行政院國家科學委員會專題研究計畫 成果報告

具封閉性質的距離正則圖之研究(3/3)
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行政院國家科學委員會補助專題研究計畫

■ 成果報告
□ 期中進度報告

(具封閉性質的距離正則圖之研究 (3/3))

計畫類別: ■個別型計畫 □ 整合型計畫　　計畫編號:NSC 98-2115-M-009-002-

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執行機構及系所: 國立交通大學應用數學系

計畫主持人: 翁志文

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行政院國家科學委員會補助專題研究計畫結案報告

計畫編號:NSC 98-2115-M-009-002-　計畫類別: 個別型計畫

執行期間: 98 年 8 月 1 日至 99年 7 月 31 日

計畫名稱: 具封閉性質的距離正則圖之研究 (3/3)

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99 年 9 月 3 日

一、中文摘要

令 Γ 爲一直徑爲 D, 相交參數 $a_2 > a_1 = 0$ 的距離正則圖且假設 Γ 無長度爲 $d+1$ 的平行四邊形, 此處 $1 \le d \le D-1$。我們證明 Γ 具 d-封閉性。利用此結果, 我們證明不存在 $D \ge 4$, 具古典參數 $(D, b, \alpha, \beta) = (D, -2, -2, ((-2)^{D+1} - 1)/3)$ 且 $b < -1$ 的距離正則圖。

關鍵詞: 距離正則圖、古典參數、平行四邊形、d-封閉性

二、英文摘要

Let Γ denote a distance-regular graph with diameter D and intersection numbers $a_2 > a_1 = 0$. We show that for each $1 \le d \le D-1$, if Γ contains no parallelograms of lengths up to $d+1$ then Γ is d-bounded in the sense of the article [D-bounded distance-regular graphs, European Journal of Combinatorics(1997)18, 211-229]. By applying this result we show the nonexistence of distance-regular graphs with classical parameters $(D, b, \alpha, \beta) = (D, -2, -2, ((-2)^{D+1} - 1)/3)$ for any $D \ge 4$. In the end, we survey the progress on the classification of distance-regular graph with classical parameters (D, b, α, β) and $b < -1$.

Keywords: Distance-regular graph, classical parameters, parallelogram, D-bounded.

三 、緣由與目的

The D-bounded distance-regular graphs were introduced in 1997[22] by the project investigator. This became an important concept in the classification of classical distance-regular graphs of negative type in 1999[23]. There are many interesting geometric structures constructed from a D-bounded graph that need to be investigated. They also have applications to pooling designs [7, 5, 6]. Several authors also devote themselves to the study of -bounded distance-regular graphs as results shown in [25, 26, 27, 28, 17]. This lures the project investigator going back to this line of study.

四、結果與討論

Let $\Gamma = (X, R)$ be a distance-regular graph with diameter $D \ge 3$. Recall that a sequence x, z, y of vertices of Γ is *geodetic* whenever

$$\partial(x, z) + \partial(z, y) = \partial(x, y),$$

where ∂ is the distance function of Γ. A sequence x, z, y of vertices of Γ is *weak-geodetic* whenever

$$\partial(x, z) + \partial(z, y) \le \partial(x, y) + 1.$$

Definition 0.1. A subset $\Delta \subseteq X$ is *weak-geodetically closed* if for any weak-geodetic sequence x, z, y of Γ,

$$x,\ y \in \Delta \Longrightarrow z \in \Delta.$$

Weak-geodetically closed subgraphs are called *strongly closed subgraphs* in [15]. If a weak-geodetically closed subgraph Δ of diameter d is regular then it has valency $a_d + c_d = b_0 - b_d$, where a_d, c_d, b_0, b_d are intersection numbers of Γ. Furthermore Δ is distance-regular with intersection numbers $a_i(\Delta) = a_i(\Gamma)$ and $c_i(\Delta) = c_i(\Gamma)$ for $1 \leq i \leq d$ [21, Theorem 4.5].

Definition 0.2. Γ is said to be *i-bounded* whenever for all $x, y \in X$ with $\partial(x, y) \leq i$, there is a regular weak-geodetically closed subgraph of diameter $\partial(x, y)$ which contains x and y.

Note that a $(D-1)$-bounded distance-regular graph is clear to be D-bounded. The properties of D-bounded distance-regular graphs were studied in [22], and these properties were used in the classification of classical distance-regular graphs of negative type [23]. Before stating our main result we make one more definition.

By a *parallelogram of length* i, we mean a 4-tuple $xyzw$ consisting of vertices of Γ such that $\partial(x, y) = \partial(z, w) = 1$, $\partial(x, w) = i$, and $\partial(x, z) = \partial(y, w) = \partial(y, z) = i - 1$. The following theorem is our main result.

Theorem 0.3. *Let Γ denote a distance-regular graph with diameter $D \geq 3$, and intersection numbers $a_1 = 0$, $a_2 \neq 0$. Fix an integer $1 \leq d \leq D - 1$ and suppose that Γ contains no parallelograms of any length up to $d + 1$. Then Γ is d-bounded.*

Theorem 0.3 answers the problem proposed in [21, p. 299]. Many previous results deal with its complement case $a_1 \neq 0$, for examples under an additional assumption $c_2 > 1$ [21] and under the assumptions $a_2 > a_1 > c_2 = 1$ [16]. More precisely, for the case under the assumptions $a_2 > a_1$ and $c_2 = 1$, H. Suzuki proves the case $d = 2$ in Theorem 0.3 [16]; in particular Γ contains a regular weak-geodetically closed subgraph Ω of diameter 2. Since the Friendship Theorem [24, Theorem 8.6.39] asserts no such Ω in the case $a_1 = c_2 = 1$, there must be no such distance-regular graph Γ with $a_2 > a_1 = c_2 = 1$ and Γ contains no parallelograms of length 3. Note that the assumption $a_1 \neq 0$ implies $a_2 \neq 0$ [2, Proposition 5.5.1(i)]. Hence Theorem 0.3 is also true under the weaker assumptions $b_1 > b_2$ and $a_2 \neq 0$ (without assuming $a_1 = 0$). Our method in proving Theorem 0.3 also works for the case $b_1 > b_2$ and $a_2 \neq 0$ after a slight modification, but we decide not to duplicate the previous works.

On the other hand we suppose that Γ is d-bounded for $d \geq 2$. Let $\Omega \subseteq \Delta$ be two regular weak-geodetically closed subgraphs of diameters $1, 2$ respectively. Since Ω and Δ have different valency $b_0 - b_1$ and $b_0 - b_2$ respectively, we have $b_1 > b_2$. It is also easy to see that Γ contains no parallelograms of any length up to $d+1$ [21, Lemma 6.5]. With these comments, Theorem 0.3 is the final step in the following characterization of d-bounded distance-regular graphs in terms of forbidden parallelograms.

Theorem 0.4. *Let Γ denote a distance-regular graph with diameter $D \geq 3$. Suppose the intersection number $a_2 \neq 0$. Fix an integer $2 \leq d \leq D-1$. Then the following two conditions (i), (ii) are equivalent:*

(i) Γ is d-bounded.

(ii) Γ contains no parallelograms of any length up to $d+1$ and $b_1 > b_2$. □

Theorem 0.3 is a generalization of [2, Lemma 4.3.13], [13], and is also proved under an additional assumption $c_2 > 1$ by A. Hiraki [4]. To prove Theorem 0.3, we need many previous results of [4]. These will be stated independently in Section 2. Some applications of Theorem 0.3

were previously given in [4], [14]. The following is a new application of Theorem 0.3.

Theorem 0.5. *There is no distance-regular graph with classical parameters* $(D, b, \alpha, \beta) = (D, -2, -2, ((-2)^{D+1} - 1)/3)$, *where* $D \geq 4$.

A consequence of Theorem 0.5 is the following.

Corollary 0.6. *Let* Γ *denote a distance-regular graph with classical parameters* (D, b, α, β), $D \geq 4$ *and* $c_2 = 1$. *Then* $a_2 = a_1$ *and* $a_1 \neq 0$.

We prove Theorem 0.3 in Section 3, and prove Theorem 0.5, Corollary 0.6 in Section 4. We survey the progress on the classification of distance-regular graph with classical parameters (D, b, α, β) and $b < -1$ in Section 5.

五、附錄 (含出席國際會議報告)

1 Preliminaries

In this section we review some definitions, basic concepts and some previous results concerning distance-regular graphs. See Bannai and Ito [1] or Terwilliger [18] for more background information.

Let Γ=(X, R) denote a finite undirected, connected graph without loops or multiple edges with vertex set X, edge set R, distance function ∂, and diameter D:=max$\{ \partial(x,y) \mid x, y \in X\}$. By a *pentagon*, we mean a 5-tuple $u_1u_2u_3u_4u_5$ consisting of distinct vertices in Γ such that $\partial(u_i, u_{i+1}) = 1$ for $1 \leq i \leq 4$ and $\partial(u_5, u_1) = 1$.

For a vertex $x \in X$ and an integer $0 \leq i \leq D$, set $\Gamma_i(x) := \{ z \in X \mid \partial(x, z) = i\}$. The *valency* $k(x)$ of a vertex $x \in X$ is the cardinality of $\Gamma_1(x)$. The graph Γ is called *regular* (with *valency* k) if each vertex in X has valency k. A graph Γ is said to be *distance-regular* whenever for all integers $0 \leq h, i, j \leq D$, and all vertices $x, y \in X$ with $\partial(x, y) = h$, the number

$$p^h_{ij} = |\Gamma_i(x) \cap \Gamma_j(y)|$$

is independent of x, y. The constants p^h_{ij} are known as the *intersection numbers* of Γ.

From now on let $\Gamma = (X, R)$ be a distance-regular graph with diameter $D \geq 3$. For two vertices $x, y \in X$, with $\partial(x, y) = i$, set

$$\begin{aligned} B(x,y) &:= \Gamma_1(x) \cap \Gamma_{i+1}(y), \\ C(x,y) &:= \Gamma_1(x) \cap \Gamma_{i-1}(y), \\ A(x,y) &:= \Gamma_1(x) \cap \Gamma_i(y). \end{aligned}$$

Note that

$$\begin{aligned} |B(x,y)| &= p^i_{1\ i+1}, \\ |C(x,y)| &= p^i_{1\ i-1}, \\ |A(x,y)| &= p^i_{1\ i} \end{aligned}$$

are independent of x, y. For convenience, set $c_i := p^i_{1\ i-1}$ for $1 \leq i \leq D$, $a_i := p^i_{1\ i}$ for $0 \leq i \leq D$, $b_i := p^i_{1\ i+1}$ for $0 \leq i \leq D-1$ and put $b_D := 0$, $c_0 := 0$, $k := b_0$. Note that k is the valency of Γ. It is immediate from the definition of p^h_{ij} that $b_i \neq 0$ for $0 \leq i \leq D-1$ and $c_i \neq 0$ for $1 \leq i \leq D$. Moreover

$$k = a_i + b_i + c_i \qquad \text{for} \quad 0 \leq i \leq D. \tag{1.1}$$

A subset Ω of X is *weak-geodetically closed with respect to a vertex* $x \in \Omega$ if

$$C(y, x) \subseteq \Omega \ \text{ and } \ A(y, x) \subseteq \Omega \qquad \text{for all} \ \ y \in \Omega. \tag{1.2}$$

Note that Ω is weak-geodetically closed if and only if for any vertex $x \in \Omega$, Ω is weak-geodetically closed with respect to x [21, Lemma 2.3]. We list a few results which will be used later in this paper.

Theorem 1.1. *([21, Theorem 4.6]) Let Γ be a distance-regular graph with diameter $D \geq 3$. Let Ω be a regular subgraph of Γ with valency γ and set $d := \min\{i \mid \gamma \leq c_i + a_i\}$. Then the following (i),(ii) are equivalent.*

(i) Ω is weak-geodetically closed with respect to at least one vertex $x \in \Omega$.

(ii) Ω is weak-geodetically closed with diameter d.

In this case $\gamma = c_d + a_d$. □

Definition 1.2. Fix a vertex $x \in X$. A pentagon $u_1u_2u_3u_4u_5$ has *shape* i_1, i_2, i_3, i_4, i_5 *with respect to* x if $i_j = \partial(x, u_j)$ for $1 \leq j \leq 5$.

Theorem 1.3. *([21, Lemma 6.9],[16, Lemma 4.1]) Let Γ be a distance-regular graph with diameter $D \geq 3$. Suppose $a_1 = 0$, $a_2 \neq 0$ and Γ contains no parallelograms of length up to $d+1$ for some integer $d \geq 2$. Let x be a vertex of Γ, and let $u_1u_2u_3u_4u_5$ be a pentagon of Γ such that $\partial(x, u_1) = i-1$ and $\partial(x, u_3) = i+1$ for $1 \leq i \leq d$. Then the pentagon $u_1u_2u_3u_4u_5$ has shape $i-1, i, i+1, i+1, i$ with respect to x.* □

2 A few lemmas

Throughout this section, let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D \geq 3$, and intersection numbers $a_1 = 0$, $a_2 \neq 0$. Such graphs are also studied in [4, 11, 12, 13, 14]. Note that any two vertices at distance 2 are always contained in a pentagon since $a_2 \neq 0$, and two nonconsecutive vertices in a pentagon of Γ have distance 2 since $a_1 = 0$. In this section we give a few lemmas which will be used in the next section. These results were formulated by A. Hiraki in [4] under an additional assumption $c_2 > 1$, but this assumption is essentially not used in his proofs. For the completeness, we still provide the proofs.

Lemma 2.1. *Fix an integer $1 \leq d \leq D-1$, and suppose Γ does not contain parallelograms of length up to $d+1$. Then for any two vertices $z, z' \in X$ such that $\partial(x, z) \leq d$ and $z' \in A(z, x)$, we have $B(x, z) = B(x, z')$.*

Proof. By symmetry, it suffices to show $B(x,z) \subseteq B(x,z')$. Suppose there exists $w \in B(x,z) \setminus B(x,z')$. Then $\partial(w,z') \neq \partial(x,z)+1$. Note that $\partial(w,z') \leq \partial(w,x)+\partial(x,z') = 1+\partial(x,z)$ and $\partial(w,z') \geq \partial(w,z)-\partial(z,z') = \partial(x,z)$. This implies $\partial(w,z') = \partial(x,z)$ and $wxz'z$ forms a parallelogram of length $\partial(x,z)+1$, a contradiction. □

Lemma 2.2. *Fix integers $1 \leq i \leq d \leq D-1$, and suppose Γ does not contain parallelograms of any length up to $d+1$. Let x be a vertex of Γ. Then there is no pentagon of shape $i,i,i,i,i+1$ with respect to x for $1 \leq i \leq d$.*

Proof. Let $u_1u_2u_3u_4u_5$ be a pentagon of shape $i,i,i,i,i+1$ with respect to x. We derive a contradiction by induction on i. The case $i=1$ is impossible otherwise $u_1xu_2u_3$ is a parallelogram of length 2. Suppose $i \geq 2$. Note that $B(x,u_1) = B(x,u_2) = B(x,u_3) = B(x,u_4)$ by Lemma 2.1. We shall prove $C(x,u_1) = C(x,u_2) = C(x,u_3) = C(x,u_4)$. First we prove $C(x,u_1) = C(x,u_2)$. It suffices to show $C(x,u_2) \subseteq C(x,u_1)$ since both sets have the same size c_i. To the contrary suppose there exists $v \in C(x,u_2) - C(x,u_1)$. Note that $v \in A(x,u_1)$ as $B(x,u_1) = B(x,u_2)$. Then $B(u_1,x) = B(u_1,v)$ by Lemma 2.1 and hence $\partial(v,u_5) = i+1$ since $u_5 \in B(u_1,x)$. Now $u_2u_1u_5u_4u_3$ has shape $i-1,i,i+1,i+1,i$ with respect to v by Theorem 1.3, a contradiction since $v \notin B(x,u_4)$. This proves $C(x,u_2) \subseteq C(x,u_1)$ as desired. By symmetry, $C(x,u_3) = C(x,u_4)$. It remains to show $C(x,u_2) \subseteq C(x,u_4)$. To the contrary suppose there exists $u \in C(x,u_2) - C(x,u_4)$. Note that $u \in A(x,u_4)$ as $B(x,u_2) = B(x,u_4)$. Then $B(u_4,x) = B(u_4,u)$ by Lemma 2.1 and hence $\partial(u,u_5) = i+1$ since $u_5 \in B(u_4,x)$. Hence $u_2u_1u_5u_4u_3$ has shape $i-1,i,i+1,i+1,i$ with respect to u by Theorem 1.3, a contradiction since $u \notin B(x,u_4)$. Pick a vertex $v \in C(x,u_1) = C(x,u_2) = C(x,u_3) = C(x,u_4)$. Then $u_1u_2u_3u_4u_5$ is a pentagon of shape $i-1,i-1,i-1,i-1,i$ with respect to v, a contradiction to the inductive hypothesis. □

Proposition 2.3. *Fix integers $1 \leq i \leq d \leq D-1$, and suppose Γ does not contain parallelograms of any length up to $d+1$. Let x be a vertex and $u_1u_2u_3u_4u_5$ be a pentagon of shape $i,i-1,i,i-1,i$ or of shape $i,i-1,i,i-1,i-1$ with respect to x for $1 \leq i \leq d$. Then $B(x,u_1) = B(x,u_3)$.*

Proof. It suffices to show $B(x,u_3) \subseteq B(x,u_1)$ since both sets have the same size b_i. Pick $u \in B(x,u_3)$. Then $\partial(u,u_3) = i+1$, $\partial(u,u_4) = i$ and $\partial(u,u_2) = i$. Note that $\partial(u,u_1) \neq i-1$, otherwise by Theorem 1.3, the pentagon $u_1u_2u_3u_4u_5$ has shape $i-1,i,i+1,i+1,i$ with respect to u, a contradiction. Suppose $\partial(u,u_1) = i$ for this moment. Then to avoid obtaining a pentagon $u_1u_2u_3u_4u_5$ of type $i,i,i,i,i+1$ with respect to u we must have $\partial(u,u_5) = i+1$ by Lemma 2.2. Then $\partial(x,u_5) = i$ by construction. Now u_5u_1xu is a parallelogram of length $i+1$, a contradiction. Hence $\partial(u,u_1) = i+1$ or equivalently $u \in B(x,u_1)$. This proves $B(x,u_3) \subseteq B(x,u_1)$ as desired. □

Lemma 2.4. *Fix integers $1 \leq i \leq d \leq D-1$, and suppose Γ does not contain parallelograms of any length up to $d+1$. Let x be a vertex. Then there is no pentagon of shape $i,i,i,i+1,i+1$ with respect to x for $1 \leq i \leq d$.*

Proof. Suppose that $u_2u_3u_4u_5u_1$ is a pentagon of shape $i,i,i,i+1,i+1$ with respect to x. We derive a contradiction by induction on i. The case $i=1$ is impossible otherwise $u_2xu_3u_4$ is a parallelogram of length 2. Suppose $i \geq 2$. Pick $v \in C(x,u_2)$ and note that $\partial(v,u_1) = i$ by construction. In particular $v \notin B(x,u_2)$ and $B(x,u_2) = B(x,u_3) = B(x,u_4)$ by Lemma 2.1, so $v \in C(x,u_4) \cup A(x,u_4)$. In fact $v \in C(x,u_4)$; otherwise $\partial(v,u_4) = i$, $\partial(v,u_5) = i$ by Theorem 1.3, and then xvu_4u_5 is a parallelogram of length $i+1$, a contradiction. We also have $\partial(v,u_5) = i$ by construction. Note that $\partial(v, u_3) = i$; otherwise $\partial(v, u_3) = i-1$ and $u_2u_3u_4u_5u_1$ is a pentagon of shape $i-1, i-1, i-1, i, i$ with respect to v, a contradiction to inductive hypothesis. Now as $x = v$ in Proposition 2.3, we have $B(v, u_1) = B(v, u_3)$, a contradiction since $x \in B(v, u_1) - B(v, u_3)$. $\square$

3 Proof of Theorem 0.3

Let $\Gamma = (X, R)$ denote a distance-regular graph with intersection numbers $a_1 = 0$, $a_2 \neq 0$ and diameter $D \geq 3$. Fix an integer $1 \leq d \leq D-1$. Suppose Γ contains no parallelograms of length up to $d+1$. We shall prove Γ is d-bounded in this section. We first give a definition.

Definition 3.1. For any vertex $x \in X$ and any subset $\Pi \subseteq X$, define $[x, \Pi]$ to be the set

$$\{v \in X \mid \text{there exists } y' \in \Pi, \text{ such that the sequence } x, v, y' \text{ is geodetic }\}.$$

For any $x, y \in X$ with $\partial(x, y) = d$, set

$$\Pi_{xy} := \{y' \in \Gamma_d(x) \mid B(x, y) = B(x, y')\} \tag{3.1}$$

and

$$\Delta(x, y) = [x, \Pi_{xy}]. \tag{3.2}$$

We shall prove that for any vertices $x, y \in X$ with $\partial(x, y) = d$ the following statement B_d holds.

(B_d) $\Delta(x, y)$ is regular weak-geodetically closed with valency $a_d + c_d$.

By referring to Theorem 1.1, (B_d) is equivalent to the following statements (W_d) and (R_d).

(W_d) $\Delta(x, y)$ is weak-geodetically closed with respect to x, and

(R_d) the subgraph induced on $\Delta(x, y)$ is regular with valency $a_d + c_d$

for any vertices $x, y \in X$ with $\partial(x, y) = d$.

We prove (W_d) and (R_d) by induction on d. Since $a_1 = 0$, there is no edges in $\Gamma_1(x)$ for any vertex $x \in X$. If $d = 1$ in Definition 3.1, then $\Pi_{xy} = \{y\}$, and consequently $\Delta(x, y) = \{x, y\}$ is an edge; in particular $\Delta(x, y)$ is regular with valency $1 = a_1 + c_1$ and is weak-geodetically closed with respect to x since $a_1 = 0$. This proves (R_1) and (W_1). We now assume $d \geq 2$. By inductive hypothesis (W_i), (R_i) and (B_i) are assumed throughout this section for $1 \leq i \leq d-1$. The following proposition proves the statement (W_d).

Proposition 3.2. *For any vertices $x, y \in X$ with $\partial(x, y) = d$ and for any vertex $z \in \Delta(x, y) \cap \Gamma_i(x)$, where $1 \leq i \leq d$, we have the following (i), (ii).*

(i) $A(z, x) \subseteq \Delta(x, y)$.

(ii) *For any vertex $w \in \Gamma_i(x) \cap \Gamma_2(z)$ with $B(x, w) = B(x, z)$, we have $w \in \Delta(x, y)$.*

In particular (W_d) *holds.*

Proof. We prove (i), (ii) by induction on $d-i$. The case $i=d$ follows from the construction of $\Delta(x,y)$ in Definition 3.1 and by Lemma 2.1. Suppose $i<d$.

To prove (i) we note that if $i=1$ then $A(z,x)$ is an empty set, clearly contained in $\Delta(x,y)$. Hence we suppose $2\le i<d$ in this case. We pick a vertex $v\in A(z,x)$ and show $v\in\Delta(x,y)$. Pick $u\in\Delta(x,y)\cap\Gamma_{i+1}(x)\cap\Gamma_1(z)$. Note that (i), (ii) hold if we use u to replace z by inductive hypothesis. Let uu_2u_3vz be a pentagon of Γ for some $u_2,u_3\in X$. Note that uu_2u_3vz can not have shape $i+1,i,i-1,i,i$, shape $i+1,i+2,i+1,i,i$ by Theorem 1.3, can not have shape $i+1,i,i,i,i$ by Lemma 2.2, and can not have shape $i+1,i+1,i,i,i$ by Lemma 2.4 with respect to x. Hence uu_2u_3vz has shape $i+1,i+1,i+1,i,i$ or $i+1,i,i+1,i,i$ with respect to x. In the first case we have $u_2\in A(u,x)$, $u_3\in A(u_2,x)$, and this implies $u_2,u_3\in\Delta(x,y)$ by the inductive hypothesis of (i), and $v\in\Delta(x,y)$ by construction. In the latter case we have $B(x,u)=B(x,u_3)$ by Lemma 2.3, and consequently $u_3\in\Delta(x,y)$ by inductive hypothesis of (ii), $v\in\Delta(x,y)$ by construction.

To prove (ii) let $zv_2wv_4v_5$ be a pentagon for some $v_2,v_4,v_5\in X$. Note that $\Delta(x,z)$ is a regular weak-geodetically closed subgraph of diameter i by (B_i), and $\Delta(x,z)=\Delta(x,w)$ by construction in Definition 3.1 and since $B(x,w)=B(x,z)$; in particular $v_2,v_4,v_5\in\Delta(x,z)$ and $v_2,v_4,v_5\notin\Gamma_{i+1}(x)$. If $v_2\in A(z,x)$ then $v_2,w\in\Delta(x,y)$ by (i) that we just proved. Hence we assume $zv_2wv_4v_5$ has shape $i,i-1,i,a,b$ with respect to x for integers $a,b\in\{i-1,i\}$. Pick $u\in\Delta(x,y)\cap\Gamma_{i+1}(x)\cap\Gamma_1(z)$. Let $v_2zuy_3y_4$ be a pentagon for some $y_3,y_4\in X$. Then $v_2zuy_3y_4$ has shape $i-1,i,i+1,i+1,i$ with respect to x by Theorem 1.3. Let $v_2y_4w_3w_4w$ be a pentagon for some $w_3,w_4\in X$. If $w_4\in A(w,x)\cup C(w,x)$ then $w_4\in\Delta(x,w)=\Delta(x,z)$ and this forces $y_4\in\Delta(x,z)$ as $v_2,w_4\in\Delta(x,z)$ and by (B_i). By the same reason we have $y_3\in\Delta(x,z)$ as $z,y_4\in\Delta(x,z)$. We have a contradiction since $\Delta(x,z)$ has diameter i and $\partial(x,y_3)=i+1>i=\text{diam }\Delta(x,z)$. Hence $\partial(x,w_4)=i+1$ and $v_2y_4w_3w_4w$ has shape $i-1,i,i+1,i+1,i$ with respect to x by Theorem 1.3 as shown in Figure 1. Note that $B(x,u)=B(x,y_3)$ and $B(x,w_3)=B(x,w_4)$ by Lemma 2.1. If $B(x,y_3)=B(x,w_3)$ then by (i) and the inductive hypothesis of (ii) we have $y_3,w_3,w_4\in\Delta(x,y)$ in the order, and $w\in\Delta(x,y)$ by the construction in (3.2) to complete the proof. Suppose $B(x,y_3)\ne B(x,w_3)$ in the remaining. Let $y_4y_3p_3p_4w_3$ be a pentagon for some $p_3,p_4\in X$. By Lemma 2.1, Lemma 2.3 and Theorem 1.3, the pentagon $y_4y_3p_3p_4w_3$ has shape $i,i+1,i+2,i+2,i+1$ with respect to x. Now we have three pentagons and their shapes with respect to x as shown in Figure 1. Note that $B(x,y_4)\ne B(x,z)$, otherwise $\Delta(x,y_4)=\Delta(x,z)$ and $y_3\in\Delta(x,z)$, a contradiction as before. Pick $p\in B(x,y_4)-B(x,z)$. Then $\partial(p,y_4)=i+1$ and $\partial(p,z)=i-1$ or i. Suppose for this moment $\partial(p,z)=i-1$. Then $zuy_3y_4v_2$ is a pentagon of shape $i-1,i,i+1,i+1,i$ with respect to p by Theorem 1.3. Note that $\partial(p,p_3)=i+2$, otherwise $\partial(p,p_3)=i+1$ and xpy_3p_3 is a parallelogram of length $i+2\le d+1$, a contradiction. Now by applying Lemma 2.2, Lemma 2.4, we have $\partial(p,w_3)=i+2$ and consequently $v_2y_4w_3w_4w$ is a pentagon of shape $i,i+1,i+2,i+2,i+1$ with respect to p by Theorem 1.3. That is $p\in B(x,w)$, a contradiction to $B(x,z)=B(x,w)$. By symmetry, we also have $\partial(p,w)\ne i-1$. We suppose in the last case $\partial(p,z)=\partial(p,w)=i$. As $p\in A(x,z)$, we have $B(z,x)=B(z,p)$ by Lemma 2.1, in particular $\partial(p,u)=i+1$. By symmetry, $\partial(p,w_4)=i+1$. As $p\notin B(x,u)=B(x,y_3)$, we have $\partial(p,y_3)=i$ or $i+1$. We shall prove $\partial(p,y_3)=i$, and by symmetry $\partial(p,w_3)=i$. Suppose to the contrary we have $\partial(p,y_3)=i+1$. As $p_3\in B(y_3,x)=B(y_3,p)$,

$\partial(p, p_3) = i+2$. Applying Lemma 2.2, Lemma 2.4 to the pentagon $w_3y_4y_3p_3p_4$ and considering its shape with respect to p, we find $\partial(p, w_3) \neq i + 1$, and applying Theorem 1.3 to find $\partial(p, w_3) \neq i$. Now $\partial(p, w_3) = i + 2$ and pxw_4w_3 is a parallelogram of length $i + 2 \leq d + 1$, a contradiction. We conclude that $y_4y_3p_3p_4w_3$ is a pentagon of shape $i+1, i, a, i+1, i$ or of shape $i+1, i, i+1, b, i$ with respect to p for $a, b \in \{i, i+1\}$ by Lemma 1.3, Lemma 2.2, and this implies $x \in B(p, p_4) = B(p, y_4)$ in the first case or $x \in B(p, p_3) = B(p, y_4)$ in the latter case by Proposition 2.3, a contradiction since $x \in C(p, y_4)$.

$\Delta(x, y)$ is clear to be weak-geodetically closed with respect to x by (1.2) and (i). □

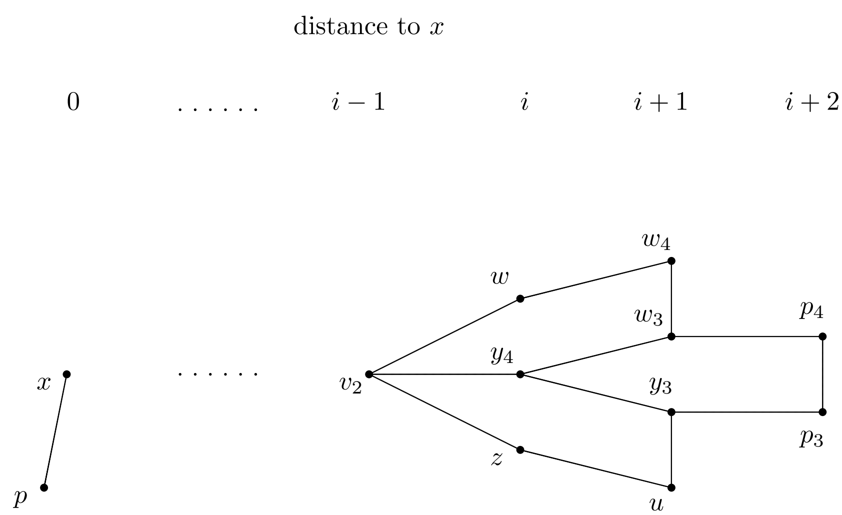

Figure 1. Three pentagons in the proof of Proposition 3.2(ii).

The following proposition proves (R_d) and hence completes the proof of Theorem 0.3.

Proposition 3.3. *For any vertices $x, y \in X$ with $\partial(x, y) = d$, $\Delta(x, y)$ is regular with valency $a_d + c_d$.*

Proof. Set $\Delta = \Delta(x, y)$. Clearly from the construction and Proposition 3.2, $|\Gamma_1(y') \cap \Delta| = a_d + c_d$ for any $y' \in \Pi_{xy}$. First we show $|\Gamma_1(x) \cap \Delta| = a_d + c_d$. Note that $y \in \Delta \cap \Gamma_d(x)$ by construction of Δ. For any $z \in C(x, y) \cup A(x, y)$,

$$\partial(x, z) + \partial(z, y) \leq \partial(x, y) + 1.$$

This implies $z \in \Delta$ since Δ is weak-geodetically closed with respect to x by Proposition 3.2. Hence $C(x, y) \cup A(x, y) \subseteq \Delta$. Suppose $B(x, y) \cap \Delta \neq \emptyset$. Choose $t \in B(x, y) \cap \Delta$. Then there exists $y' \in \Pi_{xy}$ such that $t \in C(x, y')$, a contradiction to $B(x, y) = B(x, y')$. Hence $B(x, y) \cap \Delta = \emptyset$ and $\Gamma_1(x) \cap \Delta = C(x, y) \cup A(x, y)$. This proves $|\Gamma_1(x) \cap \Delta| = a_d + c_d$.

Since each vertex in Δ appears in a sequence of vertices $x = x_0, x_1, \ldots, x_d$ in Δ, where $\partial(x, x_j) = j$, $\partial(x_{j-1}, x_j) = 1$ for $1 \leq j \leq d$, and $x_d \in \Pi_{xy}$, it suffices to show

$$|\Gamma_1(x_i) \cap \Delta| = a_d + c_d \tag{3.3}$$

for $1 \leq i \leq d-1$. For each integer $1 \leq i \leq d$, we show

$$|\Gamma_1(x_{i-1}) \setminus \Delta| \leq |\Gamma_1(x_i) \setminus \Delta| \tag{3.4}$$

by the 2-way counting of the number of the pairs (z, s) for $z \in \Gamma_1(x_{i-1}) \setminus \Delta$, $s \in \Gamma_1(x_i) \setminus \Delta$ and $\partial(z, s) = 2$. For a fixed $s \in \Gamma_1(x_i) \setminus \Delta$, we have $\partial(x, s) = i+1$ and $\partial(x_{i-1}, s) = 2$ since Δ is weak-geodetically closed with respect to x by Proposition 3.2. Hence $z \in A(x_{i-1}, s)$. The number of such pairs (z, s) is at most $|\Gamma_1(x_i) \setminus \Delta| a_2$.

On the other hand, we show this number is $|\Gamma_1(x_{i-1}) \setminus \Delta| a_2$ exactly. Fix an $z \in \Gamma_1(x_{i-1}) \setminus \Delta$. Note that $\partial(x, z) = i$ by Proposition 3.2, and $\partial(x_i, z) = 2$ since $a_1 = 0$. Pick any $s \in A(x_i, z)$. We shall prove $s \notin \Delta$. Suppose to the contrary $s \in \Delta$ in the below arguments and choose any $w \in C(s, z)$. Note that $\partial(x, s) \leq i$, otherwise $\partial(x, s) = i+1$ and the pentagon $x_{i-1}x_i swz$ has shape $i-1, i, i+1, i+1, i$ with respect to x by Theorem 1.3 to force $z \in \Delta$ by Proposition 3.2(i) and construction of Δ, a contradiction. Similarly $\partial(x, w) \leq i$. If $s \in A(x_i, x)$, $w \in A(s, x)$ and $z \in A(w, x)$, then $z \in \Delta$ by Proposition 3.2(i), a contradiction. Applying Proposition 2.3 in the remaining cases we have $B(x, z) = B(x, x_i)$ and then $z \in \Delta$ by Proposition 3.2(ii), a contradiction.

From the above counting, we have

$$|\Gamma_1(x_{i-1}) \setminus \Delta| a_2 \leq |\Gamma_1(x_i) \setminus \Delta| a_2 \tag{3.5}$$

for $1 \leq i \leq d$. Eliminating a_2 from (3.5), we find (3.4) or equivalently

$$|\Gamma_1(x_{i-1}) \cap \Delta| \geq |\Gamma_1(x_i) \cap \Delta| \tag{3.6}$$

for $1 \leq i \leq d$. We have known previously $|\Gamma_1(x_0) \cap \Delta| = |\Gamma_1(x_d) \cap \Delta| = a_d + c_d$. Hence (3.3) follows from (3.6).

□

4 Classical parameters

Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D \geq 3$. Γ is said to have *classical parameters* (D, b, α, β) whenever the intersection numbers of Γ satisfy

$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix} \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}\right) \qquad \text{for } 0 \leq i \leq D, \tag{4.1}$$

$$b_i = \left(\begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix}\right)\left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}\right) \qquad \text{for } 0 \leq i \leq D, \tag{4.2}$$

where

$$\begin{bmatrix} i \\ 1 \end{bmatrix} := 1 + b + b^2 + \cdots + b^{i-1}. \tag{4.3}$$

Applying (1.1) with (4.1), (4.2), we have

$$a_i = \begin{bmatrix} i \\ 1 \end{bmatrix}\left(\beta - 1 + \alpha(\begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix} - \begin{bmatrix} i-1 \\ 1 \end{bmatrix})\right) \tag{4.4}$$

$$= \begin{bmatrix} i \\ 1 \end{bmatrix}\left(a_1 - \alpha(\begin{bmatrix} i \\ 1 \end{bmatrix} + \begin{bmatrix} i-1 \\ 1 \end{bmatrix} - 1)\right) \tag{4.5}$$

for $1 \leq i \leq D$.

Suppose Γ has classical parameters (D, b, α, β) and $D \geq 3$. Then b is an integer, $b \neq 0$ and $b \neq -1$ [2, p. 195]. To apply Theorem 0.4 we need the following lemma.

Lemma 4.1. *([19, Theorem 2.12], [21, Lemma 7.3(ii)]) Let Γ denote a distance-regular graph with classical parameters (D, b, α, β), $b < -1$ and $D \geq 3$. Then Γ contains no parallelograms of any length.* □

More general version of Lemma 4.1 can be found in [20, 11, 12].

Theorem 4.2. *([22, Theorem 4.2]) Let Γ denote a distance-regular graph with classical parameters (D, b, α, β) and $b < -1$. Suppose that Γ is D-bounded with $D \geq 4$. Then*

$$\beta = \alpha \frac{1 + b^D}{1 - b}. \tag{4.6}$$

Proof of Theorem 0.5. Let Γ denote a distance-regular graph with classical parameters $(D, b, \alpha, \beta) = (D, -2, -2, ((-2)^{D+1} - 1)/3)$, where $D \geq 4$. Then Γ contains no parallelograms of any length by Lemma 4.1. By (4.1), (4.4), we have $c_2 = 1$ and $a_2 = 2 > 0 = a_1$. Hence Γ is D-bounded by Theorem 0.4 and since $b_1 > b_2$. By (4.6), $\beta = ((-2)^{D+1} - 2)/3)$, a contradiction. □

We quote a few previous results in the study of distance-regular graphs with classical parameters and $c_2 = 1$ for later use.

Lemma 4.3. *([22, Corollary 6.3]) There is no distance-regular graph Γ with classical parameters (D, b, α, β), $D \geq 4$, $c_2 = 1$ and $a_2 > a_1 > 1$.* □

Lemma 4.4. *([14, Theorem 2.2]) Let Γ denote a distance-regular graph with classical parameters (D, b, α, β) and $D \geq 3$. Assume the intersection numbers $a_1 = 0$, $a_2 \neq 0$, and $c_2 = 1$. Then $(b, \alpha, \beta) = (-2, -2, ((-2)^{D+1} - 1)/3)$.* □

Lemma 4.5. *([19, Theorem 2.11], [21, Lemma 7.3(ii)]) Let Γ denote a distance-regular graph with classical parameters (D, b, α, β) and $D \geq 3$. Suppose Γ contains no parallelograms of length 2. Then Γ contains no parallelograms of any length.* □

Proof of Corollary 0.6. Since $c_2 = 1$, Γ contains no parallelograms of length 2 and then contains no parallelogram of any length by Lemma 4.5. By Lemma 4.3, Lemma 4.4, Theorem 0.5, only the case $a_2 > a_1 = 1$ and the case $a_2 = a_1$ remain. The first case is impossible by Friendship Theorem as mentioned in the introduction. For the latter case, we have $\alpha = -b/(1+b)$ since $c_2 = 1$ and by (4.1). Applying this to (4.5) we find the impossibility of $a_2 = a_1 = 0$.

We close this section by proposing the following conjecture.

Conjecture 4.6. *There is no distance-regular graph* Γ *with classical parameters* (D, b, α, β), $D \geq 4$, *and* $c_2 = 1$.

There is a mistake in [2, Proposition 6.1.2] which proves the above conjecture. This mistake is corrected in [3].

Remark 4.7. (See [2, p. 194]) The Triality graph ${}^3D_{4,2}(q)$ is a distance-regular graph with classical parameters $(3, -q, q/(1-q), q^2+q)$, $c_2 = 1$ and $a_1 = a_2 = q-1$. Hence the assumption $D \geq 4$ in Conjecture 4.6 is necessary. Note that the Triality graph ${}^3D_{4,2}(q)$ is not 3-bounded by Theorem 0.4 since $b_1 = b_2$.

5 Classical parameters with $b < -1$

Let $\Gamma = (X, R)$ denote a distance-regular graph with classical parameters (D, b, α, β), $b < -1$ and $D \geq 3$. We survey the progress on the classification of such Γ in this section. Two main classes of such examples are the dual polar graphs ${}^2A_{2D-1}(-b)$ and the Hermitian forms graphs $Her_{-b}(D)$ as listed in [2, Tabel 6.1]. A.A. Ivanov and S.V. Shpectorov show that if Γ has the same intersection numbers as the dual polar graph ${}^2A_{2D-1}(-b)$ then Γ is the dual polar graph ${}^2A_{2D-1}(-b)$ [8]. They also show that if Γ does not contain parallelograms of length 2 and has the same intersection numbers as the Hermitian forms graph $Her_{-b}(D)$ then Γ is the Hermitian forms graph $Her_{-b}(D)$ [9, 10]. P. Terwilliger shows that in fact Γ does not contains parallelograms of any length [19] as also stated in Lemma 4.1. According to different assumptions on the intersection numbers of Γ, the D-bounded property of Γ are proved by different authors as stated in the introduction. Putting all these results together, if Γ has intersection numbers $b_1 > b_2$ and $a_2 \neq 0$ then Γ is D-bounded as also stated in Theorem 0.4.

We assume $b_1 > b_2$ and $a_2 \neq 0$ in Γ thereinafter. The third author shows that if $D \geq 4$ then

$$\beta = \alpha \frac{1+b^D}{1-b} \tag{5.1}$$

in [22] as also stated in (4.6), and use this to conclude in [23] that if Γ is not the dual polar graph ${}^2A_{2D-1}(-b)$ and not the Hermitian forms graph $Her_{-b}(D)$ then

$$\alpha = (b-1)/2, \quad \beta = -(1+b^D)/2, \tag{5.2}$$

where $-b$ is a power of an odd prime.

There are some results of Γ in the assumption $D \geq 3$, $a_1 = 0$ and $a_2 \neq 0$. For example in [14], the second author and the third author show that $c_2 \leq 2$, and in the case $c_2 = 1$, it must be

$$(b, \alpha, \beta) = (-2, -2, ((-2)^{D+1} - 1)/3). \tag{5.3}$$

Note that if $D \geq 4$, (5.3) does not hold by (5.2). This is essentially the proof of Theorem 0.5. Hence we have the following conjecture about the case $D = 3$.

Conjecture 5.1. *There is no distance-regular graphs with classical parameters* $(D, b, \alpha, \beta) = (3, -2, -2, 5)$.

Also in [4] A. Hiraki assume that $D \geq 3$, $a_1 = 0$, $a_2 \neq 0$, $c_2 > 1$ and show that Γ is either the Hermitian forms graph $Her_2(D)$ or α, β satisfy (5.2) with $b = -3$. Hence the following conjecture is the first step to study the unknown case of (5.2).

Conjecture 5.2. *There is no distance-regular graph with classical parameters* $(D, b, \alpha, \beta) = (3, -3, -2, 13)$.

References

[1] E. Bannai and T. Ito, Algebraic Combinatorics I: Association Schemes, Benjamin/Cummings, Menlo Park, 1984.

[2] A.E. Brouwer, A.M. Cohen, and A. Neumaier, Distance-Regular Graphs, Springer-Verlag, Berlin, 1989.

[3] A.E. Brouwer, A.M. Cohen, and A. Neumaier, Additions and Corrections to the book of BCN, http://www.win.tue.nl/ aeb/drg/index.html

[4] A. Hiraki, Distance-regular graphs with $c_2 > 1$ and $a_1 = 0 < a_2$, Graphs Combin. 25(1)(2009), 65-79.

[5] Tayuan Huang, Kaishun Wang and Chih-wen Weng, Pooling spaces associated with finite geometry, European Journal of Combinatorics, 29(2008) 1483-1491

[6] Tayuan Huang, Kaishun Wang and Chih-wen Weng, A Class of Error-Correcting Pooling Designs over Complexes, J Comb Optim, (2010) 19: 486-491 (doi 10.10071 s10878-008-9179-4)

[7] Tayuan Huang and Chih-wen Weng, Pooling Spaces and Non-Adaptive Pooling Designs, Discrete Mathematics, 282(1-3), 163-169, 2004

[8] A.A. Ivanov and S.V. Shpectorov, The association schemes of dual polar spaces of type ${}^2A_{2d-1}(p^f)$ are characterized by their parameters if $d \geq 3$, Linear Algebra Appl., 144/155(1989), 133–139.

[9] A.A. Ivanov and S.V. Shpectorov, Characterization of the association schemes of Hermitian forms over $GF(2^2)$, Geometriae Dedicata, 30(1989), 23–33.

[10] A.A. Ivanov and S.V. Shpectorov, A characterization of the association schemes of Hermitian forms, J. Math. Soc. Japan, 43(1)(1991), 25-48.

[11] Y. Liang, and C. Weng, Parallelogram-free distance-regular graphs, J. Combin. Theory Ser. B, 71(2)(1997), 231–243.

[12] Y. Pan, M. Lu, and C. Weng, Triangle-free distance-regular graphs, J. Algebraic Combin., 27(2008), 23–34.

[13] Y. Pan and C. Weng, Three bounded properties in triangle-free distance-regular graphs, European J. Combin., 29(2008), 1634–1642.

[14] Y. Pan and C. Weng, A note on triangle-free distance-regular graphs with $a_2 \neq 0$, J. Combin. Theory Ser. B, 99(2009), 266–270.

[15] H. Suzuki, On strongly closed subgraphs of highly regular graphs, European J. Combin., 16(1995), 197–220.

[16] H. Suzuki, Strongly closed subgraphs of a distance-regular graph with geometric girth five, Kyushu Journal of Mathematics, 50(2)(1996), 371–384.

[17] Hiroshi Suzuki, On Strongly Closed Subgraphs with Diameter Two and Q-Polynomial Property, preprint, 2004

[18] P. Terwilliger, The subconstituent algebra of an association scheme (Part I), Journal of Algebraic Combinatorics, 1(1992), 363–388.

[19] P. Terwilliger, Kite-free distance-regular graphs, Europ. J. of Combin., 16(4)(1995), 405–414.

[20] C. Weng, Kite-free P- and Q-polynomial schemes, Graphs and Combinatorics, 11(1995), 201-207.

[21] C. Weng, Weak-geodetically closed subgraphs in distance-regular graphs, Graphs and Combinatorics, 14(1998), 275–304.

[22] C. Weng, D-bounded distance-regular graphs, European Journal of Combinatorics, 18(1997), 211–229.

[23] C. Weng, Classical distance-regular graphs of negative type, J. Combin. Theory Ser. B, 76(1999), 93–116.

[24] D. B. West, Introduction to Graph Theory, Prentice Hall, Upper Saddle River, 1996.

[25] Suogang Gao and Jun Guo, A construction of a Distance-regular Graph with Diameter 2, submitted to Graph and Combinatorics

[26] Jun Guo and Suogagng Gao, Lattices Generated by Join of Strongly Closed Subgraphs in d-bounded Distance-regular Graphs, submitted to Discrete Mathematics

[27] Jun Guo, Wen Liu, Suogang Gao, Some counting formulae of strongly closed subgraphs in distance-regular graphs, preprint

[28] Suogang Gao, Jun Guo, Wen Lin, Lattices Generated by Strongly Closed Subgraphs in d-bounded Distance-regular Graphs, preprint

參加 2009 年釜山國際大學所舉辦的韓日代數與組合國際學術會議結及順道訪問浦項大學結案報告

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承蒙國科會的支助，於上述時間、地點參加韓國釜山國際大學所舉辦的韓日代數與組合國際學術會議。韓國與日本代數組合學者每年兩次輪流舉辦會議，這是第六屆。此會議報告主題相當廣泛，有組合矩陣理論、結合方案、代數編碼、距離正則圖、球面設計、代數表示理論及筆者的翻轉迷宮新主題。

筆者於 2/8 由中正機場搭機前釜山，及至 2/13 晚上返回中正機場，此趟旅程前後達 5 天之久。筆者主要在 2/10 當天早上以 Flipping Puzzle on a simple Graph 為題演講，這個題目是筆者的博士生黃皜文君的研究第二個成果，前一成果去年已於荷蘭報告。這一方向的演講每次都獲得相當多回應的，上次在荷蘭學到此研究與 orthogonal groups 有關，而此次聽到此研究的前身 sigma games 與 orthogonal polynomials—Chebyshev polynomials 有關。

2/11 全天是年輕學者的 Kumjung Seminar，當天結束筆者搭 Jack Koolen 教授的便車到POSTECH 訪問。 Jack Koolen 教授是荷蘭人，比我小四歲，卻在韓國領導一代數組合團隊，成果豐碩，相當令人佩服。他介紹 Delsarte clique graphs 給我，我研究的封閉距離這則圖似乎可推廣到此領域。

此次訪問由於機票直接由網路向航空公司購買，報帳麻煩，所以機票部份就自費處理，剩餘經費暑假計畫參加中國天津所舉辦的兩岸圖論與組合會議之用。

參加 2010 年浦項科技大學及浦項數學機構聯合舉辦的代數組合與幾何組合國際學術會議結案報告

時間: 99 年 7 月 11 日- 99 年 7 月 15 日

地點: 韓國慶州

報告人:翁志文

計劃編號: NSC 98-2115-M-009-002

承蒙國科會的支助，於上述時間、地點參加韓國浦項科技大學及浦項數學機構聯合舉辦的代數組合與幾何組合國際學術會議。此次會議除了邀請許多韓日學者外，也有多位遠從荷蘭、比利時及美國的學者。會議報告主題繞著結合方案為中心，有距離正則圖、球面設計、區族設計、量子數學、值譜理論、代數編碼等。筆者演講在 7/12 下午，題目為 「Pooling designs with d-disjunct property and block weight d+1」，是此計劃所延生應用方面的主題。 參考文末資料。

此次會議認識一位比利時Ghent 大學的博士生，他在有限幾何 hemisystem 上的研究，也出現一類筆者 10 多年前提出但懸而未決的距離正則圖，這激發筆者想去學 hemisystem。

Pooling designs with d-disjunct property and block weight $d+1$

Chih-wen Weng (翁志文)

(with Yu-pei Huang and Wu, Hsin-Jung)

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July 12, 2010

Definition

An incidence structure $(P, \mathcal{B})$ is called *d-disjunct* if any block in $\mathcal{B}$ is not covered by the union of d other blocks.

Assume $P = \{1, 2, \ldots, v\}$, $\mathcal{B} = \{B_1, B_2, \ldots, B_b\}$ and M is be the incidence matrix of $(P, \mathcal{B})$, i.e.

$$M_{ij} = \begin{cases} 1, & i \in B_j; \\ 0, & i \notin B_j \end{cases}$$

for $1 \leq i \leq v$ and $0 \leq j \leq b$.

The incidence matrix M of a d-disjunct incidence structure can be used in non-adaptive group testing programming, in which $v << b$ is preferred.

1. Let M be a $v \times b$ incidence matrix of an incidence structure and set $F_2 = \{0, 1\}$. Define the output function $o_M : F_2^b \to F_2^v$ by
$$o_M(P) := M \star P = \bigcup_{P_i=1} M_i,$$
where $\star$ is the matrix product by using Boolean sum to replace addition.
2. If the incidence structure is d-disjunct, then $o_M \upharpoonright F_2^b(\leq d)$ is known to be injective, where $F_2^b(\leq d)$ is the set of binary vectors of length b and Hamming weight at most d.
3. This means that for each element u in the image of o_M on $F_2^b(\leq d)$, we know which $P \in F_2^b$ to have $o_M(P) = u$.
4. In application, P is interpreted as the unknown infected subset $\{j \mid P_j = 1\}$ of a given set of b items, and u is interpreted as the sequence of test results. Then the injective property of o_M implies that the infected subset can be determined from the sequence of test results if the number of infected items is known in advance to be at most d.

Example

The following 4×6 binary matrix is used to detect the infected item in $\{1,2,3,4,5,6\}$, if the infected item is known to be at most one in advance (but do not know which one):

$$\left(\begin{array}{c|cccccccc}
\text{Tests/Items} & 1 & 2 & \color{red}{\mathbf{3}} & 4 & 5 & 6 & & o_M((0,0,1,0,0,0)^T) \\
\hline
\text{one} & 1 & 1 & 1 & 0 & 0 & 0 & \rightarrow & 1 \\
\text{Two} & 1 & 0 & 0 & 1 & 1 & 0 & \rightarrow & 0 \\
\text{Three} & 0 & 1 & 0 & 1 & 0 & 1 & \rightarrow & 0 \\
\text{Four} & 0 & 0 & 1 & 0 & 1 & 1 & \rightarrow & 1
\end{array}\right)$$

If there are two infected items, the above 4×6 matrix does not work for detecting them. For example, both the infected sets $\{3,4\}$ and $\{1,6\}$ have the same output $(1,1,1,1)^T$. So it is impossible to recover the infected set from the output $(1,1,1,1)^T$.

Relation to t-design

Definition

An incidence structure $(P, \mathcal{B})$ is called a t-(v, k, λ) design if

1. $|P| = v$,
2. $|B| = k$ for and $B \in \mathcal{B}$, and
3. any t-subset of P is contained in exactly λ blocks in $\mathcal{B}$.

Remark

1. A 2-$(v, k, 1)$ design is $(k-1)$-disjunct since a block has k points and it intersects another block in at most one point, so $k-1$ other blocks can cover at most $k-1$ points of a block, leaving at least one point uncovered.
2. If any point is incidence in at least two blocks, then any block in a d-disjunct matrix has size at least $d+1$.
3. A d-disjunct incidence structure is called a pooling design.

First result

Theorem

Let $(P, \mathcal{B})$ be a d-disjunct pooling design with constant block size $d+1$, and define $v = |P|$ and $b = |\mathcal{B}|$. Then $b \leq \max\{v(v-1)/d(d+1), v-d\}$. Moreover if $v - d \leq v(v-1)/d(d+1)$, then the above upper bound of b is reached if and only if $(P, \mathcal{B})$ is a 2-$(v, d+1, 1)$ design.

The $v \times b$ incidence matrix

$$M = \begin{pmatrix} I_b \\ J_d \end{pmatrix}$$

satisfies the equality $b = v - d$, where I_b is the $b \times b$ identity matrix and J_d is the $d \times d$ all 1's matrix.

The following example gives the equality in previous theorem for $d = q - 1$.

Example

$(2 - (q^2, q, 1)$ design$)$ Let q be a prime power. The affine plane F_q^2 over F_q has q^2 points and $q^2 + q$ lines. Of course any line has q points and any two lines intersect at at most 1 point. Hence the points-lines incidence matrix is $v \times b$ d-disjunct with with constant weight w, where $v = q^2$, $b = q^2 + q$ and $w = q = d + 1$ satisfy

$$b = q^2 + q = v(v - 1)/d(d + 1).$$

The first q which is not a prime power is when $q = 6 = d + 1$. In this case the equality does not hold by the Bruck-Ryser-Chowla Theorem. Then there is no 5-disjunct pooling design with 36 points, 42 blocks and constant bock size 6. We will construct a 5-disjunct pooling design with 36 points, 37 blocks and constant block size 6.

Forward difference property

1. Let q be a prime power and $m \geq q$ be an integer.
2. Let $F_q := \{0, a^0, a^1, \ldots, a^{q-2}\}$ denote the finite field of q elements, where a is a generator of the cyclic multiplication group $F_q^* := F_q - \{0\}$.
3. Let $m \geq q$ be an integer. Let $\mathbb{Z}_m := \{0, 1, \ldots, m-1\}$ be the addition group of integers modulo m. We use the order of integers to order the elements in $\mathbb{Z}_m$, e.g. $0 < 1$.
4. A subset $T \subseteq \mathbb{Z}_m \times F_q$ is said to have the forward difference distinct property in $\mathbb{Z}_m \times F_q$ if the forward difference set

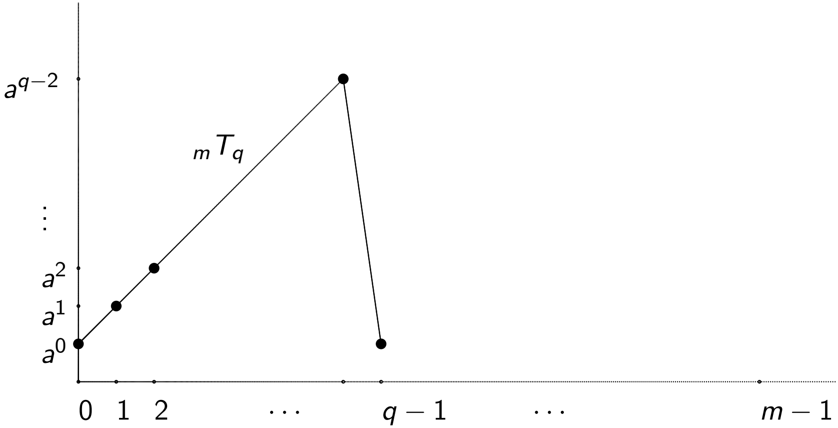
$$FD_T := \{(j, y) - (i, x) \mid (i, x), (j, y) \in T \text{ with } i < j\}$$

consists of $\frac{|T|(|T|-1)}{2}$ elements.

The Set ${}_mT_q$

Let ${}_mT_q \subseteq \mathbb{Z}_m \times F_q$ be defined by

$$ {}_mT_q = \{(i, a^i) \mid i \in \mathbb{Z}_m, 0 \le i \le q-1\}. $$



The Set ${}_5T_5$

For $q = 5$, $a = 2$,

$$ {}_5T_5 = \{(0,1),(1,2),(2,4),(3,3),(4,1)\} $$

and

$$ \begin{aligned} FD_{{}_5T_5} = \{ \quad & (1,1),(1,2),(1,4),(1,3) \\ & (2,3),(2,1),(2,2) \\ & (3,2),(3,4) \\ & (4,0) \quad \}. \end{aligned} $$

Since $|FD_{{}_5T_5}| = 10$, the set ${}_5T_5$ has the forward difference distinct property in $\mathbb{Z}_5 \times F_5$.

${}_mT_q$ has the forward difference distinct property

Lemma

The set ${}_mT_q$ has the forward difference distinct property in $\mathbb{Z}_m \times T_q$.

Proof.

Given any pair $(c, d) \in \mathbb{Z}_m \times F_q$, solve the equations

$$(c, d) = (j, a^j) - (i, a^i)$$

for $0 \le i < j \le q-1$. Note that $1 \le c \le q-1$ to have a solution. If $c = q-1$ then $j = q-1$ and $i = 0$. If $c \ne q-1$ then $a^i = d/(a^{j-i} - 1) = d/(a^c - 1)$ and $j = c + i$. In each case the pair $(i, a^i), (j, a^j)$ is unique determined by the element $(c, d) \in \mathbb{Z}_m \times F_q$. □

Difference Property

A subset $T \subseteq \mathbb{Z}_m \times F_q$ is said to have the difference distinct property in $\mathbb{Z}_m \times F_q$ if the difference set $D_T := -FD_T \cup FD_T$ consists of $|T|(|T|-1)$ elements.

Since ${}_mT_q$ intersects a vertical line in at most one point, we find $(0, x) \notin D_{{}_mT_q}$ for any $x \in F_q$.

Non-example ($m = q = 5$)

We have seen

$$\begin{aligned} FD_{{}_5T_5} = \{ \quad & (1,1),(1,2),(1,4),(1,3) \\ & (2,3),(2,1),(2,2) \\ & (3,2),(3,4) \\ & (4,0) \quad \}. \end{aligned}$$

Hence

$$\begin{aligned} -FD_{{}_5T_5} = \{ \quad & (4,4),(4,3),(4,1),(4,2) \\ & (3,2),(3,4),(3,3) \\ & (2,3),(2,1) \\ & (1,0) \quad \}. \end{aligned}$$

Since $|D_{{}_5T_5}| = 16 \neq 20$, the set ${}_5T_5$ does not have the difference distinct property in $\mathbb{Z}_5 \times F_5$.

Example ($m-1=q=5$)

$$\begin{aligned} FD_{{}_6T_5} = \{ \quad & (1,1),(1,2),(1,4),(1,3) \\ & (2,3),(2,1),(2,2) \\ & (3,2),(3,4) \\ & (4,0) \quad \}. \end{aligned}$$

Hence considering as the negative in $\mathbb{Z}_6 \times F_5$, we have

$$\begin{aligned} -FD_{{}_6T_5} = \{ \quad & (5,4),(5,3),(5,1),(5,2) \\ & (4,2),(4,4),(4,3) \\ & (3,3),(3,1) \\ & (2,0) \quad \}. \end{aligned}$$

Since $|D_{{}_6T_5}| = 20$ now, the set ${}_6T_5$ has the difference distinct property in $\mathbb{Z}_6 \times F_5$.

${}_{2q-1}T_q$ has the difference distinct property

Lemma

For $m \geq 2q-1$, the set ${}_mT_q$ has the difference distinct property in $\mathbb{Z}_m \times T_q$.

Proof.

We have $|FD_{{}_mT_q}| = |-FD_{{}_mT_q}| = q(q-1)/2$. The first coordinate of an element in $FD_{{}_{2q-1}T_q}$ runs from 1 to $q-1$, and the first coordinate of an element in $-FD_{{}_{2q-1}T_q}$ from $m+1-q$ to $m-1$. The assumption $m \geq 2q-1$ implies $-FD_{{}_{2q-1}T_q} \cap FD_{{}_{2q-1}T_q} = \emptyset$. □

${}_{2q-3}T_q$ has the difference distinct property

Lemma

The set ${}_mT_q$ has the difference distinct property for $m = 2q - 3$.

Proof.

We have $|FD_{T_{m,q}}| = |-FD_{T_{m,q}}| = q(q-1)/2$. Let $(c, d) \in FD_{T_{m,q}}$. If $m = 2q - 3$, then $1 \leq c \leq q - 1$ and $q - 2 \leq -c \leq 2q - 4$. Thus the repetition of differences occurs only when $c = q - 2$ or $c = q - 1$. Note that $d = 0$ iff $c = q - 1$, and $-d = 0$ iff $-c = q - 2$. For $c = q - 2$, suppose $(c', d') \in -FD_{{}_mT_q}$ and $(c', d') = (c, d)$. Then we have $c' = q - 2$ and $d' = 0$. Hence $d = 0$, a contradiction. Similarly for $c = q - 1$, we have $d = 0$ but $(q - 1, 0) \notin -FD_{T_{m,q}}$.

$\square$

${}_{2q-4}T_q$ has the difference distinct property

Lemma

The set ${}_mT_q$ has the difference distinct property for $m = 2q - 4$.

Proof.

Let $(c, d) \in FD_{T_{m,q}}$. Since $m = 2q - 4$, we have $1 \le c \le q - 1$ and $q - 3 \le -c \le 2q - 5$. Thus the repetition of differences occurs only when $c = q - 3$, $q - 2$ or $q - 1$. Note that $d = 0$ iff $c = q - 1$, and $-d = 0$ iff $-c = q - 3$. For $c = q - 1$ or $c = q - 3$, similar process as the above $m = 2q - 3$ case can be applied to get contradictions. For $c = q - 2$, $-c = q - 2$. Thus a repetition implies that there are $(q - 2, d_1), (q - 2, d_2) \in FD_{T_{m,q}}$ such that $d_1 = -d_2$. Note that the only two elements of $FD_{T_{m,q}}$ with the first coordinate $q - 2$ are $(q - 2, a^{q-2} - 1)$ and $(q - 2, a^{q-1} - a)$, where a is the generator chosen for F_q^*. So we have $a^{q-2} - 1 = -(a^{q-1} - a)$ and this implies $a = -1$, also a contradiction. □

Lines with any two intersecting in at most a point

Proposition

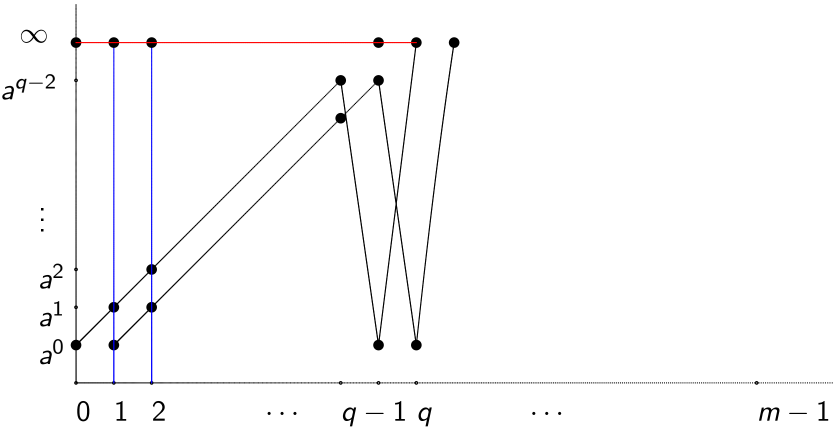
Suppose that ${}_mT_q \subseteq \mathbb{Z}_m \times F_q$ *has the difference distinct property in* $\mathbb{Z}_m \times F_q$*. Set* $\mathcal{B} = \{u +_m T_q \mid u \in \mathbb{Z}_m \times F_q\}$*. Then* $|L \cap L'| \leq 1$ *for any distinct* $L, L' \in \mathcal{B}$*.*

Proof.

Routine. $\square$

1. Note that there are mq lines and mq points in $\mathbb{Z}_m \times F_q$, and a line has $q = |T|$ points with q different first coordinates.
2. Apparently more lines can be added to $\mathcal{B}$ still having the conclusion of the above proposition, for example, adding vertical lines to $\mathcal{B}$.
3. We will add m more points to P, add $m+1$ lines to $\mathcal{B}$, and add one more point to each original line in $\mathcal{B}$.

A picture for the finial result



Lines in $Z_m \times (F_q \cup \{\infty\})$

Second and final result

Theorem

There exists a q-disjunct pooling design $(P, \mathcal{B})$ *with* $|P| = m(q+1)$, $|\mathcal{B}| = m(q+1)+1$ *and constant block weight* $q+1$, *where* q *is a prime power, and* m *is an integer at least three satisfying* $m = 2q-4$, $m = 2q-3$ *or* $m \geq 2q-1$.

By choosing $q = 5$ and $m = 2q - 4 = 6$, there exists a 5-disjunct pooling design with 36 points, 37 blocks and constant block size 6.

The end

Thank you for your attention.

無研發成果推廣資料

98 年度專題研究計畫研究成果彙整表

計畫主持人：翁志文	計畫編號：98-2115-M-009-002-
計畫名稱：具封閉性質的距離正則圖之研究(3/3)	

成果項目			量化			單位	備註（質化說明：如數個計畫共同成果、成果列為該期刊之封面故事...等）
			實際已達成數（被接受或已發表）	預期總達成數(含實際已達成數)	本計畫實際貢獻百分比		
國內	論文著作	期刊論文	0	0	100%	篇	
		研究報告/技術報告	0	0	100%		
		研討會論文	0	0	100%		
		專書	0	0	100%		
	專利	申請中件數	0	0	100%	件	
		已獲得件數	0	0	100%		
	技術移轉	件數	0	0	100%	件	
		權利金	0	0	100%	千元	
	參與計畫人力（本國籍）	碩士生	0	0	100%	人次	
		博士生	0	0	100%		
		博士後研究員	0	0	100%		
		專任助理	0	0	100%		
國外	論文著作	期刊論文	0	2	100%	篇	
		研究報告/技術報告	0	0	100%		
		研討會論文	0	0	100%		
		專書	0	0	100%	章/本	
	專利	申請中件數	0	0	100%	件	
		已獲得件數	0	0	100%		
	技術移轉	件數	0	0	100%	件	
		權利金	0	0	100%	千元	
	參與計畫人力（外國籍）	碩士生	0	0	100%	人次	
		博士生	0	2	100%		
		博士後研究員	0	0	100%		
		專任助理	0	0	100%		

其他成果 (無法以量化表達之成果如辦理學術活動、獲得獎項、重要國際合作、研究成果國際影響力及其他協助產業技術發展之具體效益事項等，請以文字敘述填列。)	無

	成果項目	量化	名稱或內容性質簡述
科教處計畫加填項目	測驗工具(含質性與量性)	0	
	課程/模組	0	
	電腦及網路系統或工具	0	
	教材	0	
	舉辦之活動/競賽	0	
	研討會/工作坊	0	
	電子報、網站	0	
	計畫成果推廣之參與（閱聽）人數	0	

國科會補助專題研究計畫成果報告自評表

請就研究內容與原計畫相符程度、達成預期目標情況、研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）、是否適合在學術期刊發表或申請專利、主要發現或其他有關價值等，作一綜合評估。

1. 請就研究內容與原計畫相符程度、達成預期目標情況作一綜合評估

■達成目標

□未達成目標（請說明，以 100 字為限）

□實驗失敗

□因故實驗中斷

□其他原因

說明：

2. 研究成果在學術期刊發表或申請專利等情形：

論文：□已發表 ■未發表之文稿 □撰寫中 □無

專利：□已獲得 □申請中 ■無

技轉：□已技轉 □洽談中 ■無

其他：（以 100 字為限）

3. 請依學術成就、技術創新、社會影響等方面，評估研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）（以 500 字為限）

完成某類距離正則圖的刻劃並進而證明其中一類的不存在性,

能讓後人免於懷疑