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Optimal Control of an M/G/1/K Queueing System with Combined *F* Policy and Startup Time

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Abstract We investigate the optimal management problem of an M/G/1/K queueing system with combined F policy and an exponential startup time. The F policy queueing problem investigates the most common issue of controlling the arrival to a queueing system. We present a recursive method, using the supplementary variable technique and treating the supplementary variable as the remaining service time, to obtain the steady state probability distribution of the number of customers in the system. The method is illustrated analytically for exponential service time distribution. A cost model is established to determine the optimal management F policy at minimum cost. We use an efficient Maple computer program to calculate the optimal value of F and some system performance measures. Sensitivity analysis is also investigated.

Keywords F policy, M/G/1/K queue · Optimization · Recursive methods · Sensitivity analyses · Startup times · Supplementary variables

1 Introduction

A supplementary variable technique is used to study the optimal management problem of the F policy M/G/1/K queueing system where the server needs a startup time before allowing customers in the system and $K < \infty$ denotes the maximum number

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of customers in the system. The method of controlling arrivals focuses on reducing the number of customers in the system. The model presented in this paper is very useful in real-life situations since the control of arriving customers is considered.

Gupta [1] developed analytic closed-form solutions for the F policy M/M/1/K queueing system with an exponential startup time. However, steady-state analytic solutions of the F policy queueing systems with interarrival times or service times distribution of the general type have rarely been found. It is extremely difficult, if not possible, to develop an explicit expression for the probability distributions of the number of customers in the system. However, it will become particularly helpful to use the supplementary variable technique for the non-Markovian queueing system having general interarrival times or general service times. The supplementary variable technique was first introduced by Cox [2]. Based on this technique, Gupta and Rao [3, 4] provided a recursive method to develop the steady-state probability distribution of the number of failed machines in the M/G/1 machine repair problem with no spares and the cold-standby M/G/1 machine repair problem, respectively.

Past work regarding queues may be divided into two categories: (i) the case of controlling the service and (ii) the case of controlling the arrivals. In the case of controlling the service, the N policy M/M/1 queueing system without startup was first introduced by Yadin and Naor [5]. This model was extended by Bell [6, 7], Heyman [8], Kimura [9], Teghem [10], Wang and Ke [11], and others. Wang and Ke [11] presented a recursive method and applied the supplementary variable technique to develop the steady-state probability distributions of the number of customers for the N policy M/G/1 queueing system. Recently, Ke and Wang [12] developed a recursive method and used the supplementary variable technique to compute the steady-state probability distributions of the number of customers for the N policy G/M/1 queueing system. The server startup corresponds to the preparatory work of the server before beginning the service. In some real-life situations, the server often needs a startup time before starting the service. The research of several authors on queueing systems with startup time focus mainly on the N policy M/G/1 queues. The N policy M/M/1 queueing system with exponential startup time was first proposed by Baker [13]. Borthakur et al. [14] extended the Baker model to the general startup time. The N policy M/G/1 queueing system with startup time was analyzed by Medhi and Templeton [15], Takagi [16], Lee and Park [17], Hur and Paik [18], and so on. Recently, Ke [19] presented a recursive method and used the supplementary variable technique to investigate the operating characteristics for the N policy G/M/1 queueing system with exponential startup time. In the case of controlling the arrivals, so far very few researchers have examined the F policy M/G/1/K queueing system with server startup or the F policy G/M/1/K queueing system with server startup. Steady-state analytic solutions for the F policy M/M/1/K queueing system with an exponential startup time was first derived by Gupta [1]. Through a series of propositions, the relationship between the N policy and the F policy are established by Gupta [1].

The primary objective of this paper is threefold. First, we develop a recursive method, using the supplementary variable technique and treating the supplementary variable as the remaining service time, to develop the steady-state probability distributions of the number of customers for the F policy M/G/1/K queueing system. Second, to illustrate a recursive method, we present one simple example for exponential service time distribution. Third, we use an efficient Maple computer program to determine the optimal management F policy to minimize the total expected cost per customer per unit time.

2 Description of the System

We consider controlling the arrivals to a finite capacity M/G/1/K queueing system with combined F policy exponential startup time. It is assumed that customers arrive according to a Poisson process with parameter λ, and the service time of the successive customers are independent and identically distributed (i.i.d.) random variables having a distribution $S(u)$ $(u \geq 0)$, a probability density function $s(u)$ $(u \geq 0)$ and mean service time s_1. The arrival process is independent of the service process. We assume that arriving customers form a single waiting line based on the order of their arrivals; that is, the first-come, first-served discipline is followed. Suppose that the server can serve only one customer at a time. Customers entering into the service facility and finding that the server is busy have to wait in the queue until the server is available. Gupta [1] first introduced the concept of a F policy. The definition of a F policy is described as follows: When the number of customers in the system reaches its capacity K (i.e. the system becomes full), no further arriving customers are allowed to enter the system until enough customers in the system have been served so that the number of customers in the system decreases to a threshold value F $(0 \leq F < K - 1)$. At that time, the server requires to take an exponential startup time with parameter β to start allowing customers in the system. Thus, the system operates normally until the number of customers in the system reaches its capacity at which time the above process is repeated all over again.

3 Steady State Results

We use the following supplementary variable: $U \equiv$ remaining service time for the customer in service. The state of the system at time t is given by

$$N(t) \equiv \text{number of customers in the system},$$

$$U(t) \equiv \text{remaining service time for the customer being served}.$$

Let us define

$$P_{0,n}(u,t)\mathrm{d}u = \Pr\{N(t) = n, u < U(t) \leq u + \mathrm{d}u\}, \quad u \geq 0,\ n = 0, 1, \ldots, K,$$

$$P_{1,n}(u,t)\mathrm{d}u = \Pr\{N(t) = n, u < U(t) \leq u + \mathrm{d}u\}, \quad u \geq 0,\ n = 0, 1, \ldots, K-1,$$

$$P_{0,n}(t) = \int_0^\infty P_{0,n}(u,t)\mathrm{d}u, \quad n = 0, 1, \ldots, K,$$

$$P_{1,n}(t) = \int_0^\infty P_{1,n}(u,t)\mathrm{d}u, \quad n = 0, 1, \ldots, K-1.$$

Relating the state of the system at time t and $t+\mathrm{d}t$, we obtain

$$(\mathrm{d}/\mathrm{d}t)P_{0,0}(t) = -\beta P_{0,0}(t) + P_{0,1}(0,t), \tag{1}$$

$$(\partial/\partial t - \partial/\partial u)P_{0,n}(u,t) = -\beta P_{0,n}(u,t) + P_{0,n+1}(0,t)s(u), \quad 1 \le n \le F, \tag{2}$$

$$(\partial/\partial t - \partial/\partial u)P_{0,n}(u,t) = P_{0,n+1}(0,t)s(u), \quad F+1 \le n \le K-1, \tag{3}$$

$$(\partial/\partial t - \partial/\partial u)P_{0,K}(u,t) = \lambda P_{1,K-1}(u,t), \tag{4}$$

$$(\mathrm{d}/\mathrm{d}t)P_{1,0}(t) = -\lambda P_{1,0}(t) + \beta P_{0,0}(t) + P_{1,1}(0,t), \tag{5}$$

$$\begin{aligned}(\partial/\partial t - \partial/\partial u)P_{1,1}(u,t) = &-\lambda P_{1,1}(u,t) + \beta P_{0,1}(u,t)\\ &+ \lambda P_{1,0}(t)s(u) + P_{1,2}(0,t)s(u),\end{aligned} \tag{6}$$

$$\begin{aligned}(\partial/\partial t - \partial/\partial u)P_{1,n}(u,t) = &-\lambda P_{1,n}(u,t) + \beta P_{0,n}(u,t) + \lambda P_{1,n-1}(u,t)s(u)\\ &+ P_{1,n+1}(0,t)s(u), \quad 2 \le n \le F,\end{aligned} \tag{7}$$

$$\begin{aligned}(\partial/\partial t - \partial/\partial u)P_{1,n}(u,t) = &-\lambda P_{1,n}(u,t) + \lambda P_{1,n-1}(u,t)s(u)\\ &+ P_{1,n+1}(0,t)s(u), \quad F+1 \le n \le K-2,\end{aligned} \tag{8}$$

$$(\partial/\partial t - \partial/\partial u)P_{1,K-1}(u,t) = -\lambda P_{1,K-1}(u,t) + \lambda P_{1,K-2}(u,t). \tag{9}$$

In steady state, let us define

$$\begin{aligned}
P_{0,n} &= \lim_{t\to\infty} P_{0,n}(t), \quad n = 0,1,\ldots,K,\\
P_{1,n} &= \lim_{t\to\infty} P_{0,n}(t), \quad n = 0,1,\ldots,K-1,\\
P_{0,n}(u) &= \lim_{t\to\infty} P_{0,n}(u,t), \quad n = 1,2,\ldots,F,\\
P_{1,n}(u) &= \lim_{t\to\infty} P_{1,n}(u,t), \quad n = 0,1,\ldots,K-1
\end{aligned}$$

and further define

$$P_{0,n}(u) = P_{0,n}s(u), \quad n = 1,2,\ldots,F. \tag{10}$$

From (1–10), we can obtain easily the following steady state equations:

$$0 = -\beta P_{0,0} + P_{0,1}(0), \tag{11}$$

$$-(\mathrm{d}/\mathrm{d}u)P_{0,n}(u) = -\beta P_{0,n}s(u) + P_{0,n+1}(0)s(u), \quad 1 \le n \le F, \tag{12}$$

$$-(\mathrm{d}/\mathrm{d}u)P_{0,n}(u) = P_{0,n+1}(0)s(u), \quad F+1 \le n \le K-1, \tag{13}$$

$$-(\mathrm{d}/\mathrm{d}u)P_{0,K}(u) = \lambda P_{1,K-1}(u), \tag{14}$$

$$0 = -\lambda P_{1,0} + \beta P_{0,0} + P_{1,1}(0), \tag{15}$$

$$-(\mathrm{d}/\mathrm{d}u)P_{1,1}(u) = -\lambda P_{1,1}(u) + \beta P_{0,1}s(u) + \lambda P_{1,0}s(u) + P_{1,2}(0)s(u), \tag{16}$$

$$\begin{aligned}-(\mathrm{d}/\mathrm{d}u)P_{1,n}(u) = &-\lambda P_{1,n}(u) + \beta P_{0,n}s(u) + \lambda P_{1,n-1}(u) + P_{1,n+1}(0)s(u),\\ &2 \le n \le F,\end{aligned} \tag{17}$$

$$-(\mathrm{d}/\mathrm{d}u)P_{1,n}(u) = -\lambda P_{1,n}(u) + \lambda P_{1,n-1}(u) + P_{1,n+1}(0)s(u),$$
$$F+1 \le n \le K-2, \tag{18}$$
$$-(\mathrm{d}/\mathrm{d}u)P_{1,K-1}(u) = -\lambda P_{1,K-1}(u) + \lambda P_{1,K-2}(u). \tag{19}$$

Further define

$$S^*(\theta) = \int_0^\infty e^{-\theta u}\mathrm{d}S(u) = \int_0^\infty e^{-\theta u}s(u)\mathrm{d}u,$$
$$P^*_{0,n}(\theta) = \int_0^\infty e^{-\theta u}P_{0,n}(u)\mathrm{d}u,$$
$$P^*_{1,n}(\theta) = \int_0^\infty e^{-\theta u}P_{1,n}(u)\mathrm{d}u,$$
$$P_{0,n} = P^*_{0,n}(0) = \int_0^\infty P_{0,n}(u)\mathrm{d}u,$$
$$P_{1,n} = P^*_{1,n}(0) = \int_0^\infty P_{1,n}(u)\mathrm{d}u,$$
$$\int_0^\infty e^{-\theta u}\frac{\partial}{\partial u}P_{0,n}(u)\mathrm{d}u = \theta P^*_{0,n}(\theta) - P_{0,n}(0),$$
$$\int_0^\infty e^{-\theta u}\frac{\partial}{\partial u}P_{1,n}(u)\mathrm{d}u = \theta P^*_{1,n}(\theta) - P_{1,n}(0).$$

Therefore, if the LST is taken of both sides of (12–14) and (16–19), it is found that

$$-\theta P^*_{0,n}(\theta) = -\beta P_{0,n}S^*(\theta) + P_{0,n+1}(0)S^*(\theta) - P_{0,n}(0), \quad 1 \le n \le F, \tag{20}$$
$$-\theta P^*_{0,n}(\theta) = P_{0,n+1}(0)S^*(\theta) - P_{0,n}(0), \quad F+1 \le n \le K-1, \tag{21}$$
$$-\theta P^*_{0,K}(\theta) = \lambda P^*_{1,K}(\theta) - P_{0,K}(0), \tag{22}$$
$$(\lambda-\theta)P^*_{1,1}(\theta) = \beta P_{0,1}S^*(\theta) + \lambda P_{1,0}S^*(\theta) + P_{1,2}(0)S^*(\theta) - P_{1,1}(0), \tag{23}$$
$$(\lambda-\theta)P^*_{1,n}(\theta) = \beta P_{0,n}S^*(\theta) + \lambda P^*_{1,n-1}(\theta) + P_{1,n+1}(0)S^*(\theta) - P_{1,n}(0),$$
$$2 \le n \le F, \tag{24}$$
$$(\lambda-\theta)P^*_{1,n}(\theta) = \lambda P^*_{1,n-1}(\theta) + P_{1,n+1}(0)S^*(\theta) - P_{1,n}(0),$$
$$F+1 \le n \le K-2, \tag{25}$$
$$(\lambda-\theta)P^*_{1,K-1}(\theta) = \lambda P^*_{1,K-2}(\theta) - P_{1,K-1}(0). \tag{26}$$

3.1 Recursive Method

A recursive method is developed to obtain $P^*_{0,n}(0)$ and $P^*_{1,n}(0)$. Our solution algorithm will first obtain $P_{0,n}(0)$ $(1 \le n \le K)$ which are then used for finding $P^*_{0,n}(0)$.

Using (11) and setting $\theta = 0$ in (20) and (21), we get

$$P_{0,n}(0) = \beta \sum_{i=0}^{\zeta_{n-1}} P_{0,i}, \quad 1 \le n \le K, \quad \zeta_n = \begin{cases} n, \ 0 \le n \le F-1, \\ F, \ F \le n \le K, \end{cases} \tag{27}$$

and

$$\begin{aligned} P_{0,n+1}(0) &= -\beta \varphi_{n,F} P_{0,n} + P_{0,n}(0), \quad 1 \le n \le K-1, \\ \varphi_{n,F} &= \begin{cases} 1, \ 1 \le n \le F, \\ 0, \ \text{otherwise}. \end{cases} \end{aligned} \tag{28}$$

Using (28) in (20) and (21), we get

$$P^*_{0,n}(\theta) = \{[1 - S^*(\theta)]/\theta\} P_{0,n}(0), \quad 1 \le n \le K-1. \tag{29}$$

Taking $\lim_{\theta \to 0}$ in (29) and using L'Hôspital's rule once gives

$$P^*_{0,n}(0) = s_1 P_{0,n}(0), \quad 1 \le n \le K-1, \tag{30}$$

where $s_1 = -S^{*(1)}(0)$ is the mean service time.

Using (27) in (30), we have

$$P^*_{0,n}(0) = \phi_n P_{0,0}, \quad 1 \le n \le K-1, \tag{31}$$

$$\phi_n = \begin{cases} 1, & n = 0, \\ s_1 \beta (1 + s_1 \beta)^{\zeta_{n-1}}, & 1 \le n \le K. \end{cases} \tag{32}$$

Thus, $P^*_{0,1}(0), P^*_{0,2}(0), \dots, P^*_{0,K-1}(0)$ can be obtained by using (31).

Next, we derive the expressions of $P_{1,n}(0)$ $(1 \le n \le K)$ in terms of $P_{1,0}$ and $P_{0,0}$. Using (31) in (23–24) and then setting $\theta = \lambda$ in (23–26), we finally obtain

$$P_{1,2}(0) = [P_{1,1}(0) - \beta \phi_1 P_{0,0} S^*(\lambda) - \lambda P_{1,0} S^*(\lambda)]/S^*(\lambda), \tag{33}$$

$$P_{1,n+1}(0) = [P_{1,n}(0) - \beta \varphi_{n,F} \phi_n P_{0,0} S^*(\lambda) - \lambda P^*_{1,n-1}(\lambda)]/S^*(\lambda),$$
$$2 \le n \le K-2, \tag{34}$$

$$P_{1,K-1}(0) = \lambda P^*_{1,K-2}(\lambda). \tag{35}$$

To obtain $P^*_{1,n-1}(\lambda)$ $(1 \le n \le K-1)$ in (34–35), using (31) in (23–24) again, differentiating (23–26) $(l-1)$ times with respect to θ and setting $\theta = \lambda$, we finally get

$$P^{*(l-1)}_{1,1}(\lambda) = -(S^{*(l)}(\lambda)/l)[\lambda P_{1,0} + \beta \phi_1 P_{0,0} + \lambda P_{1,2}(0)],$$
$$l = 1, \dots, K-2, \tag{36}$$

$$P_{1,n}^{*(l-1)}(\lambda) = -(1/l)[P_{1,n+1}(0)S^{*(l)}(\lambda) + \beta\varphi_{n,F}\phi_n P_{0,0}S^{*(l)}(\lambda) + \lambda P_{1,n-1}^{*(l)}(\lambda)],$$
$$2 \le n \le K-2, \quad l = 1, \dots, K-n-1, \tag{37}$$
$$P_{1,K-1}^{*}(\lambda) = -\lambda P_{1,K-2}^{*(1)}(\lambda), \tag{38}$$

where $P_{1,n}^{*(0)}(\lambda) = P_{1,n}^{*}(\lambda)$ and $S^{*(l)}(\theta) = (d^l/d\theta^l)S^*(\theta)$ denote the lth derivative of $S^*(\theta)$.

Solving (36–38) recursively, we obtain

$$P_{1,n}^{*}(\lambda) = -\ell_n S^*(\lambda)P_{1,0} - \sum_{i=1}^{\zeta_n}[\beta\ell_{n-i+1}\phi_i S^*(\lambda)/\lambda]P_{0,0}$$
$$- \sum_{i=1}^{n}[\ell_{n-i+1}S^*(\lambda)/\lambda]P_{1,i+1}(0), \quad 1 \le n \le K-1, \tag{39}$$

where

$$\ell_n = \begin{cases} -[(-\lambda)^n S^{*(n)}(\lambda)/n!S^*(\lambda)], & 1 \le n \le K-1, \\ 0, & \text{otherwise.} \end{cases} \tag{40}$$

Using (39) in (34), we obtain

$$P_{1,n}(0) = [1/S^*(\lambda)]P_{1,n-1}(0) + \sum_{i=1}^{n-2}\ell_{n-i-1}P_{1,i+1}(0)$$
$$+ \beta\left[\sum_{i=1}^{\zeta_{n-2}}\ell_{n-i-1}\phi_i - \varphi_{n-1,F}\phi_{n-1}\right]P_{0,0}$$
$$+ \lambda\ell_{n-2}P_{1,0}, \quad 3 \le n \le K-1. \tag{41}$$

We further define

$$\Psi_n = \begin{cases} 1, \quad n = 0, \\ \displaystyle\sum_{1\le k\le n}\ \sum_{\tau_1+\tau_2+\cdots+\tau_k=n} \kappa_{\tau_1}\kappa_{\tau_2}\cdots\kappa_{\tau_k}, \\ \quad n = 1, 2, \dots, K-3,\ \tau_1, \tau_2, \dots, \tau_k \in \{1, 2, \cdots, n\} \\ 0, \quad \text{otherwise,} \end{cases} \tag{42}$$

where

$$\kappa_n = \begin{cases} [1/S^*(\lambda)] + \ell_1, & n = 1, \\ \ell_n, & n = 2, 3, \dots, K-3, \\ 0, & \text{otherwise.} \end{cases} \tag{43}$$

Remark 3.1 The representative meaning of the above formulation (42) is to sum up all possible products of k κs in which the total of subscript values of κ equals n. We give an easily understood example for $n = 4$:

$$\begin{aligned}\Psi_4 &= \kappa_4 + \kappa_3\kappa_1 + \kappa_2\kappa_2 + \kappa_1\kappa_3 + \kappa_1\kappa_1\kappa_2 + \kappa_1\kappa_2\kappa_1 + \kappa_2\kappa_1\kappa_1 + \kappa_1\kappa_1\kappa_1\kappa_1 \\ &= \kappa_4 + 2\kappa_3\kappa_1 + \kappa_2^2 + 3\kappa_1^2\kappa_2 + \kappa_1^4.\end{aligned}$$

Using (42) and (43) to solve (41) recursively, and including (15) and (33), we finally get

$$P_{1,1}(0) = A(1)P_{1,0} + B(1)P_{0,0}, \tag{44}$$

$$P_{1,n}(0) = \sum_{i=2}^{n} \Psi_{n-i}[A(i)P_{1,0} + B(i)P_{0,0}], \quad 2 \le n \le K-1, \tag{45}$$

where

$$A(n) = \begin{cases} \lambda, & n = 1, \\ \lambda[1 - S^*(\lambda)]/S^*(\lambda), & n = 2, \\ \lambda \ell_{n-2}, & 3 \le n \le K-1, \end{cases} \tag{46}$$

$$B(n) = \begin{cases} -\beta, & n = 1, \\ -\beta[1 + \varphi_{1,F}\phi_1 S^*(\lambda)]/S^*(\lambda), & n = 2, \\ \beta \sum_{i=1}^{\zeta_{n-2}} \ell_{n-i-1}\phi_i - \beta\varphi_{n-1,F}\phi_{n-1}, & 3 \le n \le K-1. \end{cases} \tag{47}$$

Substituting (45), (44), and (35) into (39) finally yields

$$\begin{aligned} P_{1,0} = -\Bigg\{ \Bigg[&\sum_{i=1}^{K-2} \ell_{K-i-1} \sum_{j=2}^{i+1} \Psi(i-j+1)B(j) \\ &+ \sum_{i=2}^{K-1} [\Psi(K-i-1)B(i)/S^*(\lambda)] \\ &+ \sum_{i=1}^{\zeta_{K-2}} \beta\ell_{K-i-1}\phi_i \Bigg] \Bigg/ \Bigg[\sum_{i=1}^{K-2} \ell_{K-i-1} \sum_{j=2}^{i+1} \Psi(i-j+1)A(j) \\ &+ \sum_{i=2}^{K-1} [\Psi(K-i-1)A(i)/S^*(\lambda)] + \lambda\ell_{K-2} \Bigg] \Bigg\} P_{0,0}. \end{aligned} \tag{48}$$

Finally, we develop the steady-state probabilities $P^*_{1,n}(0)$ in terms of $P_{0,0}$. Setting $\theta = 0$ in (23–26) we have

$$P^*_{1,n}(0) = (1/\lambda)\left[\beta \sum_{i=0}^{\zeta_n} \phi_i P_{0,0} + P_{1,n+1}(0)\right], \quad 0 \le n \le K-2, \tag{49}$$

$$P^*_{1,K-1}(0) = (\beta/\lambda) \sum_{i=0}^{F} \phi_i P_{0,0}. \tag{50}$$

As $P_{1,1}(0), P_{1,2}(0), \ldots, P_{1,K-1}(0)$ and $P_{1,0}$ are known, $P^*_{1,1}(0), P^*_{1,2}(0), \ldots, P^*_{1,K-1}(0)$ can be determined recursively using (49) and (50) in terms of $P_{0,0}$.

Now the only unknown quantity is $P^*_{0,K}(0)$, which can be obtained from (22). To find it, differentiating (22) with respect to θ and setting $\theta = 0$, we have

$$P^*_{0,K}(0) = -\lambda P^{*(1)}_{1,K-1}(0). \tag{51}$$

To find $\lambda P^{*(1)}_{1,K-1}(0)$, differentiating (23–26) with respect to θ and setting $\theta = 0$, we finally obtain

$$P^{*(1)}_{1,1}(0) = [P_{1,1} + \beta\phi_1 P_{0,0}S^{*(1)}(0) + \lambda P_{1,0}S^{*(1)}(0) + P_{1,2}(0)S^{*(1)}(0)]/\lambda, \tag{52}$$

$$P^{*(1)}_{1,n}(0) = [P_{1,n} + \beta\varphi_{n,F}\phi_n P_{0,0}S^{*(1)}(0) + \lambda P^{*(1)}_{1,n-1}(0) + P_{1,n+1}(0)S^{*(1)}(0)]/\lambda,$$
$$2 \le n \le K-2, \tag{53}$$

$$P^{*(1)}_{1,K-1}(0) = [P_{1,K-1} + \lambda P^{*(1)}_{1,K-2}(0)]/\lambda. \tag{54}$$

As $P^{*(1)}_{1,1}(0)$ is known completely from (52), the values $P^{*(1)}_{1,n}(0)$ for $n = 2, 3, \ldots, K-1$ can be found recursively from (53) and (54). Therefore, we obtain

$$P^{*(1)}_{1,K-1}(0) = (1/\lambda)\left[\sum_{i=1}^{K-1} P_{1,i} + \beta S^{*(1)}(0)\sum_{i=1}^{F}\phi_n P_{0,0} + S^{*(1)}(0)\sum_{i=2}^{K-1} P_{1,i}(0) + \lambda P_{1,0}S^{*(1)}(0)\right]. \tag{55}$$

Substituting (55) into (51), we have

$$P^*_{0,K}(0) = -\left[\sum_{i=1}^{K-1} P_{1,i} + \beta S^{*(1)}(0)\sum_{i=1}^{F}\phi_n P_{0,0} + S^{*(1)}(0) \times \sum_{i=2}^{K-1} P_{1,i}(0) + \lambda P_{1,0}S^{*(1)}(0)\right]. \tag{56}$$

So $P^*_{0,1}(0), P^*_{0,2}(0), \ldots, P^*_{0,K}(0)$ is known in terms of $P_{0,0}$, which can be determined using the normalizing condition

$$\sum_{i=0}^{K} P_{0,i} + \sum_{i=0}^{K-1} P_{1,i} = 1. \tag{57}$$

To demonstrate the working of the recursive method, we first describe the solution algorithm for calculating the steady state probabilities $P^*_{0,n}(0)$ $(0 \le n \le K)$ and $P^*_{1,n}(0)$ $(0 \le n \le K-1)$. Next, to illustrate the solution algorithm, we provide one simple example for the exponential service time distribution.

3.2 Solution Algorithm

Let F be the threshold, let K be the maximum capacity of the system, and let $S^{*(l)}(\theta)$ be the lth derivative of $S^*(\theta)$, where $l = 1, 2, \ldots, K$. We set the values of F, K, and the LST expression of the service time distribution, namely $S^*(\theta)$. The steps of the solution algorithm are stated as follows:

Step 1. For each $n = 0, 1, \ldots, K$, compute ϕ_n using (32).
Step 2. For each $n = 1, 2, \ldots, K-1$, compute $P^*_{0,n}(0)$ using (31) in terms of $P_{0,0}$.
Step 3. Compute $\ell_n (1 \le n \le K-2)$ and $\kappa_n (1 \le n \le K-3)$ using (40) and (43), respectively.
Step 4. For each $n = 0, 1, \ldots, K-3$, compute Ψ_n using (42).
Step 5. For each $n = 1, 2, \ldots, K-1$, compute $A(n)$ and $B(n)$ using (46) and (47).
Step 6. For each $n = 1, 2, \ldots, K-1$, compute $P_{1,n}(0)$ using (44) and (45) in terms of $P_{1,0}$ and $P_{0,0}$.
Step 7. Compute $P_{1,0}$ using (48) in terms of $P_{0,0}$. Thus, $P_{1,n}(0)(1 \le n \le K-1)$ are determined from Step 6.
Step 8. For each $n = 1, 2, \ldots, K-1$, compute $P^*_{1,n}(0)$ using (49) and (50) in terms of $P_{0,0}$.
Step 9. For $n = K$, compute $P^*_{0,n}(0)$ using (56) in terms of $P_{0,0}$.
Step 10. Determine $P_{0,0}$ using (57). Thus $P^*_{0,n}(0)(n = 1, 2, \ldots, K)$ are obtained from Steps 2 and 9, and $P^*_{1,n}(0)(n = 0, 1, \ldots, K-1)$ are obtained from Steps 7 to 8.

3.3 Simple Example

For the F policy M/M/1/K queueing system, we set the mean service time $s_1 = 1/\mu$, where μ is the service rate. Assume that $F = 1$ and $K = 4$. In this case, we have

$$S^*(\theta) = \mu/(\mu + \theta).$$

Step 1. For each $n = 0, 1, \ldots, 4$, compute ϕ_n.
Using (32), we obtain

$$\phi_0 = 1, \quad \phi_1 = (1-\alpha)/\alpha, \qquad \phi_2 = \phi_3 = \phi_4 = (1-\alpha)/\alpha^2,$$
$$\text{where } \alpha = \mu/(\mu + \beta).$$

Step 2. For each $n = 1, 2, 3$, compute $P^*_{0,n}(0)$ using (31) in terms of $P_{0,0}$.
From (31), we finally get

$$P^*_{0,1}(0) = \phi_1 P_{0,0} = [(1-\alpha)/\alpha] P_{0,0},$$
$$P^*_{0,2}(0) = P^*_{0,3}(0) = \phi_2 P_{0,0} = [(1-\alpha)/\alpha^2] P_{0,0}.$$

Step 3. For each $n = 1, 2$, compute ℓ_n and κ_n using (40) and (43), respectively.
For each $n = 1, 2$, using (40) yields $\ell_1 = -1/(1+\sigma)$ and $\ell_2 = -1/(1+\sigma)^2$, where $\sigma = \mu/\lambda$.
For each $n = 1$, we find from (43) that $\kappa_1 = (1 + \sigma + \sigma^2)/\sigma(1+\sigma)$.

Step 4. For each $n = 0, 1$, compute Ψ_n.
It implies from (42) that $\Psi_0 = 1$ and $\Psi_1 = (1 + \sigma + \sigma^2)/\sigma(1 + \sigma)$.
Step 5. For each $n = 1, 2, 3$, compute $A(n)$ and $B(n)$.
Using (46) and (47), it follows that

$$A(1) = \mu/\sigma, \qquad A(2) = \mu/\sigma^2, \qquad A(3) = -\mu/\sigma(1+\sigma).$$
$$B(1) = -(1-\alpha)\mu/\alpha, \qquad B(2) = -(\alpha+\sigma)(1-\alpha)\mu/\sigma\alpha^2,$$
$$B(3) = -(1-\alpha)^2\mu/(1+\sigma)\alpha^2.$$

Step 6. For $n = 1, 2, 3$, using (44–45), we compute $P_{1,n}(0)$ in terms of $P_{1,0}$ and $P_{0,0}$.
It yields from (44) and (45) that

$$P_{1,1}(0) = A(1)P_{1,0} + B(1)P_{0,0},$$
$$P_{1,2}(0) = \Psi_0[A(2)P_{1,0} + B(2)P_{0,0}],$$
$$P_{1,3}(0) = \Psi_1[A(2)P_{1,0} + B(2)P_{0,0}] + \Psi_0[A(3)P_{1,0} + B(3)P_{0,0}].$$

Step 7. Compute $P_{1,0}$ using (48) in terms of $P_{0,0}$. Thus, $P_{1,n}(0)(1 \le n \le 3)$ are obtained from Step 6.
From (48), we finally have

$$P_{1,0} = [\sigma(1-\alpha)(\alpha+\sigma+\sigma^2+\sigma^3)/\alpha^2]P_{0,0} \quad (P^*_{1,0}(0) = P_{1,0}),$$
$$P_{1,1}(0) = [\sigma\mu(1-\alpha)(1+\sigma+\sigma^2)/\alpha^2]P_{0,0},$$
$$P_{1,2}(0) = [\sigma\mu(1-\alpha)(1+\sigma)/\alpha^2]P_{0,0},$$
$$P_{1,3}(0) = [\sigma\mu(1-\alpha)/\alpha^2]P_{0,0}.$$

Step 8. For each $n = 1, 2, 3$, compute $P^*_{1,n}(0)$ using (49) and (50) in terms of $P_{0,0}$.
Using (49) and (50) yields

$$P^*_{1,1}(0) = [\sigma(1-\alpha)(1+\sigma+\sigma^2)/\alpha^2]P_{0,0},$$
$$P^*_{1,2}(0) = [\sigma(1-\alpha)(1+\sigma)/\alpha^2]P_{0,0},$$
$$P^*_{1,3}(0) = [\sigma(1-\alpha)/\alpha^2]P_{0,0}.$$

Step 9. For $n = 4$, compute $P^*_{0,n}(0)$ using (56) in terms of $P_{0,0}$.
Using (56), it follows that

$$P^*_{0,4}(0) = [(1-\alpha)/\alpha^2]P_{0,0}.$$

Step 10. Determine $P_{0,0}$ using (57). Thus $P^*_{0,n}(0)$ $(n = 0, 1, \dots, 4)$ are obtained from Steps 2 and 9, and $P^*_{1,n}(0)$ $(n = 0, 1, 2, 3)$ are obtained from Steps 7 to 8.

$$P_{0,0} = \alpha^2/[\alpha^2 + \alpha(1-\alpha) + 3(1-\alpha) + \sigma(1-\alpha)(3+\alpha+3\sigma+2\sigma^2+\sigma^3)].$$

We note that these results are the same as those given in Gupta [6, p. 1006].

4 System Performance Measures

Our analysis is based on the following system performance measures of the F policy M/G/1 queueing system with exponential startup time. Let

$L_s \equiv$ average number of customers in the system;
$P_b \equiv$ probability that the server is busy;
$P_s \equiv$ probability that the server requires a startup time before starting service;
$P_{bl} \equiv$ probability that the server is blocked.

The expressions for L_s, P_b, P_s, P_{bl} are given by

$$L_s = \sum_{n=1}^{K} n P_{0,n} + \sum_{n=1}^{K-1} n P_{1,n},$$

$$P_b = \sum_{n=0}^{K} P_{0,n} + \sum_{n=0}^{K-1} P_{1,n},$$

$$P_s = \sum_{n=0}^{F} P_{0,n},$$

$$P_{bl} = \sum_{n=0}^{K} P_{0,n}.$$

5 Optimal *F* Policy

We develop the total expected cost function per unit time for the F policy M/G/1 queueing system with startup times, in which F is a management decision variable. The main purpose of this paper is to determine the optimum management F policy so as to minimize this total expected cost function. Let

$C_h \equiv$ holding cost per unit time for each customer present in the system;
$C_b \equiv$ busy cost per unit time for a busy server;
$C_s \equiv$ startup cost per unit time for the preparatory work of the server before starting the service;
$C_{bl} \equiv$ fixed cost for every lost customer when the system is blocked.

Utilizing the definitions of each cost element listed above, the total expected cost function per unit time is given by

$$TC(F) = C_h L_s + C_b P_b + C_s P_s + C_{bl} \lambda P_{bl}. \tag{58}$$

The optimal value of F, F^* is determined by satisfying the following inequality:

$$TC(F^* - 1) \geq TC(F^*) \quad \text{and} \quad TC(F^* + 1) \geq TC(F^*). \tag{59}$$

Table 1 The optimal value of F and its minimum expected cost (the F policy M/M/1 queueing system)

		$\lambda(\mu,\beta)=(1.0,3.0)$			$\mu(\lambda,\beta)=(0.8,3.0)$			$\beta(\lambda,\mu)=(0.8,1.0)$		
		0.5	0.6	0.7	1.0	1.1	1.2	2.0	4.0	5.0
Case1	F^*	9	7	5	4	7	10	5	4	4
	$TC(F^*)$	105.00	127.49	151.42	177.60	158.45	143.31	177.68	177.56	177.54
Case2	F^*	12	11	9	6	10	12	6	6	6
	$TC(F^*)$	105.00	127.50	151.55	178.29	158.66	143.37	178.36	178.25	178.23
Case3	F^*	12	11	8	6	10	12	6	6	6
	$TC(F^*)$	105.00	127.50	151.56	178.31	158.67	143.37	178.40	178.27	178.24
Case4	F^*	11	9	7	4	8	11	5	5	5
	$TC(F^*)$	117.50	142.50	169.00	197.99	176.77	160.02	198.07	197.94	197.92
Case5	F^*	5	4	3	2	4	6	2	2	2
	$TC(F^*)$	122.47	149.93	180.05	213.87	189.10	169.77	213.96	213.83	213.80

6 Numerical Results

We now perform a sensitivity analysis on the optimum value F^* based on changes in the specific values of the system parameter and fix the system capacity at $K=15$. We present two simple examples for two different service time distributions such as exponential and 3-stage Erlang. We employ the following cost elements:

Case 1: $C_h=5$, $C_b=200$, $C_s=250$, $C_{bl}=300$.
Case 2: $C_h=5$, $C_b=200$, $C_s=250$, $C_{bl}=350$.
Case 3: $C_h=5$, $C_b=200$, $C_s=300$, $C_{bl}=350$.
Case 4: $C_h=5$, $C_b=225$, $C_s=300$, $C_{bl}=350$.
Case 5: $C_h=10$, $C_b=225$, $C_s=300$, $C_{bl}=350$.

The optimal value of F, F^*, and its minimum expected cost $TC(F^*)$ for the above five cases are shown in Tables 1, 2. We first fix (μ,β) and vary the values of λ. Tables 1, 2 reveal that: (i) $TC(F^*)$increases as λ increases for any cases; (ii) F^*decreases as λ increases for any cases. Next, we fix (λ,β) and vary the values of μ. We observe from Tables 1, 2 that: (i) $TC(F^*)$decreases as μ increases for any cases; (ii) F^*increases as μ increases for any cases. Finally, we fix (λ,μ) and vary the values of β. It appears from Tables 1, 2 that: (i) $TC(F^*)$slightly decreases as β increases for any cases; (ii) F^*does not change at all when β changes from 2.0 to 5.0 for any cases. Intuitively, F^* is insensitive to changes in β.

It can be easily seen from Tables 1, 2 that (i) F^* increases as C_h decreases or C_{bl} increases (see Case 4–5 and Case 1–2) and that (ii) $C_h$and C_{bl} have a larger effect on F^* than C_b and C_s (see Case 3–4 and Case 2–3).

7 Conclusions

In this paper, we have provided a recursive method for computing the steady state probability distribution of the number of customers in a finite system. We also illus-

Table 2 The optimal value of F and its minimum expected cost (the F policy M/$E_3/1$ queueing system)

		$\lambda(\mu,\beta)=(1.0,3.0)$			$\mu(\lambda,\beta)=(0.8,3.0)$			$\beta(\lambda,\mu)=(0.8,1.0)$		
		0.5	0.6	0.7	1.0	1.1	1.2	2.0	4.0	5.0
Case1	F^*	9	7	6	4	7	10	4	4	4
	$TC(F^*)$	104.17	126.00	148.91	174.00	155.50	141.11	174.02	173.99	173.98
Case2	F^*	12	11	9	6	10	12	6	6	6
	$TC(F^*)$	104.17	126.00	148.93	174.22	155.54	141.12	174.24	174.20	174.20
Case3	F^*	12	11	9	6	10	12	6	6	6
	$TC(F^*)$	104.17	126.00	148.93	174.23	155.54	141.12	174.26	174.21	174.20
Case4	F^*	11	9	7	5	9	12	5	5	5
	$TC(F^*)$	116.67	141.00	166.42	194.12	173.71	157.78	194.15	194.11	194.10
Case5	F^*	6	4	3	2	4	6	2	2	2
	$TC(F^*)$	120.83	147.00	175.28	207.57	183.63	165.53	207.60	207.56	207.55

trated a recursive method by a study of the exponential service time distribution. In addition, we derived the optimum value of the control parameter F so as to minimize an expected cost function. We performed a sensitivity analysis among the optimal value of F, specific values of system parameters, and the cost elements. Based on the numerical results, we could make an effective decision based on exact solutions for practical and general queueing system with quantitative measurement.

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