claim for the case where $\Psi(N)\to\infty$ as $N\to\infty$ (otherwise, the result is an easy consequence of the Borel-Cantelli lemma). Next, denote by N_k the largest integer with $\Psi(N_k) < k$. It is easy to see that we only have to prove the result for the subsequence N_k.

We are going to need some notation. First, put

$$k = \sum_{j=0}^{l} a_j 2^j, \quad a_l \ne 0,\ a_j\in\{0,1\}\ \forall j.$$

Define the following set

$$S(k) = \left\{ (i,m) : \; a_i = 1, \; m = \sum_{j=i+1}^{l} a_j 2^{j-i} \right\}.$$

Moreover, denote by

$$u_t = u_t(i,m) = \max\left\{ n \in \mathbb{N} : \; \Psi(n) < (m+t)2^i \right\},$$

where $t \in \{0,1\}$. Finally, with the notation of Lemma 3, we put

$$E(i,m;D_1,\ldots,D_d) = \sum_{u_0 < n \leq u_1} \sum_{\deg Q = n, \; [D_1,\ldots,D_d] | Q} \left(s(Q;D_1,\ldots,D_d) - \frac{1}{|D_1 \cdots D_d|} \cdot \frac{1}{q^{l_n}} \right).$$

Then, we obviously have

$$E(N_k;D_1,\ldots,D_d) = \sum_{(i,m)\in S(k)} E(i,m;D_1,\ldots,D_d).$$

Now, set

$$E(l) := \sum_{\deg(D_1),\ldots,\deg(D_d) \leq \Gamma_1(N_{2^{l+1}})} |D_1 \cdots D_d| \sum_{0 \leq i \leq l, \; m < 2^{l-i+1}} E(i,m;D_1,\ldots,D_d)^2.$$

Then, with the estimate from Lemma 3

$$\mathbb{E}E(i,m;D_1,\ldots,D_d)^2 \ll \frac{\Gamma_2(N_{2^{l+1}})}{|D_1 \cdots D_d| \cdot |[D_1,\ldots,D_d]|} \sum_{u_0 < n \leq u_1} q^{n-l_n},$$

we obtain

$$E(l) \ll 2^l l^2 \sum_{\deg(D_1),\ldots,\deg(D_d) \leq \Gamma_1(N_{2^{l+1}})} \frac{1}{|[D_1,\ldots,D_d]|} \ll 2^{l(1+\bar{\epsilon})},$$

where the last step follows from Lemma 2 and $\bar{\epsilon}$ will be chosen below. This in turn implies that

$$P\left(E(l) \geq 2^{l(1+\epsilon)} \right) \ll \frac{1}{2^{l(\epsilon-\bar{\epsilon})}},$$

where we choose $\bar{\epsilon} < \epsilon$. Hence, the Borel-Cantelli lemma yields that

$$E(l) < 2^{l(1+\epsilon)}, \qquad \text{a.s.}$$

for l large enough.

Finally consider

$$\sum_{\deg(D_1),\ldots,\deg(D_d) \leq \Gamma_1(N_k)} E(N_k;D_1,\ldots,D_d)$$
$$\leq \left(\sum_{\deg(D_1),\ldots,\deg(D_d) \leq \Gamma_1(N_k)} \frac{1}{|D_1 \cdots D_d|} \sum_{(i,m)\in S(k)} 1 \right)^{1/2} \cdot (E(r))^{1/2}$$
$$\ll 2^{l(1/2+\epsilon)} l^{d+1} \ll 2^{l(1/2+\epsilon)}.$$

From this the assertion follows. ▮

Now, we can prove our second main result.

Proof of Theorem 2. As in the proof of the proposition, we can assume w.l.o.g. that $\Psi(N) \to \infty$ as $N \to \infty$. Then, we again choose N_k as the largest integer with $\Psi(N_k) < k$. As before, it is easy to see that it is suffices to prove our claim for the sequence N_k.

Next, we introduce the notation $S(N_k; D_1, \dots, D_d)$ for the number of solutions of (6) with $\deg Q \leq N_k$ (here, $(f_1, \dots, f_d)$ is fixed). Then, by an inclusion-exclusion argument, the number of solutions of (2) with $\deg Q \leq N_k$ is given by

$$\sum_{\deg(D_1),\dots,\deg(D_d)\leq\Gamma_2(N_k)} \mu(D_1)\cdots\mu(D_d)S(N_k; D_1, \dots, D_d),$$

where $\mu(\cdot)$ denotes the Moebius function. We split the sum into two parts A and B according to whether there is an D_i with $\deg D_i > \Gamma_1(N_k)$ or not, respectively.

First, we will consider A. Note that

$$\begin{aligned}
\mathbb{E}|A| &\leq \sum_{\substack{\deg(D_1),\dots,\deg(D_d)\leq\Gamma_2(N_k)\\ \deg D_i>\Gamma_1(N_k) \text{ for some } i}} \mathbb{E}S(N_k; D_1, \dots, D_d)\\
&\ll \Psi(N_k) \sum_{\substack{\deg(D_1),\dots,\deg(D_d)\leq\Gamma_2(N_k)\\ \deg D_i>\Gamma_1(N_k) \text{ for some } i}} \frac{1}{|D_1\cdots D_d|}\cdot\frac{1}{|[D_1, \dots, D_d]|}\\
&\ll \Psi(N_k)\left(\sum_{\deg D_1>\Gamma_1(N_k)} \frac{1}{|D_1|^2}\right)\cdot\left(\sum_{\deg D_2\leq\Gamma_2(N_k)} \frac{1}{|D_2|}\right)\cdots\left(\sum_{\deg D_d\leq\Gamma_2(N_k)} \frac{1}{|D_d|}\right)\\
&\ll \frac{(\log\Psi(N_k))^{d-1}}{\Psi(N_k)},
\end{aligned}$$

where we have used Lemma 3. Consequently,

$$P(|A| > (\log\Psi(N_k))^{d+1}) \ll \frac{1}{\Psi(N_k)(\log\Psi(N_k))^2} \ll \frac{1}{k(\log k)^2}.$$

Hence, the Borel-Cantelli lemma implies that for almost all $(f_1, \dots, f_d)$,

$$A = \mathcal{O}((\log\Psi(N_k))^{d+1}).$$

So, in view of our claimed result, the main contribution will come from B. Here, we can use the above proposition and obtain

$$B = \Psi(N_k) \sum_{\deg(D_1),\dots,\deg(D_d)\leq\Gamma_1(N_k)} \frac{\mu(D_1)\cdots\mu(D_d)}{|D_1\cdots D_d|\cdot|[D_1, \dots, D_d]|} + \mathcal{O}\left(\Psi(N_k)^{1/2+\epsilon}\right).$$

Now, as in the proof of Lemma 1

$$\sum_{\deg(D_1),\dots,\deg(D_d)\leq\Gamma_1(N_k)} \frac{\mu(D_1)\cdots\mu(D_d)}{|D_1\cdots D_d|\cdot|[D_1, \dots, D_d]|} = c_0 + \Psi(N_k)^{\epsilon-2}.$$

Combining all the estimates proves the claimed result. ∎

Acknowledgments

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Participation in conferences within NSC 98-2115-M-009-009

by

Michael Fuchs

This is a short report concerning participation in international conferences within my national science counsel project NSC 98-2115-M-009-009.

I participated in the 21st International Meeting on Probabilistic, Combinatorial and Asymptotic Methods for the Analysis of Algorithms which took place in Vienna, Austria from June 28th to July 2nd, 2010. This was the first joint meeting of previously two different conference and seminar series on "Mathematics and Computer Science" and "Analysis of Algorithms". It will be held every year and is the major conference in the area of analysis of algorithms which is currently my main field of interest.

I gave a talk on July 1st, 2010 based on my conference paper entitled "The Variance of Partial Match Retrievals in k-dimensional Bucket Digital Trees" (attached to this report). This paper was accepted after the usual scientific peer review process and is about to be published in one of the forthcoming issues of the journal *Discrete Mathematics and Theoretical Computer Science Proceedings*.

My paper was concerned with digital trees, which was a popular topic at the conference with overall four papers dedicated to it. The others were written by Svante Janson (Uppsala University), Philippe Flajolet (INRIA Rocquencourt) and Stephan Wagner (Stellenbosch University), all experts in the field. Their papers and talks gave me further input for possible future research directions.

The variance for partial match retrievals in k-dimensional bucket digital trees

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received January 13, 2010, *revised May 19, 2010*, *accepted ???*.

The variance of partial match queries in k-dimensional tries was investigated in a couple of papers in the mid-nineties, the resulting analysis being long and complicated. In this paper, we are going to re-derive these results with a much easier approach. Moreover, our approach works for k-dimensional PATRICIA tries, k-dimensional digital search trees and bucket versions as well.

Keywords: k-dimensional digital trees, partial match retrieval, variance, JS-admissibility, Mellin transform

1 Introduction and Results

Data structures for storing and retrieving multidimensional data are of vital importance in several areas of computer science such as design of data base systems and graphics algorithms. One possible class of such data structures was introduced in [10] and is based on digital data, i.e., data which is composed of infinite 0-1 strings. We assume throughout this work that every bit in these strings is generated independently and with the same probability (symmetric Bernoulli model).

We will first describe the above data structure in more details. Therefore, assume that we have given a set of multidimensional data. Then, we apply a " regular shuffling" procedure to transform the multidimensional data into one-dimensional data. Finally, this data is stored in a digital tree. To be more precise, let $R_1, \ldots, R_n$ denote k-dimensional records, i.e.,

$$R_{i,1} = \left(R_{i,1}^{[1]}, R_{i,1}^{[2]}, R_{i,1}^{[3]}, \ldots\right),$$
$$\vdots$$
$$R_{i,k} = \left(R_{i,k}^{[1]}, R_{i,k}^{[2]}, R_{i,k}^{[3]}, \ldots\right).$$

After shuffling, we obtain the one-dimensional string $\tilde{R}_i$

$$\tilde{R}_i = \left(R_{i,1}^{[1]}, \ldots, R_{i,k}^{[1]}, R_{i,1}^{[2]}, \ldots, R_{i,k}^{[2]}, R_{i,1}^{[3]}, \ldots, R_{i,k}^{[3]}, \ldots\right).$$

Then, $\tilde{R}_1, \ldots, \tilde{R}_n$ are used to construct a digital tree.

As digital trees, we use the three standard types (see [8]). First, for the *k-dimensional trie* the underlying digital tree is the trie data structure. For the readers convenience, we recall how a trie is constructed: if we only have one record, then we place it into the root which is considered an external node; if we have more than one record, then the root becomes an (empty) internal node and the records are either directed to the left or to right subtree according to whether their first bit is 0 or 1; finally, the subtrees are build recursively by the same procedure, where the first bits are removed; see Figure 1 for an example.

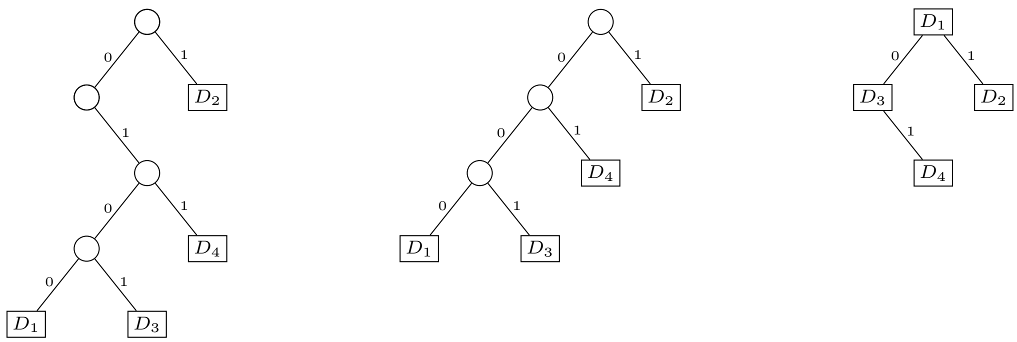

Fig. 1: A 2-dimensional trie, PATRICIA trie and digital search tree built from the data

data	D_1	D_2	D_3	D_4
$R_{i,1}$	$0010\cdots$	$1001\cdots$	$0001\cdots$	$0111\cdots$
$R_{i,2}$	$1000\cdots$	$1010\cdots$	$1101\cdots$	$1011\cdots$

A variant of k-dimensional tries are *k-dimensional PATRICIA tries*, where the shuffled data is stored in a PATRICIA trie. Recall that a PATRICIA trie is constructed as a trie only multiple one-way branching is suppressed; again see Figure 1 for an example. This yields a more balanced tree improving the overall performance of tries.

A final type is given by the *k-dimensional digital search tree* which is based on the digital search tree data structure. Recall that a digital search tree is constructed as follows: the first record is placed in the root; all other records are directed to the left or right subtree according to whether their first bit is 0 or 1; finally, the subtrees are constructed by the same principle again with the first bits removed; see Figure 1 for an example. So, in contrast to tries and PATRICIA tries, no distinction between internal and external nodes is necessary for digital search trees.

Note that all trees introduced above have the common feature that nodes only hold at most one record. If we allow nodes to hold up to $b \geq 1$ records, then the resulting trees are called *k-dimensional bucket digital trees*.

In this paper, we will study the cost of partial match retrievals in k-dimensional bucket digital trees. Here, a partial match query will ask for the retrieval of all records matching certain criteria. Formally, a partial match query is a k-dimensional vector $R = (R_1, \ldots, R_k)$ with some of its coordinates a string of

0-1 bits and others unspecified. For instance, for $k = 2$, we might have

$$R_1 = (\star, \star, \star, \star, \ldots)\,,$$
$$R_2 = (0, 1, 1, 0, \ldots)\,,$$

Then, a shuffled record $\tilde{R}$ is produced as before. For the above example this yields

$$\tilde{R} = (\star, 0, \star, 1, \star, 1, \star, 0, \ldots)\,.$$

The partial match query asks now for the retrieval of all data in the tree with $\tilde{R}$ being used as search query. Here, 0 or 1 in $\tilde{R}$ means either going to the left or right subtree of the current node, whereas $\star$ means that we have to proceed with our search in both subtrees. The cost of such a partial match query will be measured by the number of nodes visited (where we only consider internal nodes for tries and PATRICIA tries); so for the query $\tilde{R}$ above, we have a cost of 2 for the trie and Patricia trie of Figure 1 and a cost of 3 for the digital search tree.

Before going on, we will fix some notation which we are going to use throughout the work. First, it should be clear that under our random model, the cost of a partial match query only depends on the *partial match pattern* q which is a k-tuple of symbols from $\{S, \star\}$, where the i-th coordinate is S if R_i is specified and $\star$ otherwise, We will fix such a q throughout this work and denote the number of $\star$ entries in q by u, where we assume that $0 < u < k$. Furthermore, we will consider cyclic shifts of the entries of q by one position to the left which will be denoted by q'; more generally, $q^{(l)}$ will denote the cyclic shift of the entries of q by l position to the left. Also, we will associate to a partial match pattern q an infinite sequence $(\delta_1, \delta_2, \delta_3, \ldots)$ with $\delta_i = 1$ if $q_{i \bmod k} = S$ and $\delta_i = 2$ otherwise. Finally, we denote the random variable describing the cost of the partial match query by $X_{q,n}$ for all three types of bucket digital trees of size n and bucket size $b \geq 1$ (for the sake of simplicity, we suppress the index b).

In this paper, we will be concerned with stochastic properties of $X_{q,n}$. Therefore, let us recall what is known about this random variable. First, the mean value of $X_{q,n}$ for k-dimensional tries was investigated in [2] where the authors proved that

$$\mathbb{E}(X_{q,n}) \sim n^{u/k} P_1(\log_2 n^{1/k})$$

with $P_1(z)$ a one-periodic function whose Fourier expansion was given in [2] as well (for a comparison of this result with other data structure for multidimensional search see [8] and [10]). Similar results were subsequently proved in [6] for k-dimensional bucket digital tries, k-dimensional PATRICIA tries, and k-dimensional digital search trees, too. As for the variance, it was conjectured in [7] that for k-dimensional tries

$$\mathrm{Var}(X_{q,n}) \sim n^{u/k} P_2(\log_2 n^{1/k}) \tag{1}$$

with $P_2(z)$ a one-periodic function. The authors proved this conjecture for $k = 2$ in [7]. The general case was then settled in [11]. Note that this result implies that $X_{q,n}/\mathbb{E}(X_{q,n})$ converges to 1 in probability. Hence, the distribution of $X_{q,n}$ is concentrated around its mean.

As for the method of proof of (1), the authors of [7] applied the analytic approach from [2] to derive asymptotic expansions of mean and second moment. Then, they used these expansions to compute the variance, where they had to cope with highly non-trivial cancellations. Here, their proof crucially rested on an identity of Ramanujan which only works in the case $k = 2$ and does not seem to have an analogue for $k > 2$. In [11], a new and mainly elementary approach was devised to settle the general case.

In this paper, we will re-derive the above result with a more simpler approach. Our approach will be analytic and use some standard tools from the analysis of algorithms (the same approach was already used in other contexts; see [4],[5],[12]). The crucial difference to the approach from [7] is that we incorporate the cancellations at a much earlier stage, making the resulting analysis more easier. Moreover, our approach will work for k-dimensional bucket tries as well.

Before explaining our approach in more details, we are going to state our result. Therefore, we need some notation. Set

$$\tilde{h}_q(z) = 2\delta_1 e_b(z)e^{-z}\tilde{L}_{q'}(z/2) + e_b(z)e^{-z} - e_b(z)^2e^{-2z},$$

where $e_b(z) = 1 + z + z^2/2! + \cdots + z^b/b!$ and $\tilde{L}_q(z) = \exp\{-z\}\sum_{n\geq 0}\mathbb{E}(X_{q,n})z^n/n!$.

Theorem 1 *The cost of a partial match query with u non-specified coordinates in a k-dimensional trie of size n satisfies*

$$\mathrm{Var}(X_{q,n}) = n^{u/k}P_2(\log_2 n^{1/k}) + \mathcal{O}(n^{2u/k-1})$$

with one-periodic function

$$P_2(z) = \sum_{r=-\infty}^{\infty} c_r e^{2\pi i r z}$$

and Fourier coefficients

$$c_r = \frac{1}{kL}\sum_{l=0}^{k-1}\delta_1\cdots\delta_l 2^{-\omega_r l}\int_0^\infty z^{-\omega_r-1}\tilde{h}_{q^{(l)}}(z)\mathrm{d}z,$$

where $\omega_r = u/k + 2\pi i r/(kL)$ and $L = \log 2$.

With slightly more work, the Fourier coefficients can be further simplified.

Corollary 1 *The Fourier coefficient in the above theorem can be expressed as*

$$\begin{aligned} c_r = \frac{\Gamma(-\omega_r)}{kL}\Bigg(&\delta(2^{-\omega_r})\left(\binom{-\omega_r+b}{b} - 2^{\omega_r}\sum_{j_1,j_2=0}^{b}\binom{j_1+j_2}{j_1}\binom{-\omega_r+j_1+j_2}{j_1+j_2}2^{-j_1-j_2}\right) \\ &- \sum_{l\geq b+1}\binom{-l+b}{b}\binom{-\omega_r+l+b}{b}\binom{\omega_r}{l}\frac{2^{1-l}\sigma(2^{-\omega_r},2^{-l}))}{1-2^{-lk+u}}\Bigg), \end{aligned} \qquad (2)$$

where

$$\delta(z) = \sum_{j=0}^{k-1}\delta_1\cdots\delta_j z^j, \qquad \sigma(z_1,z_2) = \sum_{j_1,j_2=0}^{k-1}\delta_1\cdots\delta_{j_1+j_2+1}z_1^{j_1}z_2^{j_2}.$$

For instance, for $k = 2$, $s = 1$, and $q = (\star, S)$, the value of c_0 becomes

$$\frac{(1+\sqrt{2})\sqrt{\pi}}{2\ln 2}\left(\frac{7\sqrt{2}}{8} - 1 - 4\sqrt{2}\sum_{l\geq 2}\binom{1/2}{l}\frac{(l-1)(l+1/2)2^{-l}}{1-2^{-l+1/2}}\right) \approx 2.09184\cdots,$$

where the last approximation was computed with Maple. This value coincides with the one given in [7]. Note that the expression given in the latter paper is slightly different; we leave it as an exercise to the reader to show that they are the same.

Next, our approach can also be straightforwardly applied to k-dimensional bucket PATRICIA tries. Here, we have the same result as above only $\tilde{h}_q(z)$ replaced by

$$\begin{aligned}\tilde{h}_q(z) =& 2\delta_1(e_b(z)e^{-z} - \delta_1 e_b(z/2)e^{-z} + (\delta_1 - 1)e^{-z/2})\tilde{L}_{q'}(z/2)\\ &+ e_b(z)e^{-z} - \delta_1 e_b(z/2)e^{-z} + \delta_1 e^{-z/2} - (e_b(z)e^{-z} - \delta_1 e_b(z/2)e^{-z} + \delta_1 e^{-z/2})^2.\end{aligned}$$

Also, a similar explicit expression for the Fourier coefficients as in Corollary 1 can be given. Since, the resulting formula is more messy we do not give details.

Finally, k-dimensional bucket digital search trees are slightly more involved. Here, we will use a variant of the above approach which was introduced in [3]. In order to state our result, we again need some notation. Therefore, set

$$Q(s) = \prod_{j\geq 1}\left(1 - \frac{s}{2^j}\right), \qquad Q_l = \prod_{j=1}^{l}\left(1 - 2^{-j}\right)$$

and

$$\tilde{h}_q(z) = \left(\sum_{j=0}^{b}\binom{b}{j}\tilde{L}_q^{(j)}(z)\right)^2 - \sum_{j=0}^{b}\binom{b}{j}\left(\tilde{L}_q(z)^2\right)^{(j)}.$$

Theorem 2 *The cost of a partial match query with u non-specified coordinates in a k-dimensional digital search tree of size n satisfies*

$$\mathrm{Var}(X_{q,n}) = n^{u/k}P_2(\log_2 n^{1/k}) + \mathcal{O}(n^{2u/k-1})$$

with one-periodic function

$$P_2(z) = \sum_{r=-\infty}^{\infty} c_r e^{2\pi i r z}$$

and Fourier coefficients

$$c_r = \frac{1}{kL\Gamma(1+\omega_r)}\sum_{l=0}^{k-1}\delta_1\cdots\delta_l 2^{-\omega_r l}\int_0^{\infty}\frac{s^{\omega_r}}{Q(-2s)^b}\left(\int_0^{\infty}e^{-zs}\tilde{h}_{q^{(l)}}(z)\mathrm{d}z + p(s)\right)\mathrm{d}s, \qquad (3)$$

where

$$p(s) = \frac{(1+s)^{b-1} + (-1)^b}{s+2}.$$

Moreover, the Fourier coefficients can be further simplified here, too. We will state the result for $b = 1$. Therefore, set

$$\varphi(\omega; x) = \begin{cases}\pi(x^{\omega} - 1)/(\sin(\pi\omega)(x-1)) & \text{if } x \neq 1;\\ \pi\omega/\sin(\pi\omega), & \text{if } x = 1.\end{cases}$$

Then, we have the following corollary.

Corollary 2 *If the bucket size equals one, then the Fourier coefficient in the above theorem can be expressed as*

$$c_r = \frac{1}{kLQ(1)\Gamma(1+\omega_r)} \sum_{l=0}^{k-1} \delta_1 \cdots \delta_l 2^{-\omega_r l} \sum_{j_1,j_2,j_3\geq 0} \frac{(-1)^{j_1} \bar{\delta}_{q^{(l)},j_2} \bar{\delta}_{q^{(l)},j_3} 2^{-\binom{j_1}{2}+(1-\omega_r)j_1}}{2^{j_2+j_3} Q_{j_1} Q_{j_2} Q_{j_3}} \varphi(\omega_r; 2^{j_1-j_2} + 2^{j_1-j_3}),$$

where

$$\bar{\delta}_{q,j} = \sum_{l\geq 0} \frac{(-1)^l 2^{-\binom{l+1}{2}}}{Q_l} \prod_{h=1}^{l+j} \delta_h.$$

We conclude the introduction by giving a short sketch of the paper. In the next section, we will treat k-dimensional bucket tries. Then, in Section 3, we will briefly discuss k-dimensional bucket PATRICIA tries. Finally, in Section 5, we will prove the results for k-dimensional bucket digital search trees.

2 k-dimensional Bucket Tries

Note that from the definition of k-dimensional bucket tries, we have

$$X_{q,n} \stackrel{d}{=} \begin{cases} X_{q',I_n} + X^*_{q',n-I_n} + 1, & \text{if } q = (\star, \ldots); \\ X_{q',I_n} + 1, & \text{if } q = (S, \ldots), \end{cases} \qquad (n \geq b+1),$$

where $I_n = \text{Binom}(n, 1/2)$, (X_n^*) is an independent copy of (X_n) with $X_n \stackrel{d}{=} X_n^*$, and $X_{q,0} = X_{q,1} = \cdots = X_{q,b} = 0$.

From this recurrence we will proceed as follows. First, we are going to apply the poissonization-depoissonization procedure from [5]. This will allow us to entirely focus on the Poisson model. Next, we will define a *poissonized variance* which is not really a variance, but asymptotically behaves like one (this idea was probably first used in [4]). This will be the crucial step leading to a much more simplified derivation. The remaining analysis is then carried out by using Mellin transform, a standard tool from the analysis of algorithm (for an excellent introduction see [1]).

Poissonization. Let $\tilde{P}_q(z,y)$ denote the Poisson generating function of $\mathbb{E}(\exp\{X_{q,n}y\})$, i.e.,

$$\tilde{P}_q(z,y) = e^{-z} \sum_{n\geq 0} \mathbb{E}(e^{X_n y}) \frac{z^n}{n!}.$$

Then, we obtain from the above distributional recurrence

$$\tilde{P}_q(z,y) = e^y \tilde{P}_{q'}(z/2,y)^{\delta_1} + e_b(z) e^{-z}(1-e^y).$$

Next, by taking first and second derivatives with respect to y and setting $y=0$, we obtain the following functional equation for the Poisson generating function of the mean (denoted by $\tilde{L}_q(z)$)

$$\tilde{L}_q(z) = \delta_1 \tilde{L}_{q'}(z/2) + 1 - e_b(z) e^{-z}$$

and for the Poisson generating function of the second moment (denoted by $\tilde{M}_q(z)$)

$$\tilde{M}_q(z) = \begin{cases} 2\tilde{M}_{q'}(z/2) + 4\tilde{L}_{q'}(z/2) + 2\tilde{L}_{q'}(z/2)^2 + 1 - e_b(z)e^{-z}, & \text{if } q = (\star, \ldots); \\ \tilde{M}_{q'}(z/2) + 2\tilde{L}_{q'}(z/2) + 1 - e_b(z)e^{-z}, & \text{if } q = (S, \ldots). \end{cases}$$

Going from these Poisson generating functions back to the original quantity is done via the depoissonization tools from [5]. We will use here the language from [3], where we coined the term Jacquet-Szpankowski admissibility (or JS-admissibility for short). Recall that $\tilde{f}(z)$ is called JS-admissible if the following two conditions hold (where here and throughout this work, ϵ will denote a small constant whose value might change from one appearance to the next).

(I) There exists an $\alpha \in \mathbb{R}$ such that uniformly for $|\arg(z)| \le \epsilon$

$$\tilde{f}(z) = \mathcal{O}\left(|z|^{\alpha}\right).$$

(O) We have, uniformly for $\epsilon \le |\arg(z)| \le \pi$,

$$f(z) := e^z \tilde{f}(z) = \mathcal{O}\left(e^{(1-\epsilon)|z|}\right).$$

The importance of this notation is due to the following proposition which is proved by a standard application of the saddle point method (see [5] for many more such results).

Proposition 1 *Let $\tilde{f}(z)$ be the Poisson generating function of f_n. If $\tilde{f}(z)$ is JS-admissibility, then*

$$f_n = \sum_{0 \le j < 2l} \frac{\tilde{f}^{(j)}(n)}{j!} \tau_j(n) + \mathcal{O}(n^{\alpha - l})$$

with $\tau_j(n) = n![z^n](z-n)^j e^z$

In our context, JS-admissible is easily checked via the following result.

Proposition 2 *Assume that we have*

$$\tilde{f}_{q^{(l)}}(z) = \delta_{l+1} \tilde{f}_{q^{(l+1)}}(z/2) + \tilde{g}_{q^{(l)}}(z), \qquad (0 \le l < k),$$

where all involved functions are entire. Moreover, assume that $\tilde{g}_{q^{(l)}}(z)$ is JS-admissible for $0 \le l < k$. Then, $\tilde{f}_{q^{(l)}}(z)$ is JS-admissible for $0 \le l < k$.

Proof: We only show how to prove **(I)**. Therefore, we start by iterating the recurrence. This yields

$$\tilde{f}_q(z) = 2^u \tilde{f}_q(z/2^k) + \sum_{l=0}^{k-1} \delta_1 \cdots \delta_l \tilde{g}_{q^{(l)}}(z/2^l).$$

Now set

$$\tilde{B}_q(r) := \max_{|z|=r, |\arg(z)| \le \epsilon} |\tilde{f}_q(z)|.$$

Then, by the assumptions, we obtain

$$\tilde{B}_q(r) \leq 2^u \tilde{B}_z(r/2^k) + \mathcal{O}\left(r^\alpha\right).$$

Next, we define $\tilde{K}_q(r)$ by

$$\tilde{K}_q(r) = 2^u \tilde{K}_q(r/2^k) + \mathcal{O}\left(r^\alpha\right).$$

Then, $\tilde{B}_q(r) \leq \tilde{K}_q(r)$. Moreover, we immediately obtain by iteration

$$\tilde{K}_q(r) = \begin{cases} r^\alpha, & \text{if } \alpha > u/k; \\ r^{u/k}\log r, & \text{if } \alpha = u/k; \\ r^{u/k}, & \text{if } \alpha < u/k. \end{cases}$$

This proves our claim. □

Using this result together with the closure properties from [3] proves that both $\tilde{L}_q(z)$ and $\tilde{M}_q(z)$ are JS-addmissible. Also, note that we have

$$\tilde{L}_q(z) = \mathcal{O}\left(|z|^{u/k}\right), \qquad \tilde{M}_q(z) = \mathcal{O}\left(|z|^{2u/k}\right) \tag{4}$$

uniformly as $|z| \to \infty$ and $|\arg(z)| \leq \epsilon$.

Next, we define the poissonized variance as

$$\tilde{V}_q(z) = \tilde{M}_q(z) - \tilde{L}_q(z)^2.$$

Then, by a straightforward computation

$$\tilde{V}_q(z) = \delta_1 \tilde{V}_{q'}(z/2) + \tilde{h}_q(z),$$

where $\tilde{h}_q(z)$ was defined in the introduction.

Note that $\tilde{V}_q(z)$ is not the Poisson generating function of a variance but only mimicks the defintion of the variance. However, it behaves asymptotically like the variance as proved in the following proposition (see also Theorem 6 in [5]).

Proposition 3 *As $n \to \infty$,*

$$\mathrm{Var}(X_{q,n}) = \tilde{V}_q(n) + \mathcal{O}\left(n^{2u/k-1}\right).$$

Proof: From Proposition 2 and (4), we have

$$\begin{aligned} \mathrm{Var}(X_{q,n}) &= \mathbb{E}(X_{q,n}^2) - (\mathbb{E}(X_{q,n}))^2 \\ &= \tilde{M}_q(n) + \mathcal{O}\left(n^{2u/k-1}\right) - \left(\tilde{L}_q(n) + \mathcal{O}\left(n^{u/k-1}\right)\right)^2 \\ &= \tilde{V}_q(n) + \mathcal{O}\left(n^{2u/k-1}\right). \end{aligned}$$

This proves the claim. □

Asymptotic Expansion of $\tilde{L}_q(z)$. We will first look at the mean value (in the Poisson model) which is needed in the proof of Corollary 1. Therefore, by using iteration as in the proof of Proposition 2, we obtain

$$\tilde{L}_q(z) = 2^u \tilde{L}_q(z/2^k) + \sum_{l=0}^{k-1} \delta_1 \cdots \delta_l \tilde{g}_{q^{(l)}}(z/2^l),$$

where $\tilde{g}_{q^{(l)}}(z) = 1 - e_b(z)e^{-z}$. Our goal is to derive an asymptotic expansion of $\tilde{L}_q(z)$. A standard tool for that purpose is the Mellin transform which we are going to apply next.

First, we have to clarify existence of the Mellin transform of $\tilde{L}_q(z)$. Therefore, note that by (4) and the trivial bound $\tilde{L}_q(z) = \mathcal{O}\left(z^{b+1}\right)$ as $z \to 0$. Hence, the Mellin transform of $\tilde{L}_q(z)$ exists in the strip $\langle -b-1, -u/k \rangle$. Applying Mellin transform to the above functional equation then yields

$$\mathscr{M}[\tilde{L}_q(z); \omega] = \frac{-\Gamma(w)}{1 - 2^{\omega k + u}} \binom{w+b}{b} \sum_{l=0}^{k-1} \delta_1 \cdots \delta_l 2^{\omega l}, \qquad \Re(\omega) \in \langle -b-1, -u/k \rangle. \tag{5}$$

Moreover, by inverse Mellin transform and shifting the line of integration to the right (see the converse mapping theorem in [1]), we have

$$\tilde{L}_q(z) \sim z^{u/k} P_1(\log_2 z^{1/k}), \qquad (z \to \infty), \tag{6}$$

where

$$P_1(z) = \sum_{r=-\infty}^{\infty} c_r e^{2\pi i r z}, \qquad c_r = \frac{-\Gamma(-\omega_r)}{kL} \binom{-\omega_r + b}{b} \sum_{l=0}^{k-1} \delta_1 \cdots \delta_l 2^{-\omega_r l}.$$

Note that due to the fast decay of (5) along vertical lines, (6) more generally holds uniformly for $|z| \to \infty$ and $|\arg(z)| \le \pi/2 - \epsilon$.

Asymptotic Expansion of $\tilde{V}_q(n)$. Here, we proceed as for the mean. First, by using iteration as above and applying Mellin transform to the resulting functional equation, we have

$$\mathscr{M}[\tilde{V}_q(z); \omega] = \frac{1}{1 - 2^{\omega k + u}} \sum_{l=0}^{k-1} \delta_1 \cdots \delta_l 2^{\omega l} \mathscr{M}[\tilde{h}_{q^{(l)}}(z); \omega], \qquad \Re(\omega) \in \langle -b-1, -u/k \rangle.$$

Now, from (4), we have that as $z \to \infty$

$$\tilde{h}_{q^{(l)}}(z) = \mathcal{O}(z^{-\beta})$$

for any $\beta > 0$ and $0 \le l < k$. Obviously, $\tilde{h}_q^{(l)}(z) = \mathcal{O}(z^{b+1})$ as $z \to 0$ for $0 \le l < k$. Hence, the Mellin transform of $\tilde{h}_{q^{(l)}}(z)$ exists in the strip $\langle -b-1, \infty \rangle$. Our claimed result follows from this by inverse Mellin transform and shifting the line of integration to the right.

Simplification of the Fourier Coefficients. The main task is the evaluation of

$$\int_0^{\infty} z^{-\omega_r - 1} \left(2\delta_{l+1} e_b(z) e^{-z} \tilde{L}_{q^{(l+1)}}(z/2) + e_b(z) e^{-z} - e_b(z)^2 e^{-2z} \right) \mathrm{d}z.$$

Therefore, we concentrate on

$$\int_0^\infty z^{-\omega_r-1}e_b(z)e^{-z}\tilde{L}_{q^{(l+1)}}(z/2)\mathrm{d}z,$$

the remaining parts being easy. Note that due to (5), we have

$$\mathscr{M}[\tilde{L}_{q^{(l+1)}}(z/2);\sigma]=\frac{-2^\sigma\Gamma(\sigma)}{1-2^{\sigma k+u}}\binom{\sigma+b}{b}\delta_{q^{(l+1)}}(2^\sigma),$$

where

$$\delta_{q^{(l+1)}}(z)=\sum_{j=0}^{k-1}\delta_{l+2}\cdots\delta_{l+j+1}z^j.$$

Now, by inverse Mellin transform

$$\begin{aligned}\int_0^\infty & z^{-\omega_r-1}e_b(z)e^{-z}\tilde{L}_{q^{(l+1)}}(z/2)\mathrm{d}z\\ &=\int_{(-b)}\frac{-2^\sigma\Gamma(\sigma)}{1-2^{\sigma k+u}}\binom{\sigma+b}{b}\delta_{q^{(l+1)}}(2^\sigma)\int_0^\infty z^{-\omega_r-\sigma-1}e_b(z)e^{-z}\mathrm{d}z\mathrm{d}\sigma\\ &=\int_{(-b)}\binom{\sigma+b}{b}\binom{-\omega_r-\sigma+b}{b}\frac{-2^\sigma\Gamma(\sigma)\Gamma(-\omega_r-\sigma)}{1-2^{\sigma k+u}}\delta_{q^{(l+1)}}(2^\sigma)\mathrm{d}\sigma,\end{aligned}$$

where the outer integral is along the line $\Re(\sigma)=-b$. Finally, by shifting the line of integration to the left and collecting residues, we obtain the absolute convergent series

$$\begin{aligned}\int_0^\infty & z^{-\omega_r-1}e_b(z)e^{-z}\tilde{L}_{q^{(l+1)}}(z/2)\mathrm{d}z\\ &=-\Gamma(-\omega_r)\sum_{l\ge b+1}\binom{-l+b}{b}\binom{-\omega_r+l+b}{b}\binom{\omega_r}{l}\frac{2^{-l}\delta_{q^{(l+1)}}(2^{-l})}{1-2^{-lk+u}}.\end{aligned}$$

Collecting everything and standard computation yields the claimed result.

3 k-dimensional Bucket PATRICIA Tries

Here, from the definition of Patricia tries, we have for $q=(\star,\ldots)$

$$X_{q,n}=\begin{cases}X_{q',I_n}+X^*_{q',n-I_n}, & \text{if } I_n\in\{0,n\},\\ X_{q',I_n}+X^*_{q',n-I_n}+1, & \text{otherwise,}\end{cases}\qquad (n\ge b+1)$$

and for $q=(S,\ldots)$

$$X_{q,n}=\begin{cases}X_{q',I_n}, & \text{if } I_n=n,\\ X_{q',I_n}+1, & \text{otherwise,}\end{cases}\qquad (n\ge b+1),$$

where notation and initial conditions are as in the previous section.

From this we then obtain for the Poisson generating function of the mean (with notation as before)

$$\tilde{L}_q(z) = \delta_1 \tilde{L}_{q'}(z/2) + 1 + \tilde{g}_q(z)$$

where

$$\tilde{g}_q(z) = -e_b(z)e^{-z} + \delta_1 e_b(z/2)e^{-z} - \delta_1 e^{-z/2},$$

and the Poisson generating function of the second moment

$$\tilde{M}_q(z) = \begin{cases} 2\tilde{M}_{q'}(z/2) + 2\tilde{L}_{q'}(z/2)^2 + 4(1-e^{-z/2})\tilde{L}_{q'}(z/2) + 1 + \tilde{g}_q(z), & \text{if } q = (\star, \dots); \\ \tilde{M}_{q'}(z/2) + 2(1-e^{-z/2})\tilde{L}_{q'}(z/2) + 1 + \tilde{g}_q(z), & \text{if } q = (S, \dots). \end{cases}$$

Moreover, we have for the poissonized variance

$$\tilde{V}_q(z) = \delta_q \tilde{V}_{q'}(z/2) + \tilde{h}_q(z),$$

where $\tilde{h}_q(z)$ was defined in the introduction. The remaining analysis now proceeds from these functional equational equations as in the previous section.

4 k-dimensional Bucket Digital Search Trees

Again, we start from a distributional recurrence for $X_{q,n}$ which for the current situation reads as follows

$$X_{q,n+b} \stackrel{d}{=} \begin{cases} X_{q',I_n} + X^*_{q',n-I_n} + 1, & \text{if } q = (\star, \dots); \\ X_{q',I_n} + 1, & \text{if } q = (S, \dots), \end{cases} \qquad (n \geq 0),$$

where the notation is as before and initial conditions are given by $X_{q,0} = 0$ and $X_{q,1} = \cdots = X_{q,b-1} = 1$.

From here, we can in principle proceed as before. However, we will see that the equation satisfied by the Poisson generating function is more complicated. More precisely, we have to cope with a differential-functional equation compared with the functional equation from the trie case. Here, we will first use Laplace transform to get rid of the differential operator. Then, after suitable normalization, we will be able to proceed as before. This combined use of Laplace and Mellin transform was introduced in [3] and we direct the interested reader to that paper for more details concerning technicalities.

Poissonization. We again define

$$\tilde{P}_q(z,y) = e^{-z} \sum_{n \geq 0} \mathbb{E}(e^{X_{q,n} y}) \frac{z^n}{n!}.$$

Then,

$$\sum_{j=0}^{b} \binom{b}{j} \tilde{P}_q(z,y) = e^y \tilde{P}_{q'}(z/2,y)^{\delta_1}.$$

Taking derivatives yields for the Poisson generating function of mean and second moment (denoted as before)

$$\sum_{j=0}^{b} \binom{b}{j} \tilde{L}_q^{(j)}(z) = \delta_1 \tilde{L}_{q'}(z/2) + 1 \qquad (7)$$

and

$$\sum_{j=0}^{b}\binom{b}{j}\tilde{M}_q^{(j)}(z)=\begin{cases}2\tilde{M}_{q'}(z/2)+4\tilde{L}_{q'}(z/2)+2\tilde{L}_{q'}(z/2)^2+1, & \text{if } q=(\star,\dots);\\ \tilde{M}_{q'}(z/2)+2\tilde{L}_{q'}(z/2)+1, & \text{if } q=(S,\dots).\end{cases}$$

The first step is again to show that $\tilde{L}_q(z)$ and $\tilde{M}_q(z)$ are JS-admissible. Therefore, we need the following result which is proved by a reduction to the trie case (see [3] for similar results).

Proposition 4 *Assume that we have*

$$\sum_{j=0}^{b}\binom{b}{j}\tilde{f}_{q^{(l)}}^{(j)}(z)=\delta_{l+1}\tilde{f}_{q^{(l+1)}}(z/2)+\tilde{g}_{q^{(l)}}(z),\qquad(0\le l<k),$$

where all involved functions are entire and 0 *at* $z=0$*. Moreover, assume that* $\tilde{g}_{q^{(l)}}(z)$ *is JS-admissible for* $0\le l<k$*. Then,* $\tilde{f}_{q^{(l)}}(z)$ *is JS-admissible for* $0\le l<k$*.*

From this it then follows as in the trie case that $\tilde{L}_q(z)$ and $\tilde{M}_q(z)$ are JS-admissible.

Next, we consider the poissonized variance $\tilde{V}_q(z)=\tilde{M}_q(z)-\tilde{L}_q(z)^2$. An easy computation proves that

$$\sum_{j=0}^{b}\binom{b}{j}\tilde{V}^{(j)}(z)=\delta_1\tilde{V}_{q'}(z/2)+\tilde{h}_q(z),$$

where $\tilde{h}_q(z)$ was defined in the introduction. Then, from the JS-admissibility of $\tilde{L}_q(z)$ and $\tilde{M}_q(z)$, we obtain as for tries the following result.

Proposition 5 *As* $n\to\infty$*,*

$$\mathrm{Var}(X_{q,n})=\tilde{V}_q(n)+\mathcal{O}\left(n^{2u/k-1}\right).$$

Asymptotic Expansion of $\tilde{L}_q(z)$. Again, we first consider the mean value. Note that due to the differential operator it is not possible to iterate (7). Therefore, we first have to get rid of the differential operator which is achieved by applying Laplace transform. This yields

$$(s+1)^b\mathscr{L}[\tilde{L}_q(z);s]=2\delta_1\mathscr{L}[\tilde{L}_{q'}(z);2s]+(s+1)^{b-1}/s. \tag{8}$$

Next, we normalize with $Q(s)$ from the introduction. Therefore, set $\bar{L}_q(s)=\mathscr{L}[\tilde{L}_q(z);s]/Q(-s)^b$ and $\bar{G}(s)=(s+1)^{b-1}/(Q(-2s)^b s)$. Then,

$$\bar{L}_q(s)=2\delta_1\bar{L}_{q'}(2s)+\bar{G}(s).$$

Now, we can iterate and obtain

$$\bar{L}_q(s)=2^{k+u}\bar{L}_q(2^k s)+\sum_{l=0}^{k-1}2^l\delta_1\cdots\delta_l\bar{G}(2^l s).$$

Observe that this is a similar functional equation as in the trie case. Hence, we can proceed as before. Thus, we again apply Mellin transform. First, note that the Mellin transform of $\bar{L}_q(s)$ exists in a non-trivial strip. Moreover, due to the rapid growth of $Q(s)$ at infinity (see [3]), the Mellin transform of $\bar{G}(s)$

exists in the strip $\langle 1, \infty\rangle$. Applying Mellin transform yields

$$\mathscr{M}[\bar{L}_q(s);\omega] = \frac{\mathscr{M}[\bar{G}(s);\omega]}{1-2^{k-\omega k+u}}\sum_{l=0}^{k-1}\delta_1\cdots\delta_l 2^{l-\omega l}, \qquad \Re(\omega)\in\langle 1+u/k,\infty\rangle.$$

Next, by inverse Mellin transform and shifting the line of integration to the left, we obtain

$$\bar{L}_q(z) \sim \sum_{r=-\infty}^{\infty} c_r s^{-1-u/k-2\pi i r/(kL)}, \qquad (s\to 0).$$

Since, $Q(-s)^b = 1+\mathcal{O}(|s|)$ as $s\to 0$, the same asymptotic expansion holds for $\mathscr{L}[\tilde{L}_q(z);s]$ as well. Finally, by formal inverse Laplace transform (see [3] for technical details justifying this step), we have

$$\tilde{L}_q(z) \sim z^{u/k} P_1(\log_2 z^{1/k}), \qquad (z\to\infty),$$

where P_1 is a computable, 1-periodic function. A more careful analysis shows that the above asymptotic expansion holds uniformly for $|z|\to\infty$ and $|\arg(z)|\le \pi/2-\epsilon$.

Asymptotic Expansion of $\tilde{V}_q(z)$. Here, we proceed as above and obtain

$$\bar{V}_q(s) = 2^{k+u}\bar{V}_q(2^k s) + \sum_{l=0}^{k-1}\delta_1\cdots\delta_l 2^l \bar{H}_{q^{(l)}}(2^l s),$$

where $\bar{V}_q(s) = \mathscr{L}[\tilde{V}_q(z);s]/Q(-s)^b$ and $\bar{H}_{q^{(l)}}(s) = (\mathscr{L}[\tilde{h}_{q^{(l)}}(z);s]+p(s))/Q(-2s)^b$ with

$$p(s) = \frac{(1+s)^{b-1}+(-1)^b}{s+2}.$$

Now, observe that

$$\tilde{h}_q(z) = \begin{cases}\mathcal{O}(z^{2u/k-2}), & \text{if } z\to\infty;\\ \mathcal{O}(1), & \text{if } z\to 0^+,\end{cases}$$

where the first bound follows from the bound of the previous paragraph (which we are allowed to differentiate due to Ritt's theorem; see [9]) and the second bound is trivial. This together with the growth properties of $Q(s)$ then in turn yields

$$\bar{H}_q(s) = \begin{cases}\mathcal{O}(1/s), & \text{if } s\to\infty;\\ \mathcal{O}(s^{-\beta}), & \text{if } s\to 0^+,\end{cases}$$

where $\beta>0$ is an arbitrary constant. Consequently, the Mellin transform of $\bar{H}_q$ exists in the strip $\langle 1,\infty\rangle$. The remaining proof of Theorem 2 proceeds then as in the previous paragraph.

Simplification of the Fourier Coefficients for $b = 1$. First, by iteration of (8),

$$\mathscr{L}[\tilde{L}_q(z); s] = \frac{1}{s} \sum_{j \geq 0} \frac{\delta^*_{q,j}}{(s+1)\cdots(2^j s+1)},$$

where $\delta^*_{q,j} = \prod_{l=1}^{j} \delta_l$. Next, by partial fraction expansion,

$$\mathscr{L}[\tilde{L}_q(z); s] = \frac{1}{s} \sum_{j \geq 0} \sum_{l=0}^{j} \frac{(-1)^{j-l} 2^{-\binom{j-l+1}{2}} \delta^*_{q,j}}{(2^l s+1) Q_l Q_{j-l}} = \frac{1}{s} \sum_{l \geq 0} \frac{\bar{\delta}^*_{q,l}}{(2^l s+1) Q_l},$$

where

$$\bar{\delta}_{q,l} = \sum_{j \geq 0} \frac{(-1)^j 2^{-\binom{j+1}{2}}}{Q_j} \delta_{q,j+l}.$$

Consequently, by inverse Laplace transform

$$\tilde{L}_q(z) = \sum_{l \geq 0} \frac{\bar{\bar{\delta}}_{q,l}}{Q_l} (1 - e^{-z/2^l}).$$

This implies

$$\tilde{L}'_q(z) = \sum_{l \geq 0} \frac{\bar{\bar{\delta}}_{q,l}}{2^l Q_l} e^{-z/2^l}, \qquad \tilde{L}'_q(z)^2 = \sum_{l,h \geq 0} \frac{\bar{\bar{\delta}}_{q,l} \bar{\bar{\delta}}_{q,h}}{2^{l+h} Q_l Q_h} e^{-z/2^l - z/2^h}.$$

Plugging this into (3) (note that for $b = 1$, we have $\tilde{h}_q(z) = \tilde{L}'_q(z)^2$) and using

$$\frac{1}{Q(-2s)} = \frac{1}{Q(1)} \sum_{j \geq 0} \frac{(-1)^j 2^{-\binom{j}{2}}}{Q_j (s + 2^{-j})}$$

together with some standard computations proves the claim.

5 Conclusion

In this paper, we gave a new and simpler approach to the variance of partial match queries in k-dimensional bucket digital trees. Our method used standard tools from the analysis of algorithm such as poissonization-depoissonization and Mellin transform. The main simplification comes from the *poissonized variance* which incorporates cancellations at a much earlier stage compared to previous derivations.

Our approach allowed us to derive asymptotic expansions of the variance in k-dimensional bucket tries, k-dimensional bucket PATRICIA tries and k-dimensional bucket digital search trees. In all cases, the variance is asymptotic to $n^{u/k} P(\log_2 n^{1/k})$ where P is a 1-periodic function. Since the mean has the same order, our results show that the cost of partial match retrievals is concentrated around the mean.

We conclude by pointing out that even though we only derived the main term in the asymptotic expansions, our approach can be straightforwardly applied to derive longer asymptotic expansions, too.

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無研發成果推廣資料

98 年度專題研究計畫研究成果彙整表

計畫主持人：符麥克	計畫編號：98-2115-M-009-009-
計畫名稱：正特徵域上丟番圖逼近的賦距結果	

成果項目			量化			單位	備註（質化說明：如數個計畫共同成果、成果列為該期刊之封面故事...等）
			實際已達成數（被接受或已發表）	預期總達成數(含實際已達成數)	本計畫實際貢獻百分比		
國內	論文著作	期刊論文	0	0	100%	篇	
		研究報告/技術報告	0	0	100%		
		研討會論文	0	0	100%		
		專書	0	0	100%		
	專利	申請中件數	0	0	100%	件	
		已獲得件數	0	0	100%		
	技術移轉	件數	0	0	100%	件	
		權利金	0	0	100%	千元	
	參與計畫人力（本國籍）	碩士生	0	0	100%	人次	
		博士生	0	0	100%		
		博士後研究員	0	0	100%		
		專任助理	0	0	100%		
國外	論文著作	期刊論文	2	0	100%	篇	
		研究報告/技術報告	0	0	100%		
		研討會論文	1	0	100%		
		專書	0	0	100%	章/本	
	專利	申請中件數	0	0	100%	件	
		已獲得件數	0	0	100%		
	技術移轉	件數	0	0	100%	件	
		權利金	0	0	100%	千元	
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其他成果 (無法以量化表達之成果如辦理學術活動、獲得獎項、重要國際合作、研究成果國際影響力及其他協助產業技術發展之具體效益事項等，請以文字敘述填列。)	無

	成果項目	量化	名稱或內容性質簡述
科教處計畫加填項目	測驗工具(含質性與量性)	0	
	課程/模組	0	
	電腦及網路系統或工具	0	
	教材	0	
	舉辦之活動/競賽	0	
	研討會/工作坊	0	
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