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Optimal control policy for a standing order inventory system

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Abstract

In this paper, we consider a standing order inventory system in which an order of fixed size arrives in each period. Since demand is stochastic, such a system must allow for procurement of extra units in the case of an emergency and sell-offs of excess inventory. Assuming the average-cost criterion, Rosenshine and Obee (Operations Research 24 (1976) 1143–1155) first studied such a system and devised a 4-parameter inventory control policy that is not generally optimal. The current paper uses dynamic programming to determine the optimal control policy for a standing order system, which consists of only two operational parameters: the dispose-down-to level and order-up-to level. Either the average-cost or discounted-cost criterion can be assumed in the proposed model. Also, both the backlogged and lost-sales problems are investigated in this paper. By using a convergence theorem, we stop the dynamic programming computation and obtain the two optimal parameters.

Keywords: Inventory; Emergency order; Dynamic programming; Storage constraint; Lost-sales; Standing order

1. Introduction

Inventory control systems in the literature are generally divided into two groups: continuous-review models and periodic-review models. The former typically assumes a fixed order size, while the latter usually predetermines the period length. Since demand is stochastic in the real world, the order interval for the former is thus variable, while the order quantity for the latter varies period by period.

A standing order inventory system is a periodic-review one in which the order size is also fixed. However, as Rosenshine and Obee [15] pointed out, it must allow for procurement of extra units in the case of an emergency and sell-offs of excess inventory if necessary. Assuming that demand in different periods is independently and identically distributed and demand not satisfied at once is backlogged, Rosenshine and Obee hypothesized that the size of a standing order is greater than or equal to the mean demand of a period and devised a 4-parameter inventory control policy for such a system: the storage capacity, emergency order point, size of a standing order, and emergency order-up-to level (i.e., if inventory exceeds the storage capacity, the excess inventory is sold off, and if inventory falls below the order point, an emergency order is placed to raise inventory to the order-up-to level). Using a Markov chain approach, they determined the latter

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two operational parameters, given the former two. Consequently, the inventory policy they devised is not generally optimal.

In this paper, we use dynamic programming to derive the optimal control policy for a standing order system considered in [15]. We assume (as in [15]) that the emergency unit item cost is higher than the regular unit cost, which in turn is greater than the unit sell-off revenue. Also, we assume that the size of standing orders is predetermined by the buyer. The optimal policy derived has only two operational parameters: the dispose-down-to level and order-up-to level. If inventory at a review epoch is lower than the order-up-to level, an emergency order is placed to raise inventory to this level, and if inventory at a review epoch is higher than the dispose-down-to level, inventory is sold off down to this level.

A standing order inventory system has many attractive features compared to a base-stock periodic-review model [15]. The fixed cost for placing periodic orders is eliminated and lead-time does not exist. Also, suppliers are more likely to offer a certain form of price breaks or discounts for items delivered under a standing order. Moreover, a supplier does not suffer from the bullwhip effect if a standing order is negotiated with its buyer.

A standing order system bears resemblance to supply contracts with a fixed periodic delivery. Several studies have recently been done on this area. Anupindi and Akella [1] investigate a finite-horizon periodic commitment model with a response time to adjustments in the order quantity. Henig et al. [10] design a periodic inventory/transportation model where both downward and upward adjustments in the order quantity are permitted. Bassok et al. [3] present a supply contract problem with periodic commitments and limited flexibility to change the purchase quantity. Ehrhardt [8] considers the problem of selecting a fixed replenishment quantity to be delivered in each of n consecutive periods in the future. Janssen and de Kok [12] discuss a two-supplier periodic model where one supplier delivers a fixed quantity while the amount delivered by the other is governed by an order-up-to policy. Urban [17] describes a multi-period "recurrent" newsvendor problem where changes in the order quantity result in an additional cost to the buyer. Moinzadeh and Nahmias [13] consider a continuous-review inventory model where fixed as well as variable costs are incurred for any upward adjustments to the fixed order quantity. Chiang [6] devises an order-splitting periodic model where n fixed-size shipments (except the first one) are delivered in future time points that are evenly separated. Recently, Cheung and Yuan [4] extend the model of Anupindi and Akella [1] to an infinite-horizon one with no extra costs incurred for units ordered beyond the periodic quantity. See, e.g., Anupindi and Bassok [2] and Tsay et al. [16] for other related research on supply contracts with periodic commitments. See also, e.g., Chiang and Gutierrez [7] for a periodic-review inventory model with emergency orders.

Note that Henig et al.'s model [10] and Rosenshine and Obee's model [15] are similar in the sense that both allow for emergency orders at a review epoch. The difference between them is that when excess inventory seems to exist at a review epoch, the former [10] permits the supplier to deliver a quantity that is less than the periodic commitment (with no refunds given), while the latter [15] gives the buyer the option of disposing of excess inventory (upon receipt of a standing order). It is seen below that Henig et al.'s model is a special case of the basic dynamic program developed.

The rest of this paper is organized as follows. In Section 2, we develop a dynamic programming model for the standing order inventory system described above, which incorporates both the backlogging and lost-sales cases. In Section 3, we present a method for computing the optimal dispose-down-to level and order-up-to level. In Section 4, we conclude this paper.

2. A dynamic programming model

Let ξ be the demand of a period and $\varphi(\cdot)$ its probability density function. Demand is assumed to be non-negative and independently distributed in different periods. In addition, we use the following notation.

μ	average demand of a period
R	the standing order size
C	the unit item cost
C_e	the unit cost via the emergency mode
C_s	the unit revenue of excess inventory sold off
h	the inventory cost per unit held per period
p	the shortage cost per unit per period in the backlogging case
π	the shortage cost per unit in the lost-sales case (π should be larger than its counterpart p, for it usually includes the sales price)
L	expected holding and shortage costs of a period

α the one-period discount factor, $0 < \alpha \leqslant 1$
I net inventory (i.e., on-hand inventory minus backorder) in the backlogging case or on-hand inventory in the lost-sales case, before the receipt of R at a review epoch
$f_n(I)$ the expected discounted cost of procurement, holding, shortage, and emergency ordering (minus sell-off revenue) with n periods remaining until the end of the planning horizon, given I at a review epoch, the standing order R, and an optimal policy is used
$(X)^+$ $\max\{X, 0\}$.

We assume $C_s < C < C_e$. Thus, it is not economical to order a positive quantity via the emergency mode while disposing of some inventory in the same period. Also, immediate delivery (and negligible fixed costs) for emergency orders is assumed, as in [10] and [15]. In addition, $\pi > C_e$ is assumed (for the use of the emergency mode to be meaningful). Let $t(\cdot)$ be a transition function that represents the starting inventory of the next review period. $f_n(I)$ satisfies the recursive equation

$$f_n(I) = \min_{Q \geqslant -R}\{Y(Q) + CR + L(I + R + Q) + \alpha E f_{n-1}(t(I + R + Q - \xi))\}, \quad (1)$$

where $f_0(I) \equiv 0$, Q is the quantity ordered via the emergency mode (if positive) or the quantity disposed of (if negative) at a review epoch, and $Y(Q) = \max\{C_eQ, C_sQ\}$ is the emergency operation cost which is piecewise linear. Note that in the backlogging problem, $t(X) = X$ and $L(\cdot)$ is given by

$$L(X) = \int_0^{X^+} h(X - \xi)\varphi(\xi)\,\mathrm{d}\xi + \int_{X^+}^{\infty} p(\xi - X)\varphi(\xi)\,\mathrm{d}\xi, \quad (2)$$

while in the lost-sales problem, $t(X) = (X)^+$ and

$$L(X) = \int_0^{X} h(X - \xi)\varphi(\xi)\,\mathrm{d}\xi + \int_{X}^{\infty} \pi(\xi - X)\varphi(\xi)\,\mathrm{d}\xi. \quad (3)$$

It is assumed that $Q \geqslant -R$, i.e., the quantity sold off at a review epoch is less than or equal to the standing order size (note that this is really not a restrictive assumption, as we shall see later that a stationary policy is optimal in the long run and there is a maximum inventory level SU such that $Q \geqslant -R$ holds naturally). Notice that Rosenshine and Obee [15] use the undiscounted-cost (i.e., the average cost per period) criterion, while our model allows for both the undiscounted-cost and discounted-cost criteria (thus, the full unit emergency cost and unit sell-off revenue, rather than the marginal cost or loss as in [15], should be used). Henig et al.'s model [10] is a special case of our model with $C_s = 0$. Also, both Henig et al. and Rosenshine and Obee did not consider the lost-sales problem.

Let $Z = I + R + Q$, i.e., the inventory level after a possible emergency order or disposal is made at a review epoch. We express model (1) by

$$f_n(I) = \min_{Z \geqslant I}\{Y(Z - I - R) + C_sR + L(Z) + \alpha E f_{n-1}(t(Z - \xi))\}, \quad (4)$$

where the constant item cost $(C - C_s)R$ is excluded for simplicity. Letting

$$G_n(Z) = L(Z) + \alpha E f_{n-1}(t(Z - \xi)), \quad (5)$$

we can write model (4) by

$$f_n(I) = \min_{Z \geqslant I}\{Y(Z - I - R) + C_sR + G_n(Z)\}, \quad (6)$$

which simplifies to

$$f_n(I) = \min_{Z \geqslant I+R}\{C_eZ + G_n(Z)\} - C_e(I + R) + C_sR \quad (7)$$

or

$$f_n(I) = \min_{I \leqslant Z \leqslant I+R}\{C_sZ + G_n(Z)\} - C_sI, \quad (8)$$

depending on whether a possible emergency order or disposal is made at a review epoch.

Let Df and DDf be respectively the first and second derivatives of the function f.

Lemma 1. *$f_n(I)$ is convex.*

Proof (By induction). $f_0(I)$ is convex (and $\mathrm{D}f_0(I) \geqslant -C_e$). Assume that $f_{n-1}(I)$ is convex (and $\mathrm{D}f_{n-1}(I) \geqslant -C_e$ for the lost-sales case). In the backlogging case, $f_n(I)$ is convex since $Y(\cdot)$ is convex and the holding and shortage costs in (1) are linear (apparently, these costs in L can be allowed to be not linear but convex). In the lost-sales case, it is seen from (6) that $f_n(I)$ is convex if $G_n(Z)$ is convex. Now,

$$\begin{aligned} DG_n(Z) &= \int_0^Z h\varphi(\xi)\,d\xi - \int_Z^\infty \pi\varphi(\xi)\,d\xi \\ &\quad + \alpha\int_0^Z Df_{n-1}(Z-\xi)\varphi(\xi)\,d\xi, \\ DDG_n(Z) &= (h+\pi)\varphi(Z) \\ &\quad + \alpha\int_0^Z DDf_{n-1}(Z-\xi)\varphi(\xi)\,d\xi \\ &\quad + \alpha Df_{n-1}(0)\varphi(Z) \geqslant (h+\pi)\varphi(Z) \\ &\quad + \alpha\int_0^Z DDf_{n-1}(Z-\xi)\varphi(\xi)\,d\xi \\ &\quad - \alpha C_e\varphi(Z) > 0, \end{aligned}$$

since $\pi > C_e$. In addition, we see from (7) and (8) that as $C_e > C_s$ and $f_n(I)$ is convex, $Df_n(I) \geqslant -C_e$. □

For inventory models with convex ordering costs, see, e.g., Porteus [14] for optimal policies. Here, we include a specific analysis of model (6) with piecewise linear ordering costs (as in [10]). Let SL_n minimize $C_eZ + G_n(Z)$ and SU_n minimize $C_sZ + G_n(Z)$. Since $C_s < C_e$, SL_n is smaller than SU_n. It follows from (7) and (8) that the optimal policy is to order the amount $SL_n - I - R$ at cost C_e per unit if $I + R \leqslant SL_n$, sell the amount $I + R - SU_n$ (respectively R) at price C_s per unit if $I + R \geqslant SU_n \geqslant I$ (respectively if $I \geqslant SU_n$), and do nothing (i.e., neither order via the emergency mode, nor sell off inventory) if $SL_n \leqslant I + R \leqslant SU_n$ (see also Lemma 1 of [10]). In other words,

$$Z = I, \quad f_n(I) = L(I) + \alpha Ef_{n-1}(t(I-\xi)) \quad \text{if } I \geqslant SU_n, \tag{9}$$

$$Z = SU_n, \quad f_n(I) = C_s(SU_n - I) + L(SU_n) + \alpha Ef_{n-1}(t(SU_n-\xi)) \quad \text{if } I+R \geqslant SU_n \geqslant I, \tag{10}$$

$$Z = I+R, \quad f_n(I) = C_sR + L(I+R) + \alpha Ef_{n-1}(t(I+R-\xi)) \quad \text{if } SL_n \leqslant I+R \leqslant SU_n, \tag{11}$$

$$Z = SL_n, \quad f_n(I) = C_sR + C_e(SL_n - I - R) + L(SL_n) + \alpha Ef_{n-1}(t(SL_n-\xi)) \quad \text{if } I+R \leqslant SL_n. \tag{12}$$

We can see above that the optimal control policy for the finite-horizon model is governed by the two operational parameters: the emergency order-up-to level SL_n and the dispose-down-to level SU_n. Noticing in (4) that the total ordering cost $Y(Z - I - R) + C_sR$ is non-negative and $L(\cdot)$ is also non-negative, we have

Theorem 1. *If* $\alpha < 1$, *then as* $n \to \infty$, $\lim SL_n = SL$, $\lim SU_n = SU$, *and SL and SU minimize* $C_eZ + G(Z)$ *and* $C_sZ + G(Z)$, *respectively, where*

$$G(Z) = L(Z) + \alpha Ef(t(Z-\xi)),$$
$$f(I) = \min_{Z\geqslant I}\{Y(Z-I-R) + C_sR + G(Z)\}.$$

Proof. (It is basically the same as that of the second part of Theorem 1 of [10].) We verify that conditions (a)–(d) and (f) in Theorem 8–15 of Heyman and Sobel [11] hold here. Conditions (b), (c), and (d) of the theorem are immediate. For condition (a), consider the (non-optimal) base-stock policy and let $B(I)$ denote its (infinite-horizon discounted) expected costs when initial inventory is I. By the non-negativity of L (and $Y(Z - I - R) + C_sR$), f_n is monotone increasing, and since $f_n(I) \leqslant B(I)$ for every n, condition (a) is valid. Furthermore, for a given I we get from (1) that the optimal Q satisfies $Y(Q) + CR \leqslant B(I)$ because L and f_0 are non-negative. Thus, Q can be bounded from above and condition (*f*) is valid. □

Hence, a stationary policy (SL, SU) is optimal in the long run for the discounted-cost criterion. SU is then the maximum inventory level after a possible emergency order or disposal is made at a review epoch (if we ignore the first possible review epochs when $I > SU$).

Rosenshine and Obee [15] considered a storage capacity IMAX such that if inventory at a review epoch exceeds IMAX, the excess inventory is sold off (see Federgruen and Zipkin [9] for a related periodic problem with limited production capacity). Suppose that our basic model has such a storage constraint, i.e.,

$$f_n(I) = \min_{I\leqslant Z\leqslant \mathrm{IMAX}}\{Y(Z-I-R) + C_sR + L(Z) + \alpha Ef_{n-1}(t(Z-\xi))\}. \tag{13}$$

If the optimal SU obtained (by using Theorem 2 below) without the constraint $Z \leqslant \mathrm{IMAX}$ is less than or equal to IMAX, the storage capacity will not constitute an effective constraint. Otherwise, assume that $SU_n > \mathrm{IMAX}$ but $SL_n < \mathrm{IMAX}$ (if $\mathrm{IMAX} \leqslant SL_n$ as well, IMAX is the only operational parameter for $f_n(I)$ and the analysis is simplified and thus omitted). Then,

$$Z = \mathrm{IMAX}, \quad f_n(I) = C_s(\mathrm{IMAX} - I) + L(\mathrm{IMAX}) + \alpha Ef_{n-1}(t(\mathrm{IMAX}-\xi)) \quad \text{if } I+R \geqslant \mathrm{IMAX}, \tag{14}$$

$$Z = I + R, \quad f_n(I) = C_s R + L(I + R) + \alpha E f_{n-1}(t(I + R - \xi)) \quad \text{if } SL_n \leqslant I + R \leqslant \text{IMAX}, \tag{15}$$

$$Z = SL_n, \quad f_n(I) = C_s R + C_e(SL_n - I - R) + L(SL_n) + \alpha E f_{n-1}(t(SL_n - \xi)) \quad \text{if } I + R \leqslant SL_n. \tag{12}$$

Lemma 2. *If* IMAX $< SU_n$, $f_n(I)$ *is a convex function.*

Proof (By induction). $f_0(I)$ is convex. Assume that $f_{n-1}(I)$ is convex. $G_n(Z)$ is convex (as shown in the proof of Lemma 1). It follows from (8) that as SU_n minimizes $C_s Z + G_n(Z)$ and IMAX $< SU_n$, $Df_n(I) < -C_s$ if $I + R <$ IMAX; on the other hand, we see from (14) that $Df_n(I) = -C_s$ if $I + R \geqslant$ IMAX. Also, by (7) and (12), $Df_n(I) = -C_e$ if $I + R \leqslant SL_n$ and $Df_n(I) \geqslant -C_e$ if $I + R \geqslant SL_n$. Since $C_s < C_e$ and $f_n(I)$ is convex for $SL_n \leqslant I + R \leqslant$ IMAX by (15), it follows that $f_n(I)$ is convex. □

Also, Theorem 1 holds here (without lim $SU_n = SU$ that minimizes $C_s Z + G(Z)$).

3. Computing *SL* and *SU*

Theorem 1 does not reveal how to obtain SL and SU. Conjecturing that an (SL, SU) policy continues to be optimal over an infinite horizon for the average-cost criterion, Henig et al. used the Markov chain approach for computing SL and SU. Here, we conjecture as well that an (SL, SU) policy is optimal for our more general model if the long-run average cost is to be minimized, and suggest using Theorem 2 for computing SL and SU under either the average-cost or discounted-cost criterion.

Theorem 2. *If there exists some n such that*

(a) $SU_n = SU_{n-1}$,
(b) $Df_n(I) = Df_{n-1}(I)$ *for* $I \leqslant SU_n$, *then* $SU_i = SU_n$ *and* $SL_i = SL_n$ *for* $i \geqslant n + 1$.

Proof. If $SU_n = SU_{n-1}$ and $Df_n(I) = Df_{n-1}(I)$ for $I \leqslant SU_n$, it follows from (5) that $DG_{n+1}(Z) = DG_n(Z)$ for $I \leqslant SU_n$. As SU_n minimizes $C_s Z + G_n(Z)$, it also minimizes $C_s Z + G_{n+1}(Z)$, i.e., $SU_{n+1} = SU_n$. Also, due to $SL_n < SU_n$, SL_n minimizes $C_e Z + G_{n+1}(Z)$ as well, i.e., $SL_{n+1} = SL_n$. In addition, by expressing $f_{n+1}(I)$ as in (9)–(12), it can be easily seen that $Df_{n+1}(I) = Df_n(I)$ for $I \leqslant SU_{n+1}$. Hence, the argument continues and $SU_i = SU_n$ and $SL_i = SL_n$ for all $i \geqslant n + 1$. □

As we see from Theorem 2, if conditions (a) and (b) are satisfied, the sequences $\{SL_i\}$ and $\{SU_i\}$ converge respectively to $SL = SL_n$ and $SU = SU_n$ and thus the dynamic programming computation can be stopped (note that if a storage constraint is included and effective, Theorem 2 involves only condition (b) with SU_n replaced by IMAX). See Chiang and Gutierrez [7] and Chiang [5] for a similar theorem that is applied to a backorder model in the two-supply-mode setting and a lost-sales model in the replenishment-cycle environment, respectively. SL and SU are then optimal operational parameters for the infinite-horizon model. Condition (a) is expected to be satisfied more quickly than condition (b), which is true of the following computation. The reason is that in most cases in practice there exists a minimum divisible quantity and demand occurs in a multiple of this quantity. Since demand in a period is non-negative and bounded, it follows that the state space for I is finite. Note that even if demand can occur in any finite non-negative amount, the state space must be discretized when implemented on a digital computer. Moreover, the space for SU_n is also finite, since the order quantity is also bounded in practice and orders will be placed in a multiple of the above divisible quantity.

As for condition (b), since $Df_n(I)$ can be any real number, to facilitate the computation, we use the following *approximation*: the first derivatives of two consecutive cost functions could be regarded as equal when

$$\underset{I \leqslant SU_n}{\text{Max}} |Df_n(I) - Df_{n-1}(I)| \leqslant \varepsilon. \tag{16}$$

If $\varepsilon = 0.02$, (16) was satisfied for all the 203 problems in Tables 1–4 (the average number of periods required is about 90). If $\varepsilon = 0.01$, (16) was satisfied for all but five problems; if $\varepsilon = 0.005$ instead, (16) failed to be met for 22 problems. For these problems not solved for the infinite horizon, the dynamic programming computation stopped in a period for which SL_n and SU_n are apparently incorrect (the computation was aborted).

To illustrate, consider the base case: $C = \$100$, $C_e = \$110$, $C_s = \$90$, $\mu = 5$ (with Poisson demand), $R = 5$, $\alpha = 1$, $h = \$1$, $p = \$20$. After solving, we find that $SL = 7$ and $SU = 16$. In addition, we vary the

Table 1
Computation of the optimal operational parameters for a backlogged standing order system

Input parameters			Operational parameters		With a storage constraint	
p	C_s	C	SL	SU	SL	SU
\$2	0	110	−3	23	−4	20
		150	−5	26	−5	20
		200	−7	30	−8	20
	50	110	−1	18	Same	
		150	−3	22	−3	20
		200	−5	26	−5	20
	90	110	2	12	Same	
		150	−1	18	Same	
		200	−3	23	−3	20
20	0	110	5	28	5	20
		150	5	32	5	20
		200	5	36	4	20
	50	110	6	22	6	20
		150	6	27	5	20
		200	5	31	5	20
	90	110	7	16	Same	
		150	6	22	6	20
		200	5	28	5	20
200	0	110	9	31	9	20
		150	9	35	8	20
		200	9	39	8	20
	50	110	9	25	9	20
		150	9	30	9	20
		200	9	34	8	20
	90	110	10	18	Same	
		150	9	25	9	20
		200	9	31	9	20

Data: $\mu = 5$ (with Poisson demand), $R = 5$, $\alpha = 1$, $h = \$1$, IMAX = 20.

Table 2
Computation of the optimal operational parameters for a lost-sales standing order system

Input parameters			Operational parameters		With a storage constraint	
π	C_s	C_e	SL	SU	SL	SU
\$202	0	110	8	30	8	20
		150	7	33	7	20
		200	2	34	2	20
	50	110	8	24	8	20
		150	7	28	7	20
		200	2	30	2	20
	90	110	9	17	Same	
		150	8	23	7	20
		200	2	26	2	20
220	0	110	8	30	8	20
		150	7	33	7	20
		200	5	36	4	20
	50	110	9	24	8	20
		150	8	28	7	20
		200	5	31	5	20
	90	110	9	18	Same	
		150	8	24	8	20
		200	5	28	5	20
400	0	110	9	31	9	20
		150	9	35	9	20
		200	9	39	8	20
	50	110	10	25	10	20
		150	9	30	9	20
		200	9	34	8	20
	90	110	10	19	Same	
		150	10	25	10	20
		200	9	31	9	20

Data: $\mu = 5$ (with Poisson demand), $R = 5$, $\alpha = 1$, $h = \$1$, IMAX = 20.

value of C_e, C_s, and p in the base case to investigate the effect of these input parameters on the optimal control policy. Table 1 reports computational results for 27 problems. As we see, SL is non-increasing in C_e and SU is non-decreasing in C_e, implying that emergency operations on both ends (whether purchases or disposals) are used less and less as C_e increases. Also, SU is non-increasing in C_s and SL is non-decreasing in C_s, indicating that emergency operations on both ends are used more frequently as C_s increases. In addition, as p increases, both SL and SU tend to increase to avoid running out of goods (i.e., there would be greater use of emergency purchases and lesser use of disposals).

In Table 2, we consider the lost-sales case and design the experiment such that π is equal to $p +$ largest C_e in Table 1, and observe similar results regarding how SL and SU will change due to an increase in the value of C_e, C_s, or p. Notice that if $SL \leqslant R$, emergency orders are never placed. This is found in six problems of Table 2 where the difference between π and C_e is small. In addition, we recall that the ordinary zero-time-lag lost-sales periodic problem could be viewed as a backorder model in which a credit of αC is given to each unit of

Table 3
Computation of the optimal operational parameters for a backlogged standing order system

Input parameters			Operational parameters		With a storage constraint	
p	C_s	C_e	SL	SU	SL	SU
\$2	0	110	−4	22	−4	20
		150	−6	25	−6	20
		200	−8	28	−9	20
	50	110	−1	17	Same	
		150	−4	21	−4	20
		200	−6	25	−6	20
	90	110	2	12	Same	
		150	−2	17	Same	
		200	−5	21	−5	20
20	0	110	5	28	5	20
		150	5	31	5	20
		200	5	35	4	20
	50	110	6	22	6	20
		150	5	26	5	20
		200	5	30	5	20
	90	110	7	15	Same	
		150	6	21	6	20
		200	5	26	5	20
200	0	110	9	31	9	20
		150	9	34	8	20
		200	9	38	8	20
	50	110	9	24	9	20
		150	9	29	9	20
		200	9	33	8	20
	90	110	10	18	Same	
		150	9	24	9	20
		200	9	29	9	20

Data: $\mu = 5$ (with Poisson demand), $R = 5$, $\alpha = 0.999$, $h = \$1$, IMAX = 20.

demand actually backlogged [18]. Here, if the lost-sales standing order model yields an optimal SL that is greater than or equal to R, it could also be viewed as a backorder model where a credit of αC_e is given to each unit of demand actually backlogged, i.e., $L(\cdot)$ is given by

$$L(X) = \int_0^{X^+} h(X-\xi)\varphi(\xi)\,d\xi + \int_{X^+}^{\infty} (\pi - \alpha C_e)(\xi - X)\varphi(\xi)\,d\xi. \quad (17)$$

There are nine problems in Table 2 where $\pi - \alpha C_e$ is equal to p in Table 1. Three problems do not yield SL that is greater than or equal to R and the other six have the same SL and SU as in Table 1.

Suppose that we add a storage constraint IMAX = 20 into the problems in Tables 1 and 2. The revised SU and SL are reported in the last two columns of Tables 1 and 2. As we see, when a storage constraint is included and effective, SL may decrease. This is because if there is a storage capacity IMAX which is below SU, the buyer is more likely to have to sell off goods, and is thus more averse to spending money on an emergency order, thus lowering SL.

Assume now that $\alpha = 0.999$ (other input parameters being equal). We solve the same problems in Table 1 and observe similar results, as shown in

Table 4
Computation of the optimal operational parameters for a backlogged standing order system

Input parameters			$R = 6$		$R = 5$		$R = 4$	
p	C_s	C_e	SL	SU	SL	SU	SL	SU
\$2	0	110	−33	7	−4	22	5	111
		150	−40	7	−6	25	5	146
		200	−47	7	−8	28	5	188
	50	110	−22	7	−1	17	5	62
		150	−33	7	−4	21	5	97
		200	−42	7	−6	25	5	140
	90	110	− 6	7	2	12	5	24
		150	−22	7	−2	17	5	60
		200	−35	7	−5	21	5	102
20	0	110	1	13	5	28	8	113
		150	− 2	13	5	31	8	149
		200	− 4	13	5	35	8	191
	50	110	3	12	6	22	8	65
		150	1	13	5	26	8	100
		200	− 2	13	5	30	8	143
	90	110	5	11	7	15	8	27
		150	3	12	6	21	8	63
		200	0	12	5	26	8	105
200	0	110	7	16	9	31	11	116
		150	6	17	9	34	11	151
		200	6	17	9	38	11	194
	50	110	8	15	9	24	11	67
		150	7	16	9	29	11	103
		200	6	17	9	33	11	145
	90	110	9	14	10	18	11	30
		150	8	15	9	24	11	65
		200	7	16	9	29	11	108

Data: $\mu = 5$ (with Poisson demand), $\alpha = 0.999$, $h = \$1$.

Table 3. If we compare results in these two tables, SL and SU in Table 3 are less than or equal to their respective counterparts in Table 1. This is possibly due to the fact that shortage becomes less costly if the discounted-cost criterion is used.

Moreover, we vary R for the 27 problems in Table 3. As we see from Table 4, as R is larger, the system is enabled to operate with a smaller amount of inventory, i.e., both SL and SU tend to decrease. This is because as R is larger, the amount of inventory bought at the cheaper C (as opposed to C_e) increases, thus increasing the willingness of the system to dispose of inventory more easily (i.e., decreasing SU) as well as wait for the next shipment rather than placing an emergency order (i.e., decreasing SL). In addition, if $R < \mu$ and C_e is large, SU could be very high, indicating that the system probably will never dispose of inventory, and if $R > \mu$ and p is small, SL could be very low, implying that emergency orders are probably never placed.

4. Conclusion

In this paper, we propose a dynamic programming model for the standing order inventory system where a fixed quantity is delivered to the buyer in each period. The proposed basic model incorporates both the backlogged and lost-sales cases (note that the model can actually handle the partial backlogging case by writing $t(X) = (X)^+ - b(-X)^+$ where b is the fraction of excess demand backlogged, and expressing L appropriately). It also can include a possible storage constraint. Also, Henig et al.'s model is a special case of the basic model with the unit sell-off revenue equal to zero.

Since demand is stochastic in the real world, a standing order system must allow for sell-offs and emergency orders. It is shown that the optimal control policy is governed by the two operational parameters: the dispose-down-to level and order-up-to level, and these two parameters can be computed by using a convergence theorem. Computational results show that as the emergency unit item cost increases or as the unit sell-off revenue decreases, the optimal dispose-down-to level may increase while the optimal order-up-to level may decrease.

Notice that we assume throughout the whole paper that the fixed cost for sell-offs and emergency orders is zero or negligible. It is possible that the fixed cost for sell-offs and/or emergency orders is not negligible. This provides a future research direction. Also, it is assumed that the size of standing orders is not a decision variable of the basic model, i.e., the issue of the *optimal* standing order size is not examined in this paper. It seems that the optimal standing order size depends on the unit item cost, the emergency unit item cost, the unit sell-off revenue, and other cost parameters. This provides another research direction.

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