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附件一

行政院國家科學委員會補助專題研究計畫 ■成果報告 □期中進度報告

常數彈性變異數過程與其應用

計畫類別：■ 個別型計畫　　□ 整合型計畫
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共同主持人：
計畫參與人員：王志瑋、邱婉茜

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中　華　民　國　99 年　1　月　30　日

1.Introduction (前言與研究目的)

This study is intended to examine the empirical performance of the Constant–Elasticity-of-Variance (CEV) option pricing model by Cox (1975) and Cox and Ross (1976), especially whether and by how much the generalization of the CEV model among prevailing option pricing models improves option pricing. In order to reduce the empirical biases of the Black-Scholes (BS) (1973) option pricing model, succeeding option pricing models have to relax the restrictive assumptions made by the BS model: the underlying price process (distribution), the constant interest rate, and the dynamically complete markets. The tradeoff is, however, a more computational cost.

To examine whether these generalized models are worth the additional complexity and cost, Bakshi, Cao and Chen (1997) compared a set of nested models in which the most general model allowed volatility, interest rate, and jumps to be stochastic (SVSI-J)[1]. They examined four alternative models from three perspectives: (1) internal consistency of implied parameters/volatility with relevant time-series data, (2) out-of-sample pricing, and (3) hedging. Their research showed that modeling stochastic volatility and jumps (SVJ) is critical for pricing and internal consistency, while introducing stochastic volatility (SV) alone yields the best performance for hedging.[2] However, models not in the nested set were not evaluated in their empirical study. Accordingly, the CEV model, which introduces only one more parameter while providing the time-changing volatility feature, is not nested in the stochastic volatility model and thus not empirically tested by Bakshi, Cao and Chen (1997). Therefore, this study is to include the CEV model in the empirical investigation and examine the model performance.

Although the CEV model is not as general and flexible as the SVJ model, its simplicity may still be worth exploring since the above mentioned models are expensive to implement. In particular, the above mentioned models, when applied to American option pricing, require high-dimensional lattice models which are prohibitively expensive. On the other hand, the CEV model requires only a single dimensional lattice (Nelson and Ramaswamy (1990) and Boyle and Tian (1999)).

The CEV model proposed by Cox (1975) and Cox and Ross (1976) is complex enough to allow for changing volatility and simple enough to provide a closed form solution for options with only two parameters. The CEV diffusion process also preserves

[1] Bakshi, Cao and Chen (2009) complement the nested model tests and report the empirical performance of the four models nested in the SVSI model. According to the pricing and hedging performance measures, their results show that the SVSI and the SV models both perform much better than the stochastic interest rate (SI) and the BS models. The SI model can produce respectable pricing improvement over the BS model. However, in the presence of stochastic volatility, doing so no longer improves pricing performance much further.

[2] Bakshi, Cao and Chen (2000) expanded their samples to longer term options using LEAPS. Their empirical results still indicate that modeling stochastic volatility is the first-order of importance. Once the model has accounted for stochastic volatility, allowing interest rates to be stochastic does not improve pricing performance any further. Only for devising a hedge of LEAPS put does incorporating stochastic interest rates make a difference. However, the hedging performance is not the interest of this paper. Therefore, we will focus our analysis on pricing performance.

the property of nonnegative values of the state variables as is in the lognormal diffusion process assumed in the Black-Scholes model (Chen and Lee, 1993). The early research of the CEV model was conducted by MacBeth and Merville (1980) and Emanuel and MacBeth (1982) to test the empirical performance of the CEV model and compared it with the BS model. Recent studies of the CEV process include applications in path-dependent options and credit risk models. We will briefly review and summarize the literature in Section 2.

The empirical study of the CEV model was first conducted by MacBeth and Merville (1980). They provided results on six stock options and showed that the CEV parameter β is generally less than two, which explains the empirical evidence for the negative relationship between the sample variance of returns and stock prices. Manaster (1980) criticized the approach by MacBeth and Merville (1980) and suggested that (i) the CEV parameter β and the volatility parameter δ should be estimated jointly without using the information (implied parameter $\hat{\sigma}$ of at-the-money option) from the BS model, and (ii) post-estimation testing should be conducted to see whether the CEV model continues to fit the observed date better than the BS model for the day or week following the parameter estimation. In response, Emanuel and MacBeth (1982) tested the post-estimation performance of the CEV model but still used similar approach for parameter estimation. More recently, Lee, Wu, and Chen (2004) took S&P 500 index options as opposed to stock options to avoid the American option premium biases, but still employed the similar two-step estimation to obtain the estimated β and δ. Also using the S&P 500 index to reduce market imperfections, Jackwerth and Rubinstein (2001) compared the ability of several models including the CEV model to explain the otherwise identically observed option prices that differ by strike prices, times-to-expiration, or trade times. They found that the performance of the CEV model is similar to other models they tested, and those better performing models all incorporate the negative correlation between the index level and volatility.

In contrast to the previous empirical studies of the CEV model, first, we jointly estimate parameters β and δ by minimizing the sum of squared dollar pricing errors, absolute dollar pricing errors, and percentage pricing errors of the daily market price and the estimated price of options. Secondly, a "synchronized" dataset of stock prices and option prices by Bakshi, Cao and Chen (1997) is used[3]. We find that (i) In terms of in-sample performance, the squared sum of pricing errors of the CEV model is similar to the SV models in short-term and at-the-money options, but is worse in other categories and (ii) In terms of out-of-sample performance, the mean absolute errors and percentage errors show that the CEV model performs better than the SV model in the short term and out-of-the-money categories. In addition, the CEV model is even better than the SVJ model in a few cases in these categories.

The rest of the paper is organized as follows: Section 2 reviews the CEV model and previous empirical studies. The recent applications under the CEV process of

[3] We thank Charles Cao for providing the original data of the paper "Empirical Performance of Alternative Option Pricing Models," *Journal of Finance*, 1997.

path-dependent option pricing as well as credit risk derivative modeling are also presented. Section 3 discusses the CEV model as well as alternative models to be empirically tested in our study. Section 4 provides the empirical testing results, and Section 5 presents conclusion.

2. Literature Review (文獻探討)

We first review the CEV model and previous empirical studies in Section 2.1. In Section 2.2, the recent applications under the CEV process of path-dependent option pricing as well as credit risk derivative modeling are presented.

2.1 The Constant Elasticity Variance Option Pricing Model

2.1.1 The CEV Option Pricing Model

An important issue in option pricing is to find a stock return distribution that allows returns to stock and its volatility to be correlated with each other. There is considerable empirical evidence that the returns to stocks are heteroscedastic and the volatility of stock returns changes with the stock prices. A great deal of empirical evidence indicates that stock volatility is negatively related to the stock price, and it is so-called leverage effect first discussed by Black (1976). To accommodate this leverage effect, the Constant Elasticity of Variance (CEV) model by Cox (1975) and Cox and Ross (1976) relaxes the constant volatility assumptions of the Black-Scholes model and treats volatility as a deterministic function – as a power function of the price of the underlying asset. The rationale for an inverse relationship between the stock price and its variance of return can be explained by some simple economic arguments. Researchers use both financial and operating leverage arguments. A decline in a leveraged firm's stock price may lead to an increase in its debt-equity ratio, hence the riskiness of the stock increases. Even if a firm has no debt, the decline of the stock price can make it more difficult for the firm to meet its fixed costs and thus increases volatility (Hull, 2002).

The CEV model assumes the diffusion process for the stock is

(Eq 2.1.1) $dS = \mu S dt + \delta S^{\beta/2} dz$,

and the instantaneous variance of the percentage price change or return, σ^2, follows deterministic relationship:

(Eq 2.1.2) $\sigma^2(S,t) = \delta^2 S^{(\beta-2)}$

where the elasticity of this variance with respect to the stock price equals β.

If β =2, prices are lognormally distributed and the variance of returns is constant, which is the same as the well-known Black-Scholes model. If β <2, the stock price is inversely related to the volatility. Cox originally restricted $0 \le \beta < 2$. Emanuel and MacBeth (1982) extended his analysis to the case $\beta > 2$ and discussed its properties.

However, Jackwerth and Rubinstein (2001) found that typical values of the β can fit market option prices well for the post-crash period only when $\beta < 0$, and they called the model with $\beta < 0$ the unrestricted CEV model[4]. In their empirical study, the difference of the pricing performance of restricted CEV model ($\beta \geq 0$) and BS model is not significant.

When β<2, the nondividend-paying CEV call pricing formula is as follows:

(Eq 2.1.3)

$$C = S\left[\sum_{n=0}^{\infty} g(S' \mid n+1) G(K' \mid n+1+\frac{1}{2-\beta})\right] - Ke^{-r\tau}\left[\sum_{n=0}^{\infty} g(S' \mid n+1+\frac{1}{2-\beta}) G(K' \mid n+1)\right]$$

When β>2, the CEV call pricing formula is as follows:

(Eq 2.1.4)

$$C = S\left[1-\sum_{n=0}^{\infty} g(S' \mid n+1+\frac{1}{2-\beta}) G(K' \mid n+1)\right] - Ke^{-r\tau}\left[1-\sum_{n=0}^{\infty} g(S' \mid n+1) G(K' \mid n+1+\frac{1}{2-\beta})\right]$$

where

$$S' = \left[\frac{2re^{r\tau(2-\beta)}}{\delta^2(2-\beta)(e^{r\tau(2-\beta)}-1)}\right] S^{2-\beta}$$

$$K' = \left[\frac{2r}{\delta^2(2-\beta)(e^{r\tau(2-\beta)}-1)}\right] K^{2-\beta}$$

$g(x \mid m) = \dfrac{e^{-x} x^{m-1}}{\Gamma(m)}$ is the gamma density function

$$G(x \mid m) = \int_x^{\infty} g(y \mid m)\, dy$$

C is the call price; *S*, the stock price; τ , the time to maturity; *r*, the risk-free rate of interest; *K*, the strike price; and β and δ , the parameters of the formula.

2.1.2 Previous Empirical Studies

MacBeth and Merville (1980) were the first to empirically test the CEV option model. They tested the CEV model against the Black-Scholes (BS) model using daily

[4] The unrestricted CEV model is mathematically legitimate. However, there are some economic arguments supporting a restriction on the parameter β . For example, it is inconceivable for the stock index to have a significant probability of bankruptcy while this is likely with sufficiently negative β . See the detail in Jackwerth and Rubinstein (Page 12; 2001) and Bates (1996a).

closing prices of call options on six companies' stock from December 31, 1975 to December 31, 1976. Their estimation procedure is as follows:

From *(Eq 2.1.1)*,

(Eq 2.1.5) $$\frac{(dS-\mu S dt)^2}{S^{\beta} dt} \equiv u_t \propto \chi^2(1)$$

where the constant or proportionality is $1/\delta^2$.

Following *(Eq 2.1.5)*, for some interval of time *dt*,

(Eq 2.1.6) $$\ln[(dS-\mu S dt)^2] - \ln dt = 2\ln\delta + \beta \ln S + \ln[\chi^2(1)].$$

They first estimate μ from the sample of daily returns. Point estimates of the elasticity parameter β can then be obtained using a linear regression since the Chi-square random variables are uncorrelated through time. Note that they only choose the integer value of β and fix it the same for all options written on the same stock. The way they choose the integer value is that starting with their point estimate of β, they use a numerical routine to calculate an implied value of δ for each observed option price until they have approximately the same of value of δ for each option price. This is done on four arbitrarily selected days during a year. Finally, δ is deduced from the BS model by taking an at-the-money option on a given day. That is $\delta_t = \sigma_t S_t^{(2-\beta)/2}$ where the variance rate σ_t^2 is from the BS model.

Their empirical results show that the CEV parameter β is generally less than two and ranging from -4 for IBM to 1 for Xerox, which explains the empirical evidence for the negative relationship between the sample variance of returns and stock price. Moreover, they demonstrate that under these circumstances, the CEV model generates estimated option prices closer to the market prices than those of the BS models.

Manaster (1980) criticized the approach by MacBeth and Merville (1980) and suggested that (i) the CEV parameter β and the volatility parameter δ should be estimated jointly without using the information (implied parameter $\hat{\sigma}$ of at-the-money option) from the BS model, and (ii) post-estimation testing should be conducted to see whether the CEV model continues to fit the observed date better than the BS model for the day or week following the parameter estimation.

In response, Emanuel and MacBeth (1982) tested the post-estimation performance of the CEV model for 1 day, 5 days (a week), and 17 days (a month) including the daily closing prices of call options written on the same six stocks for each day in 1978. To perform the out-of-the-sample test, they select the best value of β on each day by searching integer values minimizing the squared deviation between model prices and market prices of option with at least 90 days to expiration. Their results showed that the CEV model yields more accurate predictions of future option prices than the BS model in nearly all cases in the period of less than one month[5].

Using time series data of underlying assets alone, several empirical studies found that estimates of β are confined to 0 and 2 as opposed to the negative estimates. Beckers (1980) estimated the CEV parameters for 47 stocks using the daily stock price data from 1972 to 1977. He found that most return distributions are less positively skewed than the lognormal ($\beta < 2$) and support the significant relationship between the level of the stock price and its volatility. Gibbons and Jacklin (1988) examined stock prices over a longer data sample during 1962 to 1985, and also almost invariably estimated β between 0 and 2.

Recently, Lee, Wu, and Chen (2004) took the S&P 500 index options as opposed to stock options to avoid the American option premium biases, and used the non-central chi-square probability functions proposed by Schroder (1989) to reduce the approximation errors. In addition, they also expanded their analysis into six moneyness and three maturity categories. They employed a similar two-step estimation to obtain the estimated β and δ as MacBeth and Merville (1980), and the difference was that they did not constrain the elasticity value β to integer values. Their results still supported the MacBeth and Merville results (1980) although the samples were not subject to the American premium biases. The CEV model in terms of the non-central chi-square distribution performs better than the Black-Scholes model in pricing the S&P 500 index call options during January 1, 1992 to June 30, 1997. Furthermore, with the estimates of $\beta < 2$ for the sample period, it is implied that a negative relation exists between the sample index value and its volatility of daily returns.

Also using the S&P 500 index to reduce market imperfections, Jackwerth and Rubinstein (2001) evaluated five kinds of option models with a total of nine models among the deterministic models, the stochastic models and the naïve trader rules. The five categories of models are: (i) the Black-Scholes model; (ii) two naïve smile-based predictions that use today's observed smile directly for the prediction; (iii) two versions of the CEV models; (iv) an implied binomial tree model; and (v) three parametric models including displace diffusion, jump diffusion, and stochastic volatility.

They performed two main types of tests for the following relations: (1) Options prices at the same time, with the same underlying asset, and the same strike price, but with different times-to-expiration; (2) Option prices with the same underlying asset, the same expiration date, and the same ratio of strike price to underlying asset price, but observed at different times. Investigating the relation (1) involves the problem of deducing short-term option prices from longer-term option prices. The volatility smile for the longer-term options is assumed known, and the volatility smile for the shorter-term options is unknown.

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[5] They also noted that the CEV model works best when β is less than two, given the empirical evidence that implied volatility is inversely related to stock price. However, for the period of April to November in 1978, the estimated values of β are larger than two. This in turn predicts that volatility and stock price move in the same direction and hence reduces the superior predictive power of option prices of the CEV model compared with the BS model during 1978.

They then fitted the alternative option models to the longer-term option prices, and compared the model values with the observed market prices for the shorter-term options and calculated pricing errors (backward-looking test). To investigate relation (2), they calibrated alternative models on current longer-term option prices, and computed the errors of the forecast prices using the underlying asset price observed 10 and 30 days later (forward-looking test). To decompose the source of any remaining pricing errors, they also conducted related experiments assuming in addition that the at-the-money implied volatility of the shorter-term options in the test (1), and the future at-the-money option price in the test (2) are known.

The database includes minute-by-minute trades and quotes from April 2, 1986 to December 29, 1995, which can be divided into a pre-crash period from April 2, 1986 to October 16, 1987, and a post-crash period from June 1, 1988 to December 29, 1995. All option models are parameterized to price the observed longer-term options best, those with times-to-expiration between 135 and 225 days, and options with 45 to 134 days to expiration are classified as shorter-term options. They then calculate the implied volatilities for these two groups each day and use the median implied volatilities as the representative daily volatility smile for a given time-to-expiration. Finally, due to the lack of liquidity for the deep out-of-the money and deep in-the-money options, they only use those with moneyness (strike price / index level ratios) between 0.79 and 1.16.

Jackwerth and Rubinstein found that in the pre-crash period, all models match the performance of the Black-Scholes model. The reason is that the volatility smiles were almost flat during this period. In the post-crash period, surprisingly, the naive trader rules perform best. Furthermore, the performances of all models are very similar, except the Black-Scholes and the restricted CEV model. The unrestricted CEV model is similar to other models they tested, and those better performing models all incorporate the negative correlation between index level and volatility.

2.2 Recent Development and Applications of the CEV Process

Recent applications of the CEV process are mainly in path-dependent option and credit derivative pricing. We first summarize the recent path-dependent option pricing studies under the CEV process in Section 2.2.1, and then present the credit risk application of the CEV process under the unified pricing framework in Section 2.2.2.

2.2.1 Path-Dependent Option Pricing

In the context of path-dependent option pricing, numerical method of the CEV process was first developed by Nelson and Ramaswamy (1990) using the binomial method. Boyle and Tian (1999) then constructed a trinomial method to approximate the CEV process and used it to price the barrier and lookback options. Boyle and Tian found that the prices of the barrier and lookback options for the CEV process deviate significantly from those for the lognormal process in the BS model, while the corresponding differences between the CEV and the Black-Scholes models are relatively small. They concluded that the model specification of options depend on extrema is much more important than for that of standard options. Later on, Detemple and Tian (2002) proposed a recursive integral equation for the valuation of American-style derivatives when the underlying asset price follows the CEV process. Using the Early Exercise Premium (EEP) representation, they derived a recursive integral equation for the exercise boundary and provide a parametric representation of the prices of American option and American capped option.

Opposed to the numerical method, Davydov and Linetsky (2001, 2003) derived the analytic closed-form formulae for the prices of the barrier and lookback options. In the former paper, a Euler numerical inversion algorithm of the Laplace transforms is used to obtain the option value, while in the latter they used a different approach by Eigenfunction expansion. Leung and Kwok (2006) derived the analytic expressions for the double Laplace transform of the density function of occupation time and the joint density function of occupation time and terminal asset value under the CEV process. They also used it to price the α -quantile options. In addition to the research mentioned above, some applications of the CEV process in path-dependent option pricing and the related papers are Lo, Yuen, and Hui, (2000), Lo, Tang, Ku and Hui (2004), and DelBaen and Sirakawa (2002).

2.2.2 Application in Credit Risk and Derivative Modeling

In credit risk modeling, the CEV process also has an advantage over the geometric Brownian motion that, intuitively, the standard CEV process can hit zero due to the increased volatility of the former process at low stock prices while the geometric Brownian motion cannot. To circumvent the estimation problem of structural credit risk models in which the leverage information is from the stale book values, these studies alternatively model the default trigger event as equity value hitting the zero barrier. In addition, the empirical evidence of the clear link between default risk and equity volatility can also be parsimoniously captured using the CEV process given its ability to model the leverage effect.

Albanese and Chen (2004) and Campi and Sbuelz (2005) used the CEV model to price the equity default swaps. Carr and Linetsky (2006) and Campi, Polbennikov, and Sbuelz (2005) further introduced the hazard process of the reduced-form models to avoid the default predictability issue. Their models assume that the stock price follows a CEV diffusion, punctuated by a possible jump to zero. Therefore, using the stock process hitting zero as the default trigger event, the default can come from either diffusion or the unpredictable Poisson jump process. They call the resulting stock price process the jump to default extended CEV process (the JDCEV model). They also showed that, by incorporating jump into the model, the JDCEV model can capture the volatility skews much better than the pure CEV diffusion model, especially for the skews across different moneyness.

Carr and Linetsky (2006) developed a unified framework under the CEV diffusion and jump to default process for pricing, trading, and risk managing corporate liabilities, credit derivatives, and equity derivatives. Their generalizations are financially relevant as they include killing (default), as well as time-dependent parameters, while retaining analytical tractability due to the remarkable properties of the Bessel processes. Campi, Polbennikov, and Sbuelz (2005) also used a similar but time homogeneity setting, which does not include correlated jump parameter, for corporate bond prices and credit default swap (CDS) prices.

Carr and Linetsky (2006) assume frictionless markets, no arbitrage, and take an equivalent martingale measure (EMM) Q. The pre-default stock dynamics under the EMM is a time-inhomogeneous diffusion process solving a stochastic differential equation

(Eq 2.2.1) $$dS_t = [r(t) - q(t) + \lambda(S_t, t)]S_t dt + \sigma(S_t, t)S_t dB_t; \quad S_0 = S > 0$$

where $r(t) \geq 0$, $q(t) \geq 0$, $\sigma(S,t) > 0$ and $\lambda(S,t) \geq 0$ are the time-dependent risk-free interest rates, time-dependent dividend yields, time- and state-dependent instantaneous stock volatilities, and time- and state-dependent default intensities, respectively.

To be consistent with the leverage effect and the implied volatility skew, Carr and Linetsky (2006) assume the instantaneous volatility as a CEV process[6] $\sigma(S,t) = a(t)S^{\beta}$. In addition, to be consistent with the empirical evidence of linkage of corporate bond yields and CDS spreads to equity volatility, the default intensity is assumed as an affine function of the instantaneous variance of the underlying stock

(Eq 2.2.2) $$\lambda(S,t) = b(t) + c\sigma^2(S,t) = b(t) + ca^2(t)S^{2\beta}.$$

where $b(t) \geq 0$ is a deterministic non-negative function of time and $c > 0$ governs the sensitivity of default intensity to instantaneous equity variance σ^2. By letting both the hazard rate and the instantaneous variance depend on the stock price, the JDCEV model accommodates large negative correlations between default indicators and stock prices, and between realized volatilities and stock prices. Moreover, by forcing the hazard rate and the instantaneous variance to depend on the stock price in the same manner, the JDCEV model induces the large positive correlation between default indicators and volatilities that have been observed in the market. The parameters β and c both play a role in determining the slope of the volatility skew, which gives more flexibility in accommodating slopes which vary with term. Note that their pre-default process is a CEV process with the additional term $ca^2(t)S^{2\beta+1}$ in the drift term.

Note that the standard CEV model of Cox (1975) is nested within their general specification. In fact, the JDCEV model nests a more general time-inhomogeneous version of Cox's model with time-dependent interest rate, dividend yield, and volatility scale parameters $r(t)$, $q(t)$, and $a(t)$, respectively. To obtain this special case, set $b = 0$ and $c = 0$, so that default can only occur when the stock price diffuses into zero[7]. When $b > 0$ is a positive constant and $c = 0$, the JDCEV model reduces to the CEV model with killing at a constant rate considered by Campi et al. (2005). In fact, the model by Campi et al. (2005)

[6] They follow the notation of Davydov and Linetsky (2001a, 2003).

[7] The CEV process with $\beta < 0$ hits zero with positive probability. In contrast, for $\beta = 0$ the limiting process of geometric Brownian motion never hits zero.

differs from the JDCEV model only in that they impose time-homogeneity and the default intensity is independent of the stock price and the return volatility.

3. The Option Pricing Models (選擇權模型)

In this section, the CEV and the stochastic volatility models to be tested in this study are presented. The general model incorporating the stochastic volatility, stochastic interest rate and random jump by Bakshi, Cao and Chen (1997) is presented in Appendix.

3.1 The CEV Option Pricing Model

The CEV model and the call option formula have been shown by Cox (1975) in Section 2.1. In this paper, the CEV formula in terms of the noncentral chi-square distribution expressed by Schroder (1989) is adopted to compute option prices. Therefore, in this section we present the work by Schroder (1989) in which the complementary noncentral chi-square distribution function can be evaluated by the iterative algorithm as well as an approximation derived by Sankaran (1963).

Schroder (1989) expressed the CEV call option pricing formula in terms of the noncentral chi-square distribution:

When β<2,

(Eq 3.1.1) $C = S_t Q(2y;2+2/(2-\beta),2x) - e^{-rt}K(1-Q(2x;2+2/(2-\beta),2y))$

When β>2,

(Eq 3.1.2) $C = S_t Q(2x;2+2/(2-\beta),2y) - e^{-rt}K(1-Q(2y;2+2/(2-\beta),2x))$

$Q(z;v,k)$ is a complementary noncentral chi-square distribution function with z, v, and k being the evaluation point of the integral, degree of freedom, and noncentrality, respectively, where

$$k = \frac{2r}{\delta^2(2-\beta)(e^{r(2-\beta)\tau}-1)}$$

$$x = kS_t^{2-\beta}e^{r(2-\beta)\tau}$$

$$y = kK^{2-\beta}$$

The complementary noncentral chi-square distribution function can be expressed as an infinite double sum of gamma functions as follows[8]:

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[8] Extensive literature exists to efficiently compute noncentral chi-square distribution (see Dyrting (2004), Benton and Krishnamoorthy (2003) and the references therein).

(Eq 3.1.3) $Q(2z;2v,2k) = 1 - \sum_{n=1}^{\infty} g(n+v,z) \sum_{i=1}^{n} g(i,k)$

Schroder also presented a simple iterative algorithm to compute the infinite sum as follows:

(1) Initializing the following variables:

$$gA = \frac{e^{-z} z^{v}}{\Gamma(1+v)}$$

$$gB = e^{-k}$$

$$Sg = gB$$

$$R = 1 - gA \cdot Sg$$

where $gA = g(1+v,z)$ and $gB = g(1,k)$

(2) Looping with n=2 and incrementing by one after each iteration until the contributions t the sum, R are becoming very small.

$$gA = gA \cdot \frac{z}{n+v-1}$$

$$gB = gB \cdot \frac{k}{n-1}$$

$$Sg = Sg + gB$$

$$R = R - gA \cdot Sg$$

where $gA = g(n+v,z)$, $gB = g(n,k)$ and $Sg = g(1,k) + \cdots + g(n,k)$

Although the CEV formula can be represented more simply in the terms of noncentral chi-square distributions that are easier to interpret, the evaluation of the infinite sum of each noncentral chi-square distribution can be computationally slow when neither z or k are too large. This study uses the approximation derived by Sankaran (1963) to compute the complementary noncentral chi-square distribution $Q(2z;2v,2k)$ when z and k are large as follows:

(Eq 3.1.4) $Q(z;v,k) \sim \dfrac{1 - hp[1 - h + 0.5(2-h)mp] - [z/(v+k)]^{h}}{h\sqrt{2p(1+mp)}}$

where $h = 1 - (2/3)(v+k)(v+3k)(v+2k)^{-2}$

$$p = \frac{v + 2k}{(v + k)^2}$$

$$m = (h - 1)(1 - 3h)$$

When neither z or k are too large[9] (i.e., $z < 1000$ and $k < 1000$ and no underflow errors occur), the exact CEV formula is used. Otherwise the approximation CEV formula is used.

3.2 The Stochastic Volatility Option Pricing Models

Unlike the CEV model, the Stochastic Volatility (SV) models consider the volatility of the stock as a separate stochastic factor. The SV models provide a flexible distribution structure of asset returns in which the correlation between the asset returns and the volatility process can be used to control the level of skewness and the volatility variation coefficient (volatility of volatility) can be used to control the amount of kurtosis. Skewness in the distribution of spot returns affects the pricing of in-the-money options relative to out-of-the-money options. Kurtosis affects the pricing of near-the-money versus far-from-the-money options.

The stochastic volatility models differ in several aspects: the process assumed for the volatility, the correlation between the Wiener process of the asset price and that of the volatility, and the method of pricing volatility risk.

First, the volatility processes are assumed in two different classes. Scott (1987), Wiggins (1987), Stein and Stein (1991), and Heston (1993) assume mean-reverting processes, while Hull and White (1987) assume a constant drift. Second, the introduction of a stochastic volatility process makes the partial differential equation (PDE) governing the options price much more complex. Some of the stochastic volatility models make the questionable assumption that this correlation is zero in order to simplify the PDE (Stein and Stein; 1991). Others develop the models under the assumption of arbitrary correlation to make it more realistic (Hull and White, 1987; Wiggins, 1987; Scott 1987; Heston, 1993).

Third, a stochastic volatility is not a tradable or hedgeable source of risk. As a result, there is no unique risk-neutral probability valuation measure to price the options, and risk premium associated with the stochastic volatility must be introduced to cope with the problem. Hull and White (1987), and Stein and Stein (1991) assume that volatility risk is uncorrelated with consumption and therefore perfectly diversifiable. Scott (1987) makes the same assumption when he applies their models, although they formulate the models in terms of an unspecified risk premium at the beginning. Instead of making an assumption that avoids the problem of pricing volatility risk, Wiggins (1987) assumes investors' preferences may be represented by a constant relative risk-aversion utility function and empirically estimate the price of volatility risk. Lastly, Heston (1993) assumes that the risk premium is proportional to the return variance.

[9] Since the computational speed of computers is much faster today, we changed the original setting of Schroder from z <200 and k <200 to z <1000 and k <1000 and found no difficulty.

Here we only present the setting of the Heston model (1993) since it is most relevant to the model we will empirically test in our study. Heston (1993) derived a closed-form solution for the price of the European call option with stochastic volatility using the technique of the characteristic function. In addition, his model allows for arbitrary correlation between the asset returns and volatility.

Heston (1993) assumes that the asset price at time t follows the diffusion equation

(Eq 3.2.1) $$dS = \mu S dt + \sqrt{v(t)} S dZ_1(t)$$

where $Z_1(t)$ is a Weiner process.

The volatility dynamic follows the Ornstein-Uhlenbeck process as

(Eq 3.2.2) $$d\sqrt{v(t)} = -\beta\sqrt{v(t)}dt + \delta dZ_2(t)$$

Using Ito's lemma, this is the square-root process used by Cox, Ingersoll, and Ross (1985):

(Eq 3.2.3) $$dv(t) = \kappa[\theta - v(t)]dt + \sigma\sqrt{v(t)}dZ_2(t)$$

where $Z_2(t)$ is a Weiner process having correlation ρ with $Z_1(t)$. The parameters κ, θ, and σ are the speed of adjustment, long-term mean, and variation coefficient of the variance of the instantaneous return $v(t)$.

Recently, Jones (2003) extends the Heston model and proposes a more general stochastic volatility models in the CEV class and a model with a time-varying leverage effect. The first model in the CEV class has been applied in interest rate by Chan et al (1992), in which the square root in the variance diffusion term is replaced by an exponent of undetermined magnitude. The second model separates power parameters on the two random shocks to instantaneous variance of the model. Thus, the elasticity of variance is no longer constant but depends on the level of the variance process. Moreover, this enables the correlation of the price and variance processes to depend on the level of instantaneous variance.

4. Empirical Tests and ResIts (研究結果與討論)

In this section, the empirical results of European option pricing are reported in Section 4.1, while the analysis of numerical methods in terms of cost-accuracy based analysis is presented in Section 4.2.

4.1 European-Style Option Pricing

In this section, the empirical results following the framework of Bakshi, Cao and Chen (1997) to facilitate the comparison of model performances is presented. The dataset is described in Section 4.1.1, and the option pricing models in Section 4.1.2. Next, we repost the empirical results of the in-sample performance in Section 4.1.3, the model misspecification in terms of volatility smile in Section 4.1.4, and the out-of-sample performance in Section 4.1.5, respectively.

4.1.1 Data Description

We use the S&P 500 call option prices for the empirical work[10]. The sample period extends from June 1, 1988 through May 31, 1991. The intradaily bid-ask quotes for S&P 500 options are originally obtained from the Berkeley Option Database. The daily Treasury-bill bid and ask discounts with maturities up to one year are from the Wall Street Journal. Note that the recorded S&P 500 index are not the daily closing index level. Rather, they are the corresponding index levels at the moment when the option bid-ask quote is recorded. Therefore, there is no nonsynchronous price issue here, except that the S&P 500 index level itself may contain stale component stock prices at each point in time.

For European options, the spot stock price must be adjusted for discrete dividends. For each option contract with τ periods to expiration from time t, Bakshi, Cao and Chen first obtain the present value of the daily dividends $D(t)$ by computing $\overline{D}(t,\tau)=\sum_{s=1}^{\tau-1} e^{-R(t,s)s} D(t+s)$, where $R(t,s)$ is the s-period yield-to-maturity. Next, they subtract the present value of future dividends from the time-t index level, in order to obtain the dividend-exclusive S&P 500 spot index series that is later used as input into the option models.

Bakshi, Cao and Chen (1997) also exclude some samples with the following filters: (1) option price quotes that are time-stamped later than 3:00pm Central Standard Time are eliminated. This ensures that the spot price is recorded synchronously with its option counterpart. (2) Options with less than six days to expiration may induce liquidity-related biases. (3) Price quotes lower than \$3/8 are not included due to the impact of price discreteness. (4) Quotes not satisfying the arbitrage restriction $C(t,\tau) \geq \max\left(0, S(t)-K, S(t)-\overline{D}(t,\tau)-KB(t,\tau)\right)$.

In light of the Black-Scholes model's moneyness- and maturity-related biases, researchers and practitioners have tried to find ways to estimate and use the "implied-volatility matrix." To see how the candidate models are compared against each other under such a matrix treatment, the option data is dividend into several categories according to either moneyness or term to expiration. Define $S(t)-K$ as the time-t intrinsic value of a call. A call option is then classified as at-the-money (ATM) if its $S/K \in (0.97, 1.03)$; out-of-the-money (OTM) if $S/K \leq 0.97$; and in-the-money (ITM) if $S/K \geq 0.97$. A finer partition resulted in six moneyness categories. By the term to expiration, an option contract can be classified as (i) short-term (<60 days); (ii) medium-term (60-180 days); and (iii) long-term (>180 days). The sample properties of the S&P 500 call prices are reported in the paper of Bakshi, Cao and Chen (1997)[11] and not repeated here.

4.1.2 Option Pricing Models

We follow the framework of Bakshi, Cao and Chen (1997) and conduct the empirical tests in the CEV model. The testing results will then be compared with those of Bakshi, Cao and Chen (1997): (i) the Black-Scholes (BS) model, (ii) the square root

[10] See footnote 4.
[11] Table 1 of Bakshi, Cao and Chen (1997), page 2013.

stochastic-volatility (SV) model, (iii) the stochastic-volatility and stochastic-interest-rate (SVSI) model, and (iv) the stochastic-volatility random-jump (SVJ) model. The empirical results will focus on the CEV and four models as described above.

In this paper, the CEV formula in terms of the noncentral chi-square distribution expressed by Schroder (1989) is adopted to compute option prices. IMSL (International Mathematical and Statistical Library) is used for the computation of the noncentral chi-square probabilities.

4.1.3 Structural Parameter Estimation and In-Sample Performance

4.1.3.1 Estimation Procedure

Step 1. Collect *N* option prices on the same stock and taken from the same point in time (or same day), for any *N* greater than or equal to one plus the number of parameters to be estimated. For each *n=1,...,N*, let τ_n and K_n be respectively the time-to-expiration and the strike price of the *n-th* option; Let $\hat{C}_n(t,\ \tau_n,\ K_n)$ be its observed price, and $C_n(t,\ \tau_n,\ K_n)$ its model price as determined by, for example, (Eq 3.1.17) with $S(t)$ and $R(t)$ taken from the market. The difference between $\hat{C}_n$ and C_n is a function of the values taken by $V(t)$ and by $\Phi \equiv \{\kappa_R, \theta_R, \sigma_R, \kappa_v, \theta_v, \sigma_v, \lambda, \mu_J, \sigma_J\}$. For each *n*, define

(Eq 4.1.1) $\varepsilon_n[V(t),\Phi] \equiv \hat{C}_n(t,\tau_n,K_n) - C_n(t,\tau_n,K_n)$

Step 2. Find $V(t)$ and parameter vector Φ, to solve

(Eq 4.1.2) $$SSE(t) \equiv \min_{V(t),\Phi} \sum_{n=1}^{N} \left|\varepsilon_n[V(t),\Phi]\right|^2$$

This step results in an estimate of the implied spot variance and the structural parameter values, for date *t*. Go back to Step 1 until the two steps have been repeated for each day in the sample.

4.1.3.2 Implied Parameters and In-Sample Pricing Fit

Before proceeding to the model comparison, we first preset the comparison between the unrestricted and the restricted ($\beta \geq 0$) CEV model as follows:

		Daily Average			
		Observation	Beta	SSE	Implied volatility
All Options	Unrestricted CEV	50.9735	-2.9290	23.9336	0.1964
	Restricted CEV		0.0031	50.1240	0.2015
Short-Term Options	Unrestricted CEV	19.4066	-4.4277	4.6479	0.2021
	Restricted CEV		0.0158	9.5121	0.2118
At-the-money Options	Unrestricted CEV	13.9563	-2.7787	3.5062	0.1852
	Restricted CEV		0.1316	4.9300	0.1856

From the testing results above, we find that out the pricing performance of the unrestricted CEV model is clearly superior to the restricted CEV model. The average daily square sum of dollar pricing error (SSE) of the unrestricted CEV model is smaller than the restricted CEV model in all of the categories we tested. This is consistent with the findings of Jackwerth and Rubinstein (2001). In addition, the elasticity parameter β is less than two, which confirms the negative correlation between index level and volatility. Thus, in this paper hereafter, we only use the unrestricted CEV model throughout all of the empirical tests.

As shown in Table 1, we compare the testing results of the CEV model with those of the BS, the SV, the SVSI, and the SVJ models obtained by Bakshi, Cao and Chen (1997). In the all-option category, the SSE of the CEV model is lower than that of the BS model, but higher than those of the SV, the SVSI, and the SVJ models. However, in short-term options category, the CEV model has lower SSE than the those of SV and the SVSI models, only higher than the SSE of the SVJ model. Furthermore, the CEV model performs best even better than the SVJ model in at-the-money options category.

4.1.4 Assessment of the Relative Model Misspecification

As Rubinstein (1985) had done, the most popular diagnostic of relative model misspecification is to compare the implied-volatility patterns of each model across both moneyness and maturity[12]. The procedure is as follows: First, substitute the spot index and interest rates of date t as well as the structural parameter values implied by all date ($t-1$) option prices, into the option pricing formula, which leaves only the spot volatility undetermined. Next, for each given call option of date t, find a spot volatility value that equates the model-determined price with the observed price of the call. Then, after repeating these steps for all options in the sample, obtain for each moneyness-maturity category an average implied-volatility value.

Using the subsample data from July 1990 to December 1990 as Bakshi, Cao and Chen (1997), the average implied volatilities of the CEV model are computed in Table 2. In Figure 1, the implied volatility graph is presented. We then compare it with the results by Bakshi, Cao and Chen (1997)[13]. For short-term calls, the CEV model still shows large U-shaped moneyness-related biases. However, the magnitude of the biases, 6.5%, is only slightly larger than that the SV model, around 6%. For medium-term and long-term calls, the moneyness-related smiles of implied volatility are greatly reduced, and the corresponding magnitudes are only 1.68% and 1.36%, respectively. We can also find that the implied volatility of the CEV model in long-term options (maturity $\geq$ 180 days) is the most stable case compared with other maturity-based options. For those options with longer than 180 days to expiration, the implied volatility of the CEV model is more stable than all of the other models including the SVJ model.

In sum, the CEV model is still subject to the model misspecification problems as all

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[12] See Hull (2002) for the discussion of volatility smile and Mayhew (1995) for the comprehensive literature review of implied volatility and volatility smile.

[13] Figure 1, Bakshi, Cao and Chen (1997), page 2022.

of the option pricing models tested in Bakshi, Cao and Chen (1997). However, in terms of the implied volatility, the extent of the moneyness-related biases is similar to the SV model, and is much better that the BS model.

4.1.5 Out-of-Sample Pricing Performance

In out-of-sample option pricing, the presence of more parameters may actually cause over-fitting and have the model penalized if the extra parameters do not improve its structural fitting. For this purpose, Bakshi, Cao and Chen rely on the previous day's option prices to back out the required parameter/volatility values and then use them as inputs to compute the current day's model-based option prices. Next, they subtract the model-determined price from its observed counterpart, to compute both the absolute and the average percentage pricing errors and their associated standard errors. This prevents the biases in the objective function (Eq 4.1.2) in favor of more expensive calls, such as long-term and in-the-money calls. To make our results comparable with those of Bakshi, Cao and Chen (1997), we also follow their approach by changing the objective function in (Eq 4.1.2) to absolute pricing errors

(Eq 4.1.3) $$APE(t) \equiv \min_{V(t),\Phi} \sum_{n=1}^{N} \left| \varepsilon_n [V(t), \Phi] \right| \quad \text{(Table 3)}$$

and percentage pricing errors

(Eq 4.1.4) $$PPE(t) \equiv \min_{V(t),\Phi} \sum_{n=1}^{N} \frac{\varepsilon_n [V(t), \Phi]}{\hat{C}_n(t,\ \tau_n,\ K_n)} \quad \text{(Table 4)} .$$

Pricing errors reported under the heading "All-Options-Based" are obtained using the parameter/volatility values implied by all of the previous day's call options; those under "Maturity-Based" are obtained using the parameter/volatility values implied by those previous-day calls whose maturities lie in the same category (short-term, medium-term, or long-term) as the option being priced; those under "Moneyness-Based" are obtained using the parameter/volatility values implied by those previous-day calls whose moneyness levels lie in the same category (OTM, ATM, or ITM) as the option being priced.

In Table 3, we compare the out-of-sample pricing errors of the CEV model with those of the BS, the SV, the SVSI, and the SVJ models from Bakshi, Cao and Chen (1997). We mark those results of the CEV model which are better or equal to the results of the SV model. In general, the out-of-sample pricing errors of the CEV model are in-between the BS model and the SVJ model. In OTM and part of the ATM option cases (S/K <1.00), the CEV model performs better than the SV model, while in the deep ATM and ATM options, the CEV model has larger pricing errors than the SV model. In Table 4, percentage pricing errors of the CEV model also show similar results as those of absolute pricing errors. However, in Table 4, the CEV model performs slightly better in short-term (maturity<60) and worse in long-term (maturity$\geq$180).

Finally, we should note that the CEV model only produces negative percentage pricing errors for short-term OTM ($S/K \le 1.0$ and days-to-expiration less than 60) options. This is slightly different from the observation of Bakshi, Cao and Chen (1997) that all models produce negative percentage pricing errors for options with moneyness $S/K \le 1.0$, and positive percentage pricing errors for options with $S/K \ge 1.03$, subject to time-to-expiration not exceeding 180 days.

4.2 Pricing Performances of Numerical Procedures

In this section, we will compare the pricing performances of the CEV and the stochastic volatility models in terms of cost-accuracy based analysis, namely, numerical accuracy and computational efficiency. The numerical accuracy is measured by the absolute pricing error between the option values generated by the numerical method and the closed-form solution, given fixed CPU time. The computational efficiency is measured by the required CPU time, given fixed pricing errors between the option values generated by numerical method and the closed-form solution.

The numerical method of the CEV process we use is the trinomial model developed by Boyle and Tian (1999). The numerical method of the stochastic volatility model we use is the finite difference algorithm by Scott (1997).[14]

4.2.1 The Trinomial Method Under the CEV Process

Boyle and Tian (1999) first transform the stochastic process under nature's probability measure into the Q-measure under which the deflated price processes of all securities are martingales. The revised process is as follows:

(Eq 4.2.1) $$dS = rSdt + \sigma S^{\beta/2} dz$$

They first transform the variable S so that the transformed process has constant volatility.

Let $y = y(t,S)$ and apply Ito's Lemma, the stochastic differential equation for y is

(Eq 4.2.2) $$dy = \left(\frac{\partial y}{\partial t} + rS\frac{\partial y}{\partial S} + \frac{1}{2}\sigma^2 S^\beta \frac{\partial^2 y}{\partial S^2} \right) dt + \left(\frac{\partial y}{\partial S} \sigma S^{\beta/2} \right) dz$$

To make the process *(Eq 4.2.2)* has constant volatility, they use the transformation such that

$$\frac{\partial y}{\partial S} \sigma S^{\beta/2} = v$$

for some positive constant v. And this is equivalent to

[14] We thank Louis Scott for providing the finite difference algorithm used in Scott (1997).

$$\frac{\partial y}{\partial S}=\frac{v}{\sigma}S^{-\beta/2}$$

Therefore, this transformation is given by

(Eq 4.2.3) $$y=\frac{v}{\sigma(1-\beta/2)}S^{1-\beta/2} \quad \text{for } \beta\neq 2$$

and the appropriate transformation $y=\frac{v}{\sigma}\log(S)$ for $\beta=2$.

For the case with $\beta\neq 2$, the transformed equation becomes

(Eq 4.2.4) $$dy=\left[rS\frac{v}{\sigma}S^{-\beta/2}+\frac{1}{2}\sigma^2S^{\beta}\left(-\frac{\beta}{2}\right)\frac{v}{\sigma}S^{-\beta/2-1}\right]dt+vdz$$

$$=\left[r\left(1-\frac{\beta}{2}\right)y-\frac{\beta v^2}{4(1-\beta/2)y}\right]dt+vdz$$

The transformed process above has constant volatility, which allows for a straightforward construction of a two-dimensional grid for trinomial trees. However, this transformed process has a complex drift term, which explodes when y approaches zero (with the only exception when $\beta=0$). This makes the standard trinomial branching process problematic for the region close to $y=0$, because the trinomial jumps and probabilities must be chosen to match not only volatility but the drift.

They then modify the standard trinomial method as suggested by Tian (1994), in which the trinomial branching process simultaneously utilizes both the transformed process (Eq 4.2.4) and the original process (Eq 4.2.1). The detailed procedures can be implemented in two steps as in the work by Boyle and Tian (1999).

4.2.2 The Finite Difference Method of the SV Model

The finite difference algorithm of Scott (1997) is briefly summarized as follows:

$S(t)$, $t\geq 0$ represents the price for a stock or a stock portfolio. Scott uses squared Gaussian diffusions under actual measure:

(Eq 4.2.5) $$dy(t)=[\kappa\theta-(\kappa+\lambda)y]dt+\gamma\sqrt{y(t)}dZ(t)\text{ ,}$$ [15]

where Z is Brownian motion and $\lambda_j=0$ for the risk-neutral process.

[15] Note that there are slight differences by Scott's notation in (Eq 4.2.5) and (Eq 4.2.6) compared with those of Bakshi, Cao and Chen (1997) in (Eq A.2) and (Eq A.1).

The price process is

(Eq 4.2.6) $$dS(t) = r(t)S(t)dt + \sigma\sqrt{y(t)}S(t)dW(t)$$

where W is a Brownian motion correlated with Z that $dWdZ = \rho dt$.

The finite difference method is then applied to the PDE for the valuation problem

(Eq 4.2.7) $$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 yS^2 \frac{\partial^2 C}{\partial S^2} + \frac{1}{2}\gamma^2 yS^2 \frac{\partial^2 C}{\partial y^2} + \rho\sigma\gamma\ yS\frac{\partial^2 C}{\partial S \partial y}$$

$$+ rS\frac{\partial C}{\partial S} + (\kappa\theta - \kappa y - \lambda y)\frac{\partial C}{\partial y} - rC = 0$$

Scott (1997) then expresses $\ln S(t)$ with the Brownian motion W as

$W(t) = \rho Z(t) + \sqrt{1-\rho^2}W'(t)$ and W' is a Brownian motion independent of Z . He first transforms from the state variables S and y to s^* and y , where $s^* = \ln S - (\rho/\gamma)y$. This step is to eliminate the correlation between two state variables. Next, y is transformed to $\sqrt{y}$. This step is to make the volatility of state variable a constant. Scott then solves the PDE by solving implicitly on s^* and explicitly on $\sqrt{y}$. The explicit method for the square root process of Hull and White (1990) is adopted. *N* is the number of steps in the time dimension and *M* is the number of steps for the transformed stock s^* . The value of *N* determined the grid and the step size for the variable $\sqrt{y}$ as in Hull and White (1990).

4.2.3 Cost-Accuracy Analysis of the CEV and the SV Models

In our comparison of numerical methods, we choose the parameters of the CEV and the SV models close to the all option categories of the in-sample test as we perform the European-style option empirical test. We set $\kappa = 1.5$, $\theta\kappa = 0.04$, $\gamma = 0.4$, $\rho = -0.65$, $\lambda = 0$, and the initial volatility $\sigma = 20\%$ for the SV model. And for comparison purposes, the CEV parameters are set to make the option prices equal to those of the SV model, fixing β as -3. The maturity of the option is 0.5 year; the interest rate is 5%; and the stock price is \$300. Our experiment is conducted using 3GHz Intel Pentium4 and 1GB DDR2 DRAM PC, and both methods are programmed in C++.

Three different moneyness of put options are studied in our experiment. We set the strike prices of at-the-money put option as \$300, deep out-of-the-money put option as \$250, and deep in-the-money put as \$350. The corresponding analyses for numerical accuracy are presented in Table 5, Table 7, and Table 9; and computational efficiencies are reported in Table 6, Table 8, and Table 10. The pricing error is defined as the option value generated by numerical method minus that of the closed-form solution.

One can observe from the Table 5, Table 6, Table 9, and Table 10 that to reach the desired precision in option pricing, the CEV model requires much less time than the SV model in order to converge to its closed-form solution. The CEV model can reach \$0.001 precision within 4 seconds for all three moneyness settings. In contrast, even to reach a precision with \$0.01 absolute pricing error, the SV model needs over 20 seconds for at-the-money case and over 40 seconds for deep in-the-money case. The SV model performs better for the cheaper deep out-of-the-money put option as shown in Table 7 and Table 8, however, it is still far away from the performance of the CEV model.

In empirical work, the number of iterations can be very large and therefore each iteration in optimization must be carried out in seconds in order to accommodate the large volume of cross-sectional and time series option data. Furthermore, there are a total of four parameters that need to be estimated in the SV model. Consequently, although the numerical methods of the SV model are applicable for pricing, its computational intensive optimization procedure makes the SV model practically infeasible for this task if one would like to conduct empirical test for American options.

Next, we investigate the pricing performance of the CEV numerical method across several values of β. Six different most common values of β — -4, -3, -2, -1, 0, and 1 — in our European option empirical test of previous section are chosen in our experiment. Another parameter δ is set to make the volatility equal 20%, i.e., $\delta = \sigma_{BS} S^{1-\frac{\beta}{2}} = (20\%) S^{1-\frac{\beta}{2}}$. A summary of computational efficiency in pricing European put options is reported in Table 11 with β values. We can find that computational efficiency of the CEV model is robust across different values of β. The longest time to reach \$0.0005 absolute pricing error in our experiment is merely 0.75 second in pricing deep in-the-money put with β equals to 0 and -1. Therefore, we can conclude that the trinomial method of the CEV process by Boyle and Tian (1999) in terms of numerical efficiency is robust enough for the empirical study.

Finally, we should note that the algorithms of American option pricing for these two numerical methods are similar to their European cases, except for the additional step for checking the early exercise possibility at each time step. Therefore, the time for pricing American style option is expected to be longer than that of the European option.

5. Conclusion and Conclusions (結論與建議)

In this study, we empirically test the CEV option pricing model and compare the results with those by Bakshi, Cao and Chen (1997). The CEV option pricing model performs better than the BS model in all cases. The empirical evidence showed that (i) In terms of in-sample performance, the squared sum of pricing errors of the CEV model is similar to the SV models in short-term and at-the-money options, and is worse in the all options category. (ii) In terms of out-of-sample performance, the mean absolute errors and percentage errors of the CEV model show that CEV performs better than the SV model in short term and out-of-the-money cases. In addition, the CEV model is even better than the SVJ model in a few cases in these categories. (iii) In terms of model misspecification, by using implied volatility graph introduced by Rubinstein(1985), the fluctuation of implied volatility of the CEV model is around 6.5% for short-term call options, which is much better than the BS model, and is only marginally higher than the SV model. For those options with larger than 60 days to expiration, the implied volatility of the CEV model is similar to all the other models. For longer-maturity options with more than 180 days to expiration, the volatility smile of the CEV model is even better than the SVJ model with only 1.36% fluctuation.

When applied to American option pricing, high-dimensional lattice models are prohibitively expensive. One distinguished feature of the CEV model as opposed to other stochastic volatility models is that it requires only a single dimensional lattice. Therefore, we also compare pricing performances of the CEV and stochastic volatility numerical methods in terms of two aspects: numerical accuracy and computational efficiency. Our experiment results clearly show that the CEV model performs much better than the SV model in terms of the speed of convergence to its closed form solution. In contrast, the implementation cost of the SV model in American option pricing is way too high and practically infeasible for empirical work.

In summary, the CEV model, introducing only one more parameter compared with the BS formula, improves the performance notably in all the tests of in-sample, out-of-sample and the stability of implied volatility. Furthermore, with a much simpler structure, the CEV model can still perform better than the SV model in short term and out-of-the-money categories. The empirical evidence also shows that the CEV model has similar stability of implied volatility as the SV, the SVSI, and the SVJ models. Therefore, with much less implementational cost and faster computational speed, the CEV option pricing model can be a better candidate than much more complex option pricing models, especially when one wants to apply the CEV process for pricing more complicated exotic options or credit risk models.

Appendix

The Stochastic Volatility, Stochastic Interest Rate and Random Jump Model by Bakshi, Cao and Chen (1997)

Scott (1997) was the first to derive a closed-form stochastic volatility, stochastic interest rates, and random jump option pricing model (SVSI-J). Here we present the version by Bakshi, Cao and Chen (1997) and follow their notations. Under the risk-neutral measure, the underlying nondividend-paying stock price $S(t)$ and its components are, for any *t*, given by

(Eq A.1) $$\frac{dS(t)}{S(t)} = [R(t) - \lambda\mu_J]dt + \sqrt{V(t)}d\omega_S(t) + J(t)dq(t)$$

(Eq A.2) $$dV(t) = [\theta_v - \kappa_v V(t)]dt + \sigma_v \sqrt{V(t)}d\omega_v(t)$$

(Eq A.3) $$\ln[1 + J(t)] \sim N\left(\ln[1 + \mu_J] - \frac{1}{2}\sigma_J^2,\ \sigma_J^2\right)$$

where $R(t)$ is the time-t instantaneous spot interest rate;

λ is the frequency of jumps per year

$V(t)$ is the diffusion component of return variance (conditional on no jump occurring);

$\omega_S(t)$ and $\omega_v(t)$ are each a standard Brownian motion, with $Cov_t[d\omega_S(t), d\omega_v(t)] \equiv \rho dt$;

$J(t)$ is the percentage jump size (conditional on a jump occurring) that is lognormally, identically, and independently distributed over time, with the unconditional mean μ_J. The standard deviation of $\ln[1 + J(t)]$ is σ_J;

$q(t)$ is a Poisson jump counter with intensity λ, that is, $\Pr\{dq(t) = 1\} = \lambda dt$ and $\Pr\{dq(t) = 0\} = 1 - \lambda dt$;

κ_v, θ_v / κ_v and σ_v are respectively the speed of adjustment, long-run mean, and variance coefficient of the diffusion volatility $V(t)$;

$q(t)$ and $J(t)$ are uncorrelated with each other or with $\omega_S(t)$ and $\omega_v(t)$

The interest rate process is assumed to follow the single-factor Cox, Ingersoll, and Ross (1985) process

(Eq A.4) $dR(t) = [\theta_R - \kappa_R R(t)]dt + \sigma_R \sqrt{R(t)}dw_R(t)$

where κ_R, θ_R / κ_R, and σ_R are the speed of adjustment, long-run mean, and volatility coefficient of the $R(t)$ process; $w_R(t)$ is a standard Brownian Motion assumed to be uncorrelated with any other process in the economy.

Note that the volatility risk $V(t)$, interest rate risk $R(t)$, and jump risk $J(t)dq(t)$ are all rewarded in their valuation framework. The factor prices for $V(t)$ and $R(t)$ are respectively $b_v V(t)$ and $b_r R(t)$ for some constants b_v and b_r. These factors are implicitly reflected in (Eq A.2) and (Eq A.4) and adjusted through κ_v and κ_R, respectively. Therefore, the risk premiums have been internalized in the stochastic structure rather than being assumed to be zero.

The European call option price with strike price K and term-to-expiration τ by Bakshi, Cao and Chen (1997) is shown as follows:

(Eq A.5) $C(t,\tau) = S(t)\Pi_1(t,\tau;S,R,V) - KB(t,\tau)\Pi_2(t,\tau;S,R,V)$,

where the risk-neutral probabilities, Π_1 and Π_2, are recovered from inverting the respective characteristic functions (see Bates (1996b, 2000) and Heston (1993) for similar treatments):

(Eq A.6)

$$\Pi_j(t,\tau;S(t),R(t),V(t)) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi \ln[K]} f_j(t,\tau,S(t),R(t),V(t);\phi)}{i\phi}\right]d\phi,$$

for j =1,2, with the characteristic functions f_j respectively given in (Eq A.7) and (Eq A.8). The price of a European put option on the same stock can be determined from the put-call parity.

(Eq A.7)
$$f_1(t,\tau) = \exp\left\{-\frac{\theta_R}{\sigma_R^2}\left[2\ln\left(1-\frac{[\xi_R-\kappa_R](1-e^{-\xi_R\tau})}{2\xi_R}\right)+[\xi_R-\kappa_R]\tau\right]\right.$$
$$-\frac{\theta_v}{\sigma_v^2}\left[2\ln\left(1-\frac{[\xi_v-\kappa_v+(1+i\phi)\rho\sigma_v](1-e^{-\xi_v\tau})}{2\xi_v}\right)\right]$$
$$-\frac{\theta_v}{\sigma_v^2}\left[2\ln(\xi_v-\kappa_v+(1+i\phi)\rho\sigma_v)\right]\tau + i\phi\ln[S(t)]$$

$$\frac{i\phi(1-e^{-\xi_R\tau})}{2\xi_R-[\xi_R-\kappa_R](1-e^{-\xi_R\tau})}R(t)$$

$$+\lambda(1+\mu_J)\tau[(1+\mu_J)^{i\phi}e^{(i\phi/2)(1+i\phi)\sigma_J^2}-1]-\lambda i\phi\mu_J\tau$$

$$\left.+\frac{i\phi(i\phi+1)(1-e^{-\xi_v\tau})}{2\xi_R-[\xi_R-\kappa_R+(1+i\phi)\rho\sigma_v](1-e^{-\xi_v\tau})}V(t)\right\},$$

and

(Eq A.8) $$f_2(t,\tau)=\exp\left\{-\frac{\theta_R}{\sigma_R^2}\left[2\ln\left(1-\frac{[\xi_R^*-\kappa_R](1-e^{-\xi_R^*\tau})}{2\xi_R^*}\right)+[\xi_R^*-\kappa_R]\tau\right]\right.$$

$$-\frac{\theta_v}{\sigma_v^2}\left[2\ln\left(1-\frac{[\xi_v^*-\kappa_v+i\phi\rho\sigma_v](1-e^{-\xi_v^*\tau})}{2\xi_v^*}\right)+[\xi_v^*-\kappa_v+i\phi\rho\sigma_v]\tau\right]$$

$$+i\phi\ln[S(t)]-\ln[B(t,\tau)]+\frac{2(i\phi-1)(1-e^{-\xi_R^*\tau})}{2\xi_R^*-[\xi_R^*-\kappa_R](1-e^{-\xi_R^*\tau})}R(t)$$

$$+\lambda\tau[(1+\mu_J)^{i\phi}e^{(i\phi/2)(i\phi-1)\sigma_J^2}-1]-\lambda i\phi\mu_J\tau$$

$$\left.+\frac{i\phi(i\phi-1)(1-e^{-\xi_v^*\tau})}{2\xi_v^*-[\xi_v^*-\kappa_R+i\phi\rho\sigma_v](1-e^{-\xi_v^*\tau})}V(t)\right\},$$

where $\xi_R=\sqrt{\kappa_R^2-2\sigma_R^2 i\phi}$, $\xi_v=\sqrt{[\kappa_v-(1+i\phi)\rho\sigma_v]^2-i\phi(i\phi+1)\sigma_v^2}$

$\xi_R^*=\sqrt{\kappa_R^2-2\sigma_R^2(i\phi-1)}$, and $\xi_v^*=\sqrt{[\kappa_v+i\phi\rho\sigma_v]^2-i\phi(i\phi-1)\sigma_v^2}$

The option valuation model in (Eq A.5) has several distinctive features. First, it incorporates stochastic interest rates, stochastic volatility, and jump risk, which means it nests all the models to be tested in our study as special cases. For example, we can obtain (i) the BS model by setting λ=0 and $\theta_R=k_R=\sigma_R=\theta_v=k_v=\sigma_v=0$; (ii) the SI model by setting λ =0 and $\theta_v=k_v=\sigma_v=0$;(iii) the SV model by setting λ =0 and $\theta_R=k_R=\sigma_R=0$; (iv) the SVSI model by setting λ=0; and (v) the SVJ model by setting $\theta_R=k_R=\sigma_R=0$, where to derive each special case from (Eq A.5) one may need to apply L'Hospital's rule. Secondly, the option pricing formula contains only identifiable variables such that all parameters can be estimated. This is also parsimonious compared to the model in Scott (1997).

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Table 1 Implied Volatility and In-Sample Fit

All Options

	CEV	BS	SV	SVSI	SVJ
Implied Volatility (%)	19.64	18.23	18.66	18.65	19.38
SSE	23.93	69.6	10.63	10.68	6.46

Short-Term Options

	CEV	BS	SV	SVSI	SVJ
Implied Volatility (%)	20.21	18.15	18.45	18.54	20.65
SSE	**4.65**	28.09	5.48	5.16	2.63

At-the-Money Options

	CEV	BS	SV	SVSI	SVJ
Implied Volatility (%)	18.52	18.74	18.48	18.36	19.03
SSE	**3.51**	25.34	5.98	5.45	5.31

Table 2 Implied Volatility

Maturity	<60		60-180		>=180	
Moneyness (S/K)	Average Volatility	Number of Obs	Average Volatility	Number of Obs	Average Volatility	Number of Obs
0.94<	23.1709%	238	23.2591%	846	22.1351%	540
0.94-0.96	21.6492%	255	22.1181%	241	21.8619%	112
0.96-0.98	21.2072%	290	22.4297%	242	22.0806%	94
0.98-1.00	21.3395%	290	22.7042%	233	22.5085%	109
1.00-1.02	22.6581%	288	22.6880%	218	22.7989%	94
1.02-1.04	23.3345%	261	22.9770%	200	22.9627%	75
1.04-1.06	24.3749%	258	23.1121%	204	22.8800%	58
1.06-1.08	25.5569%	225	22.8851%	188	22.7296%	75
>1.08	27.7188%	529	21.5806%	713	21.6021%	328
TTL		2634		3085		1485
Max	27.7188%		23.2591%		22.9627%	
Min	21.2072%		21.5806%		21.6021%	
Range	6.5116%		1.6785%		1.3606%	

Table 3 Out-of-Sample Pricing Errors (Absolute Pricing Errors)

		All-Options-Based Days-to-Expiration			Maturity-Based Days-to-Expiration			Moneyness-Based Days-to-Expiration		
Moneyness S/K	Model	<60	60-180	≥180	<60	60-180	≥180	<60	60-180	≥180
Panel A: Absolute Pricing Errors										
<0.94	CEV	**0.33**	**0.37**	**0.60**	**0.38**	**0.40**	**0.49**	**0.30**	**0.34**	**0.50**
	BS	0.78	1.39	1.89	1.02	1.48	1.78	0.41	0.63	0.78
	SV	0.42	0.43	0.61	0.38	0.42	0.58	0.32	0.36	0.53
	SVSI	0.37	0.39	0.57	0.38	0.40	0.52	0.30	0.36	0.53
	SVJ	0.37	0.40	0.59	0.27	0.40	0.58	0.33	0.36	0.54
0.94-0.97	CEV	**0.43**	**0.41**	**0.50**	0.39	0.43	**0.44**	0.41	0.39	0.54
	BS	0.76	1.02	1.16	0.73	1.07	1.15	0.45	0.53	0.69
	SV	0.46	0.41	0.54	0.33	0.41	0.54	0.34	0.38	0.53
	SVSI	0.40	0.40	0.55	0.34	0.41	0.52	0.34	0.38	0.52
	SVJ	0.38	0.38	0.53	0.25	0.39	0.53	0.33	0.38	0.51
0.97-1.00	CEV	**0.45**	0.44	0.60	**0.38**	**0.41**	**0.49**	0.44	**0.43**	0.65
	BS	0.61	0.62	0.66	0.51	0.64	0.66	0.70	0.74	0.94
	SV	0.48	0.41	0.53	0.39	0.41	0.52	0.40	0.43	0.60
	SVSI	0.47	0.41	0.54	0.39	0.42	0.51	0.39	0.42	0.60
	SVJ	0.42	0.40	0.52	0.31	0.40	0.51	0.36	0.41	0.63
1.00-1.03	CEV	0.49	0.57	0.60	**0.40**	0.44	0.58	0.44	0.45	0.68
	BS	0.52	0.69	0.81	0.45	0.65	0.84	0.47	0.50	0.69
	SV	0.41	0.43	0.53	0.40	0.41	0.51	0.38	0.43	0.54
	SVSI	0.43	0.42	0.53	0.41	0.41	0.49	0.38	0.42	0.52
	SVJ	0.40	0.42	0.51	0.37	0.41	0.50	0.37	0.41	0.51
1.03-1.06	CEV	0.63	0.68	0.61	0.55	0.51	0.54	0.47	0.51	0.69
	BS	0.76	1.21	1.30	0.77	1.14	1.37	0.51	0.85	1.76
	SV	0.45	0.47	0.55	0.41	0.41	0.51	0.48	0.48	0.67
	SVSI	0.42	0.45	0.54	0.41	0.41	0.50	0.48	0.48	0.66
	SVJ	0.39	0.44	0.53	0.39	0.41	0.51	0.39	0.42	0.53
≥1.06	CEV	0.64	0.65	1.02	0.57	0.52	0.66	0.55	0.51	0.79
	BS	0.82	1.39	1.57	0.79	1.35	1.64	0.56	0.62	0.72
	SV	0.54	0.49	0.65	0.47	0.40	0.51	0.44	0.41	0.54
	SVSI	0.52	0.51	0.51	0.48	0.42	0.47	0.43	0.41	0.52
	SVJ	0.43	0.43	0.56	0.36	0.39	0.50	0.40	0.40	0.54

Table 4 Out-of-Sample Pricing Errors (Percentage Pricing Errors)

Moneyness S/K	Model	All-Options-Based Days-to-Expiration <60	60-180	≥180	Maturity-Based Days-to-Expiration <60	60-180	≥180	Moneyness-Based Days-to-Expiration <60	60-180	≥180
		Panel B: Percentage Pricing Errors								
<0.94	CEV	**-11.40%**	**-0.69%**	3.81%	**-3.92%**	**1.09%**	**0.24%**	**-9.34%**	**-0.69%**	3.80%
	BS	-68.78%	-41.87%	-26.85%	-82.99%	-44.08%	-25.30%	-29.21%	-18.59%	-9.87%
	SV	-26.48%	-2.78%	-3.50%	6.63%	1.82%	0.32%	-15.04%	-1.21%	0.71%
	SVSI	-17.22%	-1.38%	-1.43%	5.26%	1.38%	-0.41%	-10.62%	-0.65%	-1.71%
	SVJ	-19.79%	-1.44%	2.13%	2.88%	0.14%	-0.43%	-12.39%	-1.58%	0.11%
0.94-0.97	CEV	**-8.78%**	**0.50%**	1.51%	**-0.72%**	**-0.24%**	**0.21%**	**-2.03%**	4.44%	5.41%
	BS	-38.23%	-15.14%	-7.39%	-35.52%	-15.86%	-7.18%	-15.38%	-2.34%	2.73%
	SV	-17.29%	-1.80%	0.28%	1.85%	-1.42%	-0.55%	-9.26%	0.13%	-0.06%
	SVSI	-11.59%	-0.70%	0.12%	2.14%	-1.08%	-0.17%	-8.13%	0.06%	0.08%
	SVJ	-11.99%	-1.03%	0.06%	-0.39%	-0.81%	-0.48%	-6.97%	7.00%	-0.04%
0.97-1.00	CEV	**-2.70%**	2.54%	1.34%	**2.29%**	1.63%	0.79%	**-2.22%**	3.28%	2.54%
	BS	-13.73%	-3.11%	-1.25%	-11.61%	-3.60%	-1.18%	-17.64%	-5.62%	-3.63%
	SV	-9.87%	-0.47%	-0.25%	-5.08%	-1.04%	-0.24%	-6.17%	0.97%	0.03%
	SVSI	-8.84%	-0.57%	-39.00%	-5.01%	-1.10%	-0.29%	-5.04%	0.89%	-0.02%
	SVJ	-6.87%	-0.33%	-0.32%	-0.81%	-0.63%	-0.17%	-3.11%	0.25%	-0.74%
1.00-1.03	CEV	2.75%	3.30%	1.38%	4.85%	2.41%	1.22%	**-0.34%**	0.73%	-0.81%
	BS	1.09%	2.61%	2.00%	1.58%	2.28%	2.41%	-0.55%	0.91%	0.12%
	SV	-1.16%	0.54%	-0.32%	-1.09%	-0.30%	0.16%	-0.68%	0.75%	-0.36%
	SVSI	-1.73%	0.05%	-0.50%	-1.16%	-0.46%	-0.09%	-0.79%	0.64%	-0.31%
	SVJ	-1.11%	0.31%	-0.26%	0.05%	-0.20%	0.20%	-0.35%	0.47%	-0.36%
1.03-1.06	CEV	3.40%	3.10%	0.91%	4.34%	2.34%	0.69%	1.03%	-0.03%	-1.46%
	BS	3.90%	4.57%	3.70%	4.02%	4.29%	4.01%	-0.51%	-2.76%	-5.05%
	SV	1.47%	0.92%	-0.32%	1.05%	0.09%	0.25%	-1.01%	-0.58%	0.33%
	SVSI	1.00%	0.50%	-0.42%	1.04%	-0.01%	-0.05%	-1.07%	-0.62%	0.44%
	SVJ	0.69%	0.55%	-0.20%	0.22%	0.03%	0.24%	0.26%	0.34%	-0.12%
≥1.06	CEV	1.91%	1.61%	-0.75%	2.12%	1.22%	-0.05%	1.27%	0.46%	-1.23%
	BS	2.49%	3.27%	2.85%	2.41%	3.16%	3.01%	1.45%	0.89%	-0.30%
	SV	1.46%	0.79%	-0.66%	1.18%	0.32%	-0.02%	0.80%	0.25%	-0.23%
	SVSI	1.36%	0.74%	-0.28%	1.21%	0.40%	0.03%	0.72%	0.22%	-0.16%
	SVJ	0.86%	0.50%	-0.41%	0.09%	0.17%	-0.04%	0.65%	0.36%	-0.36%

Table 5 Comparison of Numerical Accuracy Between the CEV and the SV Models (At-the-Money)

	At the Money Put Option				S=300	X=300
	CEV				SV	
Time (Sec)	N	Price	Error	N, M	Price	Error
	Analytic	29.14644		Analytic	29.14644	
0.1	1441	29.14279	-0.00365			
0.2	1975	29.14364	-0.00280	10	25.43504	-3.71140
0.5	2912	29.14460	-0.00184	25	28.31479	-0.83165
1	3639	29.14499	-0.00145	49	28.87105	-0.27539
2	4642	29.14529	-0.00115	96	29.06642	-0.08001
5	5805	29.14553	-0.00091	184	29.11721	-0.02923
10				264	29.12901	-0.01743
20				361	29.13582	-0.01061
50				522	29.13957	-0.00687
100				678	29.14158	-0.00486
200				867	29.14308	-0.00336

Table 6 Comparison of Computational Efficiency Between the CEV and the SV Models (At-the-Money)

	CEV			SV	
Put	X=300		Put	X=300	
Error	N	CPU Time	Error	N, M	CPU Time
		(Seconds)			(Seconds)
0.01	562	0.015	0.01	382	22.812
0.005	1078	0.047	0.005	668	96.031
0.001	5411	3.953	0.001	2600-2700	>6215.891

Table 7 Comparison of Numerical Accuracy Between the CEV and the SV Models (Deep Out-of-the-Money Put Option)

	Deep Out-of-the-Money Put Option (X=250)				S=300	X=250
	CEV				SV	
Time (Sec)	N	Price	Error	N, M	Price	Error
	Analytic	2.851594		Analytic	11.39016	
0.1	1433	11.38947	-0.00069			
0.2	1992	11.38803	-0.00213	10	13.51492	2.12476
0.5	2946	11.38925	-0.00091	23	11.64352	0.25336
1	3691	11.38920	-0.00096	49	11.42603	0.03587
2	4481	11.38996	-0.00020	96	11.39743	0.00727
5				184	11.38905	-0.00112
10				265	11.38919	-0.00097
20				361	11.38937	-0.00079
50				522	11.38969	-0.00047
100				678	11.38969	-0.00047
200				870	11.38964	-0.00052

Table 8 Comparison of Computational Efficiency Between the CEV and the SV Models (Deep Out-of-the-Money Put Option)

	CEV			SV	
Put	X=250		Put	X=250	
Error	N	CPU Time	Error	N, M	CPU Time
		(Seconds)			(Seconds)
0.01	472	0.016	0.01	81	1.641
0.005	873	0.031	0.005	102	2.156
0.001	4431	1.922	0.001	537	53.609

Table 9 Comparison of Numerical Accuracy Between the CEV and the SV Models (Deep In-the-Money Put Option)

	Deep In-the-Money Put Option (X=350)				S=300	X=350
	CEV				SV	
Time (Sec)	N	Price	Error	N, M	Price	Error
	Analytic	57.99747		Analytic	57.99747	
0.1	1434	57.99623	-0.00124			
0.2	1984	57.99584	-0.00163	10	48.40058	-9.59689
0.5	2953	57.99688	-0.00059	24	55.76441	-2.23305
1	3706	57.99656	-0.00091	48	57.47296	-0.52451
2	4574	57.99745	-0.00002	86	57.82273	-0.17474
5				184	57.95248	-0.04499
10				265	57.97286	-0.02461
20				361	57.98215	-0.01532
50				522	57.98858	-0.00889
100				678	57.99134	-0.00613
200				869	57.99313	-0.00434

Table 10 Comparison of Computational Efficiency Between the CEV and the SV Models (Deep In-the-Money Put Option)

	CEV			SV	
Put	X=350		Put	X=350	
Error	N	CPU Time (Seconds)	Error	N, M	CPU Time (Seconds)
0.01	348	0.016	0.01	477	39.563
0.005	423	0.016	0.005	779	146.563
0.001	4286	1.625	0.001	2800-2900	>9050.344

Table 11 Comparison of Computational Efficiency of the CEV Model

Across Different Values of β

CEV Put	S=300	δ =(20%)$S^{1-\beta/2}$	interest rate = 5%		T=0.5 year
	β=-4			β=-3	
	X=350			X=350	
Analytic	Solution	43.5759014	Analytic	Solution	43.8767725
Absolute Error	N	CPU Time (Seconds)	Absolute Error	N	CPU Time (Seconds)
0.01	83	0.016	0.01	87	0.016
0.005	235	0.016	0.005	224	0.016
0.001	1267	0.063	0.001	1367	0.094
0.0005	2567	0.344	0.0005	2589	0.375
	X=300			X=300	
Analytic	Solution	13.3942732	Analytic	Solution	13.3508924
Absolute Error	N	CPU Time (Seconds)	Absolute Error	N	CPU Time (Seconds)
0.01	137	0.016	0.01	113	0.016
0.005	252	0.016	0.005	224	0.016
0.001	1267	0.078	0.001	1069	0.063
0.0005	2515	0.328	0.0005	2208	0.235
	X=250			X=250	
Analytic	Solution	3.32127105	Analytic	Solution	2.85159417
Absolute Error	N	CPU Time (Seconds)	Absolute Error	N	CPU Time (Seconds)
0.01	103	0.016	0.01	72	0.016
0.005	190	0.016	0.005	175	0.016
0.001	1581	0.125	0.001	959	0.047
0.0005	2837	0.438	0.0005	2374	0.297

Table 11 (Cont.) Comparison of Computational Efficiency of the CEV Model

Across Different Values of β

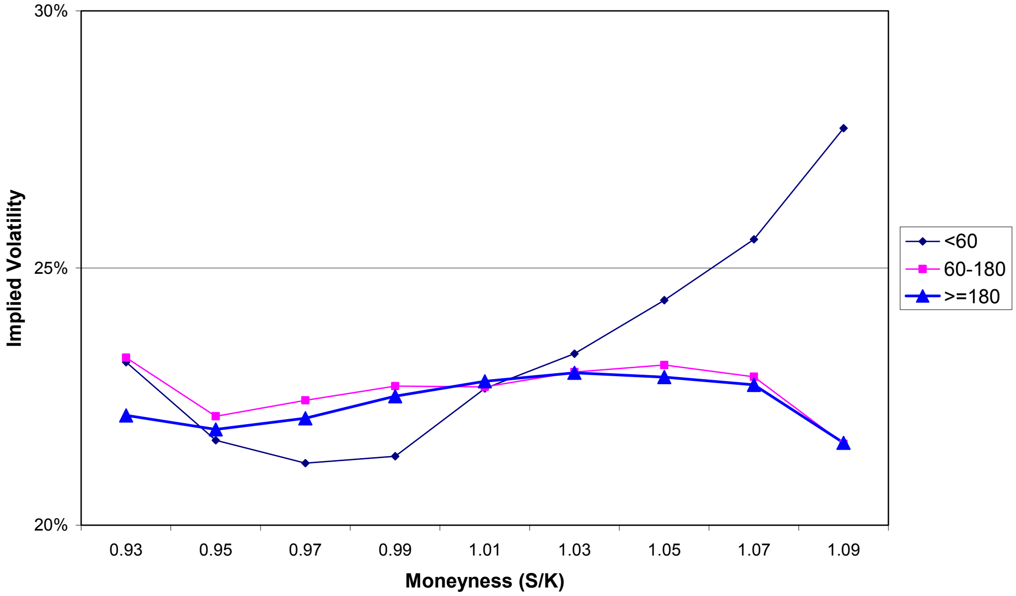
CEV Put	S=300	δ =(20%)$S^{1-\beta/2}$	interest rate = 5%		T=0.5 year
	β=-2			β=-1	
	X=350			X=350	
Analytic	Solution	44.2030637	Analytic	Solution	44.5558916
Absolute Error	N	CPU Time (Seconds)	Absolute Error	N	CPU Time (Seconds)
0.01	98	0.016	0.01	143	0.016
0.005	271	0.016	0.005	146	0.016
0.001	1413	0.094	0.001	1449	0.11
0.0005	2506	0.344	0.0005	2714	0.391
	X=300			X=300	
Analytic	Solution	13.3168835	Analytic	Solution	13.2912286
Absolute Error	N	CPU Time (Seconds)	Absolute Error	N	CPU Time (Seconds)
0.01	105	0.016	0.01	101	0.016
0.005	207	0.016	0.005	201	0.016
0.001	1005	0.047	0.001	981	0.047
0.0005	2013	0.203	0.0005	1940	0.187
	X=250			X=250	
Analytic	Solution	2.4388784	Analytic	Solution	2.07327803
Absolute Error	N	CPU Time (Seconds)	Absolute Error	N	CPU Time (Seconds)
0.01	67	0.016	0.01	59	0.016
0.005	160	0.016	0.005	146	0.016
0.001	878	0.031	0.001	629	0.032
0.0005	1413	0.093	0.0005	1805	0.156

Table 11 (Cont.) Comparison of Computational Efficiency of the CEV Model

Across Different Values of β

CEV Put	S=300	δ =(20%)$S^{1-\beta/2}$	interest rate = 5%		T=0.5 year
	β=0			β=1	
Put	X=350		Put	X=350	
Analytic	Solution	44.9365369	Analytic	Solution	45.3465076
Error	N	CPU Time (Seconds)	Error	N	CPU Time (Seconds)
0.01	158	0.016	0.01	142	0.016
0.005	317	0.016	0.005	339	0.016
0.001	1724	0.157	0.001	1605	0.14
0.0005	3397	0.75	0.0005	3413	0.75
Put	X=300		Put	X=300	
Analytic	Solution	13.2732913	Analytic	Solution	13.2626748
Error	N	CPU Time (Seconds)	Error	N	CPU Time (Seconds)
0.01	103	0.016	0.01	108	0.016
0.005	205	0.016	0.005	213	0.016
0.001	1000	0.062	0.001	1069	0.063
0.0005	2005	0.203	0.0005	2135	0.25
Put	X=250		Put	X=250	
Analytic	Solution	1.74889375	Analytic	Solution	1.46169114
Error	N	CPU Time (Seconds)	Error	N	CPU Time (Seconds)
0.01	71	0.016	0.01	65	0.016
0.005	133	0.016	0.005	121	0.016
0.001	852	0.047	0.001	724	0.032
0.0005	1650	0.125	0.0005	1429	0.11

Figure 1 Implied Volatility Graph



計畫成果自評

本研究符合原計畫進度。本研究詳盡的整理了常數彈性變異數過程的研究成果，包括近期於信用風險的應用以及路徑相關選擇權的訂價文獻。實證分析上，我們將常數彈性變異數過程在歐式選擇權的訂價結果, 與文獻中 Bakshi, Cao, and Chen (1997)著名的實證分析進行比較，我們發現在短期與價外的選擇權分類中，常數彈性變異數模型與隨機波動度模型有相近的表現。在數值分析中，相較於隨機波動度模型，常數彈性變異數模型有極佳的收斂速度以及定價表現。本研究部分已發表 — Chen, R. R., C. F. Lee, Lee, H.H.Lee, 2009, "Empirical Performance of the Constant Elasticity Variance Option Pricing Model," Review of Pacific Basin Financial Markets and Policies, Volume 12, Issue: 2 (June 2009), 177-217 (FLI and EconLit)。此外，此研究 JDCEV 模型實證部份，目前正在進行進一步的實證分析，增加 robustness test，應可於近期投稿國際學術研討會。

出席國際學術會議心得報告

計畫編號	97-2410-H-009-006-
計畫名稱	常數彈性變異數過程與其應用
出國人員姓名 服務機關及職稱	李漢星 / 國立交通大學財務金融研究所 / 助理教授
會議時間地點	自 98 年 10 月 21 日 至 98 年 10 月 24 日 / Reno, Nevada, USA
會議名稱	2009 FMA (Financial Management Association) Annual Meeting
發表論文題目	Default Prediction of Alternative Structural Credit Risk Models and Implications of Default Boundaries

一、參加會議經過

十月二十一日報到領取研討會資料。十月二十二日研討會開始之後則陸續參與 Brad Barber 的演講 "Estimating Long-run Abnormal Returns"，以及數個 academic session。本人於十月二十三下午 2:30 至 3:00 進行口頭報告與 Ren-Raw Chen 以及 Cheng-Few Lee 共同合作之 ”Default Prediction of Alternative Structural Credit Risk Models and Implications of Default Boundaries”，並與 session 參與者討論本篇研究。

二、與會心得

此次參與 FMA 國際財務管理學會會議獲益匪淺，一則有機會於研討會中發表本人之研究成果，並藉此與研討會參與者互動，獲得研討會參與者之回饋意見，對論文進行修改，再則亦於 keynote speaker 的演講中,聽取目前最新之研究發展,尤其 Brad Barber 的"Estimating Long-run Abnormal Returns"，更讓我見識到大師的丰采。最後，在各個財務的 session 與 reception 中，有機會與來自各不同國家與學校的學者認識，討論與相互交流。

Default Prediction of Alternative Structural Credit Risk Models and Implications of Default Boundaries

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Abstract

While most of the empirical studies of structural credit risk models try to test the performance of structural models in bond and credit derivatives pricing, little results are provided for default prediction. Therefore, in this study, we empirically compare four structural credit risk models – the Merton (1974), the Brockman and Turtle (2003), the Black and Cox (1976), and the Leland (1994) models – for their default prediction capabilities. Our empirical results indicate that exogenous default boundaries, flat or exponential, are not crucial in default prediction. In contrast, modeling endogenous boundary has significant improvement in long term prediction for non-financial firms. However, we should note that the performance of the Leland model compared to the Merton model is weakened as the default prediction horizon shortened.

Keywords: Default Prediction, Structural Credit Risk Model, Maximum Likelihood Estimation, Default Boundary

1. Introduction

This study is intended to examine whether and by how much the generalization of the prevailing structural credit risk models improves the performance of the default prediction. Following the seminal works of Black and Scholes (1973) and Merton (1974), the structural credit risk modeling literature has developed into an important area of research. While most of the empirical studies try to test the performance of structural models in bond and credit derivatives pricing, little results are provided for default prediction.[i] Therefore, in our study, we will compare various structural credit risk models for their default prediction capability. Moreover, the effect of default boundary modeling in default prediction can also be investigated.

Credit risk models can be divided into two main categories: credit pricing models, and portfolio credit value-at-risk (VaR) models.[ii] Credit pricing models can be subdivided into two main approaches: structural-form models and reduce-form models.[iii] Portfolio credit VaR models, developed by banks and consultants, aimed at measuring the potential loss with a predetermined confidence interval that a portfolio of credit exposures could suffer within a specified time horizon. These models typically employ simpler assumptions and address less on the causes of single firm's default. Reduced-form models are mainly represented by the Jarrow-Turnbull (1995) and Duffie-Singleton (1999) models. These models typically assume exogenous random variables drive defaults and do not condition default on the firm value and other structure features, such as asset value volatility and leverage, of the firm. In our empirical study, we limit our empirical analysis of default prediction in the single-firm structural models.[iv]

Prior empirical studies of structural models in default prediction and default boundary, even a handful, do not seem to come to a consensus. Chen, Hu, and Pan (2006) show that the Longstaff and Schwartz model (1995) performs poorly and is statistically no different from the flat barrier model without random interest rate assumption. The simpler Black-Cox (1976) outperforms the complex Longstaff and Schwartz model and they attribute the better performance to the random recovery. Devydenko (2007) finds that, in default prediction power, the simple boundary specified in terms of the face value of debt performs at least as well as more complex alternatives, the Leland and Toft (1996) or the KMV boundary. Another finding provided by Brockman and Turtle (2003) shows that the default flat barriers are significantly positive while Wong and Choi (2009) find that default barriers are positive but not significant. It seems that the above empirical results are counter intuitive to the evolution of structural credit risk modeling. Therefore, it motivates us to empirically test a more comprehensive set of the structural models and to uncover the crucial factors of default prediction.

In our empirical study, we will test various structural credit risk models extended from the Merton (1974) model. Succeeding structural models relax the restrictive assumptions originally made and seek to incorporate the most critical factors. Although these extensions introduce more realism into the model, they increase the analytical complexity and implemental difficulty. The goal of this study is, therefore, to empirically test if these complexities indeed improve the performance predicting corporate failure. Our focus is mainly put on two aspects of these extensions: the bond safety covenant in terms of continuous default, and the shareholders' discretion on the going concern decision in terms of endogenous barrier modeling. Using the Merton model as the base case, we can observe the performance enhancement, if any, through the introduction of continuous default, bankruptcy costs, and tax effect.

The European option approach by Merton (1974) ignores the possibility of failure prior to debt maturity and implicitly models corporate debt and equity as path-independent securities of the underlying asset value process. Researchers therefore introduce the default barrier to model this deficiency. In barrier models, we test the flat (or constant) default barrier model by Brockman and Turtle (2003), and the exponential barrier model of Black-Cox (1976). An arguable assumption of the above barrier models is that the default barrier is exogenously determined. As a result, Leland (1994) developed the endogenous barrier model under stationary debt structure. Therefore, we will also include endogenous barrier model in our empirical test.

Prior empirical studies indicate that structural models generate poor empirical performances. Ericsson and Reneby (2004) argue that the inferior bond pricing performance of structural models may come from the estimation approaches traditionally used in the empirical studies. As a result, the perceived advantage of reduced-form models is more a result of the estimation procedure rather that of the model structure. Therefore, we adopt a better estimation methodology, the Maximum Likelihood Estimation method proposed by Duan (1994) and Duan et al. (2004), which views the observed equity time series as a transformed data set of unobserved firm values with the theoretical equity pricing formula serving as the transformation. This method has been evaluated by Ericsson and Reneby (2005) through simulation experiments, and their result shows that the efficiency of MLE is superior to the commonly adopted volatility restriction approach in the literature. Another reason to employ MLE is that the major data required for this method in the context of structural models is the common stock prices, which have much less microstructure issues compared with bond prices.

Our paper contributes to existing literature in two aspects: First, in contrast to previous research, we adopt the theoretically superior MLE approach and empirically test the default prediction capabilities of various models under different default barrier assumptions. Second, the role of the default barrier in structural models has long been adopted by researchers in literature while its validity is not empirically investigated until the research by Brockman and Turtle (2003) and Wong and Choi (2009). One of the advantages of the MLE approach is that it can jointly estimate asset volatility and default barrier. Therefore, in addition to the flat barrier assumption, we can also explore this issue further to the exponential barrier assumptions.

Our empirical results surprisingly show that the simple Merton model has a similar capability in default prediction as that of the Black and Cox model. The Merton model even outperforms the Brockman and Turtle model, and the difference of predictive ability is statistically significant. The results are held for the in-sample, six-month and one-year out-of-sample tests for both the broad definition of bankruptcy as in Brockman and Turtle (2003) as well as the similar definition to Chen, Hu, and Pan (2006). In addition, we also find that the inferior performance of the Brockman and Turtle model may be the result of its unreasonable assumption of the flat barrier. In the one-year out-of-sample test, the Leland model outperforms the Merton model in non-financial sector and the results hold for two alternative definitions of default. Furthermore, these results are still preserved in our robustness test as we use risk-neutral default probabilities instead of physical default probabilities.

The paper is organized as follows: Section 2 is the review of prior empirical studies of structural models in default prediction. Section 3 presents the estimation method we adopt and the issues of other current estimation approaches. A simulation study of the MLE method is also reported. Section 4 reports empirical results, and Section 5 presents summary and concluding remarks.

2. Previous Empirical Studies of the Structural Credit Risk Models in Default Prediction

Brockman and Turtle (2003) investigate the bankruptcy prediction performance under down-and-out call (DOC) framework using a large-cross section of industrial firms for the period from 1989 to 1998. Brockman and Turtle (2003) use the proxy approach measuring the market value of a firm's assets as the book value of assets less the book value of shareholders' equity, plus the market value of equity as reported in Compustat. The asset volatility is measured as the square root of four times the quarterly variance measure, where the quarterly variance measure is computed by quarterly percentage changes in asset values for each firm in the sample with at least ten years of data. The promised debt payment is measured by all non-equity liabilities, computed as the total value of assets less the book value of shareholders' equity. Finally, the life span of each firm is set to be ten years, and they argued that barrier estimates are not particularly sensitive to lifespan assumption by the robustness test.

The empirical evidence shows that the failure probabilities implied by the DOC framework never underperform the well known accounting approach – Altman's Z-score. In detail, the logistic regressions by including one or both of the implied failure probability and Z-score, the DOC approach dominates Z-score in predicting corporate failure percentage of the one, three, and five year tests as well as their size or book-to-market categorized tests. In addition, in the quintile-based test, the failure probability of DOC framework also stratifies failure risks across firms and years much more effectively than the corresponding Z-score. We should note that another empirical finding by Brockman and Turtle (2003) is that implied default barriers are statistically significant for a large cross-section of industrial firms. However, Wong and Choi (2009) argue that it is the proxy approach of Brockman and Turtle (2003) that leads to barrier levels above the value of corporate liabilities. Hence, they adopt the transformed-data MLE approach and find that default barriers are positive but not very significant in the empirical study of a large sample of industrial firms during 1993 to 2002.

Bharath and Shumway (2008) examine the default predictive ability of the Merton distance to default (DD) model by studying all the non-financial firms for the period 1980 to 2003. The method they use to estimate the expected default frequency (EDF) is the same as the iterated procedure employed by Vassalou and Xing (2004). They compare the Merton DD probability with several variables — the naïve probability estimate (without implementing the iterated procedure), market equity, and past returns, and find that the Merton DD model does not produce sufficient statistics for the probability of default. Implied default probabilities form the CDSs and corporate bond yield spreads are only weakly correlated with the Merton DD probabilities after adjusting for agency ratings, bond characteristics, and their alternative predictors. Moreover, they find that the naïve probability they propose, which captures both the functional form and the same basic inputs of the Merton DD probability, performs slightly better as a predictor in hazard models and in out-of-sample forecasts. They conclude that the Merton DD probability is a marginally useful default forecaster, but it is not a sufficient statistic for default[v].

Recently, Chen, Hu, and Pan (2006) use the volatility restriction method to test five structural models including the models of Merton, Brockman and Turtle, Black-Cox, Geske (2 periods), and Longstaff-Schwartz as well as the proposed non-parametric model. The default companies in the study are those filed Chapter 11 for the period from January 1985 to

December 2002 with assets greater that $50 million. Their results indicate that the distribution characteristics of equity returns and endogenous recovery are two important assumptions. On the other hand, random interest rates, that play an important role in pricing credit derivatives, are not an important assumption in predicting default.

Lastly, Davydenko (2007) uses a unique sample of risky firms with observed market values of equity, bonds, and bank debt to investigate whether default is associated with insufficient cash reserves relative to required payments or with low market values of assets relative to debt level. Davydenko estimates the market value of firms' assets as the sum of market values of bonds, bank debt, and equity. Estimates of the market value of firms' public debts are from the monthly quotes from Merrill Lynch bond trading desks, for bonds included in the Merrill Lynch U.S. High Yield Master II Index (MLI) between December 1996 and March 2004. Estimates of bank loan prices are based on quotes provided by the LSTA/LPC Mark-to-Market Pricing service. In default prediction, his empirical results suggest that the simple boundary specified in terms of the face value of debt performs at least as well as more complex alternatives, the Leland and Toft (1996) or the KMV boundary. In addition, predictions based solely on liquidity measures, the "flow" measure in cash flow-based models such as interest coverage and quick ratio, are significantly less accurate than those based on asset values. However, his empirical observation indicates that liquidity shortages can precipitate default even by firms with high asset values when they are restricted from accessing external financing. Therefore, even though boundary-based default predictions can match observed average default frequencies, they misclassify a large number of firms in cross-section.

Leland (2004) examines the default probabilities predicted by the Longstaff and Schwartz (1995) model with the exogenous default boundary, and the Leland and Toft (1996) model with endogenous default boundary. Leland uses Moody's corporate bond default data from 1970 to 2000 in his study and follows similar calibration approach similar to Huang and Huang (2003). Rather than matching the observed default frequencies, Leland instead chooses common inputs across models to observe how well they match observed default statistics. The empirical results show that when costs and recovery rates are matched, the exogenous and endogenous default boundary models fit observed default frequencies equally well. The models predict longer-term default frequencies quite accurately, while shorter-term default frequencies tend to be underestimated. Thus, he suggests that a jump component should be included in asset value dynamics.

3. Empirical Methods

In Section 3.1, we first describe in detail the Maximum Likelihood Estimation (MLE) procedures, and we summarize in Section 3.2 the problems of other existing estimation approaches that have been pointed out in the literature. In Section 3.3, we report our results of the Monte Carlo experiments of the MLE method. In Section 3.4, we present the method we use to measure the capability of predicting financial distress.

Traditionally, structural credit risk models are estimated by the volatility restriction approach[vi] or an even simpler approach such as the proxy approach. However, these two approaches and their variants lack the statistical basis, and the empirical results they produce are less convincing. Thus, the new estimation method such as the transformed MLE has been introduced into the empirical researches of structural models.

3.1 Maximum Likelihood Estimation Method

Duan (1994) develops a transformed data MLE approach to estimate continuous time models with unobservable variables using derivative prices. The obvious advantages are that (1) the resulting estimators are known to be statistically efficient in large samples; and (2) the sampling distribution is readily available for computing confidence intervals or for testing hypotheses. In the context of structural credit risk models, equity prices are the derivative of the underlying asset value process and are readily available with large samples. In this section, we first briefly summarize the transformed-data MLE approach proposed by Duan (1994), and then turn to the implementation of this method in structural credit risk models.

Let X be an n-dimensional vector of unobserved variates. Assume that its density function, $f(x;\theta)$, exists and it is continuously twice differentiable in both arguments. A vector of observed random variates, Y, results from a data transformation of the unobservable vector X. This transformation from R^n to R^n is a function of the unknown parameter θ, and is one-to-one for every $\theta \in \Theta$, where Θ is an open subset of R^k.

Denote this transformation by $T(\cdot;\theta)$, where $T(\cdot;\cdot)$ is continuously twice differentiable in both arguments. Accordingly, $Y = T(X;\theta)$ and $X = T^{-1}(Y;\theta)$. The log-likelihood function of the observed data Y is $L(Y;\theta)$. By change of variable, the log-likelihood function for the transformed data Y can be expressed by the log-likelihood function of the unobserved random vector X, denoted as $L_X(\cdot;\theta)$, and the Jacobian, J, of a given transformation.

$$L(Y;\theta) = L_X(T^{-1}(Y;\theta);\theta) + \ln\left|J\left(T(X;\theta)^{-1}\right)\right| \quad \textbf{(1)}$$

Implementation of the Transformed-Data MLE in the Context of Structural Credit Risk Models (Duan et al. (2004)):

Step 1. Assign initial values of the parameters θ, and compute the implied asset value time series by $\hat{V}_{ih}(\hat{\theta}^{(0)}) = T^{-1}(S_{ih};\hat{\theta}^{(0)})$, where h is the length of the time period and $\hat{\theta}^{(m)}$ denotes the *m-th* iteration. Let *m=1*.

Step 2. Compute the log-likelihood function

$$L(S;\hat{\theta}^{(m)}) = L_V(\hat{V}_{ih}(\hat{\theta}^{(m)}), i = 1,\ldots,n;\hat{\theta}^{(m)}) - \sum_{i=1}^{n} \ln\left|\frac{dT(\hat{V}_{ih}(\hat{\theta}^{(m)});\hat{\theta}^{(m)})}{dV_{ih}}\right| \quad \textbf{(2)}$$

to obtain the estimated parameters $\hat{\theta}^{(m)}$.[vii]

Step 3. Compute the implied asset value time series by $\hat{V}_{ih}(\hat{\theta}^{(m)}) = T^{-1}(S_{ih};\hat{\theta}^{(m)})$, and let *m=m+1*, go back to step 2 until the maximization criterion is met.

Step 4. Use the MLE $\hat{\theta}$ to compute the implied asset value $\hat{V}_{nh}$ and the corresponding default probability.

3.2 Problems of Existing Estimation Approaches

In this section, we first summarize in Table 1 the existing empirical works of structural models in terms of their subject of research, estimation methods and input data. Next, we briefly summarize the problems of the most popular existing estimation approaches that have been pointed out in the literature[viii].

Problems of Volatility Restriction Approach

Duan (1994) addressed that the shortcoming of the volatility restriction method. The volatility relationship used in volatility restriction method is a redundant condition which provides a restriction only because the equity volatility is inappropriately treated as a constant, which is calculated from historical data. Moreover, since the volatility restriction approach is not statistical, it provides no distribution information about the parameters and cannot perform statistical inferences. In addition, Duan et al. (2003) also pointed out that the drift of the unobservable asset process could not be estimated by the JMR-RV method since the theoretical equity pricing formula does not contain the drift of the asset value process under the physical probability measure. As a result, the default probability could not be obtained.

Ericsson and Reneby (2005) also argued that the described volatility restriction effect implies that increasing stock prices result in underpriced bonds, while decreasing stock prices produce overpriced bonds. Ericsson and Reneby (2005) performed a simulation experiment and compared the performance of the transformed data maximum likelihood estimators with those of the volatility restriction method. Under the settings of four scenarios of different financial risk and business risk levels, they chose to test three structural models including the Black-Scholes-Merton model, the Briys and de Varenne (1997) model, and the Leland and Toft (1996) model. They found that the bias of the transformed-data maximum likelihood approach is negligible for practical purposes in 12 of the Monte Carlo experiments, while the VR approach exhibits an average spread error of 23%.

Problems of the KMV Approach[ix]

Duan et al. (2004) prove that the KMV method produces the point estimate identical to the transformed data ML estimate in the context of the Merton (1974) model. However, the KMV method cannot provide the sampling error of the estimate, which is crucial for statistical inference. In short, the KMV method can be regarded as an incomplete ML method. Moreover, in general, structural models may contain unknown parameters other than the firm's asset value and volatility: for example, the unknown parameters specific to the financial distress level in the barrier models. In these models, estimates of the KMV method no longer coincide with those of the EM algorithm, and therefore the KMV method cannot generate a meaning estimate for these variables.

Problems of Proxy Approach

Eom, Helwege and Huang (EHH) (2004) use the sum of the market value of equity and total debt as a proxy of the asset value of a firm. That is, $V_{proxy} = K + S$. However, Li and Wong (2008) show this assumption is unreasonable even under Merton's model. Under the option theory, assuming the true asset value V_{true}, one can find $C(V_{true}, K, T) = S = V_{proxy} - K < C(V_{proxy}, K, T)$. The inequality above comes from the fact that a call option premium must be higher than its intrinsic value before the maturity date. Since call option is an increasing function of its underlying asset, the relationship $V_{true} < V_{proxy}$ is implied by $C(V_{true}, K, T) < C(V_{proxy}, K, T)$. Therefore, we can find that the EHH approach overestimates the true asset value, and it yields biased estimation results. As the market value of assets has been overestimated, the predicted price of corporate bonds will be too high and the corresponding predicted yield spread will be underestimated. This implies the European option framework will automatically be rejected whenever the proxy approach is adopted.

Wong and Choi (2009) further criticize the proxy approach under the down-and-out call option framework of Brockman and Turtle (2003). They show that employing the proxy is equivalent to presuming that the default barrier is greater than the future promised payment of liabilities. This result holds for the arbitrary sets of input parameters including industrial sector, option maturity, and rebate level. Hence, it explains why the hypotheses test and robustness tests of Brockman and Turtle (2003) work well. Firms are presumed to have positive barriers exceeding the book value of corporate liabilities, and no doubt the implied barriers in Brockman and Turtle (2003) are significantly positive.

Table 1 Summary of Previous Empirical Studies of Structural Models

Research	Subject	Estimation Method	Main Input Data for the Estimation
Wei and Guo (1997)	Credit Spreads	Yield Curve Approach	Eurodollar and T-Bill Data
Anderson and Sundaresan (2000)	Bond Pricing and Yield Spreads	Asset Value Proxy	Yield Indices of Investment
Lyden and Saraniti (2001)	Bond Pricing and Yield Spreads	Asset Value Proxy	Matrix Bond Prices
Delianedis and Geske (2001)	Bond Pricing and Yield Spreads	Volatility Restriction	Matrix Bond Prices
Huang and Huang (2003)	Bond Pricing and Yield Spreads	Calibration	Bond Prices
Eom, Helwege, and Huang (2004)	Bond Pricing and Yield Spreads	Asset Value Proxy with Refined Volatility Estimation	Bond Prices
Hsu, Saà-Requejo, and Santa-Clara (2003)	Bond Pricing	GMM	Exchange Traded Bond Prices
Ericsson and Reneby (2004)	Yield Spreads	MLE	Stock Prices, Bond Prices, Dividend Information
Chen, Fabozzi, Pan, and Sverdlove (2005)	CDS Spreads	Minimize of Pricing Error and Absolute Pricing Error	CDS Transaction Data
Ericsson, Reneby, and Wang (2006)	CDS Premia and Bond Pricing	MLE	Credit Default Swaps, Bond Prices
Vassalou and Xing (2004)	Equity Returns	KMV (Simplified)	Equity Prices
Brockman and Turtle (2003)	Default Prediction and Default Boundary	Asset Value Proxy	Equity Prices
Bharath and Shumway (2008)	Default Prediction	KMV (Simplified)	Equity Prices
Chen, Hu, and Pan (2006)	Default Prediction	Volatility Restriction	Equity Prices
Wong and Choi (2009)	Default Barrier	MLE	Equity Prices
Davydenko (2007)	Default Prediction and Default Boundary	Market values of bond, equity, and bank loan	Bond Prices, Bank loans, Equity Prices
Leland (2004)	Default Probability Estimation	Calibration	Moody's Corporate Bond Default Data

3.3 Monte Carlo Experiment

We follow Duan et al. (2004) and set the following parameter values to perform the simulation experiment: interest rate r =0.05, asset drift μ_V =0.1, asset volatility σ_V =0.3, initial firm value V_0 =1.0, face value of debt F=1.0, and option maturity T=2. The sampling period is set to be 252 days a year, and maturity is set to be (2-$i\delta$) years for the i-th data point of the simulated time series. Finally, we change the value of the default barrier in order to examine its effect on parameter estimation.

Our results in Table 2 are based on 1,000 simulated samples following the procedure by Duan et al. (2004) to mimic the daily sample of observed equity value of a survived firm. We use the same numerical optimization algorithm of Nelder-Mead (in Matlab software package) as that in Wong and Choi (2009), and the initial value of the barrier is set as 0.5. Our experiment results in Table 2 clearly show the strength and the limitation of the MLE method. The MLE method can jointly estimate and uncover the true asset volatility and default barrier well, when the barrier hitting probability of the asset value process is not too low. However, as the true default barrier is under 0.5 in our experiment, the barrier estimates are seriously biased.

Although the default barrier estimates are biased when barrier the hitting probability of asset value process is low, this is what the statistical theory precisely predicts since the value of likelihood function is flat and not sensitive to the change of the barrier level. A low barrier relative to the firm value (or the low hitting probability of the barrier) obviously implies that the barrier is immaterial. In other words, where it is exactly located doesn't materially affect equity values. Thus, one cannot expect to pin down the barrier using the equity time series.

One important consequence regarding the estimate of the barrier parameter is that the testable hypothesis proposed by Brockman and Turtle (2003) should not be carried out by the estimates of the barriers. Brockman and Turtle (2003) use the nested concept of standard call option and down-and-out barrier option model to argue that when the default is zero, the down-and-out option collapses to the standard European call option. However, due to the nature of the likelihood function of down-and-out option framework, one cannot expect to pin down the barrier when the barrier is low relative to the asset value, i.e., the default probability is low. When the default probability is low, the low barrier estimate can vary for a wide range since it does not affect the likelihood function and equity pricing results.

Fortunately, for our empirical studies in default prediction, this should present no practical difficulties. The bias of low barrier cases could hardly affect the default probabilities of sample firms, even when the barrier estimates vary for a wide range. Furthermore, a formal test shall be carried out by the performance of default prediction capability using alternative statistical test. In our study, we adopt the Receiver Operating Characteristic Curve and Accuracy Ratio for this issue and we discuss them in the following section.

Table 2 A Monte Carlo Study of the MLE Method for the Brockman and Turtle (2003) Model

	Model Parameters		F=1	T=2
	μ_V	σ_V	H (Barrier)	Barrier Hitting Probability
True Value	0.1	0.3	0.9	67.746936%
Mean	0.36377	0.30211	0.89479	
Median	0.34914	0.29857	0.89837	
Standard Deviation	0.21523	0.04856	0.07941	
True Value	0.1	0.3	0.8	39.585685%
Mean	0.24807	0.29789	0.79156	
Median	0.22296	0.29490	0.80203	
Standard Deviation	0.21503	0.04449	0.11039	
True Value	0.1	0.3	0.75	28.074173%
Mean	0.23082	0.30232	0.69968	
Median	0.17726	0.29878	0.74795	
Standard Deviation	0.24533	0.05624	0.18828	
True Value	0.1	0.3	0.7	18.671759%
Mean	0.19528	0.29924	0.61289	
Median	0.17426	0.29643	0.69106	
Standard Deviation	0.23842	0.03912	0.22313	
True Value	0.1	0.3	0.6	6.409692%
Mean	0.11387	0.29343	0.49035	
Median	0.09683	0.29164	0.57849	
Standard Deviation	0.26237	0.03410	0.24217	
True Value	0.1	0.3	0.5	1.347824%
Mean	0.11484	0.29314	0.41125	
Median	0.11833	0.29224	0.35967	
Standard Deviation	0.28141	0.03252	0.24325	
True Value	0.1	0.3	0.4	0.127036%
Mean	0.09522	0.29244	0.41637	
Median	0.07599	0.29224	0.35732	
Standard Deviation	0.29297	0.03222	0.24788	
True Value	0.1	0.3	0.0000001	0.000000%
Mean	0.08946	0.29237	0.40017	
Median	0.08844	0.29143	0.29074	
Standard Deviation	0.29598	0.03291	0.24124	

3.4 Measuring Capability of Predicting Financial Distress — Receiver Operating Characteristic Curve and Accuracy Ratio

To analyze the capability of predicting financial distress, we adopt the accuracy ratio (AR) and Receiver Operating Characteristic (ROC) method proposed by Moody's, which is also widely used in academic literature such as the studies by Vassalou and Xing (2004) and Chen, Hu, and Pan (2006), and Duffie, Saita, and Wang (2007). Stein (2002, 2005) argues that the power of a model to predict defaults is its ability to detect "True Default," and the capability of a model to calibrate to the data is its ability to detect "True Survival."

The ROC curve in the context of bankruptcy prediction is a plot of cumulative probability of the survival group against the cumulative probability of the default group. Assuming a firm defaults when its default probability is less than a cut-off threshold, the survival sample contains true survivals and false defaults, and the default sample contains true defaults and false survivals. Thus, the probabilities within the survival (default) group of true survival (default) and false default (survival) sum to unity. Figure 1 and Figure 2 demonstrate the ROC curves; the more powerful model successfully sets apart the default and survival distribution, the more concave is the ROC curve. In contrast, a model with no differentiating power shows a 45 degree line in its ROC curve since the default and survival samples overlap completely and two distributions are, in reality, one distribution.

The key statistic in the ROC methodology, known as the Cumulative Accuracy Profile (CAP), is the Accuracy Ratio (AR). AR is defined as the ratio of the area of tested model A to the area of perfect model A_P, i.e., $AR = A / A_P$ where $0 \le AR \le 1$. Hence, the higher the AR is, the more powerful is the model.

In our study, we modified the approach by Chen, Hu, and Pan (2006)[x] as follows:

1. Rank all default probabilities (P_{Def}) from the largest to smallest.
2. Compute the 100 percentiles of default probabilities (P_{Def}).
3. Divide the sample into default and survival groups.
4. In the default group, compute the cumulative probability greater than each percentile of default probabilities. This will be plotted on the y axis.
5. In the survival group, compute the cumulative probability greater than each percentile of default probabilities. This will be plotted on the x axis.
6. Plot the ROC curve.
7. For each structural model, repeat step 1 to step 6. Calculate the Accuracy Ratio (AR) and the z-statistic by the methods of comparing the areas under ROC curves by Hanley and McNeil (1983).

Figure 1 Four Models with Different Powers

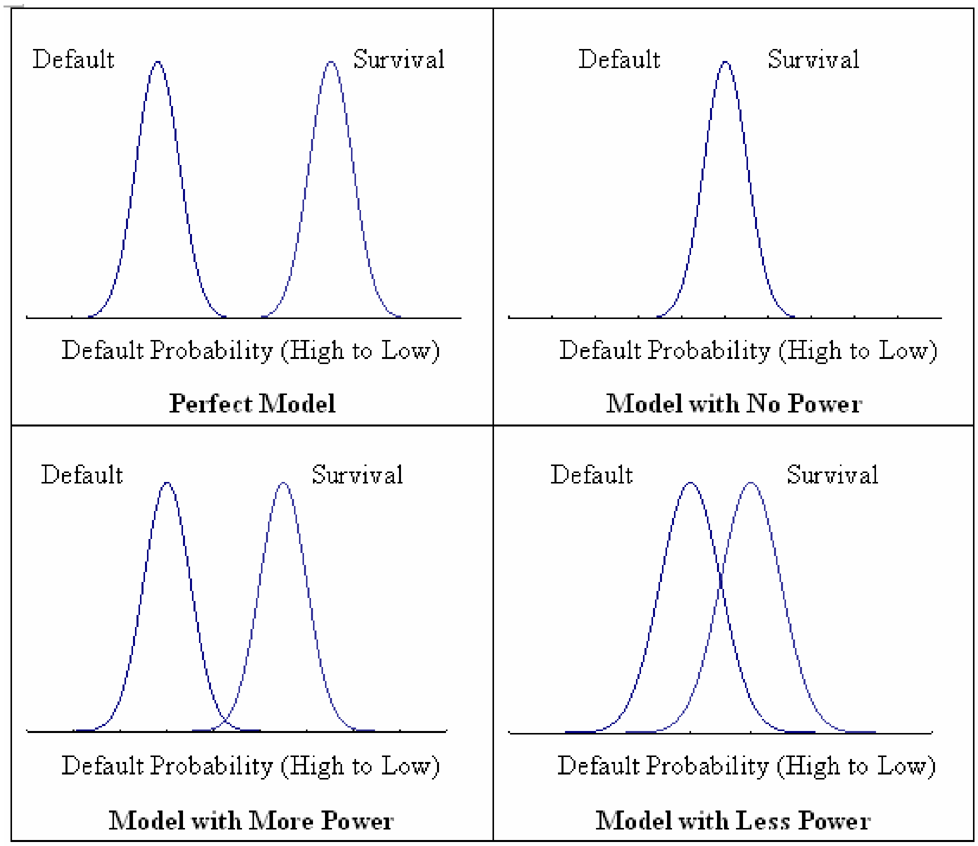
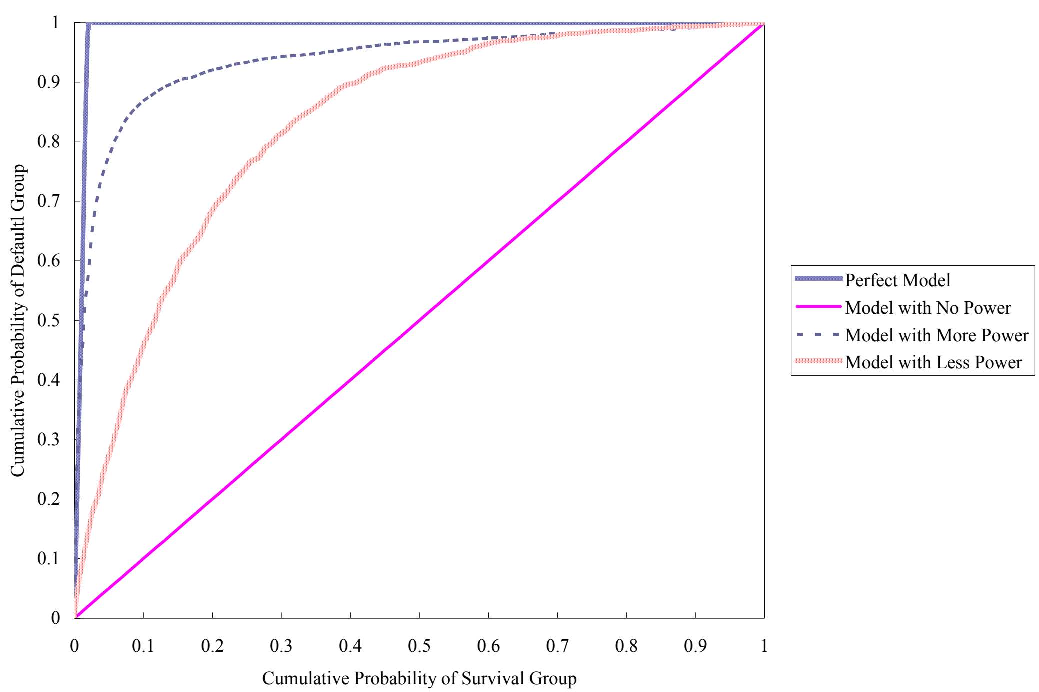


Figure 2 ROC Curves



4. Empirical Tests and Results

In this Section, we first describe the structural credit risk models to be tested in our empirical study in Section 5.1. We next present our data and the descriptive statistics in Section 5.2. In Section 5.3, the empirical results are reported and discussed. Robustness tests are presented in Section 5.4.

4.1 The Models

In our empirical study, we will test three barrier structural credit risk models extended from the Merton (1974) model. We will focus on two aspects of these extensions: the bond safety covenant in terms of continuous monitoring and default (Brockman and Turtle, 2003; Black-Cox, 1976); the shareholders' discretion on the going concern decision in terms of endogenous barrier modeling under the stationary debt structure assumption (Leland, 1994). We summarize their key features and parameters of these models in Table 3 and Table 4. All of the details of these models including close-form solutions and their corresponding default probabilities of risky debts are provided in Appendix.

5.2 Data and Summary Statistics

In our empirical test, equity prices are collected from CRSP (the Center for Research in Security Prices) and the financial statement information is retrieved from Compustat. The sampling period of the firms is from January 1986 to December 2005, while the quarterly accounting information is from 1984 to 2005 since some firms under financial distress stop filing financial reports a long time before they are delisted from the stock exchanges. The accounting information we use in our study is quarterly reports from CRSP/Compustat Merged (CCM) Database. This is to obtain the most updated debt levels and payout information, especially for those defaulted firms. In our empirical test, we consider only ordinary common shares (first digit of CRSP share type code 1) and exclude certificates, American trust components, and ADRs. Our final sample covers a 20-year period from 1986 to 2005 and includes 15,607 companies.

In our empirical test, we adopt two different definitions of default:

Definition I The broad definition of bankruptcy by Brockman and Turtle (2003), which includes firms that are delisted because of bankruptcy, liquidation, or poor performance. A firm is considered performance delisted, named by Brockman and Turtle, if it is given a CRSP delisting code of 400, or 550 to 585. Note that there are still other delisted firms due to mergers, exchanges, or being dropped by the exchange for other reasons, and they are considered as survival firms.

Definition II This definition of bankruptcy is similar to that adopted by Chen, Hu, and Pan (2006). Default firms are collected from the BankruptcyData.com database, which includes over 2,500 public and major company filings dating back to 1986. We next match the performance delisted firms with those samples collected from BankruptcyData.com, and add back the liquidated firms (with delisting code 400), to be our default group. All remaining firms are classified as survival firms. Note that one difference between our classification and Chen, Hu, and Pan (2006) is that some of the companies that filed bankruptcy petitions but were later acquired by (merged with) other companies (Delisting code: 200) are classified into survival group.

Table 3 Summary of Model Key Features

Model	Asset Value Distribution	Barrier	Interest Rate	Recovery	Bankruptcy Cost
Merton	Log-normal	None	Constant	Random	No
Brockman and Turtle (Flat Barrier)	Log-normal	Flat	Constant	Random	No
Black and Cox	Log-normal	Exponential	Constant	Random	No
Leland	Log-normal	Flat*	Constant	Fixed	Yes

* The character of endogenous flat barrier is different from the exogenously flat barrier. The endogenous barrier is derived endogenously from the optimal leverage decision, and its flat feature results from the stationary debt structure.

Table 4 Summary of Model Parameters

Model	Asset Value Process	Barrier Related	Bankruptcy Cost Related	Tax Shield Related	Other Parameters
Merton	$\theta_V = \{\mu_V, g, \sigma_V\}^{\#}$				r, F
Brockman and Turtle (Flat Barrier)	$\theta_V = \{\mu_V, g, \sigma_V\}^{\#}$	H			r, F
Black and Cox	$\theta_V = \{\mu_V, g, \sigma_V\}$	$H(C), \gamma$			r, F
Leland	$\theta_V = \{\mu_V, g, \sigma_V\}^{\#}$		α	T_C, C	r

The original Merton, Brockman and Turtle, and Leland models do not assume the asset payout, but they can be easily added into the models.

Before proceeding to the summary statistics of our final sampling firms, we first describe our sample selection criteria. First, companies with more than one share classes are excluded in our test. Second, since we also need accounting information in order to empirically test these models, firms without accounting information within two quarters going backward from the end of the estimation period are excluded. Thirdly, active firms (delisting code 100) during our sampling period while being delisted in 2006 are excluded. This is to ensure survival firms with delisting code 100 are financially healthy companies. Finally, to ensure adequate sample size for the MLE approach, we consider only those companies with over 252 days common share prices available.

Next, we report in Table 5 the main firm characteristics of our default samples in terms of market equity value, book leverage (total liabilities divided by asset value), and market leverage (total liabilities divided by market value of the firm). We can find that on average firms in default group are smaller and tend to have higher book and market leverage. In addition, the mean and median of book and market leverage of default group of default Definition II are higher than those of Definition I. This is because firms that delisted without filing Chapter 11 are considered as default firms in default Definition I, and as survival firms in Default definition II; such firms may not have debt levels as high as companies which filed Chapter 11. Finally, in Table 6, a summary of the default firms by industry and year is presented.

In the end of this section, we present our key inputs for the structural models. Determining the amount of debt for our empirical study is not an obvious matter. As opposed to the simplest approach, for example, by Brockman and Turtle (2003), to set the face value of debt equal to the total liabilities, we adopt the rough formula provided by Moody's KMV — the value of current liabilities including short-term debt, plus half of the long-term debt. This formula is also adopted by some researchers such as Vassalou and Xing (2004).[xi]

Secondly, the payout rate g captures the payouts in the form of dividends, share repurchase, and bond coupons to stock holders and bondholders. To estimate the payout rate, we adopt the weighted average method similar to those by Eom, Helwege, and Huang (2004) and Ericsson, Reneby, and Wang (2006) as

$$(Interest\ Expenses \big/ Total\ Liabilities) \times Leverage + (Equity\ Payout\ ratio) \times (1 - Leverage)$$

where $Leverage = Total\ Liabilities \big/ (Total\ Liabilities + Market\ Equity\ Value)$

For the market value of equity, we chose the number of shares outstanding times market price per share on the day closest to the financial statement date. The equity payout rate is estimated as the total equity payout, which is the sum of cash dividends, preferred dividends, and purchase of common and preferred stock, divided by the total equity payout plus market value of equity.

Table 5 Summary Statistics of Sampling Firms

Default Definition I						
	Group	Number of firms	Mean	Median	Maximum	Minimum
Market Equity Value	Survival	10729	1770.7762	173.0430	367495.1442	0.3016
	Default	4878	58.1062	8.7146	36633.7544	0.0271
Book Leverage	Survival	10729	0.5541	0.5500	4.0093	0.0008
	Default	4878	0.7628	0.6880	203.0000	0.0003
Market Leverage	Survival	10729	0.4321	0.3932	0.9961	0.0007
	Default	4878	0.5188	0.5387	0.9997	0.0001
Default Definition II						
	Group	Number of firms	Mean	Median	Maximum	Minimum
Market Equity Value	Survival	14244	1340.6120	81.9271	367495.1442	0.0271
	Default	1363	136.7769	23.4698	36633.7544	0.3356
Book Leverage	Survival	14244	0.5996	0.5666	203.0000	0.0003
	Default	1363	0.8253	0.7888	12.5273	0.0025
Market Leverage	Survival	14244	0.4389	0.4058	0.9997	0.0001
	Default	1363	0.6718	0.7577	0.9995	0.0024

Table 6 Number of the Default Firms by Industry and Year

Default Definition I												
	SIC Code											
Year	Missing	0	1	2	3	4	5	6	7	8	9	Total
1986	4	1	62	21	73	8	27	11	27	9	0	243
1987	0	5	23	16	44	9	31	10	18	4	0	160
1988	0	1	26	22	53	11	31	10	40	13	0	207
1989	1	4	23	30	63	14	29	17	30	6	0	217
1990	0	1	19	22	86	16	33	21	29	10	0	237
1991	0	2	28	33	85	10	31	24	39	13	0	265
1992	0	2	53	26	84	16	45	31	38	22	0	317
1993	0	2	15	15	52	8	11	13	15	9	0	140
1994	0	0	20	13	52	12	20	19	25	8	2	171
1995	0	1	19	21	50	12	38	21	39	16	1	218
1996	0	0	7	20	42	10	33	12	17	11	2	154
1997	0	0	16	25	52	17	44	15	36	17	0	222
1998	0	2	36	45	97	25	61	35	64	32	2	399
1999	0	4	48	53	87	22	43	28	50	34	0	369
2000	0	0	10	34	71	26	50	28	53	26	0	298
2001	0	2	15	42	87	44	58	23	131	24	1	427
2002	0	0	14	31	87	45	23	31	94	17	0	342
2003	0	1	9	20	75	20	34	16	54	18	0	247
2004	0	0	3	13	23	7	16	20	20	4	0	106
2005	0	0	5	20	43	14	8	17	25	7	0	139
Total	5	28	451	522	1306	346	666	402	844	300	8	4878

SIC Code: 0: Agriculture, Forestry, and Fishing; 1: Mining and Construction; 2 and 3: Manufacturing; 4: Transportation, Communications, Electric, Gas, and Sanitary Service; 5: Wholesale Trade, and Retail Trade; 6: Finance, Insurance, and Real Estate; 7 and 8: Service; 9: Public Administration

Table 6 Number of the Default Firms by Industry and Year (Cont.)

Default Definition II												
	SIC Code											
Year	Missing	0	1	2	3	4	5	6	7	8	9	Total
1986	1	0	5	3	11	3	4	2	3	1	0	33
1987	0	2	3	1	4	0	3	0	2	1	0	16
1988	0	0	3	1	4	6	7	1	0	0	0	22
1989	0	0	4	7	6	4	11	6	5	0	0	43
1990	0	0	3	3	12	4	8	7	2	1	0	40
1991	0	2	2	3	21	3	7	7	7	2	0	54
1992	0	1	5	2	10	6	13	7	3	5	0	52
1993	0	0	4	4	11	0	6	4	2	0	0	31
1994	0	0	2	4	14	3	5	3	4	2	2	39
1995	0	0	2	5	8	6	16	8	6	4	0	55
1996	0	0	5	6	7	3	16	2	2	0	0	41
1997	0	1	5	5	12	7	13	4	7	5	0	59
1998	0	1	6	13	28	7	20	12	13	11	1	112
1999	0	0	13	14	21	14	18	8	8	13	0	109
2000	0	0	6	17	28	15	30	12	21	13	0	142
2001	0	0	3	15	41	31	28	10	49	10	0	187
2002	0	0	6	9	33	27	11	12	23	8	0	129
2003	0	0	1	11	33	12	16	3	14	5	0	95
2004	0	0	2	7	7	3	8	5	7	0	0	39
2005	0	0	3	5	22	13	8	3	5	6	0	65
Total	1	7	83	135	333	167	248	116	183	87	3	1363

SIC Code: 0: Agriculture, Forestry, and Fishing; 1: Mining and Construction; 2 and 3: Manufacturing; 4: Transportation, Communications, Electric, Gas, and Sanitary Service; 5: Wholesale Trade, and Retail Trade; 6: Finance, Insurance, and Real Estate; 7 and 8: Service; 9: Public Administration

Thirdly, since the models in our study assume constant interest rate, one needs to feed in the appropriate interest rate for model estimation. The three month T-bill rate from the Federal Reserve website is chosen as the risk-free rate. However, the three month T-bill rate fluctuated heavily; from a high of 9.45% in March 1989, it dropped to a low of 0.81% in June 2003, and then went back to 4.08% in the end of December 2005. Therefore, to assure the proper discount rate for each firm across a 20-year sampling period, interest rates are estimated as the average of 252 daily 3-month Constant Maturity Treasury (CMT) rates for each firm during the sampling period.

Finally, the Leland model needs debt coupons and we follow Ericsson, Reneby and Wang (2006) to set the average coupon as risk-free rate times total liabilities: $Coupon = Total\ Liabilities \times Riskfree\ Rate$. In addition, the Leland model considers tax deductibility as well as bankruptcy cost. We follow Eom, Helwege, and Huang (2004) and set the tax rate to 35% and financial distress cost as 51.31%. Furthermore, we also follow Leland (1998) and Ericsson, Reneby and Wang (2006) and set the tax rate to 20% as an alternative setting. This is to reflect personal tax advantages to equity returns which reduce the advantage of debt.

5.3 Empirical Results

In our empirical test, we use the same numerical optimization algorithm of Nelder-Mead (in Matlab software package) as that adopted by Wong and Choi (2009). The inputs of parameters for debt level, asset payouts, interest rates, coupons, tax rate, and financial distress cost are as described in Section 5.1, and the option time to maturity is two years. The original Merton, Brockman and Turtle, and Leland models do not assume the asset payout rate, but they can be easily added into the models. For comparison purposes, we choose to estimate default barriers, $H = \alpha F$, instead of discount rates, γ , of each firms in the Black and Cox model, and the discount rates are assumed to be the average risk-free rates for those firms during the equity time series sampling period.

The delisting date of a delisted firm is simply the very last security trading day, while the delisting date of an active firm (delisting code 100) is set as the last trading day in year 2005. Inputs of equity time series for in-sample estimation are the equity values, ending on the delisting date and travelling back 252 trading days. The six-month (one-year) out-of-sample estimation uses equity time series from 377 to 126 (503 to 252) trading days before the delisting date. The sample sizes of the in-sample, six-month out-of-sample, and one-year out-of-sample tests are 15,607, 14,775, and 13,750, respectively. The differences in the sample sizes come from the availability of equity trading data. As we push the estimation period backward in time, we lose some firms due to the relatively shorter lives of these companies. After numerical optimization, final samples for in-sample test, six-month out-of-sample, and one-year out-of-sample tests include 15,598, 14,765, and 13,744 firms.[xii]

5.3.1 Testing Results of Default Definition I

We first present in Table 7 the performance of default prediction by decile-based analysis and provide the percentages of performance delisting in each decile. Defaulting firms are sorted into deciles by corresponding physical default probability estimates of each model, where the physical default probabilities of firms for the in-sample and out-of-sample tests are computed using the estimated firm values one week (5 trading days), six months (126 trading days), and one year (252 trading days) before the delisting date, respectively. One can clearly find that the Merton and the Black and Cox models outperform the Brockman and Turtle model, especially in the out-of-sample predictions.

We next present in Figure 3, Figure 4, and Figure 5, respectively, the in-sample, out-of-sample (six-month) and out-of-sample (one-year) ROC curves of the tested models. Formal statistical tests are carried out by the Accuracy Ratios (ARs) and the z statistics. Z statistics, compared with the Merton model, for the tested models are reported in the parentheses in Table 8. We find that in accordance with the results in the decile-based analysis, the Brockman and Turtle model is clearly inferior to the Merton and the Black and Cox models. The Leland model of in-sample test in both tax rate settings also underperforms the Merton model.

Figure 3 ROC Curves – One Week In-Sample Test (All Sample)

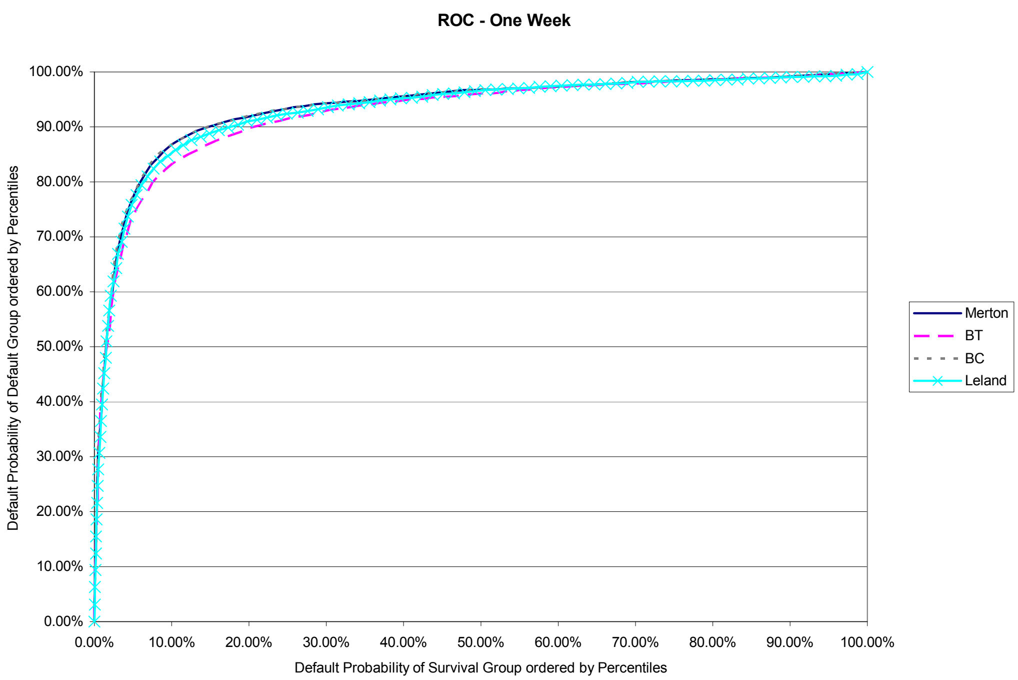


Figure 4 ROC Curves – Six-Month Out-of-Sample Test (All Sample)

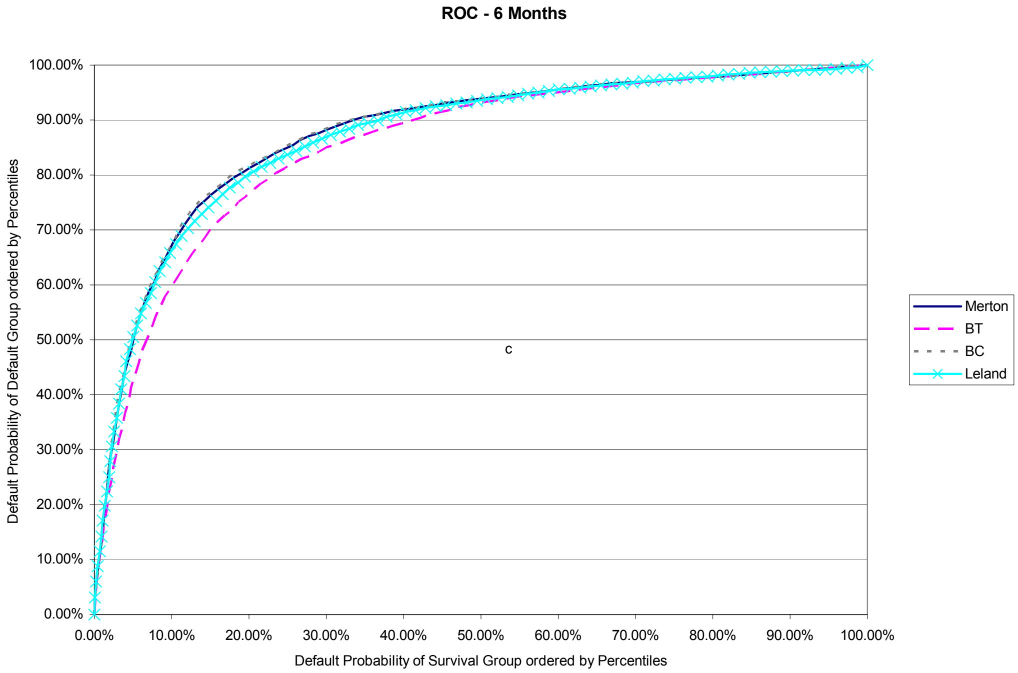
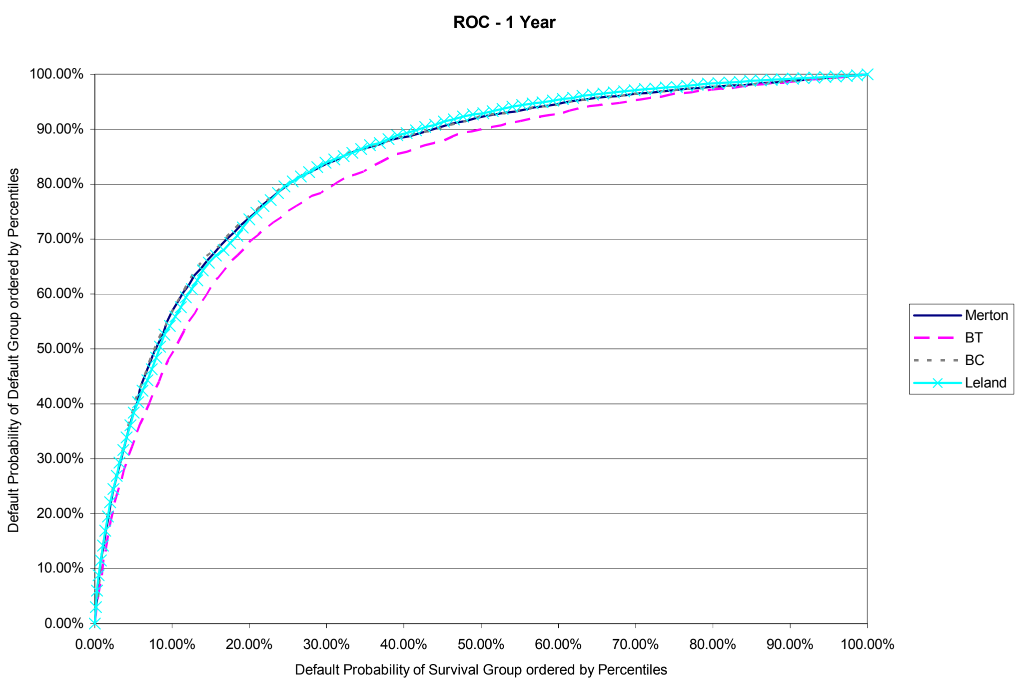


Figure 5 ROC Curves – One-Year Out-of-Sample Test (All Sample)



Our empirical result shows that the simple Merton model surprisingly outperforms the flat barrier model in default prediction. Furthermore, the performance of the Merton model is also similar to that of the Black and Cox model in all tests. The Black and Cox model has slightly higher ARs than those of Merton's model, however, the differences are not statistically significant based on the z test. Moreover, Merton's model also performs significantly better than the Leland model of the in-sample test.

The results of z test indicate that the difference of prediction capability between the Merton and the flat barrier models is statistically significant and the results hold for both in-sample and out-of-sample tests. Although theoretically the down-and-out option framework should nest the standard call option model, practically it may not perform better in the default prediction. Several possible reasons may explain our empirical results.

One of the possible explanations is that the continuous monitoring assumption of the flat barrier model makes it possible to default before debt maturity, and thus increases the estimated default probabilities of the survival firms. One may argue that the implied default probabilities of the default firms increase as well. However, the magnitude of the increments may not be the same, and we do observe this in our empirical results.

For example, the case of Alfacell Corporation, a survival firm, (CRSP permanent company number 35) clearly reflects this issue as shown in Figure 6. Alfacell experienced a drastic downfall of share prices in year 2005. However, it still survived through the end of 2006. In Figure 6, we present the one-year market equity, the estimated firm value of the Merton model, the estimated firm value of the Brockman and Turtle model, the implied barrier, and the debt level of the KMV formula, respectively. Both models generate reasonable firm value estimates based on the corresponding model assumptions. Estimated firm values of the flat barrier model are higher than those of the Merton model due to the existence of the claims of the bondholders modeled as the down-and-in option. The implied default probability of Alfacell Corporation is merely 0.04% by the Merton model, while the default probability of the flat barrier model is as high as 61.21%. The gigantic difference comes from the implied default barrier. The debt level by the KMV formula is \$1.75 million, but the implied barrier from the Brockman and Turtle model is \$31.37 million! Such a high implied barrier leads to a high default probability by the flat barrier model. In contrast, default in Merton's model is only related to the debt level at debt maturity and thus the default probability is very low. Note that to prevent from the local optimum problem of the barrier estimate, we also use another optimization routine, the fmincon function in Matlab, to re-estimate the Alfacell case but still obtain the same implied default barrier.

One may argue that imposing constraints on the default barrier can solve this issue. However, the high implied default barrier is a result of the return distribution of the equity value process. Imposing constraints clearly violates the fundamental of the maximum likelihood estimation method and hinders the MLE method from searching the global optimum. In the case of Alfacell Corporation, the likelihood function of the Brockman and Turtle model and the Merton model are 566.397 and 562.288, respectively. This indicates that the introduction of the barrier does improve the fitting of the return distribution of the equity value process. Furthermore, the equity pricing function of the flat barrier model in Equation (A.3) does not pre-specify the location of the barrier. The flat default barrier can be higher than the debt level, as assumed in the Brockman and Turtle model. Accordingly, the fundamental issue is that the flat barrier assumption itself might be unreasonable and unrealistic. Finally, we should note that the extraordinarily high implied default barrier cannot happen in the Black-Cox model since it assumes that the default barrier is lower than the debt level. As a result, the implied default probability of Alfacell Corporation is only 0.06% by the Black and Cox model.

Another possible explanation is from our measure of the default prediction capability. The AR only preserves the ranking information of the default probabilities in our empirical test. The flat barrier model may generate the default probability distribution closer to the true default probability distribution, compared with that of the Merton model. It is the tails of the default probability distributions of survival and default groups that truly determine the ARs. Nonetheless, one can clearly observe from the decile-based results in Table 7 that the Brockman and Turtle model does not have the same differentiating power for default and survival groups as that of the Merton model.

Finally, we cannot completely rule out the local optimum possibility, since it is well known that high dimensional optimization may not uncover the global optimum. The superior default prediction capability of the Merton model may come from the better estimates of model parameters due to its simpler likelihood function and lower dimension in the optimization procedure.

We next turn to the sub-sample analysis by financial (Table 9) versus non-financial (Table 10) firms. Financial companies have industry-specific high leverage ratios and thus cannot be modeled well in finance literature. Consistent with the findings by Chen, Hu, and Pan (2006), we find that the Brockman and Turtle model perform much better in finance sector than in the industrial sector, while the Merton and the Black and Cox models perform better in the industrial sector. Accordingly, the difference of default prediction power of the flat barrier and the Merton model in finance sector is no longer significant.

Another important finding is that the Leland model outperforms Merton's model in the Non-financial sector, and the differences are significant in the six-month and one-year out-of-sample tests. The Leland model shows large differences of default predictability between financial and non-financial sectors. This difference leads to its superior power of prediction in non-financial sector.

Finally, we turn to the discussion of default barriers. Unlike the Wong and Choi (2009), we do not present the barrier-to-debt ratio and have our inference based on it. This is because of the nature of likelihood function of down-and-out option framework, one cannot expect to pin down the barrier when the barrier is low relative to the asset value, i.e., the default probability is low. Therefore, to get rid of this bias, we present the differences of default probabilities between barrier models and Merton's model. Our results in Table 11, Table 12, and Table 13 show that the introduction of default barriers does influence default probabilities, especially on the default group. However, for most of the survival firms and around 30% of the firms in the default group, the impact is small. This in turn indicates that exogenous flat or exponential barriers do not have significant impact in equity pricing for these companies. Thus, our empirical finding is consistent with the results by Wong and Choi (2009) and does not support the finding by Brockman and Turtle (2003) that default barriers are significantly positive.

Table 7 Percentages of Performance Delisting firms in Each Decile (Default Definition I)

In Sample Test One Week					
15,598 firms – 10,727 survival and 4,871 performance delisting firms					
Decile (P_{Def})	**Merton**	**Brockman and Turtle**	**Black and Cox**	**Leland (TC=20%)**	**Leland (TC=35%)**
1 (Large)	30.86%	30.82%	31.02%	30.65%	30.63%
2	28.27%	28.04%	28.33%	28.68%	28.80%
3	22.34%	20.80%	22.28%	21.64%	21.58%
4	9.46%	8.91%	9.16%	8.97%	8.89%
5	3.37%	4.52%	3.49%	3.63%	3.55%
6-10 (Small)	5.71%	6.92%	5.73%	6.43%	6.55%
Out of Sample Test Six Months					
14,765 firms - 10,232 survival and 4,533 performance delisting firms					
Decile (P_{Def})	**Merton**	**Brockman and Turtle**	**Black and Cox**	**Leland (TC=20%)**	**Leland (TC=35%)**
1 (Large)	28.04%	27.05%	28.08%	27.91%	27.95%
2	24.69%	22.81%	24.88%	24.73%	24.75%
3	18.47%	17.36%	18.60%	17.67%	17.63%
4	11.03%	12.11%	11.01%	11.21%	11.23%
5	7.32%	7.54%	7.10%	6.86%	6.93%
6-10 (Small)	10.46%	13.13%	10.32%	11.63%	11.52%
Out of Sample Test 1 Year					
13,744 firms - 9,637 survival and 4,107 performance delisting firms					
Decile (P_{Def})	**Merton**	**Brockman and Turtle**	**Black and Cox**	**Leland (TC=20%)**	**Leland (TC=35%)**
1 (Large)	26.83%	25.66%	26.91%	26.86%	26.86%
2	22.23%	20.50%	22.40%	21.50%	21.48%
3	17.09%	16.78%	17.39%	17.34%	17.36%
4	12.25%	12.15%	11.91%	12.66%	12.86%
5	8.01%	8.13%	7.74%	8.04%	7.94%
6-10 (Small)	13.59%	16.78%	13.66%	13.61%	13.51%

Table 8 Accuracy Ratios and z Statistics of Physical Probabilities (Default Definition I; All Sample)

Accuracy Ratio	**Merton**	**Brockman and Turtle**	**Black and Cox**	**Leland (TC=0.2)**	**Leland (TC=0.35)**
One Week (In Sample)	0.9357	**0.9253 (-5.8513)**	0.9365 (0.7667)	**0.9314 (-2.3810)**	**0.9315 (-2.1933)**
Six Months (Out of Sample)	0.8749	**0.8531 (-8.5565)**	0.8768 (1.5632)	0.8705 (-1.6938)	0.8711 (-1.3984)
One Year (Out of Sample)	0.8422	**0.8156 (-8.8537)**	0.8433 (0.8055)	0.8442 (0.6621)	0.8449 (0.8316)
In-Sample One-Week (15,598 firms – 10,727 survival and 4,871 performance delisting firms) Out-of-Sample 6-Month (14,765 firms - 10,232 survival and 4,533 performance delisting firms) Out-of-Sample 1-Year (13,744 firms - 9,637 survival and 4,107 performance delisting firms)					

Table 9 Accuracy Ratios and z Statistics of Physical Probabilities (Default Definition I; Financial Firms)

Accuracy Ratio	**Merton**	**Brockman and Turtle**	**Black and Cox**	**Leland (TC=0.2)**	**Leland (TC=0.35)**
One Week (In Sample)	0.8939	0.8900 (-0.5698)	0.8926 (-0.3598)	**0.8750 (-3.1532)**	**0.8744 (-3.0896)**
Six Months (Out of Sample)	0.8496	0.8539 (0.5305)	0.8520 (0.5858)	**0.8209 (-3.9062)**	**0.8199 (-3.7674)**
One Year (Out of Sample)	0.8319	0.8240 (-0.8894)	0.8333 (0.3162)	**0.8083 (-2.7714)**	**0.8097 (-2.6578)**
In-Sample One-Week (2,809 firms – 2,409 survival and 400 performance delisting firms) Out-of-Sample 6-Month (2,694 firms – 2,313 survival and 381 performance delisting firms) Out-of-Sample 1-Year (2,556 firms – 2,195 survival and 361 performance delisting firms)					

Table 10 Accuracy Ratios and z Statistics of Physical Probabilities (Default Definition I; Non-Financial Firms)

Accuracy Ratio	**Merton**	**Brockman and Turtle**	**Black and Cox**	**Leland (TC=0.2)**	**Leland (TC=0.35)**
One Week (In Sample)	0.9371	**0.9255 (-6.2231)**	0.9380 (0.8838)	0.9373 (0.0707)	0.9376 (0.2090)
Six Months (Out of Sample)	0.8714	**0.8437 (-10.0585)**	0.8729 (1.1717)	**0.8777 (2.2951)**	**0.8786 (2.5352)**
One Year (Out of Sample)	0.8379	**0.8054 (-9.8963)**	0.8385 (0.3975)	**0.8543 (5.1790)**	**0.8555 (5.2588)**
In-Sample One-Week (12,789 firms – 8,318 survival and 4,471 performance delisting firms) Out-of-Sample 6-Month (12,071 firms – 7,919 survival and 4,152 performance delisting firms) Out-of-Sample 1-Year (11,188 firms – 7,442 survival and 3,746 performance delisting firms)					

Figure 6 An Illustration of the Problem of the Brockman and Turtle Model by Alpacell Corporation

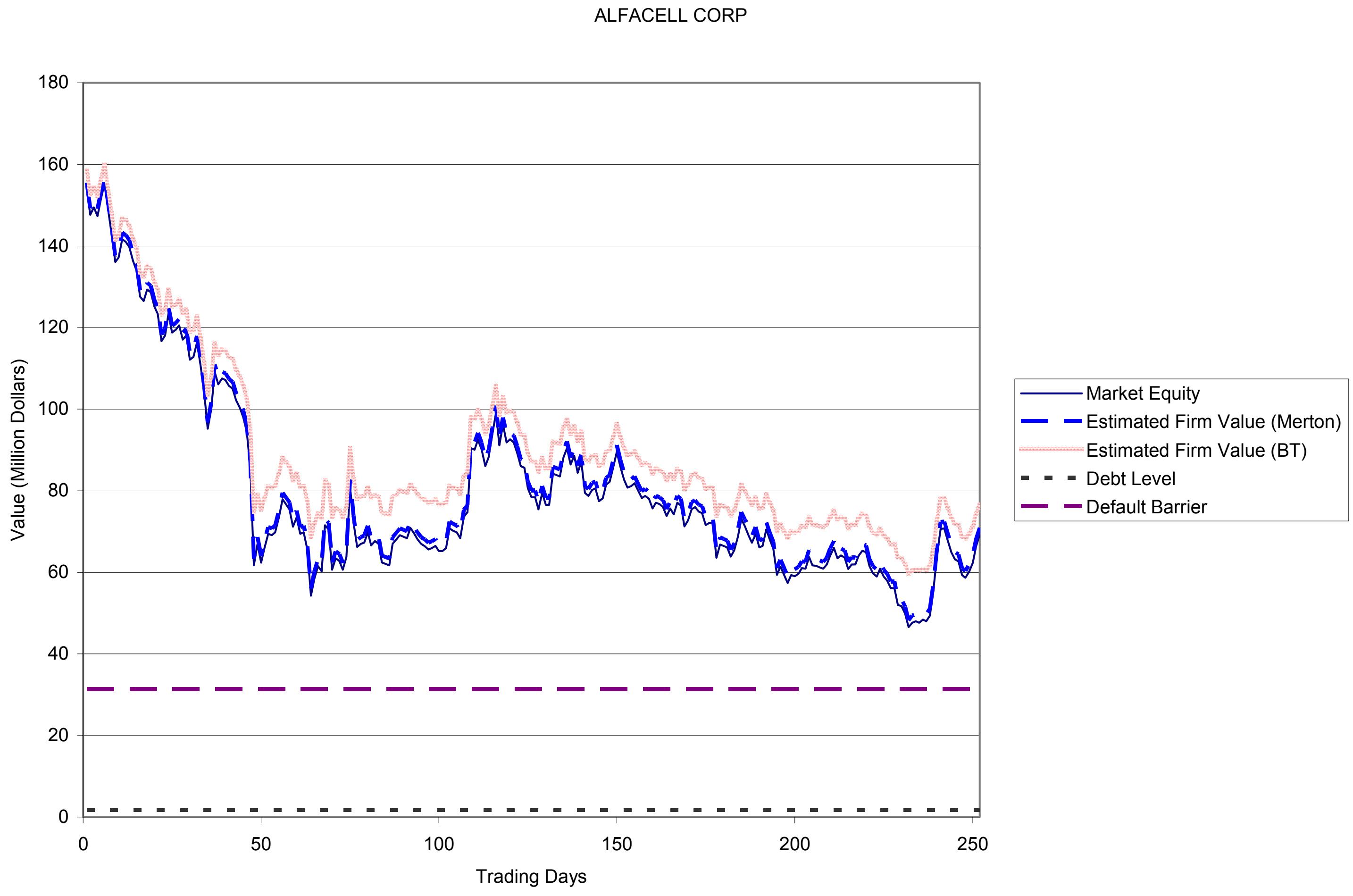


Table 11 The Effect of Default Barriers in Terms of the Default Probabilities (In-Sample Test)

Difference of Default Probabilities			BT: PDef(BT)-PDef(Merton)			BC: PDef(BC)-PDef(Merton)		
All			Survival			Default		
Percentile	BT	BC	Percentile	BT	BC	Percentile	BT	BC
5%	-0.010%	0.000%	5%	-0.010%	0.000%	5%	-0.009%	-0.001%
10%	0.000%	0.000%	10%	0.000%	0.000%	10%	-0.001%	0.000%
20%	0.000%	0.000%	20%	0.000%	0.000%	20%	0.000%	0.001%
30%	0.000%	0.000%	30%	0.000%	0.000%	30%	0.001%	0.045%
40%	0.000%	0.000%	40%	0.000%	0.000%	40%	0.226%	0.277%
50%	0.003%	0.000%	50%	0.000%	0.000%	50%	1.562%	0.879%
60%	0.415%	0.005%	60%	0.015%	0.000%	60%	5.293%	2.090%
70%	3.321%	0.141%	70%	0.808%	0.001%	70%	12.572%	4.295%
80%	12.113%	1.294%	80%	6.233%	0.024%	80%	23.580%	7.785%
90%	30.705%	5.947%	90%	23.732%	0.760%	90%	39.896%	11.195%
95%	47.584%	10.034%	95%	42.673%	3.838%	95%	53.779%	13.118%
Mean	7.540%	1.421%	Mean	5.685%	0.504%	Mean	11.729%	3.491%
Standard Deviation	17.314%	3.789%	Standard Deviation	16.407%	2.099%	Standard Deviation	18.537%	5.538%
Absolute Difference								
<0.1%	51.673%	67.057%	<0.1%	59.548%	83.278%	<0.1%	33.885%	30.419%
<0.5%	57.465%	74.028%	<0.5%	65.012%	88.155%	<0.5%	40.419%	42.119%
<1%	60.920%	77.788%	<1%	67.875%	90.334%	<1%	45.210%	49.448%
<5%	71.156%	88.348%	<5%	76.681%	95.700%	<5%	58.675%	71.744%

Table 12 The Effect of Default Barriers in Terms of the Default Probabilities (Six–Month Out-of-Sample Test)

Difference of Default Probabilities			BT: PDef(BT)-PDef(Merton)			BC: PDef(BC)-PDef(Merton)		
All			Survival			Default		
Percentile	BT	BC	Percentile	BT	BC	Percentile	BT	BC
5%	-0.002%	0.000%	5%	0.000%	0.000%	5%	-0.252%	-0.160%
10%	0.000%	0.000%	10%	0.000%	0.000%	10%	-0.004%	0.000%
20%	0.000%	0.000%	20%	0.000%	0.000%	20%	0.000%	0.001%
30%	0.000%	0.000%	30%	0.000%	0.000%	30%	0.002%	0.054%
40%	0.000%	0.000%	40%	0.000%	0.000%	40%	0.164%	0.282%
50%	0.001%	0.000%	50%	0.000%	0.000%	50%	0.936%	0.852%
60%	0.197%	0.002%	60%	0.002%	0.000%	60%	2.806%	1.928%
70%	1.681%	0.081%	70%	0.310%	0.000%	70%	7.644%	4.020%
80%	6.862%	0.925%	80%	3.002%	0.004%	80%	17.188%	7.569%
90%	21.244%	5.191%	90%	13.707%	0.273%	90%	34.410%	11.534%
95%	36.824%	9.767%	95%	27.255%	1.829%	95%	48.125%	13.906%
Mean	5.779%	1.287%	Mean	4.109%	0.330%	Mean	9.455%	3.393%
Standard Deviation	13.405%	3.747%	Standard Deviation	11.173%	1.801%	Standard Deviation	16.757%	5.601%
Absolute Difference								
<0.1%	55.166%	68.382%	<0.1%	65.772%	87.081%	<0.1%	31.822%	27.226%
<0.5%	61.851%	75.471%	<0.5%	70.936%	91.023%	<0.5%	41.855%	41.239%
<1%	65.754%	79.336%	<1%	74.012%	93.018%	<1%	47.579%	49.220%
<5%	77.157%	89.431%	<5%	82.802%	97.204%	<5%	64.731%	72.323%

Table 13 The Effect of Default Barriers in Terms of the Default Probabilities (One-Year Out-of-Sample Test)

Difference of Default Probabilities			BT: PDef(BT)-PDef(Merton)			BC: PDef(BC)-PDef(Merton)		
All			Survival			Default		
Percentile	BT	BC	Percentile	BT	BC	Percentile	BT	BC
5%	-0.001%	0.000%	5%	0.000%	0.000%	5%	-0.026%	-0.002%
10%	0.000%	0.000%	10%	0.000%	0.000%	10%	-0.001%	0.000%
20%	0.000%	0.000%	20%	0.000%	0.000%	20%	0.000%	0.000%
30%	0.000%	0.000%	30%	0.000%	0.000%	30%	0.000%	0.006%
40%	0.000%	0.000%	40%	0.000%	0.000%	40%	0.055%	0.082%
50%	0.002%	0.000%	50%	0.000%	0.000%	50%	1.160%	0.389%
60%	0.210%	0.003%	60%	0.014%	0.000%	60%	4.384%	1.239%
70%	1.957%	0.064%	70%	0.435%	0.001%	70%	12.333%	2.900%
80%	8.712%	0.774%	80%	3.309%	0.022%	80%	24.328%	5.991%
90%	28.719%	4.472%	90%	17.572%	0.668%	90%	41.073%	10.129%
95%	45.118%	8.937%	95%	37.494%	3.165%	95%	54.392%	12.344%
Mean	2.137%	0.140%	Mean	5.095%	0.488%	Mean	11.752%	2.797%
Standard Deviation	8.053%	1.180%	Standard Deviation	13.467%	2.298%	Standard Deviation	18.977%	5.387%
Absolute Difference								
<0.1%	55.530%	69.856%	<0.1%	63.474%	83.543%	<0.1%	36.888%	37.740%
<0.5%	61.940%	76.979%	<0.5%	69.783%	88.627%	<0.5%	43.535%	49.647%
<1%	65.221%	80.348%	<1%	72.990%	90.661%	<1%	46.993%	56.148%
<5%	75.662%	90.403%	<5%	82.090%	96.223%	<5%	60.579%	76.747%

5.3.2 Testing Results of Default Definition II

In this section, we regroup our survival and default groups using the definition of bankruptcy similar to that adopted by Chen, Hu, and Pan (2006). Following their approach, we collect default firms from the BankruptcyData.com database, which includes over 2,500 public and major company filings dating back to 1986. Next, we match the performance delisted samples with companies collected from BankruptcyData.com, and add back the liquidated firms (with delisting code 400) to form our default group. All of the remaining firms are classified as survival firms. Note that a difference between our classification and Chen, Hu, and Pan (2006) is that some of the companies that filed bankruptcy petitions but were later acquired by (or merged with) other companies (Delisting code: 200) are classified in the survival group. The numbers of default firms following this approach greatly reduce from 4,871 to 1,325 for the in-sample test and from 4,533 to 1,260 (4,107 to 1,183) for the six-month (one-year) out-of-sample tests.

To conserve space, ROC curves are not reported. The decile-based analysis as well as the accuracy ratios and z statistics are reported in Table 14 and Table 15, respectively. From Table 15, our results still show that the Merton model outperforms the flat barrier model, and the difference of default prediction capability is statistically significant as that in Section 5.3.1. The prediction capabilities of the Merton and the Black and Cox model are similar as well.

In addition, one can observe that all these models perform slightly worse than the broad definition of bankruptcy. The differences are around 2% across different models and tests. The reason may be the uncertainty of bankruptcy filings of companies been delisted from the stock exchange. One can use the MLE approach to capture information from the market equity values of those poorly performed and delisted firms, and obtain default probabilities of these firms. However, if those firms will eventually file bankruptcy may be subject to a lot of firm-specific human and company potential issues. These issues may not easily be captured just by the dynamics of the firms' market equity values.

In Table 16 and Table 17, the financial versus non-financial sector analysis are reported. The performances among models are also similar to those of the broad definition of bankruptcy in Section 5.2.1. Unlike the performances in broad definition of default, not only the Brockman and Turtle model performs much better in the finance sector, but the Merton, the Black and Cox, as well as the Leland models also perform better in the financial sector. However, the accuracy ratios of the flat barrier are even higher than those of the Merton and the Black and Cox models in the finance sector, although the differences are not statistically significant. In the non-financial sector, the Leland model still performs better than Merton's model, but a difference is that the Leland model no longer significantly outperforms the Merton model in six-month out-of-sample test.

Table 14 Percentages of Default Firms in Each Decile (Default Definition II)

In Sample Test One Week					
15,598 firms – 14,273 survival and 1,325 default firms					
Decile (P_{Def})	**Merton**	**Brockman and Turtle**	**Black and Cox**	**Leland (TC=20%)**	**Leland (TC=35%)**
1 (Large)	56.98%	52.53%	56.38%	58.87%	58.94%
2	26.42%	26.64%	26.26%	24.30%	24.38%
3	9.36%	12.23%	9.96%	9.06%	9.06%
4	3.40%	3.62%	3.32%	3.32%	3.17%
5	0.98%	1.74%	1.21%	1.13%	1.21%
6-10 (Small)	2.87%	3.25%	2.87%	3.32%	3.25%
Out of Sample Test Six Months					
14,765 firms - 13,498 survival and 1,267 default firms					
Decile (P_{Def})	**Merton**	**Brockman and Turtle**	**Black and Cox**	**Leland (TC=20%)**	**Leland (TC=35%)**
1 (Large)	44.20%	37.73%	43.25%	46.25%	46.41%
2	24.07%	23.60%	23.52%	22.81%	22.73%
3	14.13%	15.47%	15.47%	13.81%	13.81%
4	7.73%	9.71%	8.05%	6.08%	6.08%
5	4.26%	6.08%	4.10%	4.81%	4.89%
6-10 (Small)	5.60%	7.42%	5.60%	6.24%	6.08%
Out of Sample Test 1 Year					
14,744 firms – 12,561 survival and 1,183 performance delisting firms					
Decile (P_{Def})	**Merton**	**Brockman and Turtle**	**Black and Cox**	**Leland (TC=20%)**	**Leland (TC=35%)**
1 (Large)	36.01%	30.35%	35.42%	37.95%	37.79%
2	21.98%	20.20%	21.56%	20.88%	20.80%
3	14.88%	17.41%	16.74%	16.40%	16.40%
4	12.09%	12.00%	10.99%	9.72%	10.14%
5	6.34%	8.12%	6.59%	6.76%	6.51%
6-10 (Small)	8.71%	11.92%	8.71%	8.28%	8.37%

Table 15 Accuracy Ratios and z Statistics of Physical Probabilities (Default Definition II; All Sample)

Accuracy Ratio	**Merton**	**Brockman and Turtle**	**Black and Cox**	**Leland (TC=0.2)**	**Leland (TC=0.35)**
One Week (In Sample)	0.9152	**0.9006 (-5.3136)***	0.9121 (-1.5613)	0.9145 (-0.2014)	0.9147 (-0.1159)
Six Months (Out of Sample)	0.8574	**0.8278 (-7.5183)**	0.8561 (-0.6197)	0.8587 (0.2765)	0.8589 (0.3166)
One Year (Out of Sample)	0.8166	**0.7811 (-7.6036)**	0.8147 (-0.7825)	0.8242 (1.5007)	0.8243 (1.4354)
In-Sample One-Week (15,598 firms – 14,273 survival and 1,325 default firms) Out-of-Sample 6-Month (14,765 firms – 13,498 survival and 1,267 default firms) Out-of-Sample 1-Year (13,744 firms – 12,561 survival and 1,183 default firms)					

* Numbers in the parentheses are the Z-statistic of each model compared with the Merton model

Table 16 Accuracy Ratios and z Statistics of Physical Probabilities (Default Definition II; Financial Firms)

Accuracy Ratio	**Merton**	**Brockman and Turtle**	**Black and Cox**	**Leland (TC=0.2)**	**Leland (TC=0.35)**
One Week (In Sample)	0.8969	0.8999 (0.3203)	0.8879 (-1.2743)	0.8814 (-.14651)	0.8814 (-1.3993)
Six Months (Out of Sample)	0.8669	0.8712 (0.3806)	0.8695 (0.3352)	0.8523 (-1.2010)	0.8515 (-1.1895)
One Year (Out of Sample)	0.8582	0.8600 (0.1276)	0.8595 (0.1715)	0.8460 (-0.9036)	0.8444 (-0.9682)
In-Sample One-Week (2,809 firms – 2,698 survival and 111 default firms) Out-of-Sample 6-Month (2,694 firms – 2,588 survival and 106 default firms) Out-of-Sample 1-Year (2,556 firms – 2,453 survival and 103 default firms)					

Table 17 Accuracy Ratios and z Statistics of Physical Probabilities (Default Definition II; Non-Financial Firms)

Accuracy Ratio	**Merton**	**Brockman and Turtle**	**Black and Cox**	**Leland (TC=0.2)**	**Leland (TC=0.35)**
One Week (In Sample)	0.9117	**0.8945 (-5.8440)**	0.9088 (-1.4072)	0.9133 (0.4425)	0.9137 (0.5129)
Six Months (Out of Sample)	0.8482	**0.8128 (-8.2882)**	0.8457 (-1.0880)	0.8556 (1.5270)	0.8561 (1.5317)
One Year (Out of Sample)	0.8041	**0.7614 (-8.4366)**	0.8012 (-1.1194)	**0.8209 (3.0848)**	**0.8215 (3.0167)**
In-Sample One-Week (12,789 firms – 11,575 survival and 1,214 default firms) Out-of-Sample 6-Month (12,071 firms - 10,910 survival and 1,161 default firms) Out-of-Sample 1-Year (11,188 firms – 10,108 survival and 1,080 default firms)					

5.4 Robustness Test

In this section, we conduct a similar analysis using risk-neutral default probabilities instead of physical default probabilities. To conserve space, we report only Accuracy Ratios and z statistics. The results from Table 18 to Table 23 show that the default prediction capabilities of the four models we tested are still preserved: the Brockman and Turtle model is inferior to the Merton model in all tests. The Black and Cox model performs no statistically different from the Merton model. The Leland model is superior to the Merton model in one-year out-of-sample prediction of non-financial sector.

We next examine the default probability estimates of physical versus risk-neutral probability measures. Duan et al. (2003) claim that the transformed-data MLE approach can estimate the default probability under physical probability measure[xiii]. From our empirical results, we do observe higher ARs for all the models of in-sample test under two alternative definitions of default. In out-of-sample tests, the ARs are in general higher under physical default probabilities in the default definition II. Those of the Brockman and Turtle model are the exceptions. However, for default definition I of out-of-sample test, the ARs show an entirely opposite pattern in the common sample and non-financial sectors – the ARs of physical default probabilities are lower than those of risk-neutral probabilities. We should note that the only difference of survival and default group classification in two alternative settings is that those firms being delisted without filing Chapter 11 are assumed to be default firms by Brockman and Turtle (2003). In other words, the drift estimates using equity time series a certain period of time before delisting date cannot help improve default prediction for those firms being delisted without filing Chapter 11.

We conclude that estimating asset drift can improve default prediction, which can be seen from the in-sample testing results. Nonetheless, the out-of-sample drift estimate itself, using equity time series 6 months or one year before the delisting date, may not help improve default prediction, especially for those firms delisted without filing Chapter 11. The effect of asset drift estimation in default prediction may be confined to a relatively short forecasting horizon.

Table 18 Accuracy Ratios and z Statistics of Risk-Neutral Probabilities (Default Definition I; All Sample)

Accuracy Ratio	**Merton**	**Brockman and Turtle**	**Black and Cox**	**Leland (TC=0.2)**	**Leland (TC=0.35)**
One Week (In Sample)	0.9323	0.9295 (-1.7646)	0.9327 (0.3760)	0.9294 (-1.5612)	0.9286 (-1.8842)
Six Months (Out of Sample)	0.8802	**0.8720 (-3.5490)**	0.8812 (0.7942)	0.8775 (-1.0479)	0.8764 (-1.4379)
One Year (Out of Sample)	0.8500	**0.8349 (-5.5357)**	0.8498 (-0.1482)	**0.8561 (2.0522)**	0.8548 (1.5248)
In-Sample One-Week (15,598 firms – 10,727 survival and 4,871 performance delisting firms) Out-of-Sample 6-Month (14,765 firms - 10,232 survival and 4,533 performance delisting firms) Out-of-Sample 1-Year (13,744 firms - 9,637 survival and 4,107 performance delisting firms)					

Table 19 Accuracy Ratios and z Statistics of Risk-Neutral Probabilities (Default Definition I; Financial Firms)

Accuracy Ratio	**Merton**	**Brockman and Turtle**	**Black and Cox**	**Leland (TC=0.2)**	**Leland (TC=0.35)**
One Week (In Sample)	0.8842	0.8858 (0.2952)	0.8824 (-0.3354)	0.8717 (-1.6515)	0.8694 (-1.9267)
Six Months (Out of Sample)	0.8424	0.8550 (1.7231)	0.8422 (-0.0427)	0.8271 (-1.6603)	0.8242 (-1.9525)
One Year (Out of Sample)	0.8310	0.8328 (0.2221)	0.8188 (-1.8348)	0.8187 (-1.2027)	0.8147 (-1.5616)
In-Sample One-Week (2,809 firms – 2,409 survival and 400 performance delisting firms) Out-of-Sample 6-Month (2,694 firms – 2,313 survival and 381 performance delisting firms) Out-of-Sample 1-Year (2,556 firms – 2,195 survival and 361 performance delisting firms)					

Table 20 Accuracy Ratios and z Statistics of Risk-Neutral Probabilities (Default Definition I; Non-Financial Firms)

Accuracy Ratio	**Merton**	**Brockman and Turtle**	**Black and Cox**	**Leland (TC=0.2)**	**Leland (TC=0.35)**
One Week (In Sample)	0.9340	**0.9302 (-2.2255)**	0.9341 (0.1072)	0.9332 (-0.3717)	0.9330 (-0.4715)
Six Months (Out of Sample)	0.8787	**0.8651 (-5.4658)**	0.8777 (-0.7374)	0.8812 (0.9294)	0.8813 (0.9283)
One Year (Out of Sample)	0.8467	**0.8253 (-7.2235)**	0.8440 (-1.9933)	**0.8603 (4.3644)**	**0.8605 (4.2420)**
In-Sample One-Week (12,789 firms – 8,318 survival and 4,471 performance delisting firms) Out-of-Sample 6-Month (12,071 firms – 7,919 survival and 4,152 performance delisting firms) Out-of-Sample 1-Year (11,188 firms – 7,442 survival and 3,746 performance delisting firms)					

Table 21 Accuracy Ratios and z Statistics of Risk-Neutral Probabilities (Default Definition II; All Sample)

Accuracy Ratio	**Merton**	**Brockman and Turtle**	**Black and Cox**	**Leland (TC=0.2)**	**Leland (TC=0.35)**
One Week (In Sample)	0.9017	**0.8980 (-1.3487)**	0.9030 (0.5618)	0.9009 (-0.2301)	0.9012 (-0.1218)
Six Months (Out of Sample)	0.8483	**0.8308 (-4.6383)**	0.8464 (-0.7817)	0.8513 (0.6300)	0.8512 (0.5894)
One Year (Out of Sample)	0.8140	**0.7885 (-5.7531)**	0.8094 (-1.6651)	0.8237 (1.8650)	0.8231 (1.6810)
In-Sample One-Week (15,598 firms – 14,273 survival and 1,325 default firms) Out-of-Sample 6-Month (14,765 firms – 13,498 survival and 1,267 default firms) Out-of-Sample 1-Year (13,744 firms – 12,561 survival and 1,183 default firms)					

Table 22 Accuracy Ratios and z Statistics of Risk-Neutral Probabilities (Default Definition II; Financial Firms)

Accuracy Ratio	**Merton**	**Brockman and Turtle**	**Black and Cox**	**Leland (TC=0.2)**	**Leland (TC=0.35)**
One Week (In Sample)	0.8894	0.9026 (1.6017)	0.8921 (0.2569)	0.8811 (-0.6130)	0.8794 (-0.7295)
Six Months (Out of Sample)	0.8554	0.8738 (1.7844)	0.8405 (-1.2319)	0.8485 (-0.4297)	0.8469 (-0.5182)
One Year (Out of Sample)	0.8433	0.8548 (0.9589)	0.8109 (-2.5509)	0.8342 (-0.5323)	0.8307 (-0.7190)
In-Sample One-Week (2,809 firms – 2,698 survival and 111 default firms) Out-of-Sample 6-Month (2,694 firms – 2,588 survival and 106 default firms) Out-of-Sample 1-Year (2,556 firms – 2,453 survival and 103 default firms)					

Table 23 Accuracy Ratios and z Statistics of Risk-Neutral Probabilities (Default Definition II; Non-Financial Firms)

Accuracy Ratio	**Merton**	**Brockman and Turtle**	**Black and Cox**	**Leland (TC=0.2)**	**Leland (TC=0.35)**
One Week (In Sample)	0.8963	0.8909 (-1.7799)	0.8975 (0.5070)	0.8963 (-0.0183)	0.8970 (0.1638)
Six Months (Out of Sample)	0.8377	**0.8141 (-5.7225)**	0.8357 (-0.8163)	0.8441 (1.3148)	0.8447 (1.3744)
One Year (Out of Sample)	0.8013	**0.7683 (-6.8383)**	0.7974 (-1.4218)	**0.8165 (2.7452)**	**0.8171 (2.7272)**
In-Sample One-Week (12,789 firms – 11,575 survival and 1,214 default firms) Out-of-Sample 6-Month (12,071 firms - 10,910 survival and 1,161 default firms) Out-of-Sample 1-Year (11,188 firms – 10,108 survival and 1,080 default firms)					

5. Summary and Conclusions

In our empirical investigation, we adopt the Maximum Likelihood Estimation method proposed by Duan (1994) and Duan et al. (2004), which views the observed equity time series as a transformed data set of unobserved firm values with the theoretical equity pricing formula serving as the transformation. This method has been shown by Ericsson and Reneby (2005) through simulation experiments to be superior to the commonly adopted volatility restriction approach in the literature. Since the default boundary is unknown, the barrier model shall have three unknown parameters – asset drift, asset volatility, and a level of default boundary. One of the advantages of the MLE approach is that it can estimate these three model parameters simultaneously.

In our simulation experiment, we uncover the limitation of the MLE method. The MLE method cannot pin down the barrier using the equity time series when the default boundary, relative to the firm value, is low (or the low hitting probability of the default boundary). This is what the statistical theory precisely predicts since the value of likelihood function is flat and not sensitive to the change of the boundary level. However, for default prediction, this should present no practical difficulties. The bias of low barrier cases could hardly affect the default probabilities of sample firms, even when the barrier estimates vary for a wide range.

Our empirical results surprisingly show that the simple Merton model has a similar capability in default prediction as that of the Black and Cox model. The Merton model even outperforms the Brockman and Turtle model, and the difference of predictive ability is statistically significant. The results are held for the in-sample, six-month and one-year out-of-sample tests for both the broad definition of bankruptcy as in Brockman and Turtle (2003) as well as the similar definition to Chen, Hu, and Pan (2006). In addition, we also find that the inferior performance of the Brockman and Turtle model may be the result of its unreasonable assumption that the flat barrier itself can be over the face value of debt. In the one-year out-of-sample test, the Leland model outperforms the Merton model in non-financial sector and the results hold for two alternative definitions of default. These results are still preserved in our robustness test as we use risk-neutral default probabilities instead of physical default probabilities.

In addition, in terms of the differences of default probabilities between barrier models and Merton's model, our results indicate that the introduction of default barriers has little influence on default probabilities for a large portion of the survival firms and as many as 30% of the firms in default group. This is consistent with the results by Wong and Choi (2009) and does not support the finding by Brockman and Turtle (2003) that default barriers are significantly positive. We should note that the models investigated in our study incorporate only net-worth covenant, and firms default only when the market value of its assets fall below a certain boundary. A recent empirical study by Davydenko (2007) finds a much more complex picture of financial distress. Default of distressed firms may be triggered by either low asset value or liquidity shortage [xiv], and the importance of liquidity varies cross-sectionally depending on costs of external financing. Moreover, there are many low-value and low-liquidity firms that are able to avoid default.

In summary, our empirical results indicate that exogenous default barriers, flat or exponential, are not crucial in default prediction. In contrast, endogenous barrier modeling has significant improvement in long term prediction for non-financial firms. However, we should note that the performance of the Leland model compared to the Merton model is weakened as the default prediction horizon shortened.

Appendix. The Models

In Appendix, we summarize the model to be tested in our empirical study. Next, we present the default probability of risky debt of these models.

A.1 The Merton Model

In the Merton (1974) model, the firm's assets are assumed to be financed by equity and a zero-coupon bond with a face value of K and maturity T. Following the accounting identity, the firm's asset value equals the sum of equity and debt, i.e., $V_t = S_t + D_t$, and it holds for every time point. Since equity can be treated as the call option with strike price K, the value of equity at time $t \le T$ can be expressed as call option value under Black-Scholes framework:

$$S_t(V_t, \sigma_V, T-t) = V_t N(d_1) - e^{-r(T-t)} K N(d_2) \qquad \textbf{(A.1)}$$

where $N(\cdot)$ is a standard normal distribution function, and

$$d_1 = \frac{\ln\left(\frac{V_t}{K}\right) + \left(r + \frac{\sigma_V^2}{2}\right)(T-t)}{\sqrt{\sigma_V^2 (T-t)}}$$

$$d_2 = d_1 - \sqrt{\sigma_V^2 (T-t)}$$

Therefore, the value the risky debt at time t is $D_t = V_t - S_t$.

__**Default Probability of Risky Debt**__

The default probability of the risky debt at time T is the probability that the firm value at time T is lower than the face value of bond K, i.e., $P_{def} = P(V_T \le K)$. Note that an implicit assumption of Merton's model is that the firm can only default at time T. Since the firm's asset value process can be rewritten as $d \ln V_t = \left(\mu_V - \frac{\sigma_V^2}{2}\right) dt + \sigma_V dW_t$, the transition density of logarithm asset value is normally distributed as $\ln V_T \sim N\left[\ln V_t + \left(\mu_V - \frac{1}{2}\sigma_V^2\right)(T-t), \sigma_V^2 (T-t)\right]$.

Therefore,

$$P_{def} = P(V_T \le K) = P\left(Z \le \frac{\ln K - \left[\ln V_t + \left(\mu_V - \frac{1}{2}\sigma_V^2\right)\right]}{\sqrt{\sigma_V^2 (T-t)}} \right)$$

$$= P\left(Z \le -\frac{\ln V_t - \ln K - \left(\mu_V - \frac{1}{2}\sigma_V^2\right)}{\sqrt{\sigma_V^2 (T-t)}} \right) = N\left(-d_2\right) = 1 - N\left(d_2{}'\right) \qquad \textbf{(A.2)}$$

where $d_2{}' = \dfrac{\ln\left(\frac{V_t}{K}\right) + \left(\mu_V - \frac{\sigma_V^2}{2}\right)(T-t)}{\sqrt{\sigma_V^2 (T-t)}}$

A.2 The Flat Barrier Model: The Brockman and Turtle Model

Brockman and Turtle (2003) adopt the barrier option formula as a tool to understand the Down-and-Out call (DOC) approach to the corporate security valuation. In the context of structural model, the market value of firm's equity, S, can be express as

$$S = VN(a) - Ke^{-r(T-t)}N(a - \sigma_V \sqrt{T-t})$$

$$- V(H/V)^{2\eta} N(b) + Ke^{-r(T-t)}(H/V)^{2\eta-2} N(b - \sigma_V \sqrt{T-t})$$

$$+ R(H/V)^{2\eta-1} N(c) + R(V/H)N(c - 2\eta\sigma_V \sqrt{T-t}) \qquad \textbf{(A.3)}$$

where V is the market value of the firm's assets; K is the promised future debt payment required on the pure discount bonds issued by the corporation and due at time T; H is the value of the firm's assets that triggers bankruptcy (default barrier); R is the rebate paid to the firm's owners if the firm's asset value reaches the barrier; $T-t$ is the time until the option expires; r is the continuously compounded riskless rate of return; and $N(\cdot)$ is the standard normal cumulative distribution function.

$$a = \begin{cases} \dfrac{\ln(V/K) + \left(r + (\sigma_V^2/2)\right)(T-t)}{\sigma_V \sqrt{T-t}} & \mathit{for}\ K \geq H \\ \dfrac{\ln(V/H) + \left(r + (\sigma_V^2/2)\right)(T-t)}{\sigma_V \sqrt{T-t}} & \mathit{for}\ K < H \end{cases}$$

$$b = \begin{cases} \dfrac{\ln(H^2/VK) + \left(r + (\sigma_V^2/2)\right)(T-t)}{\sigma_V \sqrt{T-t}} & \mathit{for}\ K \geq H \\ \dfrac{\ln(H/V) + \left(r + (\sigma_V^2/2)\right)(T-t)}{\sigma_V \sqrt{T-t}} & \mathit{for}\ K < H \end{cases}$$

$$c = \frac{\ln(H/V) + \left(r + (\sigma_V^2/2)\right)(T-t)}{\sigma_V \sqrt{T-t}} \text{ and } \eta \equiv \frac{r}{\sigma_V^2} + \frac{1}{2}$$

<u>**Default Probability of Risky Debt**</u>

$$N\left(\frac{(h-v) - (\mu_V - \sigma_V^2/2)(T-t)}{\sqrt{\sigma_V^2 (T-t)}}\right)$$

$$+ \exp\left(\frac{2(\mu_V - \sigma_V^2/2)(h-v)}{\sigma_V^2}\right)\left[1 - N\left(\frac{-(h-v) - (\mu_V - \sigma_V^2/2)(T-t)}{\sqrt{\sigma_V^2 (T-t)}}\right)\right] \qquad \textbf{(A.4)}$$

where $h = \ln H$ and $v = \ln V$ [xv]

A.3 The Exponential Barrier Model: The Black and Cox Model

Black and Cox (1976) were the first to propose a barrier option model for default. Instead of only the maturity date of debt, they assume a continuous barrier function over time. Black and Cox (1976) assume that the contractual provisions allow the stockholders to receive a continuous dividend payment, gV, proportional to the value of the firm. In addition, they let the time dependence of specified bankruptcy level, C, of the safety covenant to take the exponential form $Ce^{-\gamma(T-t)}$. The closed-form solution of corporate risky bond value with safety covenants as

$$D(V,t) = Fe^{-r(T-t)}[N(z_1) - y^{2\theta-2}N(z_2)] + Ve^{-g(T-t)}[N(z_3) + y^{2\theta}N(z_4)$$
$$+ y^{\theta+\zeta}e^{g(T-t)}N(z_5) + y^{\theta-\zeta}e^{g(T-t)}N(z_6) - y^{\theta+\eta}N(z_7) - y^{\theta-\eta}N(z_8)]^{\text{xvi}} \qquad \textbf{(A.5)}$$

where $y = Ce^{-\gamma(T-t)}/V$, $\theta = (r - g - \gamma + \frac{1}{2}\sigma_V^2)/\sigma_V^2$

$\omega = (r - g - \gamma - \frac{1}{2}\sigma_V^2)^2 + 2\sigma_V^2(r-\gamma)$,

$\zeta = \sqrt{\omega}/\sigma^2$, $\eta = \sqrt{\omega - 2\sigma_V^2 g}/\sigma_V^2$,

$z_1 = [\ln V - \ln F + (r - g - \frac{1}{2}\sigma_V^2)(T-t)]/\sqrt{\sigma_V^2(T-t)}$,

$z_2 = [\ln V - \ln F + 2\ln y + (r - g - \frac{1}{2}\sigma_V^2)(T-t)]/\sqrt{\sigma_V^2(T-t)}$,

$z_3 = [\ln F - \ln V - (r - g + \frac{1}{2}\sigma_V^2)(T-t)]/\sqrt{\sigma_V^2(T-t)}$,

$z_4 = [\ln V - \ln F + 2\ln y + (r - g + \frac{1}{2}\sigma_V^2)(T-t)]/\sqrt{\sigma_V^2(T-t)}$,

$z_5 = [\ln y + \zeta\sigma_V^2(T-t)]/\sqrt{\sigma_V^2(T-t)}$,

$z_6 = [\ln y - \zeta\sigma_V^2(T-t)]/\sqrt{\sigma_V^2(T-t)}$,

$z_7 = [\ln y + \eta\sigma_V^2(T-t)]/\sqrt{\sigma_V^2(T-t)}$,

$z_8 = [\ln y - \eta\sigma_V^2(T-t)]/\sqrt{\sigma_V^2(T-t)}$.

This formula holds for all $Ce^{-\gamma(T-t)} \le Fe^{-r(T-t)}$.

An interesting choice by Black and Cox (1976) is $Ce^{-\gamma(T-t)} = \alpha Fe^{-r(T-t)}$, with $0 \le \alpha \le 1$, so that the reorganization value specified in the safety covenant is a constant fraction of the present value of the promised final payment.

Default Probability of Risky Debt

The probability that $V(\tau) \geq K$ and has not reached the boundary is given as

$$N\left(\frac{\ln V - \ln K + (r - g - \sigma_V^2/2)(\tau - t)}{\sqrt{\sigma_V^2(\tau - t)}}\right)$$

$$-\left(\frac{V}{Ce^{-\gamma(T-t)}}\right)^{1-(2(r-g-\gamma)/\sigma_V^2)} N\left(\frac{2\ln Ce^{-\gamma(T-t)} - \ln V - \ln K + (r - g - \sigma_V^2/2)(\tau - t)}{\sqrt{\sigma_V^2(\tau - t)}}\right) \qquad \textbf{(A.6)}$$

where $N(\cdot)$ is the standard normal distribution function. Setting $K = Ce^{-\gamma(T-t)}$ gives the probability in a risk neutral world that has not been reorganized or defaulted. Thus, the default probability is just one minus the survival probability in (A.6).

A.4 The Leland Model

In the Leland (1994) model, financial distress is triggered when shareholders no longer find that running a company is profitable, given the revenue produced by the assets, to continue servicing debt. Bankruptcy is determined endogenously rather than by the imposition of a positive net worth condition or by a cash flow constraint.

Denote any claim $F(V,t)$ on the firm that continuously pays a nonnegative coupon, C, per instant of time when the firm is solvent. Leland (1994) provides the solution of the perpetual debt. Let V_B denote the constant level of asset value at which bankruptcy is declared. If bankruptcy occurs, a fraction $0 \leq \alpha \leq 1$ of asset value will be lost to bankruptcy cost. The closed-form solution of risky debt is

$$D(V) = \frac{C}{r} + \left[(1-\alpha)V_B - \frac{C}{r}\right]\left(\frac{V}{V_B}\right)^{-X} \text{ where } X = 2r/\sigma^2 \qquad \textbf{(A.7)}$$

Next, Leland (1994) derives the total value of the firm, $v(V)$, which reflects three terms: the firm's asset value, plus the value of the tax deduction of coupon payments, less the value of bankruptcy costs.

$$v(V) = V + TB(V) - BC(V)$$

$$= V + (T_C C/r)[1 - (V/V_B)^{-X}] - \alpha V_B (V/V_B)^{-X} \qquad \textbf{(A.8)}$$

The value of equity is the total value of the firm less the value of debt as follows:

$$E(V) = v(V) - D(V) = V - (1-T_C)(C/r) + [(1-T_C)(C/r) - V_B](V/V_B)^{-X}. \qquad \textbf{(A.9)}$$

By maximize the value of equity at any level of V, the equilibrium bankruptcy-triggering asset value V_B is determined endogenously by the smooth-pasting condition $\left.\frac{\partial E(V;V_B,T)}{\partial V}\right|_{V=V_B} = 0$.

$$V_B = [(1 - T_C)C / r][X / (1 + X)] = (1 - T_C)C / (r + 0.5\sigma^2) \quad \textbf{(A.10)}$$

Note that V_B in (A.10) is independent of time and it confirms the assumption of the constant bankruptcy-triggering asset level V_B.

<u>Default Probability of Risky Debt</u>

The cumulative probability of the firm going bankrupt over the period $(t, T]$ is

$$N\left(\frac{-b - \lambda(T - t)}{\sigma_V \sqrt{T - t}}\right) + e^{-\frac{2\lambda b}{\sigma_V^2}} N\left(\frac{-b + \lambda(T - t)}{\sigma_V \sqrt{T - t}}\right) \quad \textbf{(A.11)}$$

where $b = \ln\left(\frac{V}{V_B}\right)$ and $\lambda = \mu_V - \frac{\sigma_V^2}{2}$ [xvii]

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[i] See Section 2 for the summary of these empirical studies.

[ii] For the comprehensive analysis of these models, see Crouhy, Galai, and Mark (2000) and Saunders and Allen (2002).

[iii] See Lando (2004) for the review of reduced-form models.

[iv] There exists an extensive literature on default or bankruptcy prediction, readers who are interested in this subject can refer to the papers by Shumway (2001) and Duffie, Saita, and Wang (2007). In this paper, we have no intension to incorporate those previously identified variables, such as firm's trailing stock return, trailing S&P 500 returns, and U.S. interest rates, into our analysis but rather focus on various default boundary assumptions in the structural models.

[v] Campbell, J. Y., J. Hilscher, and J. Szilagyi (2004) also show similar results that failure risk cannot be adequately summarized by a measure of distance to default by the KMV-Merton model.

[vi] In the context of structural credit risk modeling, the equity value is an option of asset value V_t. Therefore, for example, under Merton's model, $S_t = V_t N(d_t) - Ke^{-rT} N(d_t - \sigma_V \sqrt{T})$. Also by Itô's Lemma,

$\sigma_S = \frac{\partial S_t}{\partial V_t} \frac{V_t}{S_t} \sigma_V$. Since $S_t = g(V_t; \sigma_V)$ is a one to one function of V_t, the inverse exists. One can then first estimate the equity volatility σ_S using historical data, and the two unknown variables asset value V_t and asset volatility σ_V left can be solved by the above two-equation system.

[vii] Duan and Simonato (2002) further develop a MLE method for the two unobserved variables, namely, the firm value V_t and the instantaneous interest rate r_t. In this case, θ also contains parameters of interest process and its correlation with firm value process. Thus, one needs to modify the log-likelihood function in Step 2 to incorporate this change.

[viii] See Bruche (2005) for the issues of some other estimation methods not presented here.

[ix] The KMV method is a simple two-step iterative algorithm which begins with an arbitrary value of the asset volatility and repeats the two steps until the convergent criterion is reached. The default barrier of the KMV method is assumed as the sum of short-term liabilities plus one-half long-term liabilities. See Crosbie and Bohn (2003) and Vasslou and Xing (2004) for details.

[x] A similar approach is adopted by Chen, Hu, and Pan (2006) using distant to default (DD) instead of default probability. However, this relationship cannot be applied in the barrier option framework since the default probability is not merely a transformation of distant of default. Therefore, we use the default probability directly in our study. The same argument is also addressed by Leland (2004).

[xi] There are several reasons choosing this "default point": First, KMV has observed from a large sample of several hundreds companies that firms default when the asset value reaches a level somewhere between the value of total liabilities and the value of short-term debt. Therefore, as argued in Crouhy, et al. (2000), the probability of the asset value falling below the total face value may not be an accurate measure of the actual default probability. Secondly, as pointed out by Vassalou and Xing (2004), it is important to include long-term debt in the calculation because firms need to service the long-term debt, and these interest payments are part of the short-term liabilities. Furthermore, the size of the long-term debt may affect the ability of a firm to roll over its short-term debt, and in turn affect the default risk

[xii] We lost some samples due to convergent issue in the MLE maximization process of the Brockman and Turtle, the Black and Cox, and the Leland models. We lost 9, 10, and 6 firms of the in-sample, six-month out-of-sample, and one-year out-of-sample tests, respectively.

[xiii] The physical default probability here is under the assumption of constant asset risk premium.

[xiv] See Kim, Ramaswamy, and Sundaresan (1993) and Anderson and Sundaresan (1996) for the models assuming cash-flow or liquidity covenant.

[xv] Note that for the Brockman and Turtle and the Black and Cox models, a firm can default either before maturity or at maturity. Therefore, we also need to use incorporate the default probability at maturity for each firm.

[xvi] See Lin (2006) for the correction of the typographical error in this formula. The corrected formula is presented in (A.5).

[xvii] To incorporate payout, one needs to modify x as $\frac{(r-g-\sigma^2/2)+\sqrt{(r-g-\sigma^2/2)^2+2\sigma^2 r}}{\sigma^2}$ in (A.7) and λ as $\mu_V - g - \frac{\sigma_V^2}{2}$ in (A.11).