行政院國家科學委員會專題研究計畫 成果報告

矩陣指數跳躍下 Levy 過程通過時間之研究
研究成果報告(精簡版)

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成果報告

中文摘要：

我們考慮一般指數矩陣型 Lêvy 過程第一次離開開集合的問題。利用 ODE 技巧，我們得到關於第一次離開時間泛函的一般表示式。進一步，我們利用這個結果考慮可贖回型債券的定價問題。

關鍵字：Lêvy 過程、第一次離開時間、可贖回型債券

英文摘要：

Given a two-sided matrix-exponential Lêvy process, we consider a unction of the first exit of this process from an open set. By a standard result of ODE, the function can be written as a linear combination of known functions. In particular, when both sides of the jump distribution are linear combinations of exponentials, we obtain a semi-explicit solution for this function by using the corresponding integro-differential equation as a "sifter". For earlier results and related works, see Asmussen, Avram and Pistorius(2004), Jacobsen(2005), Chen, Lee and Sheu(2007), and many others.

Keywords: Lêvy process, matrix-exponential distribution, first exit, two-sided exit problem

First exit of two-sided matrix-exponential Lévy processes from open sets

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Abstract

Given a two-sided matrix-exponential Lévy process, we consider a function of the first exit of this process from an open set. By a standard result of ODE, the function can be written as a linear combination of known functions. In particular, when both sides of the jump distribution are linear combinations of exponentials, we obtain a semi-explicit solution for this function by using the corresponding integro-differential equation as a "sifter". For earlier results and related works, see Asmussen, Avram and Pistorius(2004), Jacobsen(2005), Chen, Lee and Sheu(2007), and many others.

Keywords: Lévy process, matrix-exponential distribution, first exit, two-sided exit problem
2000 Mathematics Subject Classification: 60J75,60G51,60G99
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1 Introduction

Consider a Markov process $X = (X_t, t \geq 0)$ on $\mathbb{R}$. For $x \in \mathbb{R}$, denote by $\mathbb{P}_x$ the law of X under which $X_0 = x$ and write simply that $\mathbb{P}_0 = \mathbb{P}$. Given an arbitrary open set $G \subset \mathbb{R}$, a bounded function g on $G^{\complement} = \mathbb{R} \backslash G$ and a killing rate $r > 0$, we consider the function Φ which measures the position of X by a value function g at the first time it leaves G:

$$\Phi(x) \equiv \mathbb{E}_x \left[e^{-r\tau_G} g(X_{\tau_G}) \right] \tag{1.1}$$

where $\tau_G = \inf\{t \geq 0; X_t \notin G\}$ and $\mathbb{E}_x(Y) \equiv \int_\Omega Y(\omega)\mathbb{P}_x(d\omega)$. Clearly, $\Phi(x) = g(x)$ for $x \notin G$. Finding solutions of Φ in G which are sufficiently explicit is not only a classical problem in probability theory but also a critical issue in applied sciences such as mathematical finance and insurance mathematics. However, except in few cases, little progress was made for general jump-diffusion processes.

Many studies have been conducted on solutions to Φ in a one-sided exit problem, that is $G = (0, \infty)$, with the underlying process X as a Lévy process. Besides the classical case of Lévy processes with no negative jumps, various authors have found that by choosing the Lévy measure of X in the family of matrix-exponential distributions (see Section 2 below), sufficiently explicit solutions may be produced. For example, Asmussen, Avram, and Pistorius [3] studied the Russian and American put options given that the logarithm of the underlying stock price is a phase-type Lévy process. (Phase-type distributions are a special case of matrix-exponential distributions.) They showed in Proposition 2 of [3] that the solution Φ is a linear combination of some semi-explicit exponentials and that the coefficients satisfy a system of linear equations. Their approach relies heavily on the probabilistic interpretation of phase-type distributions.

Inspired by the works of Gerber and Landry in [16], Asmussen, Avram, and Pistorius in [3] and many others, Chen, Lee, and Sheu first considered the function Φ in [11] under the assumption that the process follows a two-sided phase-type Lévy process. They obtained a general form for the function Φ under this simplifying assumption. Next, by observing that the solution structurally depends only on the downward jumps, they obtained a semi-explicit solution for Φ even if the downward jump distribution is a hyper-exponential distribution, namely a convex combination of exponential distributions (and upward jumps are determined by a general Lévy measure). As an application, the authors determined the optimal endogenous default level for Leland's model with jumps (cf. [18]).

Very recently, Chen and Sheu [13] reconsidered the model in [11] and gave a semi-explicit solution for Φ, with a method completely different from the one in [11], even if the downward jump distribution is a matrix-exponential distribution. Their result depends on an identity for the joint Laplace transform of the first-exit time and the undershoot(see Alili and Kyprianou [1]) and a semi-explicit solution of the negative Wiener-Hopf factor obtained by Lewis and Mordecki [22](see also [3]). (For a recent advance in the study of a generalization of the function Φ when X is a spectrally negative Lévy process, see Biffis and Kyprianou [5].)

In addition to the classical Lévy model, some authors have devoted to the study and application of regime-switching Lévy models in insurance mathematics. For example, Jacobsen [19] studied the time to ruin for a class of Markov additive process with two-sided jumps, which is a special case of regime-switching Lévy processes. The author determined explicitly the joint Laplace transform of the time to ruin and the undershoot at ruin under the assumption that the downward jump distribution is a matrix-exponential distribution. The martingale method used in [19] is based on the explicit partial eigenfunctions for the generator of the Markov additive process.

The technique in [19] was further exploited in Jacobsen and Jensen [20] for which they considered Ornstein-Uhlenbeck processes driven by a compound Poisson process and by a perturbed compound Poisson process. The downward jumps are determined by a distribution on $(0, \infty)$ which is a generalized hyper-exponential distribution, namely a *linear* combination of exponentials. (See [8].) Besides calculating the joint Laplace transform of the first-exit time and the undershoot as in [19], the authors considered the two-sided exit problem (assuming the jump distribution is a two-sided matrix-exponential distribution). The two-sided exit problem, a nontrivial extension of the one-sided exit problem, aims at identifying the law of the pair (τ_G, X_{τ_G}), where $G = (a, b)$ is a bounded interval. Till the present, very few results are available when X has two-sided jumps. For surveys when X is a spectrally one-sided Lévy process, see Bertoin [4] and Kyprianou [21].

Our main objective in this paper is to derive sufficiently explicit solutions for the function Φ, when G is a general open set and X takes the following form

$$X_t = X_0 + ct + \sigma W_t - \sum_{n=1}^{N_t} Y_n, \qquad t \geq 0. \tag{1.2}$$

Here, $c \in \mathbb{R}$, $\sigma > 0$, $W = (W_t, t \geq 0)$ is a standard Brownian motion, $N = (N_t; t \geq 0)$ is a Poisson process with rate $\lambda > 0$ and $(Y_n, n \in \mathbb{N})$ are independently and identically distributed random variables with two-sided matrix-exponential distribution F. The random elements W, N and Y are mutually independent.

We first characterize the function Φ as a solution of some ODE in G. (Our approach for this result is in the same spirit as that in [11]. However, our present result covers a wider class of processes, as well as a general open set G instead of the restrictive case $(0, \infty)$ considered in [11]). By a standard result of ODE, we write Φ as a linear combination of known functions in each component of $G^{\complement}$. Then, in the special case that F is a two-sided generalized hyper-exponential distribution, we characterize the totality of the coefficients in all components of $G^{\complement}$ as a solution of a system of linear equations. Moreover, when G is a bounded interval, our approach solves a special case of the two-sided exit problem, as shown in Example 4.1 below. Meanwhile, it is plausible that our methodology can be applied to regime-switching Lévy processes as those considered in [19], except that one would need to solve a system of ODE's instead of a single ODE.

The rest of the paper is organized as follows. In Section 2, we give a characterization of matrix-exponential distributions and show in Theorem 2.1 the transformation of an integro-differential equation into an ODE. The result in Theorem 2.1 enjoys generality more than our need in this paper and should be of interest itself. In Section 3, we prove the second-order regularity of Φ. By results of Boyarchenko and Levendorskiǐ [9] and Chen and Sheu [12], Φ satisfies an integro-differential equation and hence an ODE which guarantees Φ is a linear combination of complex exponentials. (Note that the knowledge of regularity of Φ is indispensable to ensure that Φ takes such a form, if one does not yet have a solution to the integro-differential equation.) In Section 4, we consider the case that the jump distribution is a two-sided generalized hyper-exponential distribution. Then, by using the corresponding integro-differential equation as a "sifter", we determine the coefficients for the function Φ by solving a system of linear equations. We close this paper in Section 5 by considering an application of our result to defaultable bond pricing.

2 Transformation from integro-differential equation to ODE

Throughout this section we consider an integro-differential operator $\mathcal{L}$ given by

$$\mathcal{L}\phi(x) = a(x)\phi''(x) + b(x)\phi'(x) + c(x)\phi(x) + \lambda \int \phi(x-y)dF(y). \tag{2.1}$$

Here, $\lambda > 0$ is a constant, and given a, b, and c all of which have sufficient regularities. To transform the corresponding integro-differential equation into an ODE, we first recall a definition of the class of matrix-exponential distributions.

Suppose that dF is a probability distribution on $(0, \infty)$ such that its Laplace transform takes the form of rational function:

$$\int_0^\infty e^{-zy} dF(y) = \frac{P(z)}{Q(z)},$$

where P and Q are two polynomials with no common zeros. Note that $\lim_{z\to\infty} \int_0^\infty e^{-zy} dF(y) = 0$, and hence we must have $\mathrm{Order}(P) < \mathrm{Order}(Q)$. By partial fraction decomposition, we obtain

$$\int_0^\infty e^{-zy} dF(y) = \sum_{j=1}^N \frac{A_j}{(z + a_j)^{n_j}},$$

for some $N \in \mathbb{N}$, $a_j \in \mathbb{C}$, $A_j \in \mathbb{C}$, and $n_j \in \mathbb{N}$. It follows from the uniqueness theorem for Laplace transforms that F has a probability density f given by

$$f(y) = \sum_{j=1}^N A_j y^{n_j - 1} e^{-a_j y} = \sum_{j=1}^M R_j(y) e^{-b_j y}, \qquad y > 0, \tag{2.2}$$

where $M \geq 1$, each R_j is a polynomial in y, and $b_j \in \mathbb{C}$ are distinct. Based on the argument in [11] Proposition 3.6, we deduce that $\Re a_j > 0$ for all j. Conversely, whenever a probability density function f taking the form (2.2), its Laplace transform is also a rational function.

Definition 2.1 *A distribution F on $(0,\infty)$ with a probability density function f is called a* **matrix-exponential** *distribution if its Laplace transform is a rational function, or equivalently, if f takes the form (2.2) with $\Re(a_j) > 0$, $A_j \in \mathbb{C}$, and $n_j \in \mathbb{N}$. In general, we say that a distribution dF on $\mathbb{R}\backslash\{0\}$ is a* **two-sided matrix-exponential distribution** *if it has a probability density function f given by*

$$f(y) = pf_{(+)}(y)\mathbf{1}_{y>0} + qf_{(-)}(-y)\mathbf{1}_{y<0}, \qquad y \in \mathbb{R}, \tag{2.3}$$

where (p,q) is a probability vector and both $f_{(-)}$ and $f_{(+)}$ are matrix-exponential distributions on $(0,\infty)$.

The two dense classes of distributions on $(0,\infty)$, phase-type distributions and generalized hyper-exponential distributions, are both subclasses of matrix-exponential distributions and have found many applications in applied probability. See Asmussen [2] and Botta and Harris [8].

In this section and section 3, we assume that F is a two-sided matrix-exponential distribution with a probability density f given by (2.3). Clearly, under this assumption, we have

$$\int e^{-\xi y} dF(y) = \frac{P(\xi)}{Q(\xi)}, \qquad \xi \in i\mathbb{R}, \tag{2.4}$$

where the order of the polynomial P is smaller than the order $\mathcal{O}$ of the polynomial Q, and P and Q have no common zeros. Note that Q has no zeros on $i\mathbb{R}$.

Let D be the differential operator: $D\phi = \phi'$. Also, given a polynomial $Y(x) = a_n x^n + \cdots + a_1 x + a_0$ over $\mathbb{C}$, we follow the convention that

$$Y(D) = a_n D^n + \cdots + a_1 D + a_0 I,$$

where $D^n\phi(z) = \phi^{(n)}(z)$ and $I\phi(z) = \phi(z)$. $Y(x)$ is called the **characteristic polynomial** of the differential operator $Y(D)$.

Proposition 2.1 *Suppose that ϕ is in the space $\mathcal{C}_c^\infty(\mathbb{R})$ of infinitely differentiable function with compact support. Then*

$$Q(D)\int \phi(\cdot - y) dF(y) = \int Q(D)\phi(\cdot - y) dF(y) = P(D)\phi \qquad \text{on } \mathbb{R}.$$

Proof. Since ϕ has compact support, the derivative of each order is bounded. By dominated convergence, the first equality follows.

We show the second equality. Write $T\kappa(x) = \int \kappa(x-y) dF(y)$. Observe that if $\kappa \in \mathcal{C}_c^\infty$, then $T\kappa \in L_2 = L_2(\mathbb{R})$. Indeed, since $f \in L_2$, we deduce that

$$\begin{aligned}
\int [T\kappa(x)]^2 dx &= \int \left(\int \kappa(x-y) f(y) dy \right)^2 dx = \int \left(\int_{\text{supp}(\kappa)} \kappa(y) f(x-y) dy \right)^2 dx \\
&\leq \int \left(\int \kappa(y)^2 dy \right) \left(\int_{\text{supp}(\kappa)} f(x-y)^2 dy \right) dx = \|\kappa\|_{L_2}^2 \int_{\text{supp}(\kappa)} \int f(x-y)^2 dx dy \\
&\leq \|\kappa\|_{L_2}^2 \|f\|_{L_2}^2 \int_{\text{supp}(\kappa)} dy < \infty.
\end{aligned}$$

Also, $T\kappa \in L_1 = L_1(\mathbb{R})$ since

$$\int |T\kappa(x)| dx \leq \int \int |\kappa(x-y)| dx f(y) dy \leq \|\kappa\|_{L_1} \|f\|_{L_1} < \infty.$$

Next, we show that the Fourier transform $\mathcal{F}(TQ(D)\phi)$ coincides with the one $\mathcal{F}(P(D)\phi)$, where $\mathcal{F}h(\theta) = \int e^{-2\pi i\theta y}h(y)dy$. Recall that $\mathcal{F}(Q(D)\phi)(\theta) = Q(2\pi i\theta)\mathcal{F}(\phi)(\theta)$ for all $\theta \in \mathbb{R}$. Since $T(Q(D)\phi) \in L_1 \cap L_2$ by the above results, we have, for all θ,

$$\begin{aligned}
\mathcal{F}(TQ(D)\phi)(\theta) =& \int e^{-2\pi i\theta x}\left(\int Q(D)\phi(x-y)f(y)dy\right)dx \\
=& \int\left(\int Q(D)\phi(x-y)e^{-2\pi i\theta(x-y)}dx\right)e^{-2\pi i\theta y}f(y)dy \\
=& \frac{P(2\pi i\theta)}{Q(2\pi i\theta)}Q(2\pi i\theta)\mathcal{F}(\phi)(\theta) \\
=& P(2\pi i\theta)\mathcal{F}(\phi)(\theta) \\
=& \mathcal{F}(P(D)\phi)(\theta).
\end{aligned}$$

By the Fourier inversion formula, we deduce that $TQ(D)\phi = P(D)\phi$ almost everywhere. By continuity, we conclude that the equality actually holds everywhere. The proof is now complete. □

In the following, we consider a special class of two-sided matrix-exponential distributions and obtain the same result as above by elementary calculus.

Example 2.1. Assume the probability density function f in (2.3) is of the form:

$$f(y) = \begin{cases} \sum_{j=1}^{m_{(+)}} p_j\eta_j^+ e^{-\eta_j^+ y}, & y > 0, \\ \sum_{j=1}^{m_{(-)}} q_j\eta_j^- e^{\eta_j^- y}, & y < 0, \end{cases} \tag{2.5}$$

where η_j^+ and η_j^- are positive real numbers, $\sum_{j=1}^{m_{(+)}} p_j + \sum_{j=1}^{m_{(-)}} q_j = 1$, and $p_j, q_j > 0$.

Assume ϕ is in $\mathcal{C}_c^\infty$. Then we have

$$\begin{aligned}
\int \phi(x-y)f(y)dy =& \sum_{j=1}^{m_{(+)}} p_j\eta_j^+ \int_0^\infty \phi(x-y)e^{-\eta_j^+ y}dy + \sum_{j=1}^{m_{(-)}} q_j\eta_j^- \int_{-\infty}^0 \phi(x-y)e^{\eta_j^- y}dy \\
=& \sum_{j=1}^{m_{(+)}} p_j\eta_j^+ e^{-\eta_j^+ x}\int_{-\infty}^x \phi(y)e^{\eta_j^+ y}dy + \sum_{j=1}^{m_{(-)}} q_j\eta_j^- e^{\eta_j^- x}\int_x^\infty \phi(y)e^{-\eta_j^- y}dy.
\end{aligned}$$

Note that

$$\begin{aligned}
&\left(\frac{d}{dx} + \eta_j^+\right)e^{-\eta_j^+ x}\int_{-\infty}^x \phi(y)e^{\eta_j^+ y}dy \\
=& -\eta_j^+ e^{-\eta_j^+ x}\int_{-\infty}^x \phi(y)e^{\eta_j^+ y}dy + \phi(x) + \eta_j^+ e^{-\eta_j^+ x}\int_{-\infty}^x \phi(y)e^{\eta_j^+ y}dy = \phi(x),
\end{aligned}$$

and similarly

$$\left(\frac{d}{dx} - \eta_j^-\right)e^{\eta_j^- x}\int_x^\infty \phi(y)e^{-\eta_j^- y}dy = -\phi(x).$$

The last equations imply that

$$\begin{aligned}
&\left(\frac{d}{dx} + \eta_1^+\right)\cdots\left(\frac{d}{dx} + \eta_{m_{(+)}}^+\right)\left(\frac{d}{dx} - \eta_1^-\right)\cdots\left(\frac{d}{dx} - \eta_{m_{(-)}}^-\right)\int \phi(x-y)f(y)dy \\
=& \sum_{j=1}^{m_{(+)}} p_j\eta_j^+ \prod_{k=1}^{m_{(-)}}\left(\frac{d}{dx} - \eta_k^-\right)\prod_{l=1,l\neq j}^{m_{(+)}}\left(\frac{d}{dx} + \eta_l^+\right)\phi(x) \\
&- \sum_{j=1}^{m_{(-)}} q_j\eta_j^- \prod_{k=1}^{m_{(+)}}\left(\frac{d}{dx} + \eta_k^+\right)\prod_{l=1,l\neq j}^{m_{(-)}}\left(\frac{d}{dx} - \eta_l^-\right)\phi(x).
\end{aligned} \tag{2.6}$$

On the other hand, note that

$$\begin{aligned}\int e^{-\zeta y}dF(y) &= \int_0^\infty e^{-\zeta y}f(y)dy + \int_{-\infty}^0 e^{-\zeta y}f(y)dy \\ &= \sum_{j=1}^{m_{(+)}} \frac{p_j\eta_j^+}{\zeta+\eta_j^+} + \sum_{j=1}^{m_{(-)}} \frac{-q_j\eta_j^-}{\zeta-\eta_j^-} = \frac{P(\zeta)}{Q(\zeta)}, \qquad \zeta \in i\mathbb{R},\end{aligned}$$

where $Q(\zeta) = \prod_{j=1}^{m_{(+)}}(\zeta+\eta_j^+)\prod_{j=1}^{m_{(-)}}(\zeta-\eta_j^-)$ and

$$P(\eta) = \sum_{j=1}^{m_{(+)}} p_j\eta_j^+ \prod_{k=1}^{m_{(-)}}(\zeta-\eta_k^-)\prod_{l=1,l\neq j}^{m_{(+)}}(\zeta+\eta_l^+) - \sum_{j=1}^{m_{(-)}} q_j\eta_j^- \prod_{k=1,k\neq j}^{m_{(-)}}(\zeta-\eta_k^-)\prod_{l=1}^{m_{(+)}}(\zeta+\eta_l^+).$$

From these and (2.6), we obtain that $Q(D)\int\phi(x-y)f(y)dy = P(D)\phi(x)$. □

The following theorem tells us how to transform an integro-differential equation into a linear differential equation whose characteristic polynomial can be easily identified.

Theorem 2.1 *Let $\Phi:\mathbb{R}\to\mathbb{R}$ be a bounded Borel measurable function twice continuously differentiable on some open set G and $\mathcal{L}\Phi = h$ on G for some Borel measurable function h. Given $\ell\in\mathbb{N}$. Suppose further the coefficient functions a, b, c and the function h are $\ell-$th continuously differentiable on G and $a(x)\neq 0$ for all $x\in G$. Then we have $\Phi\in\mathcal{C}^{\ell+2}(G)$. Moreover, if $\ell\geq\mathcal{O}$(the order of the polynomial Q in (2.4)), the function Φ satisfies the ODE:*

$$Q(D)\mathcal{L}_0\Phi + \lambda P(D)\Phi = Q(D)h$$

on G. Here $\mathcal{L}_0\Phi(x) = a(x)\Phi''(x) + b(x)\Phi'(x) + c(x)\Phi(x)$.

Proof. Recall the density of F is given by the function f. For the representation (2.3) of f, we set

$$f_{(\pm)}(y) = \sum_{j=1}^{M^{(\pm)}} R_j^{(\pm)}(y)e^{-b_j^{(\pm)}y}, \quad y>0.$$

We first show that if $\phi:\mathbb{R}\to\mathbb{R}$ is bounded and is continuously differentiable in G up to order k, then $\int\phi(x-y)dF(y)$ is continuously differentiable in G up to order $k+1$. By the definition of F, we have

$$\begin{aligned}\int\phi(x-y)dF(y) &= \int\phi(y)f(x-y)dy \\ &= \int_{-\infty}^x \phi(y)f(x-y)dy + \int_x^\infty \phi(y)f(x-y)dy \\ &= p\sum_{j=1}^{M^{(+)}}\int_{-\infty}^x \phi(y)R_j^{(+)}(x-y)e^{-b_j^{(+)}(x-y)}dy + q\sum_{j=1}^{M^{(-)}}\int_x^\infty \phi(y)R_j^{(-)}(y-x)e^{b_j^{(-)}(x-y)}dy.\end{aligned}$$

Since each $R_j^{(+)}$ is a polynomial, once expanding it to the form $\sum_{m=1}^n d_m x^{n-m}y^m$, we see that each $\int_{-\infty}^x\phi(y)R_j^{(+)}(x-y)e^{-b_j^{(+)}(x-y)}dy$ can be written as a linear combination of integrals of the form

$$e^{-b_j^{(+)}x}x^{n-m}\int_{-\infty}^x y^m\phi(y)e^{b_j^{(+)}y}dy. \tag{2.7}$$

Clearly, the term in (2.7) will be continuously differentiable in G up to order $k+1$ and hence $x\longmapsto\int_{-\infty}^x\phi(y)R_j^{(+)}(x-y)e^{-b_j^{(+)}(x-y)}dy$ will be, whenever ϕ is up to order k. Similar results hold for the integrals $\int_x^\infty\phi(y)R_j^{(-)}(y-x)e^{b_j^{(-)}(x-y)}dy$. This proves that $x\longmapsto\int\phi(x-y)dF(y)$ is continuously differentiable up to order $k+1$, whenever ϕ is up to order k.

Suppose now a, b, c, and h are in $\mathcal{C}^\ell(G)$ and $a \neq 0$ on G. Rewrite the integro-differential equation $\mathcal{L}\Phi = h$ as

$$\Phi''(x) = -\frac{b(x)}{a(x)}\Phi'(x) - \frac{c(x)}{a(x)}\Phi(x) - \frac{1}{a(x)}\int \Phi(x-y)dF(y) + \frac{h(x)}{a(x)}.$$

Since the right hand side is continuously differentiable, it follows that Φ is $\mathcal{C}^3(G)$. Recall we have shown in the above that $x \mapsto \int \Phi(x-y)dF(y)$ will be continuously differentiable in G up to order $k+1$ if Φ is up to order k. Hence, continuing the argument in this fashion, we deduce from the last equation that $\Phi \in \mathcal{C}^{\ell+2}(G)$.

To complete the proof, it remains to show that, if $\ell \geq \mathcal{O}$,

$$Q(D)\int \Phi(\cdot - y)dF(y) = P(D)\Phi \text{ on } G.$$

(Note that the right hand side makes sense since the order of $P < \mathcal{O}$.)

Recall the operator T in the proof of Proposition 2.1. Let T^* be its adjoint operator, that is, $T^*\kappa(x) = \int \kappa(x+y)dF(y)$. Then $T^*\kappa(x) = \int \kappa(x-y)dF^*(y)$, where $dF^*(y) = f(-y)dy$ and $\int e^{-\xi y}dF^*(y) = P(-\xi)/Q(-\xi)$. Therefore, by Proposition 2.1, we have $T^*Q(D)^*\phi = P(D)^*\phi$ for any $\phi \in \mathcal{C}_c^\infty(G)$. Hence,

$$\begin{aligned}\langle Q(D)T\Phi, \phi\rangle_{L_2} =& \langle \Phi, T^*Q(D)^*\phi\rangle_{L_2}\\ =& \langle \Phi, P(D)^*\phi\rangle_{L_2}\\ =& \langle P(D)\Phi, \phi\rangle_{L_2}.\end{aligned}$$

Since $\phi \in \mathcal{C}_c^\infty(G)$ is arbitrary and $P(D)\Phi$ is continuous on G, we have $P(D)\Phi = Q(D)\int \Phi(\cdot - y)dF(y)$ on G. □

Example 2.2. Consider the integro-differential operator $\mathcal{L}$ given by $\mathcal{L}\phi(x) = \frac{\sigma^2}{2}\phi''(x) + kx\phi'(x) + \lambda \int \phi(x-y)dF(y) - \lambda\phi(x)$. Assume that Φ satisfy $(\mathcal{L} - r)\Phi = 0$ in G. By Theorem 2.1, we know that Φ satisfies the following linear differential equation

$$\begin{aligned}0 =& Q(D)\left(\frac{\sigma^2}{2}D^2 + \kappa x D - (\lambda + r)I\right)\Phi(x) + \lambda P(D)\Phi(x)\\ =& Q(D)(\kappa x D)\Phi(x) + \left[Q(D)\left(\frac{\sigma^2}{2}D^2 - (\lambda + r)I\right) + \lambda P(D)\right]\Phi(x).\end{aligned}$$

Note that

$$\left[Q(D)\left(\frac{1}{2}\sigma^2 D^2 - (\lambda + r)I\right) + \lambda P(D)\right]\Phi(x)$$

is a linear differential equation with constant coefficients and the order of this equation is $\mathcal{O} + 2$. On the other hand, observe that $D^n(xD\Phi(x)) = \sum_{j=0}^{n+1}(\alpha_j x + \beta_j)D^j\Phi(x)$ for some constants α_j and β_j. In conclusion, Φ satisfies a Laplace equation. Namely,

$$\sum_{n=0}^{\mathcal{O}+2}(a_n x + b_n)D^n\Phi(x) \equiv 0 \quad \text{on } G, \tag{2.8}$$

where a_n, b_n are constants in $\mathbb{C}$ and $a_{\mathcal{O}+2} = 0$. □

Remark. Novikov et al. [24] considered an Ornstein-Uhlenbeck process X with the generator $\mathcal{L}$ in Example 2.2. They assumed that $\sigma = 0$ and F is an exponential distribution or a uniform distribution. They were interested in the function Φ given by (1.1) for which $G = (0, \infty)$, $r > 0$, and $g \equiv 1$. By direct differentiation, they transformed the integro-differential equation $(\mathcal{L} - r)\Phi = 0$ into a second-order linear ODE which admits known basis functions. Then they plugged in the general solution into the boundary value problem itself to find the coefficients. □

3 Function Φ as linear combination of known functions

Recall that X is a two-sided matrix-exponential Lévy process of the form in (1.2) and F is a two-sided matrix-exponential distribution with a probability density f given by (2.3). Also, given an open set $G \subset \mathbb{R}$ and a bounded function g on G^c, the function Φ is defined in (1.1). To derive an ODE for Φ, we first study the regularity of Φ.

Notation. Write $h \in \mathcal{C}^2([a,b])$ if $h^{(i)}(x), i = 0,1,2$, are continuous on the interval $[a,b]$. We say $h \in \mathcal{C}_0^2([a,\infty))$ if $h^{(i)}(x), i = 0,1,2$, are continuous in $[a,\infty)$ and they all converge to zero as x tends to infinity. Functions in $\mathcal{C}^2((-\infty,b])$ are defined in the similar way. Write G as the disjoint union of the intervals:

$$G = \bigcup_{q \in \mathfrak{Q}} I_q.$$

Here each $I_q = (a_q, b_q)$ is of the largest possible interval contained in G and $\mathfrak{Q}$ is either of the finite set $\{1, 2, \cdots, n\}$ or $\mathbb{N}$. We write $h \in \mathcal{H}(G)$ if for every $q \in \mathfrak{Q}$, h is in $\mathcal{C}^2([a_q, b_q])$ if $|b_q - a_q| < \infty$ or h is in $\mathcal{C}_0^2(\overline{(a_q, b_q)})$ if $|b_q - a_q| = \infty$. $\square$

In the following, we set J as the first jump time of X and $X_t^c = X_0 + ct + \sigma W_t$ for all $t \geq 0$. We will show in Proposition 3.1 below that $\Phi \in \mathcal{C}^2([a,b])$. Without loss of generality, we may assume that $\sigma = 1$ from now on up to Proposition 3.1. For in the general case, if we set $\Phi^*(x) = \Phi(\sigma x)$ for $x \in \overline{(\sigma^{-1}a, \sigma^{-1}b)}$, then $\Phi^*(x)$ is the functional in (1.1) with $\sigma = 1$ and the continuous differentiability of Φ^* is equivalent to that of Φ.

Lemma 3.1 *Let $(a,b) \subset G = E^c$ be maximal (that is, $a, b \in \partial E$) and of finite length. For every $x \in [a,b]$, we have*

$$\mathbb{E}_x\left[e^{-r\tau_G}; \tau_G < J, X_{\tau_G}^c = a\right] = e^{c(a-x)} \frac{\sinh\left((b-x)\sqrt{2(\lambda+r)+c^2}\right)}{\sinh\left((b-a)\sqrt{2(\lambda+r)+c^2}\right)} \tag{3.1}$$

and

$$\mathbb{E}_x\left[e^{-r\tau_G}; \tau_G < J, X_{\tau_G}^c = b\right] = e^{c(b-x)} \frac{\sinh\left((x-a)\sqrt{2(\lambda+r)+c^2}\right)}{\sinh\left((b-a)\sqrt{2(\lambda+r)+c^2}\right)} \tag{3.2}$$

and

$$\mathbb{E}_x\left[e^{-r\tau_G} g(X_{\tau_G}); \tau_G \geq J\right] = e^{-cx}\Gamma\left[H_1(x) - H_2(x)\right]. \tag{3.3}$$

Here,

$$\Gamma = \frac{\lambda}{\sqrt{2(\lambda+r)+c^2}\sinh\left[(b-a)\sqrt{2(\lambda+r)+c^2}\right]}, \tag{3.4}$$

$$H_1(x) = \int_a^b dz \int dF(y) \cosh\left[(b-a-|z-x|)\sqrt{2(\lambda+r)+c^2}\right] e^{cz}\Phi(z-y), \tag{3.5}$$

and

$$H_2(x) = \int_a^b dz \int dF(y) \cosh\left[(b+a-z-x)\sqrt{2(\lambda+r)+c^2}\right] e^{cz}\Phi(z-y). \tag{3.6}$$

Proof. (3.1), (3.2) and (3.3) follows immediately if $x = a$ or b. So, we may assume $x \in (a,b)$.

First, we deal with (3.1). Observe that on $[\tau_G < J]$, the first exit of X from E^c is caused by diffusion: $\tau_G = \rho$ and $X_{\tau_G} = X_\rho^c$, where $\rho = \inf\{t \geq 0; X_t^c \notin (a,b)\}$. Also, $[\tau_G < J] = [\rho < J]$.

Hence, by the independence of W and J, we deduce that

$$\begin{aligned}\mathbb{E}_x\left[e^{-r\tau_G};\tau_G<J,X_{\tau_G}=a\right]=&\mathbb{E}_x\left[e^{-r\rho};\rho<J,X^c_\rho=a\right]\\=&\int_0^\infty \lambda e^{-\lambda t}dt\mathbb{E}_x\left[e^{-r\rho};\rho<t,X^c_\rho=a\right]\\=&\int_0^\infty \lambda e^{-\lambda t}dt\int_0^t e^{-rs}\mathbb{P}_x\left[\rho\in ds,X^c_\rho=a\right]\\=&\int_0^\infty e^{-(\lambda+r)s}\mathbb{P}_x\left[\rho\in ds,X^c_\rho=a\right]\qquad\text{(Fubini's Theorem)}\\=&\mathbb{E}_x\left[e^{-(r+\lambda)\rho};X^c_\rho=a\right].\end{aligned}$$

Equation (3.1) now follows from Formula 3.0.5(a) in [7] page 309. Similarly, (3.2) follows from Formula 3.0.5(b) in [7] page 309.

To complete the proof, we show (3.3). Observe that J is a stopping time, and by Strong Markov property of X it follows that

$$\begin{aligned}\mathbb{E}_x\left[e^{-r\tau_G}g(X_{\tau_G});\tau_G\geq J\right]=&\mathbb{E}_x\left[e^{-rJ}\Phi(X^c_J-Y_1);a<\min_{0\leq s\leq J}X^c_s\leq\max_{0\leq s\leq J}X^c_s<b\right]\\=&\int dF(y)\int dt\lambda e^{-(\lambda+r)t}\mathbb{E}_x\left[\Phi(X^c_t-y);a<\min_{0\leq s\leq t}X^c_s\leq\max_{0\leq s\leq t}X^c_s<b\right],\end{aligned}$$

by independence of W, J and Y_1. If we let J' be an exponential random variable with mean $(\lambda+r)^{-1}$ independent of W and Y_1, the last equation can be written as

$$\begin{aligned}&\mathbb{E}_x\left[e^{-r\tau_G}g(X_{\tau_G});\tau_G\geq J\right]\\=&\frac{\lambda}{\lambda+r}\int dF(y)\mathbb{E}_x\left[\Phi(X^c_{J'}-y);a<\min_{0\leq s\leq J'}X^c_s\leq\max_{0\leq s\leq J'}X^c_s<b\right].\end{aligned}\tag{3.7}$$

Using the density of $\mathbb{P}_x\left[a<\min_{s\leq J'}X^c_s\leq\max_{s\leq J'}X^c_s<b,X^c_{J'}\in dz\right]$ given by Formula 1.15.6 in [7] page 271, equation (3.3) now follows from Fubini's Theorem. The proof is complete. □

It is clear from Lemma 3.1 that the functions on the right hand side of (3.1) and (3.2) are both in $\mathcal{C}^2([a,b])$. To have the one on the right hand side of (3.3) also in $\mathcal{C}^2([a,b])$, we need the following lemma.

Lemma 3.2 *Let (a,b) be given as in Lemma 3.1. The function*

$$H_0(x)=\int_a^b dz\int dF(y)e^{C|z-x|}e^{cz}\Phi(z-y)$$

is in $\mathcal{C}^2([a,b])$ for any constant $C\in\mathbb{R}$.

Proof. Write

$$H_0(x)=\int_a^x dz\int dF(y)e^{C(x-z)}e^{cz}\Phi(z-y)+\int_x^b dz\int dF(y)e^{C(z-x)}e^{cz}\Phi(z-y),\qquad\text{whenever }x\in[a,b].$$

On the other hand, a slight modification of [25] Proposition 2.5 Chapter 2 shows a convolution of an $L_1(\mathbb{R})$ function with a bounded function is continuous. So, by the facts that F has a density and Φ is bounded, we deduce that H_0 is differentiable on (a,b) and its first order derivative is given by

$$H_0'(x)=C\int_a^x dz\int dF(y)e^{C(x-z)}e^{cz}\Phi(z-y)-C\int_x^b dz\int dF(y)e^{C(z-x)}e^{cz}\Phi(z-y).$$

Similar argument shows the left and right hand derivatives of Φ at b and a exist and are given by the right hand side of the last equation with $x=b$ and a, respectively. This gives first order regularities. The proof of second order regularity of Φ follows the same as that of the first order one, and we omit the proof.

□

Proposition 3.1 *The function Φ defined in (1.1) is in $\mathcal{H}(G)$.*

Proof. Let I be a maximal interval in G. Assume I is unbounded above. Assume without loss of generality the left hand boundary is 0. Then by following exactly the same argument as in [12] Theorem 2.1, Φ satisfies the integral equation (2.8) in [12]. Since the subsequent proof of regularities of Φ on I depends only on this functional equation instead of on Φ itself, it is clear that $\Phi \in \mathcal{C}_0^2(\bar{I})$. If I is unbounded below, the same result follows by considering the dual process of X.

We consider the case that $I = (a, b)$ is bounded. First, for $x \in [a, b]$, write

$$\Phi(x) = g(a)\mathbb{E}_x\left[e^{-r\tau_G}; \tau_G < J, X_{\tau_G} = a\right] + g(b)\mathbb{E}_x\left[e^{-r\tau_G}; \tau_G < J, X_{\tau_G} = b\right] + \mathbb{E}_x\left[e^{-r\tau_G} g(X_{\tau_G}); \tau_G \geq J\right].$$

Then by Lemma 3.1, it suffices to show $\mathbb{E}_{\cdot}\left[e^{-r\tau_G} g(X_{\tau_G}); \tau_G \geq J\right] \in C^2([a, b])$. Observe that since $\cosh z = \frac{e^z + e^{-z}}{2}$, $H_2 \in \mathcal{C}^2([a, b])$. In addition, using Lemma 3.2, $H_1 \in \mathcal{C}^2([a, b])$. It follows from (3.3) that $\mathbb{E}_{\cdot}\left[e^{-r\tau_G} g(X_{\tau_G}); \tau_G \geq J\right] \in \mathcal{C}^2([a, b])$, and the proof is complete.

□

To write down the integro-differential equation for Φ, note that, for every $\xi \in i\mathbb{R}$, we have

$$\mathbb{E}\left[e^{\zeta X_1}\right] = e^{\psi(\zeta)}, \tag{3.8}$$

where

$$\psi(\zeta) = \frac{\sigma^2}{2}\zeta^2 + c\zeta + \lambda \int e^{-\zeta y} dF(y) - \lambda.$$

(ψ is called the **Laplace exponent** of X.) Under the assumption of the distribution F, the Laplace exponent ψ can be written as the form

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 + c\xi + \lambda\psi_1(\xi) - \lambda, \qquad \xi \in i\mathbb{R}. \tag{3.9}$$

Here $\psi_1(\xi) = \int e^{-\xi y} f(y) dy = \frac{P(\xi)}{Q(\xi)}$ by (2.4). As noted before, the right hand side of (3.9) is actually a rational function on $\mathbb{C}$ with a finite number of poles in $\mathbb{C}\backslash i\mathbb{R}$. Accordingly, we consider ψ and ψ_1 on $\mathbb{C}$ as analytic functions except at the poles in $\mathbb{C}\backslash i\mathbb{R}$. Besides we put

$$R(\zeta) = Q(\zeta)(\psi(\zeta) - r). \tag{3.10}$$

On the other hand, the infinitesimal generator $\mathcal{L}_X$ of X has a domain containing $\mathcal{C}_0^2(\mathbb{R})$ and for any $h \in \mathcal{C}_0^2(\mathbb{R})$,

$$\mathcal{L}_X h(x) = \frac{\sigma^2}{2} h''(x) + c h'(x) + \lambda \int h(x - y) dF(y) - \lambda h(x). \tag{3.11}$$

(For details, see [4].)

Let $\mathcal{Z} = (\rho_j; 1 \leq j \leq m)$ be the distinct zeros of $\psi(\zeta) - r$ and each ρ_j be a zero of multiplicity m_j of $\psi(\zeta) - r$. If $m_j = 1$ for all j, then $\mathcal{Z}$ is said to be **separable**. In terms of these ρ_j, we show below that the function Φ is a linear combination of known functions.

Theorem 3.1 *Assume G is an open set and g is a bounded Borel measurable function on $E = G^{\complement}$. Then the function Φ defined in (1.1) is infinitely differentiable on G and $R(D)\Phi \equiv 0$ on G. ($R(\zeta)$ is the polynomial defined in (3.10).) Moreover, on each maximal open interval $I_q = (a_q, b_q)$ in G, we have*

$$\Phi(x) = \sum_{j=1}^{m} \boldsymbol{Q}_j^q(x) e^{\rho_j x}. \tag{3.12}$$

Here, for every $1 \leq j \leq m$, $\boldsymbol{Q}_j^q(x)$ is a polynomial of degree less than m_j. In particular, if $\mathcal{Z}$ is separable, then $\boldsymbol{Q}_j^q(x) = \boldsymbol{Q}_j^q$ are constants. (The polynomial vector $\boldsymbol{Q} = \left(\boldsymbol{Q}_1^1, \boldsymbol{Q}_2^1, \cdots, \boldsymbol{Q}_m^1, \cdots, \boldsymbol{Q}_1^q, \cdots, \boldsymbol{Q}_m^q, \cdots\right)$ is called the coefficient vector for the solution Φ.)

Proof. By Proposition 3.1, the function Φ is in $\mathcal{H}(G)$. So, by Theorems 4.1 and 4.2 in [12] (see also [9]), we obtain that Φ is a (strong) solution of the integro-differential equation $(\mathcal{L}_X - r)\Phi(x) = 0$ in G. Then the first part of our results follows from Theorem 2.1. As for the second part, note that the zeros of $\psi(\zeta) - r$ coincide with those of the polynomial $R(\zeta)$ defined in (3.10), counting multiplicity. Since $R(D)\Phi = 0$ in G, by standard ODE theory (see [15] Theorem 2.32 and Theorem 2.33), the result follows. □

Remarks. **(a)** If $I_q = (a_q, \infty)$, we obtain, by Proposition 3.1, that $\Phi(x) \to 0$ as $x \to \infty$. Hence $\boldsymbol{Q}_j^q(x) = 0$ whenever $\Re\rho_j > 0$. Similarly, if $I_q = (-\infty, b_q)$, then $\boldsymbol{Q}_j^q(x) = 0$ whenever $\Re\rho_j < 0$. To fix an idea, we say that a polynomial vector $\boldsymbol{Q} = \left(\boldsymbol{Q}_1^1, \boldsymbol{Q}_2^1, \cdots, \boldsymbol{Q}_m^1, \cdots, \boldsymbol{Q}_1^q, \cdots, \boldsymbol{Q}_m^q, \cdots\right)$ satisfies the **vanishing condition** if $\boldsymbol{Q}_j^q = 0$ whenever $I_q = (a_q, \infty)$ ($I_q = (-\infty, b_q)$, respectively) and $\Re\rho_j > 0$ ($\Re\rho_j < 0$ respectively).
(b) The equation $R(\zeta) = 0$ is exactly the same as the modified Cramér-Lundberg equation (31) from [19] when the latter equation is translated using our present notations.
(c) In [19], the author considered the case that $G = (0, \infty)$ and $g(y) = e^{zy}1_{y\le 0}$ and searched for partial eigenfunctions ϕ of the form (3.12) with the boundary condition $\phi = g$. In other words, ϕ satisfies the equation $(\mathcal{L}_X - r)\phi(x) = 0$ in G and $\phi = g$ in $(-\infty, 0)$. See also [20] for related work. □

We close this section by stating the uniqueness of solutions for the boundary value problem.

Theorem 3.2 *Let $\phi \equiv g$ on $E = G^{\complement}$, $\phi \in \mathcal{H}(G)$, and $(\mathcal{L}_X - r)\phi \equiv 0$ in E^c. Then $\phi(x) = \mathbb{E}_x\left[e^{-r\tau}g(X_\tau)\right]$ for all $x \in \mathbb{R}$.*

Proof. The proof is the same as that of Proposition 4.1 in Chen et al. [11] if one replaces $\mathbb{R}_+$ by $\overline{E^c}$, and we omit the proof. □

4 Integro-differential equation as sifter

We have seen in Section 3 that for every open set $G \subset \mathbb{R}$ and every bounded measurable function g on $E = G^{\complement}$, the function Φ in (1.1) satisfies a linear ODE with constant coefficients and hence has a known functional form. However, unlike the standard ODE problem, we do not have the knowledge of boundary conditions of higher order derivatives, and hence the coefficient vector $\boldsymbol{Q}$ for Φ cannot be solved by the classical ODE method. On the other hand, we have seen from Theorem 3.2 that the integro-differential equation together with the function g on E, is sufficient to uniquely identify the solution. A natural question arises: what can we exert from the integro-differential equation to attain such a goal? We consider a special class of matrix-exponential distributions. We derive its corresponding system of linear equations for $\boldsymbol{Q}$ which determines uniquely $\boldsymbol{Q}$ by the uniqueness theorem.

Throughout this section, we assume the jump-size density function f of X is a two-sided generalized hyper-exponential distribution, that is, f_1 and f_2 in (2.3) are linear combinations of exponential distributions. Then we can write f as follows :

$$f(y) = \sum_{j=1}^{m} p_j|\eta_j|e^{-\eta_j y}[\chi_1(j)1_{y>0} + \chi_2(j)1_{y<0}]; \qquad y \in \mathbb{R}. \tag{4.1}$$

Here $\boldsymbol{p} = (p_1, \cdots, p_m)$ is a vector(not necessary a probability vector) such that $\sum_{i=1}^{m_0} p_i = p \ge 0$, $\sum_{i=m_0+1}^{m} p_i = q \ge 0$ and $p + q = 1$. Also $(\eta_1, \cdots, \eta_m) \in (0, \infty)^{m_0} \times (-\infty, 0)^{m-m_0}$ has distinct entries, and χ_1 and χ_2 are two indicator functions on integers: $\chi_1(j) = \mathbf{1}_{\{1,\cdots,m_0\}}(j)$ and $\chi_2(j) = \mathbf{1}_{\{m_0+1,\cdots,m\}}(j)$. We assume further that the zero set $\mathcal{Z}$ of $\psi(z) - r$ is separable. (This is true if $\boldsymbol{p}$ is a probability vector. See, e.g., [3] and [23].) It follows from Lemma 1.1(b) of [22] that there are $m_0 + 1$ distinct roots, say, $\{\rho_j, 1 \le j \le m_0 + 1\}$ in $\mathcal{Z}$ with $\Re\rho_j < 0$. In addition, by considering the dual process $-X$, there are $m - m_0 + 1$ distinct roots $\{\rho_j, m_0 + 2 \le j \le m + 2\}$ in $\mathcal{Z}$ with $\Re\rho_j > 0$. Since $\mathcal{Z}$ is separable, the coefficient vector $\boldsymbol{Q}$ is a constant vector. Moreover, the vector $\boldsymbol{Q}$ satisfies the vanishing condition, i.e. , if $-\infty = a_q < b_q < \infty$, then $\boldsymbol{Q}_k^q = 0$ for $k = 1, 2, \cdots, m_0 + 1$, and if $-\infty < a_q < b_q = \infty$, then $\boldsymbol{Q}_k^q = 0$ for $k = m_0 + 2, \cdots, m + 2$.

Given a candidate constant vector $\boldsymbol{Q}$ which further satisfies the vanishing condition if G is unbounded. Define a function $\phi : \mathbb{R} \to \mathbb{R}$ by

$$\phi(x) = \begin{cases} g(x) & \text{if } x \in E, \\ \sum_{j=1}^{m+2} \boldsymbol{Q}_j^q e^{\rho_j x}, & \text{if } x \in \overline{I_q}, q \in \mathfrak{Q}. \end{cases} \tag{4.2}$$

Assume further that the candidate vector $\boldsymbol{Q}$ is chosen well so that ϕ satisfies

$$(\mathcal{L}_X - r)\phi = 0 \text{ on } G.$$

In the following, we will show that $\boldsymbol{Q}$ satisfies a system of linear equation which contains sufficient information to uniquely identity $\boldsymbol{Q}$.

Fix $I_q = (a_q, b_q)$ for some $q \in \mathfrak{Q}$ and $x \in (a_q, b_q)$. We have

$$0 = (\mathcal{L}_X - r)\phi(x) = \sum_{k=1}^{m+2} \boldsymbol{Q}_k^q e^{\rho_k x} \left[\frac{\sigma^2}{2}\rho_k^2 + c\rho_k - (\lambda + r) \right] + \lambda \int \phi(y) f(x-y) dy. \tag{4.3}$$

Write

$$\int \phi(y) f(x-y) dy = \left(\int_E + \int_{E^c \setminus (a_q, b_q)} + \int_{(a_q,b_q)} \right) \phi(y) f(x-y) dy.$$

Let us compute the three integrals on the right hand side of the last equation. Firstly, since $\phi = g$ on E and $x \in (a_q, b_q)$, we obtain

$$\begin{aligned} \int_E \phi(y) f(x-y) dy &= \int_E g(y) f(x-y) dy \\ &= \sum_{j=1}^{m} p_j |\eta_j| e^{-\eta_j x} \left(\chi_1(j) \int_{(-\infty, a_q] \cap E} + \chi_2(j) \int_{E \cap [b_q, \infty)} \right) g(y) e^{\eta_j y} dy. \end{aligned} \tag{4.4}$$

Secondly, we have

$$\begin{aligned} &\int_{E^c \setminus (a_q,b_q)} \phi(y) f(x-y) dy = \sum_{q' \neq q} \int_{a_{q'}}^{b_{q'}} \phi(y) f(x-y) dy \\ =& \sum_{j=1}^{m} p_j |\eta_j| e^{-\eta_j x} \left[\left(\chi_1(j) \sum_{b_{q'} \leq a_q} + \chi_2(j) \sum_{b_q \leq a_{q'}} \right) \sum_{k=1}^{m+2} \frac{\boldsymbol{Q}_k^{q'}}{\rho_k + \eta_j} \left(e^{(\eta_j + \rho_k) b_{q'}} - e^{(\eta_j + \rho_k) a_{q'}} \right) \right]. \end{aligned} \tag{4.5}$$

And finally,

$$\begin{aligned} &\int_{(a_q,b_q)} \phi(y) f(x-y) dy \\ =& \sum_{k=1}^{m+2} \boldsymbol{Q}_k^q \int_{a_q}^{b_q} e^{\rho_k y} f(x-y) dy = \sum_{k=1}^{m+2} \boldsymbol{Q}_k^q e^{\rho_k x} \int_{x-b_q}^{x-a_q} e^{-\rho_k y} f(y) dy \\ =& \sum_{k=1}^{m+2} \boldsymbol{Q}_k^q e^{\rho_k x} \sum_{j=1}^{m} p_j |\eta_j| \left(\chi_1(j) \int_0^{x-a_q} + \chi_2(j) \int_{x-b_q}^0 \right) e^{-(\rho_k + \eta_j) y} dy \\ =& \sum_{k=1}^{m+2} \boldsymbol{Q}_k^q e^{\rho_k x} \sum_{j=1}^{m} \frac{p_j |\eta_j|}{\rho_k + \eta_j} \left[\chi_1(j) \left(1 - e^{-(\rho_k + \eta_j)(x - a_q)} \right) + \chi_2(j) \left(e^{-(\rho_k + \eta_j)(x - b_q)} - 1 \right) \right] \\ =& \sum_{k=1}^{m+2} \boldsymbol{Q}_k^q e^{\rho_k x} \sum_{j=1}^{m} \frac{p_j |\eta_j|}{\rho_k + \eta_j} [\chi_1(j) - \chi_2(j)] + \sum_{j=1}^{m} p_j |\eta_j| e^{-\eta_j x} \sum_{k=1}^{m+2} \frac{\boldsymbol{Q}_k^q}{\rho_k + \eta_j} \left(-\chi_1(j) e^{a_q(\rho_k + \eta_j)} + \chi_2(j) e^{b_q(\rho_k + \eta_j)} \right). \end{aligned} \tag{4.6}$$

Write for $1 \le j \le m$,

$$\boldsymbol{C}_j^q \boldsymbol{Q} = \sum_{k=1}^{m+2} \left[\frac{\boldsymbol{Q}_k^q \left[-\chi_1(j) e^{a_q(\rho_k+\eta_j)} + \chi_2(j) e^{b_q(\rho_k+\eta_j)} \right]}{\rho_k + \eta_j} + \left(\chi_1(j) \sum_{b_{q'} \le a_q} + \chi_2(j) \sum_{b_q \le a_{q'}} \right) \frac{\boldsymbol{Q}_k^{q'} \left(e^{(\eta_j+\rho_k) b_{q'}} - e^{(\eta_j+\rho_k) a_{q'}} \right)}{\rho_k + \eta_j} \right], \tag{4.7}$$

where $\boldsymbol{C}_j^q$ is a row vector with obvious entries. Observe in the present case, the Laplace exponent of X is given by

$$\psi(\zeta) = \frac{\sigma^2}{2}\zeta^2 + c\zeta + \lambda \sum_{j=1}^{m} \frac{p_j |\eta_j|}{\zeta + \eta_j} \left[\chi_1(j) - \chi_2(j) \right] - \lambda.$$

Summarize (4.3) and the decompositions (4.4), (4.5) and (4.6) of $\int \phi(y) f(x-y) dy$. We derive that $(\mathcal{L}_X - r)\phi(x)$, which is equal to 0, is the sum of the following two identities:

$$\sum_{k=1}^{m+2} \boldsymbol{Q}_k^q e^{\rho_k x} (\psi(\rho_k) - r)$$

and

$$\lambda \sum_{j=1}^{m} p_j |\eta_j| e^{-\eta_j x} \left[\boldsymbol{C}_j^q \boldsymbol{Q} + \left(\chi_1(j) \int_{(-\infty, a_q] \cap E} + \chi_2(j) \int_{E \cap [b_q, \infty)} \right) g(y) e^{\eta_j y} dy \right]. \tag{4.8}$$

Since $\psi(\rho_k) - r = 0$ for all k, we deduce that (4.8) is equal to zero. Further, by comparing the coefficients of $e^{-\eta_j x}$ $(1 \le j \le m)$ and combining the conditions of ϕ on ∂E, we obtain a square system S_q of linear equations. Its form is given below according to the type of I_q.

Case 1: $-\infty < a_q < b_q < \infty$. We have $m+2$ equations given by:

$$\begin{cases} \boldsymbol{C}_j^q \boldsymbol{Q} & = -\int_{(-\infty, a_q] \cap E} g(y) e^{\eta_j y} dy, \quad 1 \le j \le m_0, \\ \boldsymbol{C}_j^q \boldsymbol{Q} & = -\int_{E \cap [b_q, \infty)} g(y) e^{\eta_j y} dy, \quad m_0 + 1 \le j \le m, \\ \sum_{k=1}^{m+2} \boldsymbol{Q}_k^q e^{\rho_k a_q} & = g(a_q), \\ \sum_{k=1}^{m+2} \boldsymbol{Q}_k^q e^{\rho_k b_q} & = g(b_q). \end{cases} \tag{4.9}$$

Case 2: $-\infty < a_q < b_q = \infty$. (Note that $\boldsymbol{Q}_k^q = 0$ for $k = m_0+2, m_0+3 \cdots, m+2$ by vanishing condition.) We have $m_0 + 1$ equations given by

$$\begin{cases} \boldsymbol{C}_j^q \boldsymbol{Q} & = -\int_{(-\infty, a_q] \cap E} g(y) e^{\eta_j y} dy, \quad 1 \le j \le m_0, \\ \sum_{k=1}^{m_0+1} \boldsymbol{Q}_k^q e^{\rho_k a_q} & = g(a_q). \end{cases} \tag{4.10}$$

Case 3: $-\infty = a_q < b_q < \infty$ (Note that $\boldsymbol{Q}_k^q = 0$ for $k = 1, 2, \cdots, m_0 + 1$ by vanishing condition.) We have $m - m_0 + 1$ equations.

$$\begin{cases} \boldsymbol{C}_j^q \boldsymbol{Q} & = -\int_{E \cap [b_q, \infty)} g(y) e^{\eta_j y} dy, \quad m_0 + 1 \le j \le m, \\ \sum_{k=m_0+2}^{m+2} \boldsymbol{Q}_k^q e^{\rho_k b_q} & = g(b_q). \end{cases} \tag{4.11}$$

We will write the system $S = \{S_q, q \in \mathfrak{Q}\}$ in matrix form:

$$S : \boldsymbol{C}\boldsymbol{Q} = \boldsymbol{V}(g) \tag{4.12}$$

where $\boldsymbol{C}$ is the $k \times k-$ matrix with obvious entries and $\boldsymbol{V}(g)$ is a k-dimensional column vector that can be read off from the right hand sides of (4.9), (4.10), and (4.11). Here the positive integer k is defined by

$$k = \begin{cases} |\mathfrak{Q}| \cdot (m+2), & \text{if } G \text{ is bounded,} \\ (|\mathfrak{Q}| - 1) \cdot (m+2) + m_0 + 1, & \text{if } G \text{ is bounded below but unbounded above,} \\ (|\mathfrak{Q}| - 1) \cdot (m+2) + m - m_0 + 1, & \text{if } G \text{ is bounded above but unbounded below,} \\ (|\mathfrak{Q}| - 2) \cdot (m+2) + m + 2, & \text{if } G \text{ is unbounded both above and below.} \end{cases}$$

We summarize our results as follows.

Proposition 4.1 *Given a candidate vector $\boldsymbol{Q}$ that always satisfies the vanishing condition if G is unbounded. Define the function ϕ for $\boldsymbol{Q}$ by (4.2). If $(\mathcal{L}_X - r)\phi = 0$ on G, then the vector $\boldsymbol{Q}$ satisfies the system of equation S in (4.12). Conversely, if the vector $\boldsymbol{Q}$ is a solution of the system S in (4.12) for a given bounded function g on E, then the function ϕ satisfies the equation $(\mathcal{L}_X - r)\phi = 0$ on G.*

Proof. By the above argument, one observes that $(\mathcal{L}_X - r)\phi(x)$ is equal to the identity in (4.8). Our results follows directly from this observation. □

Theorem 4.1 *For every open set $G \subset \mathbb{R}$ and every bounded measurable function g on $E = \mathbb{R}\backslash G$, the system of equation S in (4.12) has a unique solution $\boldsymbol{Q}$ that satisfies the vanishing condition if G is unbounded. Moreover, on each maximal open interval $I_q = (a_q, b_q)$ in G, we have*

$$\Phi(x) \equiv \mathbb{E}_x\left[e^{-r\tau_G} g(X_{\tau_G})\right] = \sum_{j=1}^{m} \boldsymbol{Q}_j^q e^{\rho_j x}. \tag{4.13}$$

Proof. Let G be an open set in $\mathbb{R}$ and g a bounded measurable function on $E = \mathbb{R}\backslash G$. By Theorem 3.1, the function $\Phi(x) \equiv \mathbb{E}_x\left[e^{-r\tau_G} g(X_{\tau_G})\right]$ is of the form in (4.2) and satisfies $(\mathcal{L}_X - r)\phi = 0$ on G. It follows from Proposition 4.1 that the coefficient vector $\boldsymbol{Q}$ for Φ is a solution of the system S in (4.12). Furthermore, by the remark after Theorem 3.1, the vector $\boldsymbol{Q}$ satisfies the vanishing condition if G is unbounded.

To prove the uniqueness property, we assume that $\boldsymbol{P}$ is another solution of the system S in (4.12) satisfying the vanishing condition if G is unbounded. Define the function ϕ for $\boldsymbol{P}$ by (4.2)(with $\boldsymbol{Q}_j^q$ replaced by $\boldsymbol{P}_j^q$.) Note that $\phi = \Phi$ on E. By Proposition 4.1 and then Theorem 3.2, we obtain $(\mathcal{L}_X - r)\phi = 0$ on G and $\phi = \Phi$ on G. That $\boldsymbol{P} = \boldsymbol{Q}$ is due to the linear independence of $\{e^{\rho_i x}, 1 \le i \le m+2\}$ on each maximal interval I_q. This prove the first part of the theorem. The second part follows directly. □

Remark. **(a)** Consider the case that $|\mathfrak{Q}| < \infty$. Then, $\boldsymbol{C}$ is an invertible matrix, and we can find the vector $\boldsymbol{Q}$ by setting $\boldsymbol{Q} = \boldsymbol{C}^{-1}\boldsymbol{V}(g)$.
(b) It is interesting to compare Theorem 4.1 with Theorem 1(iii)-(iv) of Jacobsen [19]. For related works, see Jacobsen and Jensen [20] and Novikov et al. [24]. □

In the following example, we consider the two-sided exit problem. We note that when X is a spectrally negative Lévy process, formulae for solutions of Φ for special functions like $g(x) = 1_{x \le a}$ and $g(x) = 1_{x \ge b}$ are available in terms of the scale function. See Kyprianou [21] for details and remarks on the history of these formulae.

Example 4.1. Consider the case that $G = (a, b)$, where $-\infty < a < b < \infty$. Then for every bounded measurable function g on $G^{\complement}$, we have, for $x \in (a, b)$,

$$\Phi(x) = \mathbb{E}_x\left[e^{-r\tau_{(a,b)}} g(X_{\tau_{(a,b)}})\right] = \sum_{j=1}^{m+2} \boldsymbol{Q}_j e^{\rho_j x}.$$

Here the constant vector $\boldsymbol{Q} = (\boldsymbol{Q}_1, \boldsymbol{Q}_2, \cdots, \boldsymbol{Q}_{m+2})$ is the unique solution of the following system of linear equations:

$$\begin{cases} \sum_{k=1}^{m+2} \boldsymbol{Q}_k \frac{\rho_k}{\rho_k+\eta_j} e^{a\rho_k} = g(a) - e^{-a\eta_j} \int_{-\infty}^{a} g(y)\eta_j e^{\eta_j y} dy, & 0 \le j \le m_0, \\ \sum_{k=1}^{m+2} \boldsymbol{Q}_k \frac{\rho_k}{\rho_k+\eta_j} e^{b\rho_k} = g(b) + e^{-b\eta_j} \int_{b}^{\infty} g(y)\eta_j e^{\eta_j y} dy, & m_o + 1 \le j \le m+1, \end{cases} \tag{4.14}$$

where we set $\eta_0 = \eta_{m+1} = 0$.

Indeed, the system of equation (4.9) reads:

$$\sum_{k=1}^{m+2} \boldsymbol{Q}_k \frac{-1}{\rho_k+\eta_j} e^{a(\rho_k+\eta_j)} = -\int_{-\infty}^{a} g(y)e^{\eta_j y}dy, \qquad 1 \le j \le m_0, \tag{4.15}$$

$$\sum_{k=1}^{m+2} \boldsymbol{Q}_k \frac{1}{\rho_k+\eta_j} e^{b(\rho_k+\eta_j)} = -\int_{b}^{\infty} g(y)e^{\eta_j y}dy, \qquad m_0+1 \le j \le m, \tag{4.16}$$

$$\sum_{k=1}^{m+2} \boldsymbol{Q}_k e^{\rho_k a} = g(a), \tag{4.17}$$

$$\sum_{k=1}^{m+2} \boldsymbol{Q}_k e^{\rho_k b} = g(b). \tag{4.18}$$

For $0 \le j \le m_0$, to get the equation in (4.14), we first multiply both sides of (4.15) by $\eta_j e^{-\eta_j a}$ and then adding the resulting equation and (4.17). One may obtain (4.14) for $m_0+1 \le j \le m+1$ similarly. $\square$

In the example below, we write down explicitly the system of equations S in (4.12) when G is a union of two disjoint bounded intervals.

Example 4.2. Consider the case that $G = G_1 \cup G_2$, where $G_1 = (a,b), G_2 = (u,w)$, and $-\infty < a < b < u < w < \infty$. Then for every bounded measurable function g on $G^{\complement}$, we have, on each G_i,

$$\Phi(x) = \mathbb{E}_x[e^{-r\tau_G} g(X_{\tau_G})] = \sum_{j=1}^{m+2} \boldsymbol{Q}_j^i e^{\rho_j x}.$$

Here the constant vector $\boldsymbol{Q} = \{\boldsymbol{Q}_j^i, i = 1,2, 1 \le j \le m+2\}$ is an unique solution of the following system of linear equations:

$$\begin{cases} \sum_{k=1}^{m+2} \boldsymbol{Q}_k^1 \frac{\rho_k}{\rho_k+\eta_j} e^{a\rho_k} = g(a) - e^{-a\eta_j} \int_{-\infty}^{a} g(y)\eta_j e^{\eta_j y}dy, & 0 \le j \le m_0, \\ \sum_{k=1}^{m+2} [\boldsymbol{Q}_k^1 \frac{\rho_k}{\rho_k+\eta_j} e^{b\rho_k} - \boldsymbol{Q}_k^2 \frac{\eta_j e^{-b\eta_j}}{\rho_k+\eta_j} (e^{w(\rho_k+\eta_j)} - e^{u(\rho_k+\eta_j)})] & \\ = g(b) + e^{-b\eta_j} \int_{(b,u)\cup(w,\infty)} g(y)\eta_j e^{\eta_j y}dy, & m_o+1 \le j \le m+1, \\ \sum_{k=1}^{m+2} [\boldsymbol{Q}_k^2 \frac{\rho_k}{\rho_k+\eta_j} e^{u\rho_k} + \boldsymbol{Q}_k^1 \frac{\eta_j e^{-u\eta_j}}{\rho_k+\eta_j} (e^{b(\rho_k+\eta_j)} - e^{a(\rho_k+\eta_j)})] & \\ = g(u) - e^{-u\eta_j} \int_{(-\infty,a)\cup(b,u)} g(y)\eta_j e^{\eta_j y}dy, & 0 \le j \le m_0, \\ \sum_{k=1}^{m+2} \boldsymbol{Q}_k^2 \frac{\rho_k}{\rho_k+\eta_j} e^{w\rho_k} = g(w) + e^{-w\eta_j} \int_{w}^{\infty} g(y)\eta_j e^{\eta_j y}dy, & m_0+1 \le j \le m+1. \end{cases} \tag{4.19}$$

Here we set $\eta_0 = \eta_{m+1} = 0$. (By similar arguments as in Example 4.1, we get the above system of equations from (4.9).) $\square$

5 An application: pricing perpetual callable coupon bond

To illustrate our results in Section 4, we consider an application to bond pricing. For another two-sided exit problem applied to perpetual American strangles, see Boyarchenko [10].

It is commonly stipulated in a bond covenant that the bond will be redeemed for recapitalization purpose if the firm value is above a certain level. In this case, in addition to the common practice of taking into account the recovery at default in corporate bond pricing, one should also consider the recovery at redemption prior to default. This modeling of corporate bonds is well recognized and discussed in, for example, Black and Cox [6] and Goldstein and Leland [17].

We assume that the firm asset value is a jump diffusion whose logarithm is the one considered in Section 4. We will give explicit solution of a risk neutral price of a perpetual corporate bond for which both the default and redemption of bond are possible given two deterministic boundaries.

We also assume the existence of a constant risk free rate $r > 0$ for all maturities. Let $\mathbb{Q}$ be an equivalent martingale measure such that the firm's asset value takes the form

$$V_t = e^{X_t}, \qquad t \ge 0.$$

Here X, if $V_0 = 1$, is the process considered in Section 4, namely, X is given by (1.2) whose jump pdf is given by (4.1). Then by the definition of the family $(\mathbb{P}_x)$, $\mathbb{Q} = \mathbb{P}_{\log(V_0)}$.

Assume the management decides to follow the **upward capital structure strategy** throughout time (see [17]). That is, at time 0, the firm chooses two thresholds V_L^0 and V_U^0 satisfying $V_L^0 < V_0 < V_U^0$ and issues a perpetual callable coupon bond whose covenant specifies

(1) The life time of the bond ends if either of the following events occurs:

- **(a)** The firm asset value falls outside (V_L^0, V_U^0) at τ_1 by crossing the upper boundary V_U^0. Then recapitalization takes place, and the bond is called at a time-inhomogeneous callable price K.
- **(b)** The firm asset value falls outside (V_L^0, V_U^0) at τ_2 by crossing V_L^0. Then the firm declares bankruptcy, and liquidation occurs. The bondholder takes over the firm and receives the remaining value of the firm. However, a fraction α of the remaining firm value is lost due to bankruptcy costs.

(2) The bond pays a constant coupon rate $C > 0$ up to the life time of the bond.

Under the given risk neutral probability measure $\mathbb{Q}$, the no-arbitrage price of the corporate bond is given by

$$D(V_0) = \mathbb{E}_{\log V_0}\left[\int_0^{\tau_1\wedge\tau_2} C(1-\tau_{\mathrm{p}})e^{-rt}dt\right] + \mathbb{E}_{\log V_0}\left[\widehat{g}(V_{\tau_1\wedge\tau_2})e^{-r\tau_1\wedge\tau_2}\right], \tag{5.1}$$

where τ_{p} is the personal tax rate and

$$\widehat{g}(y) = \begin{cases} (1-\alpha)y, & \text{if } y \le V_L^0, \\ K, & \text{if } y \ge V_U^0. \end{cases}$$

Set $x = \log V_0$. Recall that $V_t = e^{X_t}$, and set $\tau_G = \inf\{t \in \mathbb{R}_+; X_t \notin G\}$, where $G = (\log V_L^0, \log V_U^0)$. Then $\tau_G = \tau_1 \wedge \tau_2$, and the components of the bond price (5.1) can be written as

$$\mathbb{E}_x\left[\int_0^{\tau_G} C(1-\tau_{\mathrm{p}})e^{-rt}dt\right] = \frac{C(1-\tau_{\mathrm{p}})}{r}\left(1-\mathbb{E}_x\left[e^{-r\tau_G}\right]\right) \tag{5.2}$$

$$\mathbb{E}_x\left[(1-\alpha)V_{\tau_1}e^{-r\tau_1}\mathbf{1}(\tau_1<\tau_2)\right] = (1-\alpha)\mathbb{E}_x\left[e^{-r\tau_G}e^{X_{\tau_G}}\mathbf{1}_{X_{\tau_G}\le \log V_L^0}\right] \tag{5.3}$$

and

$$\mathbb{E}\left[Ke^{-r\tau_2}\mathbf{1}(\tau_2<\tau_1)\right] = K\mathbb{E}_x\left[e^{-r\tau_G}\mathbf{1}_{X_{\tau_G}\ge \log V_U^0}\right]. \tag{5.4}$$

We use the result in Section 4 to give explicit solution for $D(V_0)$. For every bounded Borel measurable function g on G^c, we have, by Theorem 4.1,

$$\Phi(x) = \mathbb{E}_x[e^{-r\tau_G}g(X_{\tau_G})] = \sum_{j=1}^{m+2}\boldsymbol{Q}_j e^{\rho_j x}, \quad x \in (\log V_L^0, \log V_U^0) \tag{5.5}$$

for some constants $\boldsymbol{Q}_j$. Moreover, (4.14) implies that the constant vector $\boldsymbol{Q} = (\boldsymbol{Q}_1, \boldsymbol{Q}_2, \cdots, \boldsymbol{Q}_{m+2})$ satisfies the following system of linear equations:

$$\begin{cases} \sum_{k=1}^{m+2}\boldsymbol{Q}_k\frac{\rho_k}{\rho_k+\eta_j}(V_L^0)^{\rho_k} = g(\log V_L^0) - (V_L^0)^{-\eta_j}\int_{-\infty}^{\log V_L^0} g(y)\eta_j e^{\eta_j y}dy, & 0 \le j \le m_0, \\ \sum_{k=1}^{m+2}\boldsymbol{Q}_k\frac{\rho_k}{\rho_k+\eta_j}(V_U^0)^{\rho_k} = g(\log V_U^0) + (V_U^0)^{-\eta_j}\int_{\log V_U^0}^{\infty} g(y)\eta_j e^{\eta_j y}dy, & m_o+1 \le j \le m+1. \end{cases} \tag{5.6}$$

Here we set $\eta_0 = \eta_{m+1} = 0$. Consider $\boldsymbol{Q}$ as a column vector. We rewrite the system (5.6) in matrix form:

$$\boldsymbol{D}\boldsymbol{Q} = \boldsymbol{U}(g) \tag{5.7}$$

where $\boldsymbol{D}$ is the $(m+2)\times(m+2)$ matrix with the entries given by

$$\boldsymbol{D}_{j,k} = \begin{cases} \frac{\rho_k}{\rho_k+\eta_j}(V_L^0)^{\rho_k} & \text{if } 0 \le j \le m_0, 1 \le k \le m+2 \\ \frac{\rho_k}{\rho_k+\eta_j}(V_U^0)^{\rho_k} & \text{if } m_0+1 \le j \le m+1, 1 \le k \le m+2 \end{cases}, \tag{5.8}$$

and $\boldsymbol{U}(g)$ is the column vector with component $\boldsymbol{U}(g)_j$ defined by

$$\boldsymbol{U}(g)_j = \begin{cases} g(\log V_L^0) - (V_L^0)^{-\eta_j} \int_{-\infty}^{\log V_L^0} \eta_j g(y) e^{\eta_j y} dy & \text{if } 0 \le j \le m_0 \\ g(\log V_U^0) + (V_U^0)^{-\eta_j} \int_{\log V_U^0}^{\infty} \eta_j g(y) e^{\eta_j y} dy & \text{if } m_0 + 1 \le j \le m+1. \end{cases} \tag{5.9}$$

If we set

$$g_1(y) \equiv 1, \quad g_2(y) = e^y 1_{y \le \log V_L^0}, \text{ and } g_3(y) = 1_{y \ge \log V_U^0},$$

we have

$$\begin{aligned} \boldsymbol{W} \equiv & -\frac{C(1-\tau_p)}{r}\boldsymbol{U}(g_1) + (1-\alpha)\boldsymbol{U}(g_2) + K\boldsymbol{U}(g_3) \\ = & -\frac{C(1-\tau_\mathrm{p})}{r}(1,0,\cdots,0,1) + (1-\alpha)\left(V_L^0, \frac{1}{\eta_1+1}V_L^0, \cdots, \frac{1}{\eta_{m_0}+1}V_L^0, 0, \cdots, 0\right) + K(0,\cdots,0,1) \\ = & \left(-\frac{C(1-\tau_p)}{r} + (1-\alpha)V_L^0, \frac{1-\alpha}{\eta_1+1}V_L^0, \cdots, \frac{1-\alpha}{\eta_{m_0}+1}V_L^0, 0, \cdots, 0, -\frac{C(1-\tau_p)}{r} + K\right). \end{aligned}$$

Write $\boldsymbol{M}$ for the inverse matrix of $\boldsymbol{D}$. It follows from (5.1) that the price of the callable coupon bond is given by the formula

$$D(V_0) = \frac{C(1-\tau_p)}{r} + (\boldsymbol{M}\boldsymbol{W})^\top \boldsymbol{e}^{\boldsymbol{\rho}}(x) \tag{5.10}$$

where $x = \log V_0$ and $\boldsymbol{e}^{\boldsymbol{\rho}}(x) \equiv (e^{\rho_1 x}, \cdots, e^{\rho_{m+2} x})$. Therefore we obtain an explicit formula for the price of a perpetual callable coupon bond.

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