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Capability Adjustment for Normal Processes with Variance Shift Consideration

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Abstract: Process capability indices (PCIs) have been proposed in the manufacturing industry to provide numerical measures on process capability, which are effective tools for quality assurance and guidance for process improvement. PCIs are calculated under the assumption that the process is stable (the process mean and variation will not change). But, in practice, the process is dynamic. If the process parameters have a small shift, the control chart not be able to detect immediately. PCIs, in this case, will overestimate the true process capability. For this reason, the PCIs have to be adjusted. Bothe (2002) provided the adjustment method for normal processes with mean shift. In practice, the variance could change as well. In this project, we provide capability adjustment for normal processes with standard deviation change. The magnitude of adjustment is correlated to the detection power of the control chart used. We first investigate the detection powers of S^2 and S control charts under various sample subgroup sizes, and derive the magnitude of the adjustment. We add the adjustment to the formula of process capability index C_{pk} for normal processes. For illustration purpose, an application example is presented.

Keywords: Dynamic C_{pk}, Variance change, Process capability index, Normal distribution, Chi-square distribution, S^2 control chart, S control chart

1. Introduction

Process capability indices are important for any successful quality improvement activities and quality program implementation. They have been the focus of recent research in quality assurance and process capability analysis. Process capability indices establish the relationship between the actual performance and the manufacturing specifications, which provide management with a single-number summary of the process capability in a format that is easy to use and understand. Thus, the capability indices have been widely used in the manufacturing industry. The two basic, most commonly used indices for assessing process potential and performance, C_p and C_{pk} were discussed in Kane (1986). The more advanced index C_{pm} was formalized by Chan *et al.* (1988) to offset the weakness of the first-generation indices, taking the target value of the process into account. The third-generation index C_{pmk} was introduced by Pearn *et al.* (1992), which is more restrictive regarding to process mean deviation from the target value than the other two indices. Those PCIs defined as:

$$C_p = \frac{USL - LSL}{6\sigma},\ C_{pk} = \min\left\{\frac{USL-\mu}{3\sigma}, \frac{\mu - LSL}{3\sigma}\right\},\ C_{pm} = \frac{USL - LSL}{6\sqrt{\sigma^2 + (\mu - T)^2}}$$

$$C_{pmk} = \min\left\{\frac{USL-\mu}{3\sqrt{\sigma^2 + (\mu - T)^2}}, \frac{\mu - LSL}{3\sqrt{\sigma^2 + (\mu - T)^2}}\right\},$$

where USL and LSL are the upper and lower specification limits, respectively. μ is the process mean, σ is the process standard deviation, and T is the target value. The index C_p simply measures the spread of the specifications relative to the six-sigma spread in the process. The magnitude of C_{pk} relative to C_p is a direct measure of how off-center the process is operating. For a C_{pk} level of 1, statistically one would expect that the product's fraction of defectives is no more than 2700 parts per million (PPM) fall outside the specification limits. At C_{pk} =1.33, the defect rate drops to 66 PPM. To attain less than 0.544 PPM defect rate, a C_{pk} level of 1.67 is required. At a C_{pk} level of 2.0, the defective rate reduced to 0.002 PPM. Note that the PCIs are absolutely critical to the assumptions that the process is stable and their usual interpretation is based on normal distribution of process output. Unfortunately, it is fairly common practice to compute a PCI from a sample or historical process data without checking whether the process is under those assumptions. If they are not valid, then the statements about the expected process fallout attributed to a particular value of PCIs may be seriously in error. In this situation, we need to modify the PCIs to have a more accurate calculation and interpretation.

However, no process is ever truly stable. The concept of the six-sigma process is one way to accommodate the process behavior. The six-sigma quality improvement process was proposed by Motorola Inc. in 1986. Through years of process experience and data collection, Motorola Inc. has determined that processes will drift over time. Almost since that time, followers of this philosophy asserted that adding a shift to the average before estimating process capability is necessary. The range of shift typically falls between 1.4σ and 1.6σ. Acknowledging that processes will experience shifts in μ of various magnitudes. To have proper and to improve the process performance, adjusted PCIs are proposed. Bothe (2002) has provided a statistical reason to this issue, modifying the capability assessment. Since the processes are dynamic. The σ also undergoes some changes. In this project, we provide capability adjustment for normal processes with standard deviation change. The magnitude of adjustment is correlated to the detection power of the control chart used. We first investigate the detection powers of S^2 and S control charts under various sample subgroup sizes, and derive the magnitude of the adjustment. We add the adjustment to the formula of process capability index C_{pk} for normal processes. For illustration purpose, an application example is presented.

2. Literature Review

The six-sigma advocates claim it is necessary to add a 1.5σ shift to the average for most processes, with only personal experiences and three dated empirical studies as justification (see Bender (1975), Evans (1975), Gilson (1951)). In this chapter, we provide Bothe's statistical rationale regarding this issue. The data in Bothe's study was assumed to be close to normal distributions. For the process output having non-normal distributions, we also conduct some studies here.

2.1. Process Capability Adjustment for Normal Processes with Mean Shift

Shewhart control charts are very useful in phase I implementation of SPC, where the process is likely to be out of control and experiencing assignable causes that result in large shifts in the monitored parameters. Nevertheless, a major disadvantage of a Shewhart control chart is that it uses only the information about the process contained in the last sample observation and it ignores any information given by the entire sequence of points. This feature makes the Shewhart control chart relatively insensitive to small process shift. This potentially makes Shewhart control charts less useful in phase II monitoring problems, where the process tends to operate in control, reliable estimates of the process parameters (such as the mean and standard deviation) are available, and assignable causes do not typically result in large process upsets or disturbances. This is demonstrated in Table 1, displays the probabilities of detecting changes in μ versus subgroup size for $\text{shift} = 0.5(0.1)3\sigma$ with $n = 3,\ 4 \text{ and } 5$. The probabilities of detecting small shifts in μ are close to zero. As the size of the shift increases, so does the detection power of the $\bar{X}$ control chart to detect it, with sample subgroup sizes $n = 3,\ 4 \text{ and } 5$ eventually close to 100 percent for shifts in excess of 3σ .

Table 1. Probabilities of detection the mean shift versus subgroup size *n*.

Mean Shift Size	Subgroup Size		
	3	4	5
0.5σ	0.0164	0.0228	0.0299
1σ	0.1024	0.1587	0.2225
1.5σ	0.3439	0.5000	0.6384
2σ	0.6787	0.8413	0.9295
2.5σ	0.9083	0.9772	0.9952
3σ	0.9860	0.9986	0.9999

In studying the properties of control charts, the emphasis has been on determining the detection power and ARL (Average Run Length) of the chart. The ARL of a chart is the expected number of samples to be taken before the chart detects a shift in the process characteristic. The ARL should be large when there has been no change in the process, but the ARL should be small when the process having undergone a change. The value of the ARL is depending on the purpose being studied. For any Shewshart control chart, we have noted that the ARL can also be expressed as

$$ARL_0 = \frac{1}{\alpha},$$

for the in-control false alarm ARL_0 and

$$ARL_1 = \frac{1}{1-\beta},$$

for the out-of-control ARL_1, where α is the probability of detecting a shift when none has occurred, and β is the probability of failing to detect a real shift in process characteristic. In general, we set $ARL_1 = 2$ in most applications. Therefore, the detection power is

$$\text{Detection power} = 1-\beta = \frac{1}{ARL_1} = \frac{1}{2} = 0.5,$$

that is, the probability of the control chart to detect the small shift immediately within two samples is 50 percent. By this idea, Bothe set the detection power to be 50 percent and computed the several magnitude of adjustments for various sample subgroup sizes. The results are shown in Table 2, which displays shift sizes that have a 50 percent chance of failing to detect the change in μ, which we refer to as , for various sample subgroup sizes from 1 to 6.

Table 2. S_{50} values for normal distribution with various subgroup sizes.

Subgroup Size	S_{50}
1	3.00
2	2.12
3	1.73
4	1.50
5	1.34
6	1.22

Because shifts ranging in size from 0 up to S_{50} are likely to remain undetected, a conservative approach is to assume that every missed shift is as large as $S_{50}\sigma$. And Bothe made the modifications into the C_{pk} formula, called the *dynamic* C_{pk}, defined as follow:

$$Dynamic\ \ C_{pk} = \min\left\{\frac{USL-(\mu+S_{50}\sigma)}{3\sigma}, \frac{(\mu-S_{50}\sigma)-LSL}{3\sigma}\right\}.$$

2.2. Process Capability Adjustment for Non-normal Processes with Mean Shift

However, for the majority of cases, normal data seem impossible to be found in real-world situations. Pyzdek (1992) has mentioned the distributions of certain chemical processes such as zinc plating thickness of a hot-dip galvanizing process are very quite often skewed. Choi (1996) presents an example of a skewed distribution in the "active area" shaping stage of the wafer's production processes. The abundance of outputs from skewed distributions, the censoring effects induced by the finite precision of actual measurements, stratification, etc., makes the normal assumption often unreasonable. Thus, there should be more concern about how the indexes are applied.

In the recent years, several approaches to dealing with problems of PCIs for the non-normal populations have been suggested (see e.g. pal (2005), Ding (2004), Pearn and Chen (1997), Kotz and Lovelace (1998), Somerville and Montgomery (1996), Kocherlakota *et al.*(1992)). One approach to dealing with this situation is to transform the data so that in the new, transformed metric the data have normal distribution appearance. There are various graphical and analytical approaches to selecting a transformation, such as Box-Cox power transformation and Johnson's transformations. And some authors replaced the unknown distribution by a known three or four-parameter distribution. Examples include Clements (1989), Franklin and Wasserman (1992), Shore (1998) and Polansky (1998).

There have also been attempts to modify the usual capability indices so that they are appropriate for both normal and non-normal distributions. The general idea is to use appropriate quantiles of the process distribution, $x_{0.00135}$ and $x_{0.99865}$, to define a quantile-based PCIs. Good discussions of these approaches are in Kotz and Lovelace (1998).

Hsu *et al.* (2007) examine Bothe's study and find the detection power was less than 0.5 when data came from Gamma distributions, showing that Bothe's statistical rationales are inadequate when we had Gamma processes. Then, Hsu *et al.* (2007) calculate the magnitude of adjustments which called AS_{50} under various sample subgroup sizes *n* and Gamma parameter N, with power fixed to 0.5. Table 3 displays the magnitude of adjustments AS_{50} which Hsu *et al.* provided and data come from $\text{Gamma}(N,1)$ with various values of N =1(1)10 and n =2(1)6.

Table 3. AS_{50} values for various subgroup sizes n and various of $\text{Gamma}(N,1)$.

n	1	2	3	4	5	6	7	8	9	10	N(0,1)
2	3.611	3.185	2.992	2.876	2.797	2.738	2.692	2.655	2.625	2.599	2.12
3	2.732	2.443	2.313	2.236	2.182	2.143	2.113	2.088	2.067	2.050	1.73
4	2.252	2.034	1.936	1.878	1.838	1.808	1.785	1.767	1.752	1.738	1.50
5	1.944	1.769	1.690	1.644	1.612	1.588	1.570	1.555	1.543	1.532	1.34
6	1.727	1.581	1.515	1.476	1.450	1.430	1.415	1.403	1.392	1.384	1.22
7	1.565	1.439	1.383	1.350	1.327	1.310	1.297	1.286	1.278	1.270	1.13
8	1.438	1.328	1.279	1.249	1.229	1.215	1.203	1.194	1.186	1.180	1.06
9	1.336	1.237	1.194	1.168	1.150	1.137	1.127	1.118	1.112	1.106	1.00
10	1.251	1.162	1.123	1.100	1.084	1.072	1.063	1.055	1.049	1.044	0.95

Then, Hsu *et al.* (2007) used the quantile estimation to modify C_{pk} as:

$$\begin{aligned} C_{pk} &= \min\{C_{pu}, C_{pl}\} \\ &= \min\left\{\frac{USL-\text{median}}{x_{0.99865}-\text{median}}, \frac{\text{median}-LSL}{\text{median}-x_{0.00135}}\right\}. \end{aligned}$$

To consider the undetected process mean shift, Hsu *et al.* (2007) obtained *Dynamic* C_{pk} index for non-normal processes by modifying Bothe's *Dynamic* C_{pk} :

$$Dynamic\ \ C_{pk} = \min\left\{\frac{USL-(F_{0.5}+AS_{50}\sigma\ )}{F_{0.99865}-F_{0.5}}, \frac{(F_{0.5}-AS_{50}\sigma\ )-LSL}{F_{0.5}-F_{0.00135}}\right\}.$$

3. Normal Process

Normal process is applicable in many fields. Many phenomena generate random variables with probability distributions that are very well approximated by normal distribution. In this chapter, we introduce normal distribution and the sampling distributions of the statistics.

3.1. Normal Distribution

Undoubtedly, the most widely used model for the distribution of a random variable is normal distribution. There are three reasons why normal distribution plays a very important role in both the theory and application of statistics. First, normal distributions are good descriptions for some distributions of real data. Distributions that are often close to normal include scores on tests taken by many people, repeated careful measurements of the same quantity, and characteristics of biological populations. Second, normal distributions are good approximations to the results of many kinds of chance outcomes, such as the proportion of heads in many tosses of a coin. Third, we will see that many statistical inference procedures based on normal distributions work well for other roughly symmetric distributions. Normal distribution is also referred to as the Gaussian distribution.

Random variables with different means and variances can be modeled by normal probability density functions with appropriate choices of the center and width of the curve. The exact density curve for a particular normal distribution is described by giving its mean μ and its standard deviation σ. The value of $E(X)=\mu$ determines the center of the probability density function. Changing μ without changing σ moves the normal curve along the horizontal axis without changing its spread. The value of $V(X)=\sigma^2$ determines the width, controlling the spread of the normal curve.

The probability density function of the normal random variable X with mean μ and standard deviation σ is given by

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty, \quad -\infty < \mu < \infty, \quad \sigma > 0. \tag{1}$$

This distribution is commonly denoted by $N(\mu,\ \sigma^2)$. The cumulative distribution function is

$$F(a)=P\{x \le a\}=\int_{-\infty}^{a}\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}dx. \tag{2}$$

Since the mean μ is the location parameter, and the standard deviation σ is the scale parameter. See Figures 1 and 2. The visual appearance of normal distribution is a symmetric, single-peaked, and bell-shaped curve.

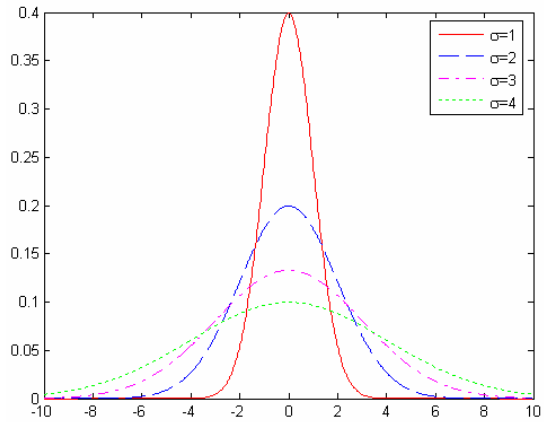

Figure 1. Normal p.d.f.s with $\mu=0$.

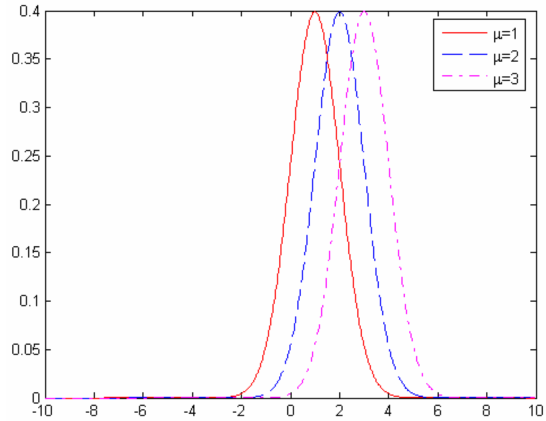

Figure 2. Normal p.d.f.s with $\sigma = 1$.

3.2. Statistics and Sampling Distributions

An important sampling distribution defined in terms of normal distribution is the Chi-square or χ^2 distribution. Let $X_1, X_2, \ldots\ldots, X_n$ are normally and independently distributed random variables with $N(0,1)$. Then the random variable

$$Y = \sum_{i=1}^{n} X_i^2 \tag{3}$$

is called the Chi-square distribution with degrees of freedom (df) n, and its probability density function is given by

$$f(y) = \frac{1}{2^{n/2}\Gamma(n/2)} e^{-y/2} y^{(n/2)-1}, \ y > 0, \ n > 0. \tag{4}$$

The mean and variance are given, respective, by

$$\mu = n, \tag{5}$$

and

$$\sigma^2 = 2n, \tag{6}$$

The Chi-square random variable with $\mathrm{df} = n$ is denoted by χ_n^2. The Chi-square distribution is a continuous asymmetrical theoretical probability distribution. The Chi-square value must fall within the range $0 \le \chi^2 \le \infty$, and thus can never be a negative number. The coefficient of skewness and kurtosis of Chi-square are given by

$$\gamma_1 = 2\sqrt{\frac{2}{n}}, \tag{7}$$

and

$$\gamma_2 = 3 + \frac{12}{n}, \tag{8}$$

Table 4 presents the values of skewness and kurtosis of the Chi-square distribution. It is reveal that χ_a^2 is stochastically larger than χ_b^2 for $a > b$.

Table 4. Values of skewness and kurtosis for various Chi-square distributions.

Distribution	Skewness	Kurtosis
N(0,1)	0.0000	3.0000
χ_5^2	1.2649	5.4000
χ_{10}^2	0.8944	4.2000
χ_{20}^2	0.6325	3.6000
χ_{30}^2	0.5164	3.4000
χ_{40}^2	0.4472	3.3000
χ_{50}^2	0.4000	3.2400

Plots in Figure 3 indicate that, the Chi-square distribution is a right-tail distribution and it can be found that for large degrees of freedom n, the Chi-square distribution is symmetric about its mean.

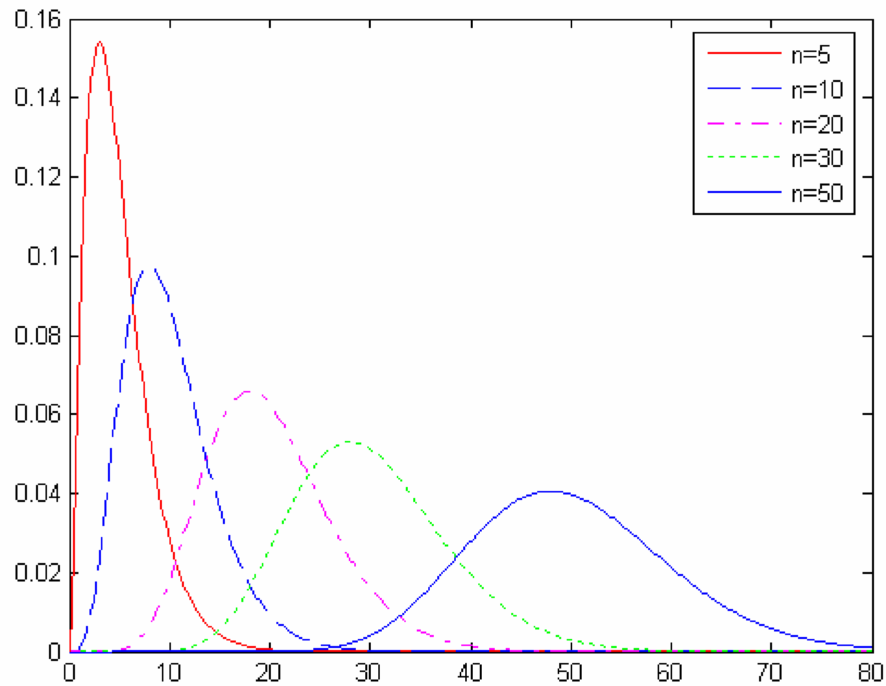

Figure 3. Chi-square distribution with various degrees of freedom n.

The Chi-square distribution is also called the variance distribution by some authors, because the variance of a random sample from normal distribution follows a Chi-square distribution. Specifically, if $X_1, X_2, \ldots\ldots, X_n$ is a random sample from an $N(\mu, \sigma^2)$ distribution. Then the probability density function of the sample variance $S^2 = \sum_{i=1}^{n}(X_i - \bar{X})^2 / n-1$ is

$$f(u = s^2) = \frac{1}{2^{(n-1)/2}\Gamma(\frac{n-1}{2})} e^{-\frac{(n-1)u/\sigma^2}{2}} \left(\frac{(n-1)u}{\sigma^2}\right)^{\frac{n-3}{2}} \frac{(n-1)}{\sigma^2},\ \mathrm{u} > 0 \tag{9}$$

and the sampling distribution of the sample standard deviation $S = \sqrt{\sum_{i=1}^{n}(X_i - \bar{X})^2 / n-1}$ is

$$f(v = s) = \frac{1}{2^{(n-1)/2}\Gamma(\frac{n-1}{2})} e^{-\frac{(n-1)v^2/\sigma^2}{2}} \left(\frac{(n-1)v^2}{\sigma^2}\right)^{\frac{n-3}{2}} \frac{(n-1)}{\sigma^2} 2v,\ \mathrm{v} > 0 \tag{10}$$

3.3. Point Estimation of Normal Processes Parameters

Suppose that the variance σ^2 and standard deviation σ of normal distribution are both unknown. If a random sample of *n* observations is taken, then the sample variance S^2 and sample standard deviation S/c_4 are point estimators of the population variance σ^2 and population standard deviation σ, respectively. It can be shown that

$$E(S^2) = \sigma^2 \quad \text{and} \quad E(S) = \left(\frac{2}{n-1}\right)^{1/2} \frac{\Gamma(n/2)}{\Gamma[(n-1)/2]}\sigma = c_4\sigma.$$

Furthermore, the standard deviation of S is $\sigma\sqrt{1-C_4{}^2}$.

4. Process Variance Change Investigation for Normal Processes Using S^2 Control Chart

The major purpose of individuals control chart is assisting on identifying shifts and drifts in processes and it is easily to be implemented. But some assumptions should be satisfied before control charts are used. The assumptions include that the process must in stationary. In practice, process is not stable. Due to above-mentioned statements, the steps in calculating the probabilities for catching various magnitude of change under various sample subgroup sizes *n* of S^2 control chart are as followings.

4.1. Detection Power of S^2 Control Chart for Normal Processes

STEP 1: Construct the limits. The parameters of the S^2 control chart are

$$\begin{aligned} \mathrm{UCL} &= \frac{\bar{S}^2}{n-1}\chi^2_{0.00135,n-1}, \\ \text{Center line} &= \bar{S}^2, \\ \mathrm{LCL} &= \frac{\bar{S}^2}{n-1}\chi^2_{0.99865,n-1}, \end{aligned} \tag{11}$$

where $\chi^2_{0.00135,n-1}$ and $\chi^2_{0.99865,n-1}$ denote the 0.00135 and 0.99865 percentage points of the Chi-square distribution with n-1 degrees of freedom, and the statistic $\bar{S}^2$ is an average sample variance obtained from the analysis of preliminary data.

STEP 2: Consider the detection power for an S^2 control chart. If σ changes from the in-control value to another value $k\sigma_0$. The probability of detecting this change is

$$\begin{aligned}\text{Detection power} &= 1-\beta \\ &= 1-P(LCL \le S^2 \le UCL \mid \sigma=\sigma_1=k\sigma_0) \\ &= 1-P(\int_{LCL}^{UCL} f(u=s^2)du \mid \sigma=\sigma_1=k\sigma_0).\end{aligned} \quad (12)$$

where $f(u)$ denotes the sampling distribution of the sample variance and σ_1 is the standard deviation after variance change (σ_0 is the standard deviation of the original process).

Table 5 presents the detection power with various magnitude of standard deviation change when data come from normal distribution under various sample subgroup sizes $n=10(1)20$. For an S^2 control chart with sample subgroup size $n=10$, σ changes greater than $2\sigma_0$ have more than a 66.1 percent chance of detecting the standard deviation changes. However, the chance of catching a $1.5\sigma_0$ change being only 21.1 percent. Such low probabilities mean that small changes to standard deviation may come and go, without us ever being aware they have negatively impacted our process.

Table 5. The detection power of the S^2 control chart for normal processes with various sample subgroup sizes.

n	variance change size($k\sigma$)					
	k=1.0	k=1.5	k=2.0	k=2.5	k=3.0	k=3.5
10	0.00270	0.21103	0.66071	0.88802	0.96388	0.98766
11	0.00270	0.23550	0.70680	0.91592	0.97636	0.99289
12	0.00270	0.26014	0.74771	0.93727	0.98465	0.99594
13	0.00270	0.28486	0.78377	0.95346	0.99009	0.99769
14	0.00270	0.30956	0.81536	0.96565	0.99365	0.99870
15	0.00270	0.33417	0.84288	0.97477	0.99595	0.99927
16	0.00270	0.35861	0.86673	0.98155	0.99743	0.99959
17	0.00270	0.38281	0.88730	0.98656	0.99838	0.99978
18	0.00270	0.40671	0.90497	0.99025	0.99898	0.99988
19	0.00270	0.43026	0.92009	0.99296	0.99936	0.99993
20	0.00270	0.45340	0.93297	0.99493	0.99960	0.99996

One way to improve the odds of catching small changes in σ is to increase the sample subgroup size. The chance of detecting a 33.4 percent when $n=15$, to almost 43 percent when $n=19$. However, this chance falls to only 21.1 percent if $n=10$.

4.2. Modified Variance Adjustments for Normal Processes

The probabilities in Table 5 are plotted on the graph displayed in Figure 4, with one curve for each sample subgroup size. Those curves are power curves, these lines portray the chances of detecting a change in standard deviation of a given size (expressed in σ units on the horizontal axis). For small change in standard deviation, all three curves are close to zero. As the size of the change increases, so does the detection power of the chart, with all three curves eventually leveling off close to 100% for shifts exceeding 3.5σ. The horizontal lines in Figure 4 show that there is a 50% chance of missing a 1.72σ change in standard deviation when $n=12$ for S^2 control chart, whereas σ must move by 1.80σ to have this same probability when $n=10$.

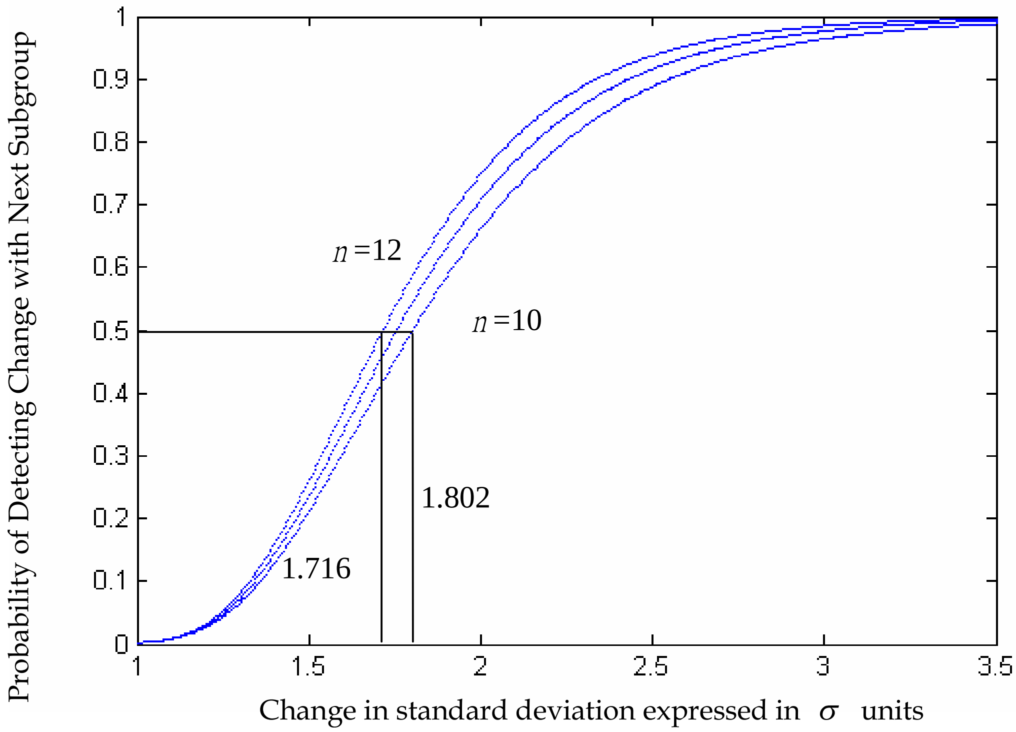

Figure 4. Power curves of S^2 control chart for subgroup sizes 10, 11 and 12.

The necessary adjustment due to the undetected standard deviation change is called AS_{power} which is the magnitude of change we need to adjust based on designated detection power and process data come from normal distributions. We develop a Matlab program to determine the adjustment AS_{power} by setting the desired detection power and the sample subgroup size *n*. Generally, the sample subgroup size of S^2 and S control charts is moderately large($n > 10$ or 12). If we set detection $\text{power} = 1/2$ and $n = 10(1)30$. The magnitude of adjustment $AS_{1/2}$ based on S^2 control chart is shown from Table 6. Temporary movements in σ smaller than $AS_{power}\,\sigma$ are more than likely to be missed by a control chart. We also provide the AS_{power} values of S^2 control charts (See Appendix A) for detection $\text{power} = 1/3,\ 1/4$ and $1/5$ versus various sample subgroup sizes $n = 2(1)30$.

Table 6. AS_{power} values of S^2 control chart for various sample subgroup sizes.

n	Power=1/2
10	1.80215
11	1.75533
12	1.71577
13	1.68158
14	1.65192
15	1.62555
16	1.60220
17	1.58119
18	1.56210
19	1.54480
20	1.52901
21	1.51445
22	1.50099
23	1.48849
24	1.47696
25	1.46611
26	1.45595
27	1.44647
28	1.43755
29	1.42903
30	1.42107

4.3. Modified Process Capability C_{pk} *for Normal Processes*

Acknowledging that processes will experience change in various magnitude of variance, and knowing that not all of these will be discovered, some allowance for them must be made when estimating outgoing quality so customers are not disappointed. Because changes ranging in size from 0 up to $AS_{power}\sigma$ are likely to remain undetected (larger moves should be caught by the chart), a conservative approach is to assume that every missed change is as large as $AS_{power}\sigma$.

Since changes can move standard deviation larger or smaller, in place of using $\hat{\sigma}$ for the process variance when estimating capability, $\hat{\sigma}$ multiplied by AS_{power} is a conservative way to evaluate the process capability. The adjustment is incorporated into the C_{pk} formula, which we refer to as "dynamic" C_{pk} index, by making the following modifications:

$$C_{pk} = \min\left\{\frac{USL-\hat{\mu}}{3\hat{\sigma}\times AS_{power}}, \frac{\hat{\mu}-LSL}{3\hat{\sigma}\times AS_{power}}\right\}. \tag{13}$$

By including an adjustment in this assessment for undetected shifts in variance, the estimate of capability will decrease and the expected total number nonconforming parts will increase. To illustrates the use of dynamic C_{pk} index, setting the detection $\text{power}=0.5$ of S^2 control chart. Then $AS_{power}=1.63$ (see Table 6) when $n=15$ from normal distribution. Factoring in the possibility missing changes in σ of up to 1.63σ drops the C_{pk} index.

5. Process Variance Change Investigation for Normal Processes Using S Control Chart

Most quality engineers use either S^2 control chart or the S control chart to monitor process variability. In this chapter, we exhibit the detection powers of S control chart by using the sampling distribution of S to find the adjustment for normal processes with standard deviations change.

5.1. Detection Power of S Control Chart for Normal Processes

Setting up and operating control charts for S requires about the same sequence of steps as those for S^2 control chart, except that for each sample we must calculate the sample standard deviation S,

$$\bar{S} = \frac{1}{m}\sum_{i=1}^{m} S_i. \tag{14}$$

STEP 1: Construct the limits. The statistic $\bar{S}/c_4$ is an unbiased estimator of σ. Therefore, the parameters of the S control chart would be

$$\begin{aligned}\text{UCL} &= B_4\bar{S},\\ \text{Center line} &= \bar{S},\\ \text{LCL} &= B_3\bar{S}.\end{aligned} \tag{15}$$

We usually define the constants

$$\begin{aligned}B_3 &= 1-\frac{3}{c_4}\sqrt{1-c_4^{\,2}},\\ B_4 &= 1+\frac{3}{c_4}\sqrt{1-c_4^{\,2}}.\end{aligned} \tag{16}$$

Note that this control chart is defined with three-sigma control chart limits.

STEP 2: Consider the detection power for an S control chart. If the standard deviation changes from the in-control value to another value $k\sigma_0$. The probability of detecting this change is

$$\begin{aligned}\text{Detection power} &= 1-\beta \\ &= 1-P(LCL \le S \le UCL \mid \sigma=\sigma_1=k\sigma_0) \\ &= 1-P(\int_{LCL}^{UCL} f(v=s)dv \mid \sigma=\sigma_1=k\sigma_0),\end{aligned} \quad (17)$$

where $f(v)$ denotes the sampling distribution of the sample standard deviation and σ_1 is the standard deviation after variance change (σ_0 is the standard deviation of the original process).

Table 7 displays the detection power of S control chart with various magnitude of standard deviation change ($k\sigma$) when data come from normal distribution under various sample subgroup sizes $n = 10(1)20$.

Table 7. The detection power of the S control chartfor normal processes with various sample subgroup sizes.

n	variance change size($k\sigma$)					
	k=1.0	k=1.5	k=2.0	k=2.5	k=3.0	k=3.5
10	0.00183	0.22585	0.67581	0.89479	0.96641	0.98860
11	0.00189	0.25147	0.72117	0.92148	0.97817	0.99348
12	0.00194	0.27709	0.76113	0.94174	0.98590	0.99630
13	0.00199	0.30264	0.79612	0.95700	0.99096	0.99791
14	0.00203	0.32803	0.82659	0.96841	0.99423	0.99883
15	0.00207	0.35319	0.85298	0.97691	0.99634	0.99347
16	0.00210	0.37806	0.87574	0.98319	0.99749	0.99964
17	0.00213	0.40257	0.89527	0.98781	0.99855	0.99980
18	0.00216	0.42668	0.91197	0.99119	0.99910	0.99989
19	0.00218	0.45035	0.92620	0.99365	0.99943	0.99994
20	0.00221	0.47352	0.93828	0.99545	0.99965	0.99997

5.2. Modified Variance Adjustments for Normal Processes

Figure 5 depicts the power curves of S control chart. Those lines portray the probability of detecting a change in standard deviation of a given size (expressed in σ units on the horizontal axis). For small change in standard deviation, all three curves are close to zero. As the size of the change increases, so does the power of the chart to detect it, with all three curves eventually leveling off close to 100% for shifts in excess of 3.5σ.

The horizontal in Figure 5 show that there is a 50% chance of missing a

1.70σ change in standard deviation when $n = 12$ for S control chart, whereas standard deviation must change by 1.78σ to have this same probability when $n = 10$.

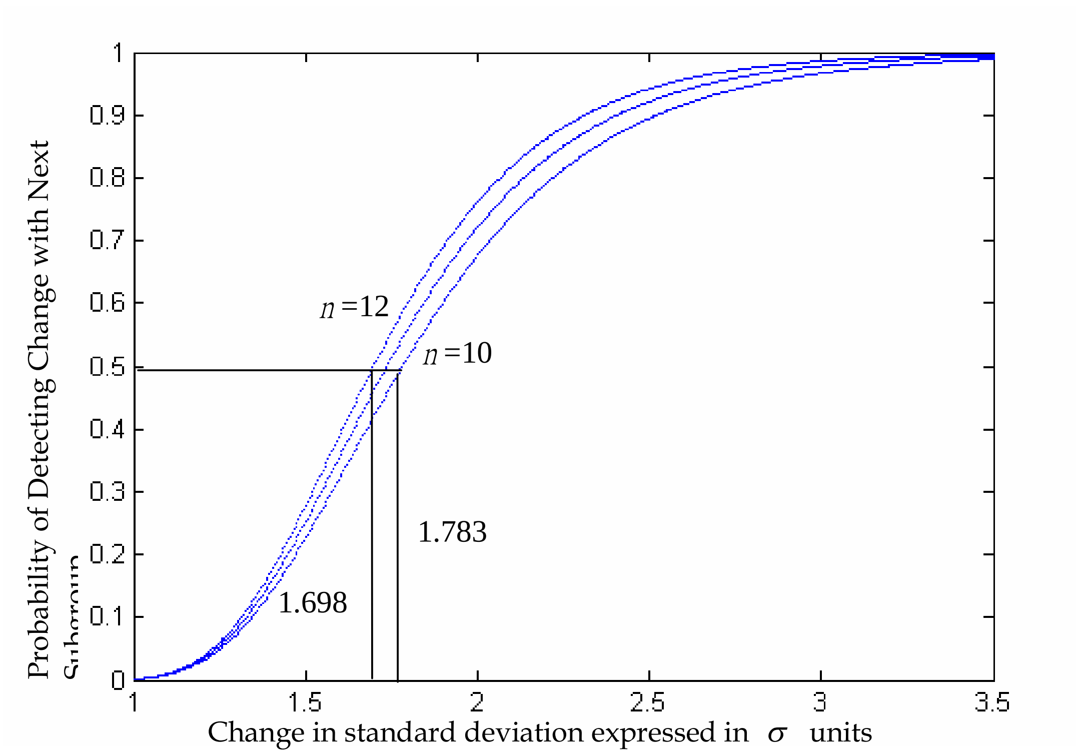

Figure 5. Power curves of S control chart for sample subgroup sizes 10, 11 and 12.

The undetected standard deviation change adjustment is called AS_{power} which is the magnitude of change we need to adjust based on designated detection power and process data come from normal distribution. We develop a Matlab program to determine the adjustment AS_{power} by setting the desired detection power and the sample subgroup size n. For example, if we set detection power=1/2 and $n = 10(1)30$. The magnitude of adjustment $AS_{1/2}$ based on S control chart is shown from Table 8.

We also provide the AS_{power} values of S control chart (See Appendix B) for detection $power = 1/3,\ 1/4\ and\ 1/5$ versus various sample subgroup sizes $n = 2(1)30$.

Table 8. AS_{power} values of S control charts for various sample subgroup sizes.

n	$power = 1/2$
	S control chart
10	1.78265
11	1.73679

12	1.69806
13	1.66483
14	1.63585
15	1.61031
16	1.58751
17	1.56705
18	1.54865
19	1.53175
20	1.51637
21	1.50237
22	1.48932
23	1.47723
24	1.46597
25	1.45547
26	1.44565
27	1.43645
28	1.42780
29	1.41956
30	1.41187

6. Application

In the previous chapter we presented the statistical reason for the magnitude of adjustment for S^2 and S control charts. We now illustrate the application of the dynamic C_{pk} to estimate process capability.

Because of advantages such as long lifetime, low power consumption, and no mercury containing, Light Emitting Didoes (LEDs) are widely used in a variety of general-purpose illumination applications. For indoor illumination, there are two different approaches for generating white light with LEDs. One way is combining a blue or UV LEDs with a down conversion phosphor, and the other way to obtain white light is mix the monochromatic LEDs with different colors. The later approach seems a better way to generate white light for indoor illumination. Figure 6 illustrates the red, green and blue (RGB) LEDs in one package. For RGB LEDs, or the white light mixed by more than three LEDs, the color rendering and luminous efficiency depend on the choice of the individual peak wavelengths of the LEDs, which will lead to very different color rendering and luminous efficiency. So the optimization of the white light formed by more than two LEDs can achieve a maximum of certain luminous efficiency and color-rendering index (CRI). To make sure the optical properties are acceptable to customers, the wavelengths we choose with highest luminous efficiency at that time are in blue 455-480 nanometer (nm), green 510-535 nm and red 610-630nm regions.

To illustrate the use of the dynamic C_{pk} to estimate process capability. Consider Table 9, which presents a part of historical data of wavelength for blue LED collected from the factory. The proposed specifications on wavelength for blue LED are USL = 480nm and LSL = 455nm, respectively. From Figure 7, note that the data lie nearly along a straight line, implying that the distribution of wavelength is normal distribution. Furthermore, Figure 8 shows the shape of the histogram implies that the distribution of wavelength is approximately normal distribution.

Table 9. The 100 observations are collected from the historical data.

463.029	466.841	463.560	462.841	467.381
465.297	463.411	462.623	463.485	470.220
464.694	463.413	464.895	467.947	465.504
462.441	464.557	465.835	463.000	464.413
467.604	464.955	464.273	464.092	466.544
464.966	465.614	463.900	463.886	461.244
466.180	467.328	464.921	468.563	469.098
463.424	466.368	464.616	465.178	467.435
463.558	461.149	462.894	462.622	462.299
465.692	466.534	467.844	462.526	463.526
465.235	466.785	465.970	468.819	467.950
469.066	466.400	467.818	469.262	465.854
464.395	467.882	463.905	468.597	465.865
469.868	465.720	462.539	462.237	463.927
468.319	463.519	466.777	464.530	465.501
460.721	464.672	465.091	464.562	466.508
461.303	459.612	463.336	465.677	463.599
463.738	466.041	464.804	463.741	462.794
464.591	465.257	460.581	462.849	464.592
465.020	463.700	467.385	464.017	462.681

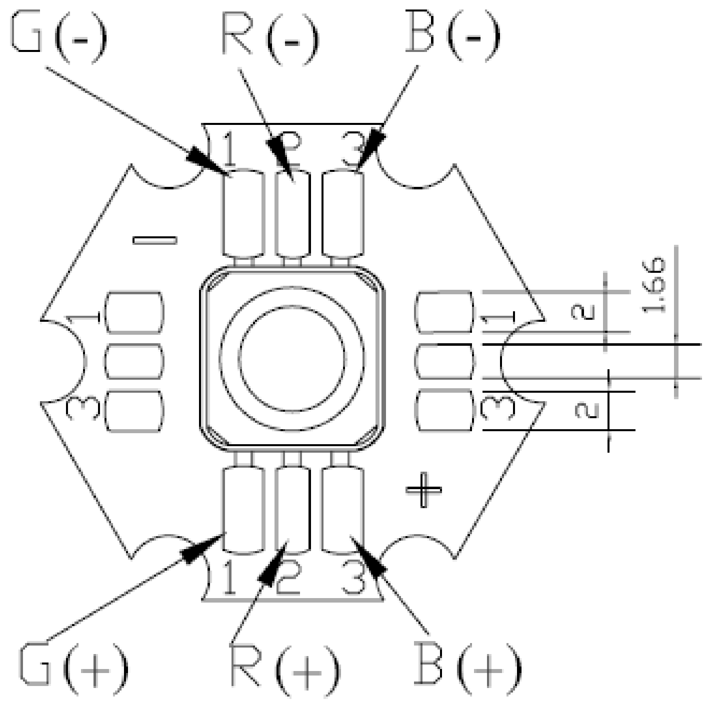

Figure 6. The RGB LEDs.

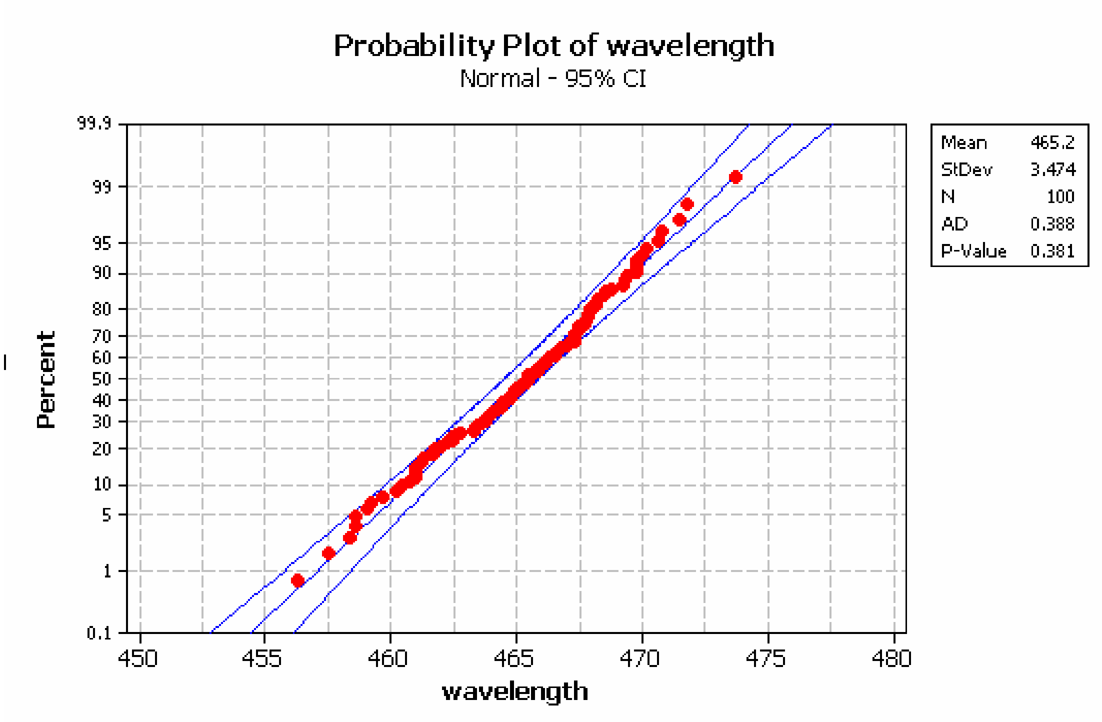

Figure 7. Normal probability plot of the historical data.

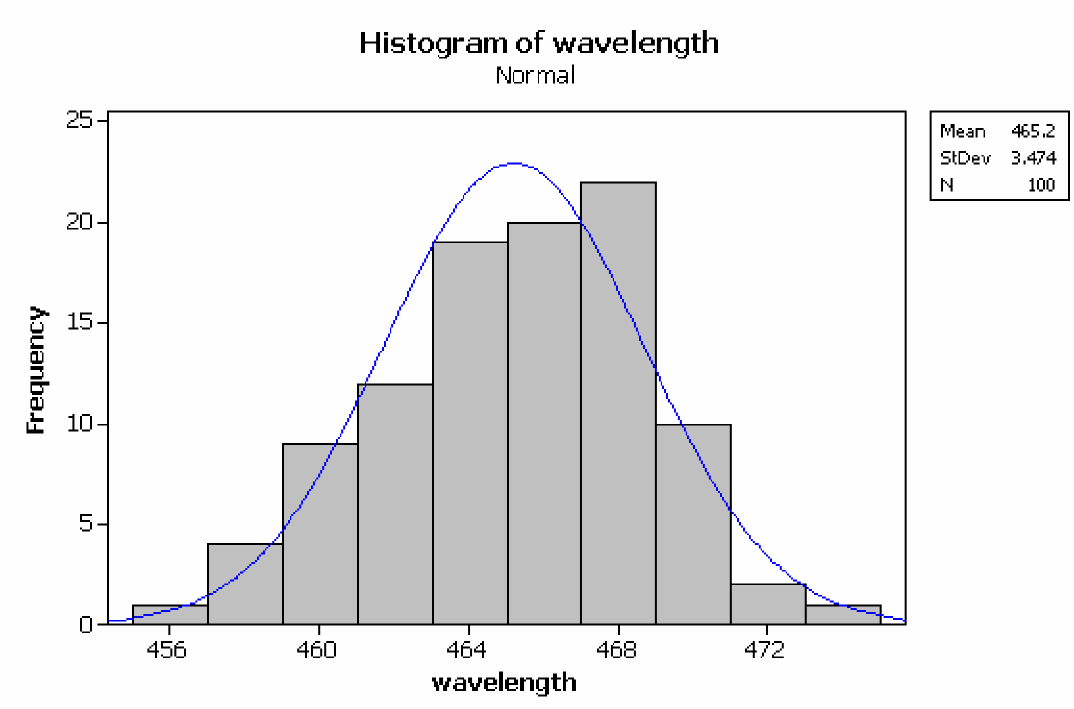

Figure 8. Histogram plot of the historical data.

The parameters μ and σ of this normal process could be estimated from the historical data, giving $\hat{\mu} = 464.98$ and $\hat{\sigma} = 2.20$. C_{pk} can be calculated as follows:

$$C_{pk} = \min\left\{\frac{USL-\hat{\mu}}{3\hat{\sigma}}, \frac{\hat{\mu}-LSL}{3\hat{\sigma}}\right\}$$

$$= \min\left\{\frac{480-465}{3(2.20)}, \frac{465-455}{3(2.20)}\right\}$$

$$= \min\{2.27, 1.52\} = 1.52.$$

Under the assumption of a stationary standard deviation. By including an adjustment in this assessment for undetected change in standard deviation, the estimate of capability will decrease and the expected total number of nonconforming parts will increase. From Table 6, $AS_{1/2}$ of S^2 control chart is 1.80 when $n = 10$. Compared C_{pk} to the value of the following index:

$$\text{Dynamic } C_{pk} = \min\left\{\frac{USL-\hat{\mu}}{3\hat{\sigma}\times AS_{power}}, \frac{\hat{\mu}-LSL}{3\hat{\sigma}\times AS_{power}}\right\}$$

$$= \min\left\{\frac{480-465}{3(2.20)(1.80)}, \frac{465-455}{3(2.20)(1.80)}\right\}$$

$$= \min\{1.26, 0.84\} = 0.84.$$

We can see that the value of the dynamic C_{pk} is much smaller. By increasing the sample subgroup size *n*, a change in σ has a higher probability to be detected. For example, if $n = 15$, the $AS_{1/2}$ would be 1.63 for normal distribution and

$$\text{Dynamic } C_{pk} = \min\left\{\frac{USL-\hat{\mu}}{3\hat{\sigma}\times AS_{power}}, \frac{\hat{\mu}-LSL}{3\hat{\sigma}\times AS_{power}}\right\}$$

$$= \min\left\{\frac{480-465}{3(2.20)(1.63)}, \frac{465-455}{3(2.20)(1.63)}\right\}$$

$$= \min\{1.39, 0.93\} = 0.93.$$

Changing sample subgroup size *n* from 10 to 15 increases the dynamic C_{pk} index value from 0.84 to 0.93.

7. Conclusion

In Bothe's study, the author provided the statistical rationale for adjusting estimates of process capability by including a possible shift in μ. But the case of standard deviation change occurs frequently in practice. This project has considered the problem for adjusting estimates of process capability by including a change in σ when the process output has normal distribution. We use a Matlab program (available on request) to compute the standard deviation change adjustments based on the detection power is 1/2($ARL_1 = 2$), 1/3($ARL_1 = 3$), 1/4($ARL_1 = 4$), 1/5($ARL_1 = 5$) percent for data come from normal distribution with various values of $n = 2(1)30$. The adjustments are incorporated into the C_{pk} formula, which we refer to as the "dynamic" C_{pk} index. It has proven to be a useful way for the engineers/practitioners to access process performance. A real-world application on LED production plant is investigated and presented to illustrate the applicability of the proposed approach.

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Appendix A. AS_{power} **for Normal Processes with Variance Change Using** S^2 **Control Chart**

n	Detection Power		
	1/3	1/4	1/5
10	1.62857	1.54233	1.48767
11	1.59479	1.51459	1.46323
12	1.56595	1.49055	1.44235
13	1.54095	1.46982	1.42409
14	1.51884	1.45142	1.40802
15	1.49934	1.43507	1.39360
16	1.48190	1.42038	1.38055
17	1.46611	1.40706	1.36888
18	1.45183	1.39497	1.35817
19	1.43864	1.38385	1.34828
20	1.42670	1.37369	1.33936
21	1.41557	1.36435	1.33098
22	1.40527	1.35556	1.32315
23	1.39580	1.34746	1.31587
24	1.38687	1.33990	1.30914
25	1.37849	1.33276	1.30283
26	1.37067	1.32603	1.29685
27	1.36339	1.31979	1.29129
28	1.35638	1.31381	1.28593
29	1.34986	1.30818	1.28085
30	1.34361	1.30289	1.27618

Appendix B. AS_{power} **for Normal Processes with Variance Change Using S Control Chart**

n	Detection Power		
	1/3	1/4	1/5
10	1.61099	1.52571	1.47147
11	1.57803	1.49866	1.44785
12	1.54988	1.47531	1.42752
13	1.52557	1.45512	1.40994
14	1.50415	1.43727	1.39429
15	1.48520	1.42148	1.38042
16	1.46831	1.40733	1.36792
17	1.45306	1.39456	1.35666
18	1.43919	1.38289	1.34636
19	1.42656	1.37218	1.33688
20	1.41489	1.36243	1.32823
21	1.40417	1.35336	1.32027
22	1.39429	1.34499	1.31285
23	1.38509	1.33716	1.30585
24	1.37657	1.32988	1.29939
25	1.36847	1.32301	1.29335
26	1.36092	1.31670	1.28765
27	1.35391	1.31065	1.28223
28	1.34718	1.30489	1.27715
29	1.34087	1.29953	1.27234
30	1.33489	1.29445	1.26781