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Chaos in a fractional order modified Duffing system

Zheng-Ming Ge *, Chan-Yi Ou

Department of Mechanical Engineering, National Chiao Tung University, 1001 Ta Hsueh Road, Hsinchu 300, Taiwan, ROC

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Abstract

In this paper, the chaotic behaviors in a fractional order modified Duffing system are studied numerically by phase portraits, Poincaré maps and bifurcation diagrams. Linear transfer function approximations of the fractional integrator block are calculated for a set of fractional orders in $(0,1]$, based on frequency domain arguments. The total system orders found for chaos to exist in such systems are 1.8, 1.9, 2.0 and 2.1.

1. Introduction

Fractional calculus is a 300-year-old mathematical topic [1–4]. Although it has a long history, the applications of fractional calculus to physics and engineering are just a recent focus of interest. Many systems are known to display fractional order dynamics, such as viscoelastic systems, dielectric polarization [5], electrode electrolyte polarization [6], and electromagnetic waves [7]. More recently, many investigations are devoted to the control [8–12] and dynamics [13–26] of fractional order dynamical systems. In [13], it is shown that the fractional order Chua's circuit of order as low as 2.7 can produce a chaotic attractor. In [14], it is shown that nonautonomous Duffing systems of order less than 2 can still behave in a chaotic manner. In [15], chaotic behaviors of the fractional order "jerk" model is studied, in which chaotic attractor can be obtained with the system order as low as 2.1, and in [16] chaos control of this fractional order chaotic system is investigated. In [17], the fractional order Wien bridge oscillator is studied, where it is shown that limit cycle can be generated for any fractional order, with a proper value of the amplifier gain.

One way to study fractional order systems is through linear approximations. By utilizing frequency domain techniques based on Bode diagrams, one can obtain a linear approximation for the fractional order integrator, the order of which depends on the desired bandwidth and the discrepancy between the actual and the approximate magnitude Bode diagrams. This approach is applied to study the behaviors of the fractional order modified Duffing equations in this paper. We use the approximate linear transfer functions for the fractional integrator of order that varies from 0.1 to 0.9, and study the resulting behavior of the entire system for each case under the effect of different types of

* Corresponding author. Tel.: +886 35712121; fax: +886 3 5720634.
E-mail address: zmg@cc.nctu.edu.tw (Z.-M. Ge).

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nonlinearities. Chaotic behaviors in the fractional order modified Duffing equations are studied by phase portraits and bifurcation diagrams. It is found that the total system orders for chaos to exist in such systems are 1.8, 1.9, 2.0 and 2.1.

This paper is organized as follows. In Section 2 the fractional derivative and its approximation are introduced. In Section 3 the system under study is described both in its integer and fractional forms. In Section 4 numerical simulation results are presented, and in Section 5 conclusions are drawn.

2. Fractional derivative and its approximation

Two commonly used definitions for the general fractional differintegral are the Grunwald definition and the Riemann–Liouville definition . The Riemann–Liouville definition of the fractional integral is given here as [27]

$$\frac{\mathrm{d}^q f(t)}{\mathrm{d}t^q} = \frac{1}{\Gamma(-q)} \int_0^t \frac{f(\tau)}{(t-\tau)^{q+1}}\,\mathrm{d}\tau, \quad q < 0 \tag{1}$$

where q can have noninteger values, and thus the name fractional differintegral. Notice that the definition is based on integration and more importantly is a convolution integral for $q < 0$. When $q > 0$, then the usual integer nth derivative must be taken of the fractional $(q-n)$th integral, and yields the fractional derivative of order q as

$$\frac{\mathrm{d}^q f}{\mathrm{d}t^q} = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left[\frac{\mathrm{d}^{q-n} f}{\mathrm{d}t^{q-n}}\right], \quad q > 0 \quad \text{and } n \text{ an integer} > q \tag{2}$$

This appears so vastly different from the usual intuitive definition of derivative and integral that the reader must abandon the familiar concepts of slope and area and attempt to get some new insight. Fortunately, the basic engineering tool for analyzing linear systems, the Laplace transform, is still applicable and works as one would expect; that is

$$L\left\{\frac{\mathrm{d}^q f(t)}{\mathrm{d}t^q}\right\} = s^q L\{f(t)\} - \sum_{k=0}^{n-1} s^k \left[\frac{\mathrm{d}^{q-1-k} f(t)}{\mathrm{d}t^{q-1-k}}\right]_{t=0} \quad \text{for all } q \tag{3}$$

where n is an integer such that $n - 1 < q < n$. If the initial conditions are considered to be zero, this formula reduces to the more expected and comforting form

$$L\left\{\frac{\mathrm{d}^q f(t)}{\mathrm{d}t^q}\right\} = s^q L\{f(t)\} \tag{4}$$

An efficient method is to approximate fractional operators by using standard integer order operators. In [27], an effective algorithm is developed to approximate fractional order transfer functions. Basically, the idea is to approximate the system behavior in the frequency domain. By utilizing frequency domain techniques based on Bode diagrams, one can obtain a linear approximation of fractional order integrator, the order of which depends on the desired bandwidth and discrepancy between the actual and the approximate magnitude Bode diagrams. In Table 1 of [13], approximations for $1/s^q$ with $q = 0.1$–0.9 in steps 0.1 are given, with errors of approximately 2 dB. These approximations are used in following simulations.

3. A fractional order modified Duffing system

The famous Duffing system is

$$\ddot{x} + a\dot{x} + x + x^3 = b\cos\omega t \tag{5}$$

where a, b are constant parameters.

It can be written as two first order ordinary differential equations:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = y \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -x - x^3 - ay + b\cos\omega t \end{cases} \tag{6}$$

Consider the following modified Duffing system:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = y \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -x - x^3 - ay + bz \\ \dfrac{\mathrm{d}z}{\mathrm{d}t} = w \\ \dfrac{\mathrm{d}w}{\mathrm{d}t} = -cz - dz^3 \end{cases} \tag{7}$$

It becomes an autonomous system with four states where a, b, c, and d are constant parameters of the system. System (7) can divide into two parts:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = y \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -x - x^3 - ay + bz \end{cases} \tag{8}$$

and

$$\begin{cases} \dfrac{\mathrm{d}z}{\mathrm{d}t} = w \\ \dfrac{\mathrm{d}w}{\mathrm{d}t} = -cz - dz^3 \end{cases} \tag{9}$$

As a nonlinear oscillator, system (9) provide the periodic time function bz to system (8) as an excitation which produces the chaos in system (8). To sum up, system (8) can be considered as a nonautonomous system with two states x, y with bz as an excitation which is a given periodic function of time, while system (8) and system (9) together can be considered as an autonomous system with four states x, y, z, w. We focus on system (8), while system (9) remains an integral order system.

Now, consider a fractional order modified Duffing system. Here, the conventional derivatives in Eq. (8) are replaced by the fractional derivatives as follows:

$$\begin{cases} \dfrac{\mathrm{d}^{q_1}x}{\mathrm{d}t^{q_1}} = y \\ \dfrac{\mathrm{d}^{q_2}y}{\mathrm{d}t^{q_2}} = -x - x^3 - ay + bz \\ \dfrac{\mathrm{d}z}{\mathrm{d}t} = w \\ \dfrac{\mathrm{d}w}{\mathrm{d}t} = -cz - dz^3 \end{cases} \tag{10}$$

where system parameter b is allowed to be varied, and q_1, q_2 are two fractional order numbers. Simulations are then performed using q_i $(i = 1,2)$ varied from 0.1 to 0.9, respectively. The approximations from Table 1 of [13] are used for the simulations of the appropriate q_ith integrals. When $q_i < 1$, then the approximations are used directly. It should further be noted that approximations used in the simulations for $1/s^{q_i}$, when $q_i > 1$, are obtained by using $1/s$ times the approximation for $1/s^{q_i-1}$ from Table 1.

4. Simulation results

In this section, all numerical simulations are run by block diagrams in Simulink environment, using ode45 solver algorithm, where the fractional integrators have been approximated by linear time invariant transfer functions following the procedure in [13]. In so far as the attractor shape is concerned, both procedures gave very similar results. In numerical simulations, three parameters $a = 0.05$, $c = 1$ and $d = 0.3$ are fixed and b is varied. The initial states of the modified Duffing system are $x(0) = 0$, $y(0) = 0$, $z(0) = 10$ and $w(0) = 10$.

Firstly, when the total order $q_1 + q_2$ is 1.8, chaos is found in the cases: $(q_1,q_2) = (1.5,0.3)$, $(q_1,q_2) = (1.3,0.5)$, $(q_1,q_2) = (0.3,1.5)$, and $(q_1,q_2) = (0.5,1.3)$. The phase portraits, Poincaré maps and the bifurcation diagrams are showed in Figs. 1–4. Secondly, when the total order $q_1 + q_2$ is 1.9, chaos is found in the cases: $(q_1,q_2) = (1.8,0.1)$,

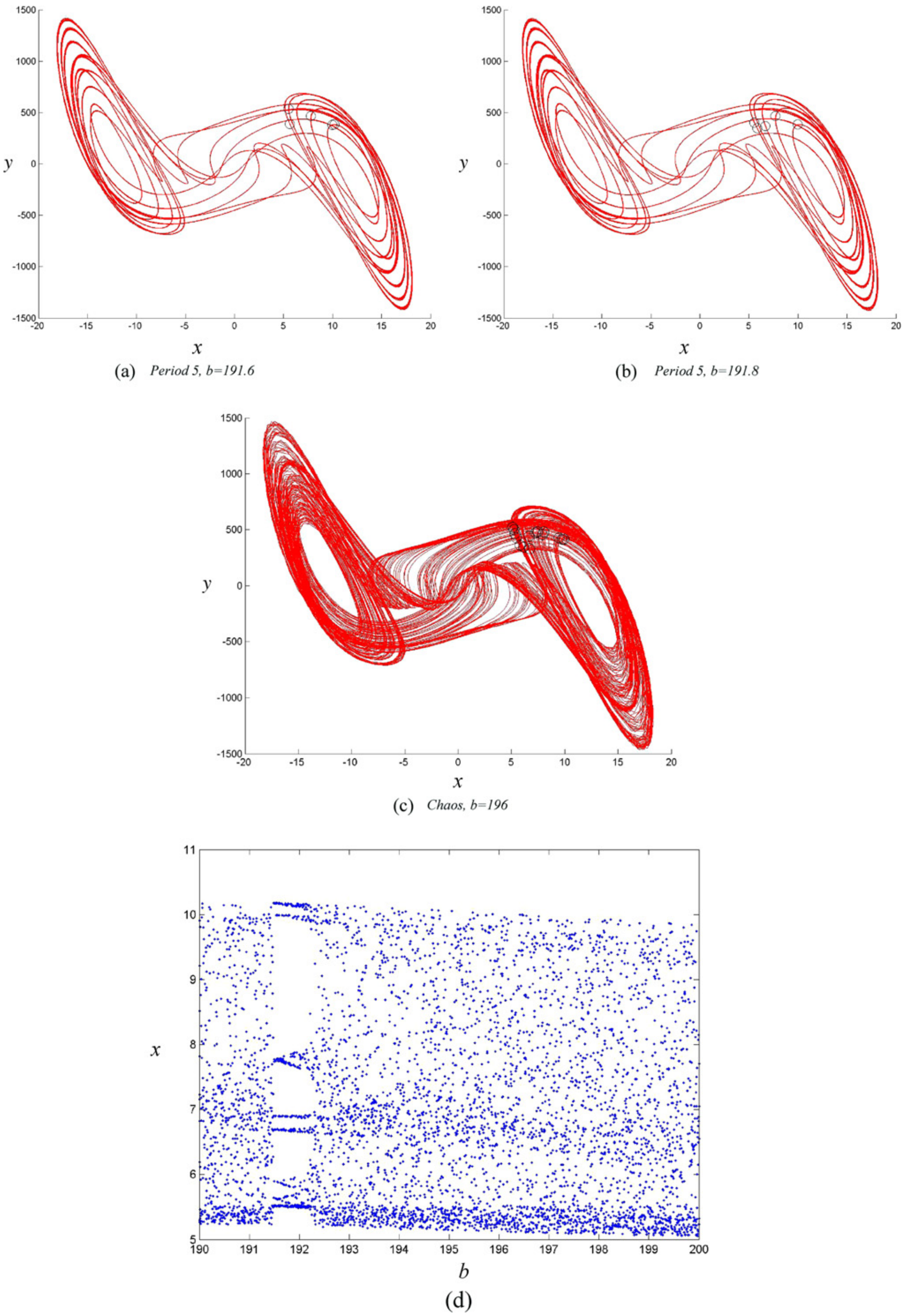


Fig. 1. The phase portraits, Poincaré maps and the bifurcation diagram for the fractional order modified Duffing system, x versus y and b versus x, $(q_1, q_2) = (1.5, 0.3)$.

$(q_1, q_2) = (1.6, 0.3)$, $(q_1, q_2) = (1.5, 0.4)$, $(q_1, q_2) = (1.4, 0.5)$, $(q_1, q_2) = (1.3, 0.6)$, $(q_1, q_2) = (1.1, 0.8)$, $(q_1, q_2) = (0.1, 1.8)$, $(q_1, q_2) = (0.3, 1.6)$, $(q_1, q_2) = (0.4, 1.5)$, $(q_1, q_2) = (0.5, 1.4)$, $(q_1, q_2) = (0.6, 1.3)$, and $(q_1, q_2) = (0.8, 1.1)$. The phase

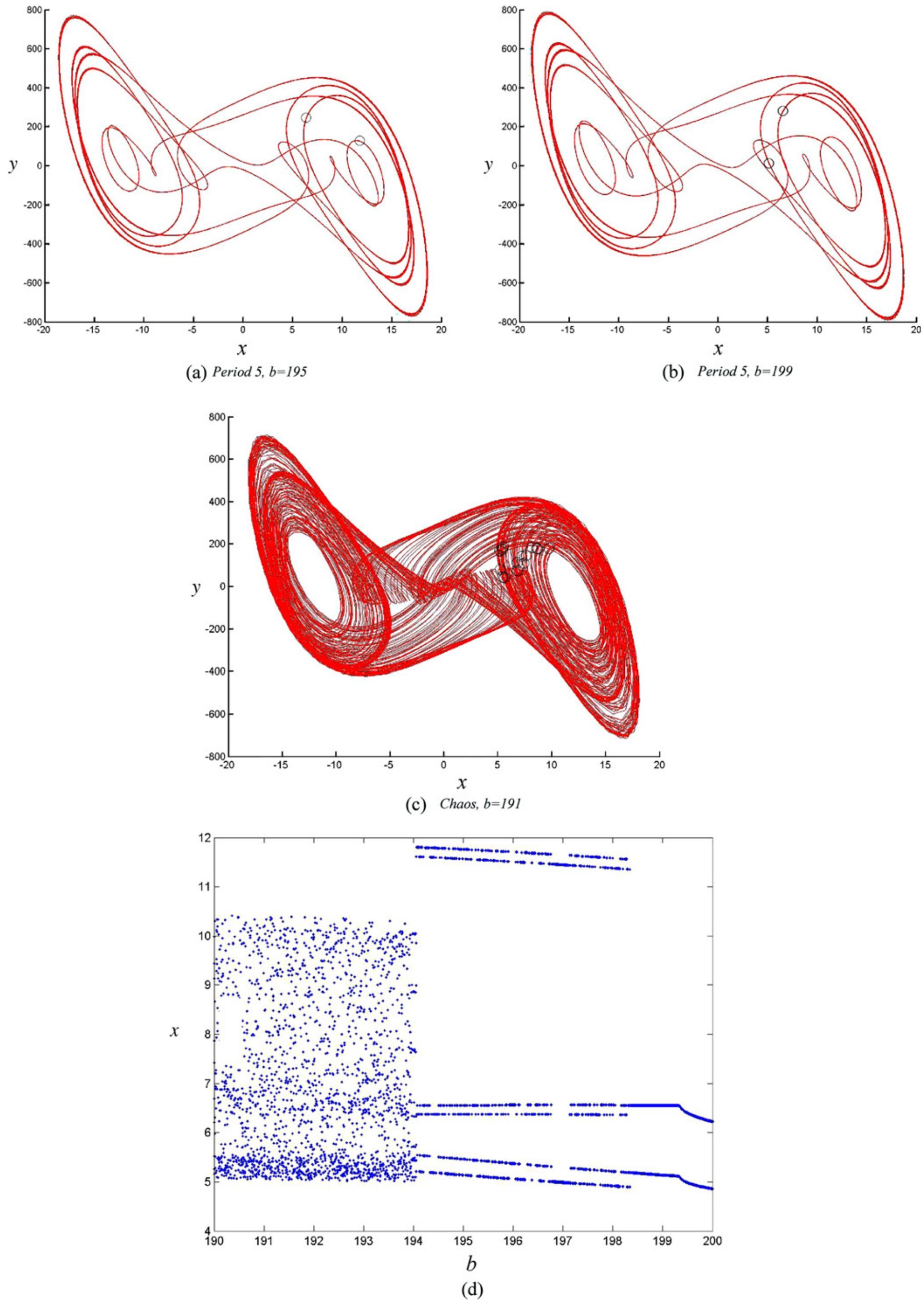


Fig. 2. The phase portraits, Poincaré maps and the bifurcation diagram for the fractional order modified Duffing system, x versus y and b versus x, $(q_1, q_2) = (1.3, 0.5)$.

portraits, Poincaré maps and the bifurcation diagrams are shown in Figs. 5–16. When the total order $q_1 + q_2$ is 2.0, chaos is found in the cases: $(q_1, q_2) = (1.9, 0.1)$, $(q_1, q_2) = (1.8, 0.2)$, $(q_1, q_2) = (1.2, 0.8)$, $(q_1, q_2) = (1.1, 0.9)$,

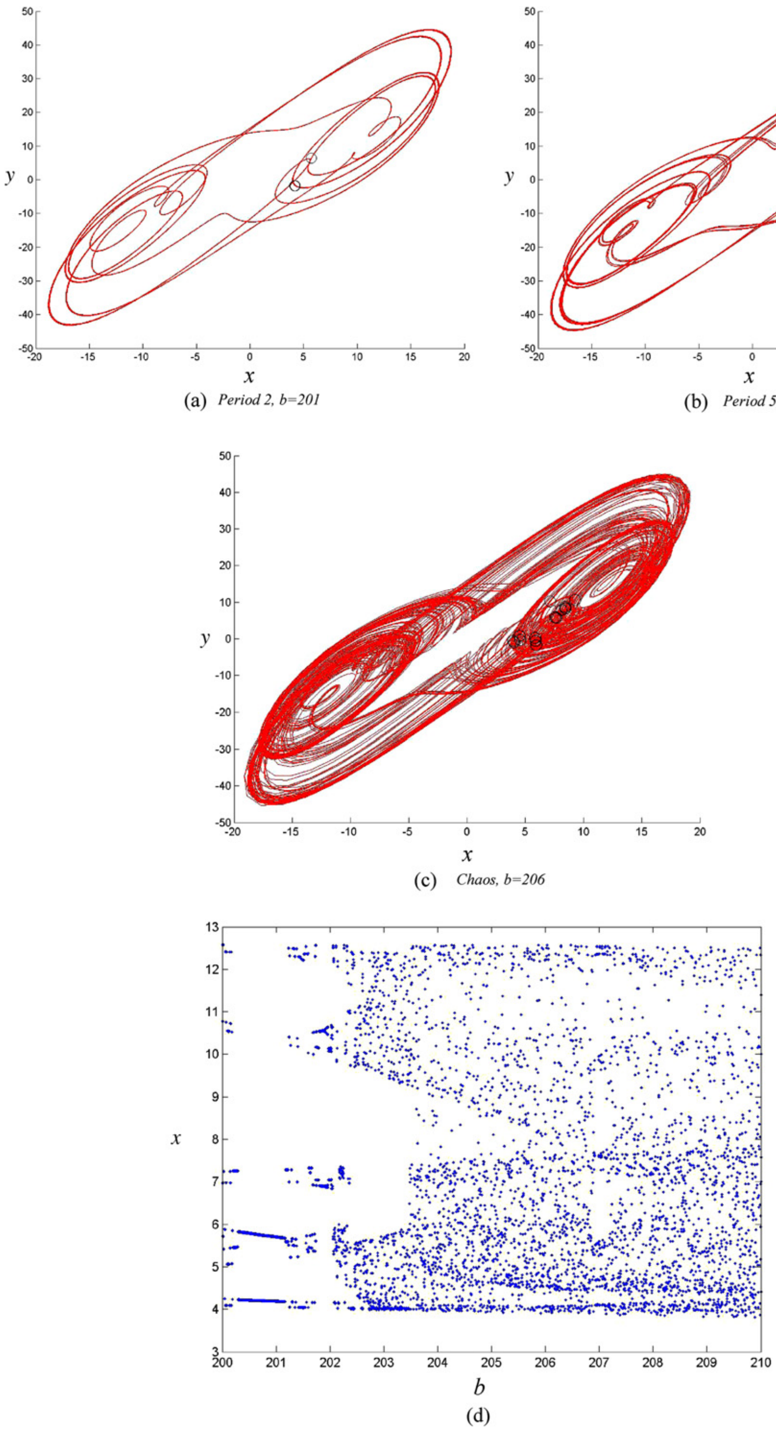


Fig. 3. The phase portraits, Poincaré maps and the bifurcation diagram for the fractional order modified Duffing system, x versus y and b versus x, $(q_1, q_2) = (0.3, 1.5)$.

$(q_1, q_2) = (0.2, 1.8)$, $(q_1, q_2) = (0.8, 1.2)$, and $(q_1, q_2) = (0.9, 1.1)$. The phase portraits, Poincaré maps and the bifurcation diagrams are shown in Figs. 17–23. Finally, when the total order 2.1, chaos is found in the cases: $(q_1, q_2) = (1.2, 0.9)$,

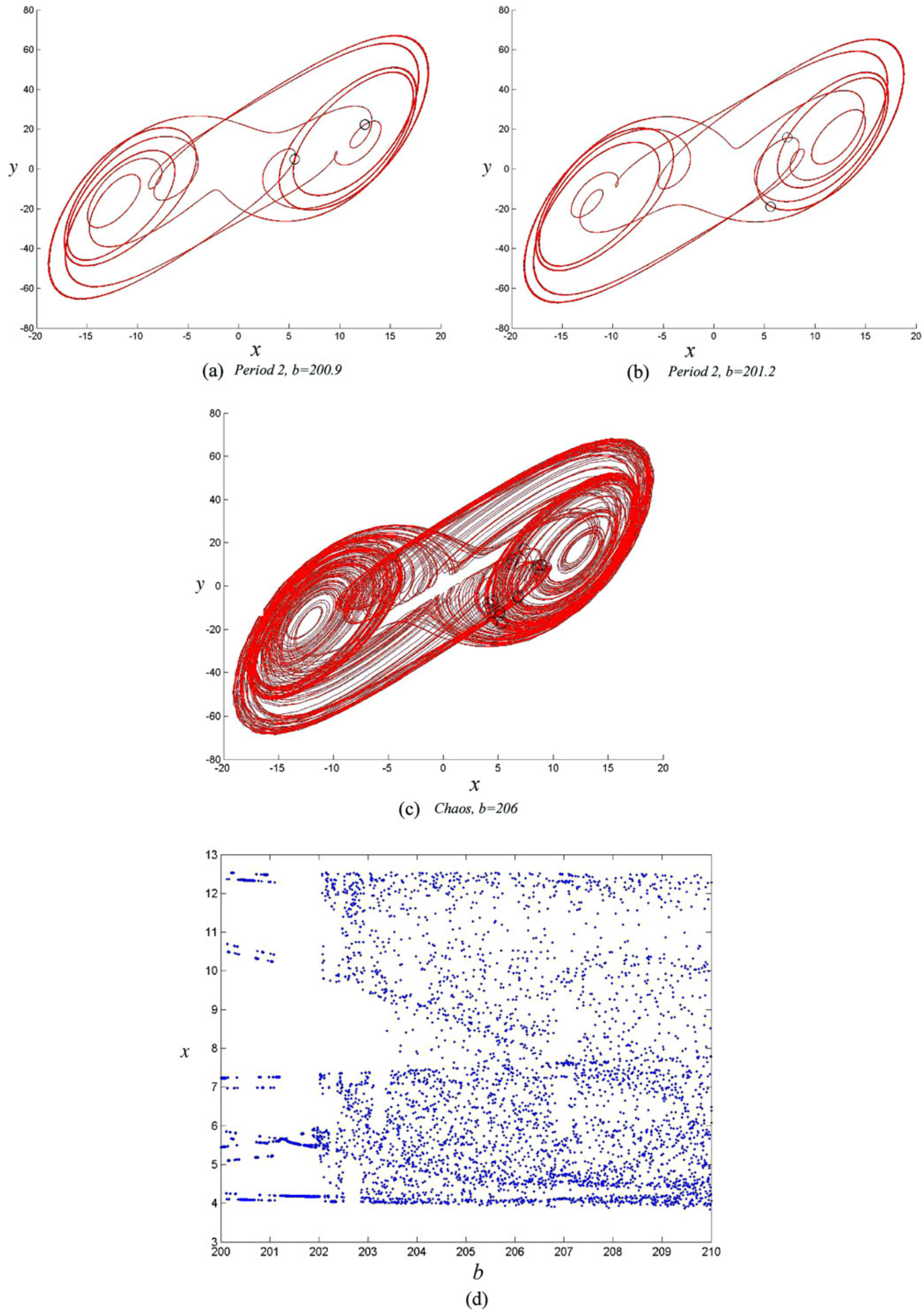


Fig. 4. The phase portraits, Poincaré maps and the bifurcation diagram for the fractional order modified Duffing system, x versus y and b versus x, $(q_1, q_2) = (0.5, 1.3)$.

$(q_1, q_2) = (0.2, 1.9)$, and $(q_1, q_2) = (1.2, 0.9)$. The phase portraits, Poincaré maps and the bifurcation diagrams are showed in Figs. 24–26. It can be seen that when q_1 is larger, the range of y state is also larger.

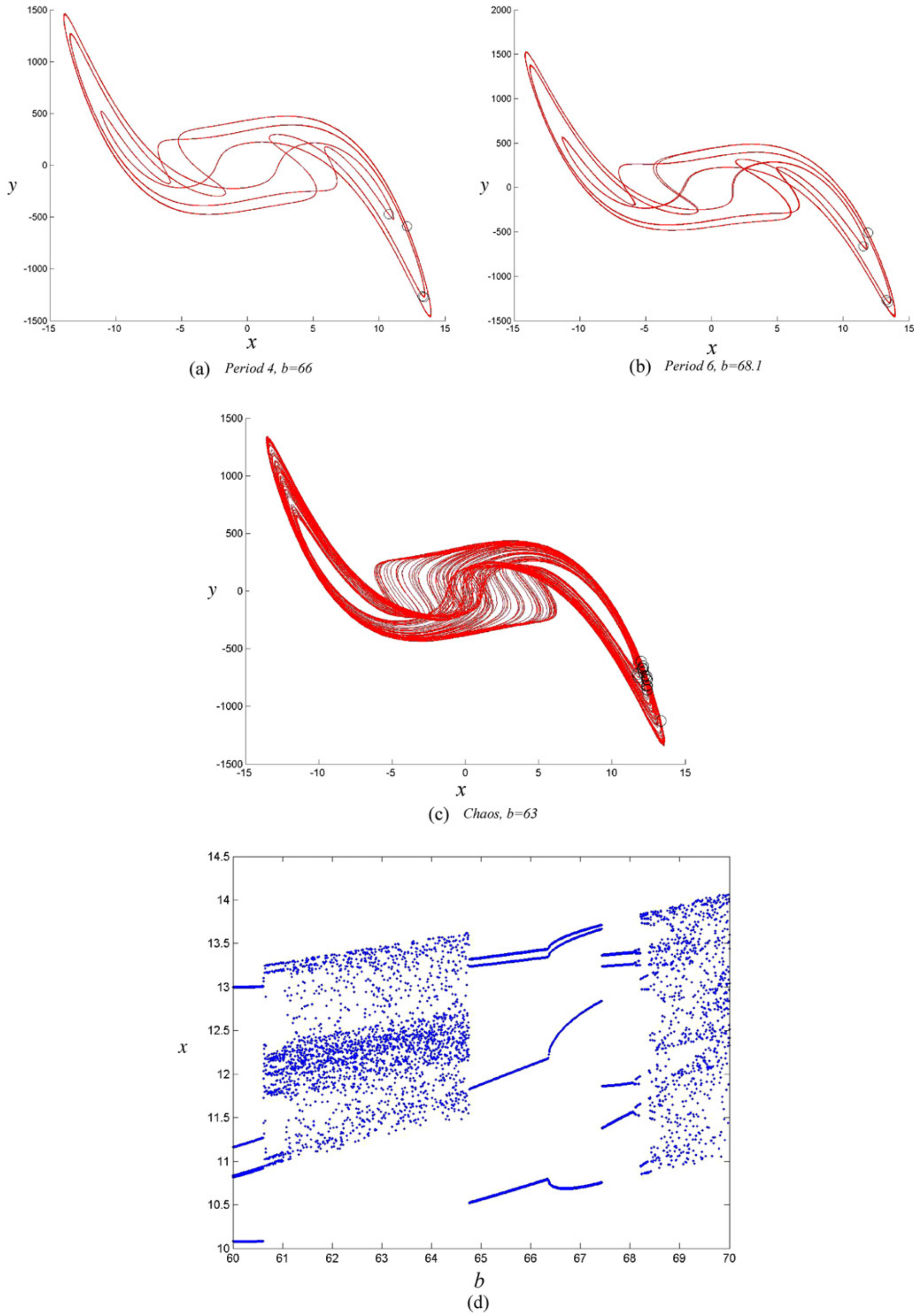


Fig. 5. The phase portraits, Poincaré maps and the bifurcation diagram for the fractional order modified Duffing system, x versus y and b versus x, $(q_1, q_2) = (1.8, 0.1)$.

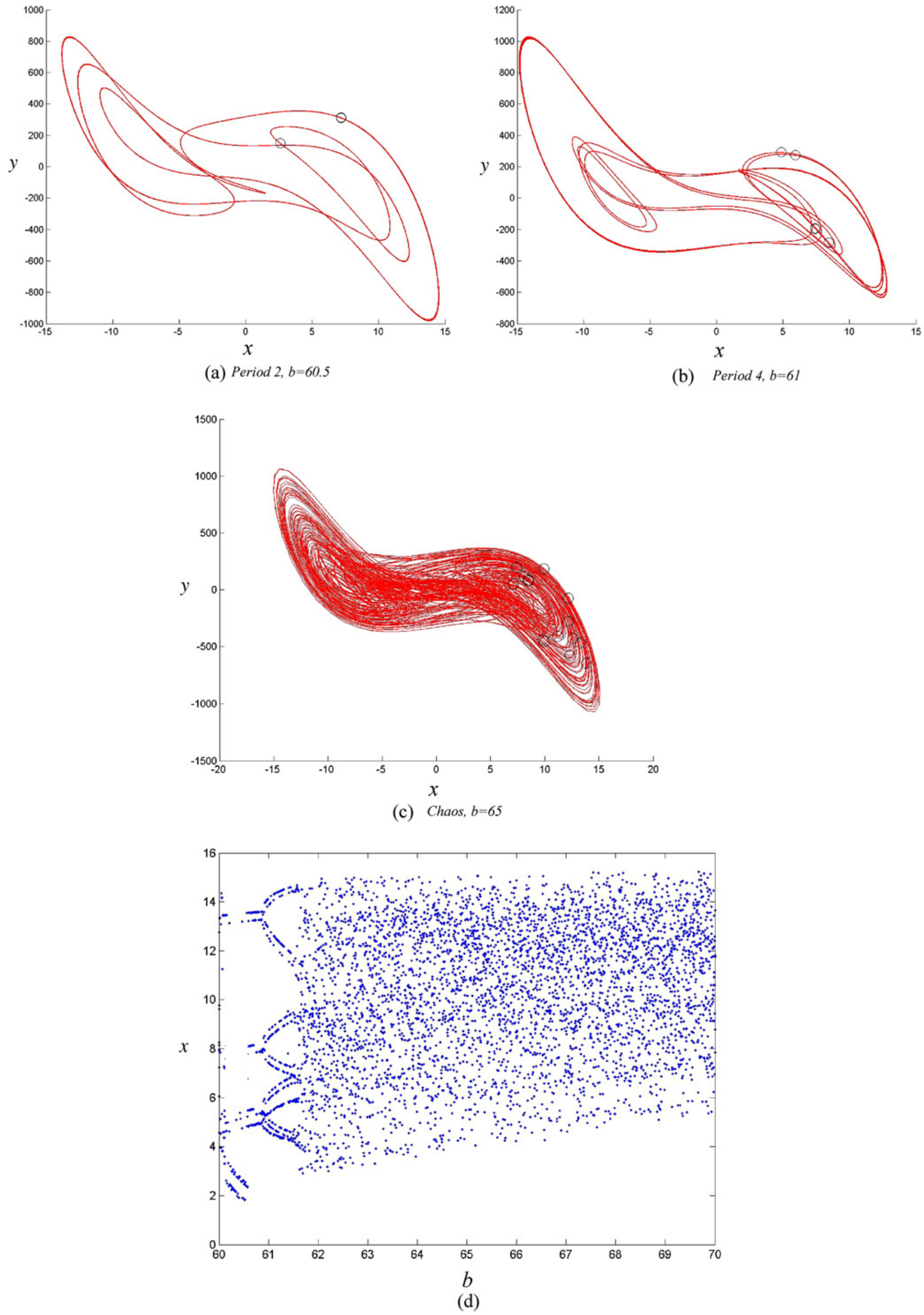


Fig. 6. The phase portraits, Poincaré maps and the bifurcation diagram for the fractional order modified Duffing system, x versus y and b versus x, $(q_1, q_2) = (1.6, 0.3)$.

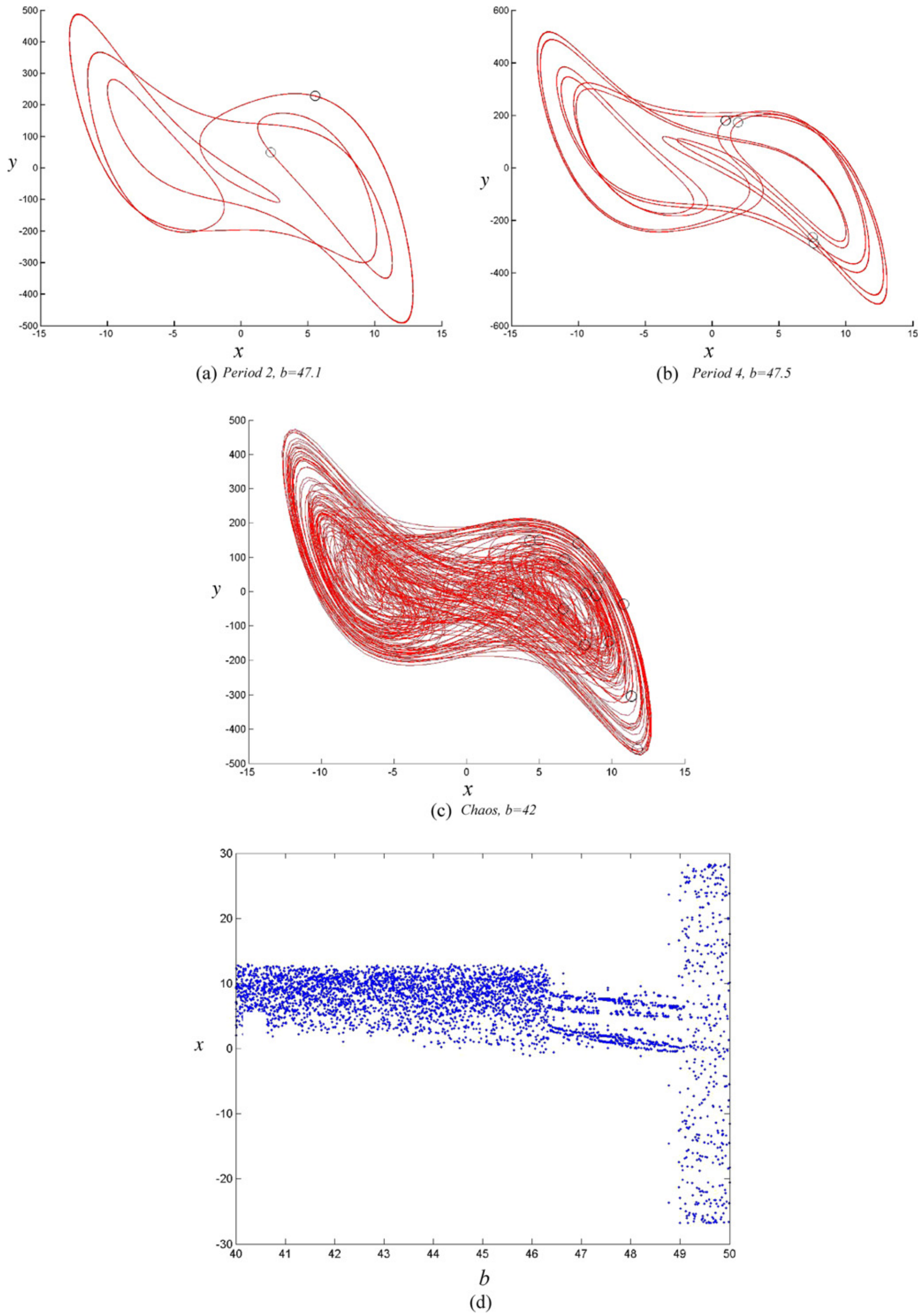


Fig. 7. The phase portraits, Poincaré maps and the bifurcation diagram for the fractional order modified Duffing system, x versus y and b versus x, $(q_1, q_2) = (1.5, 0.4)$.

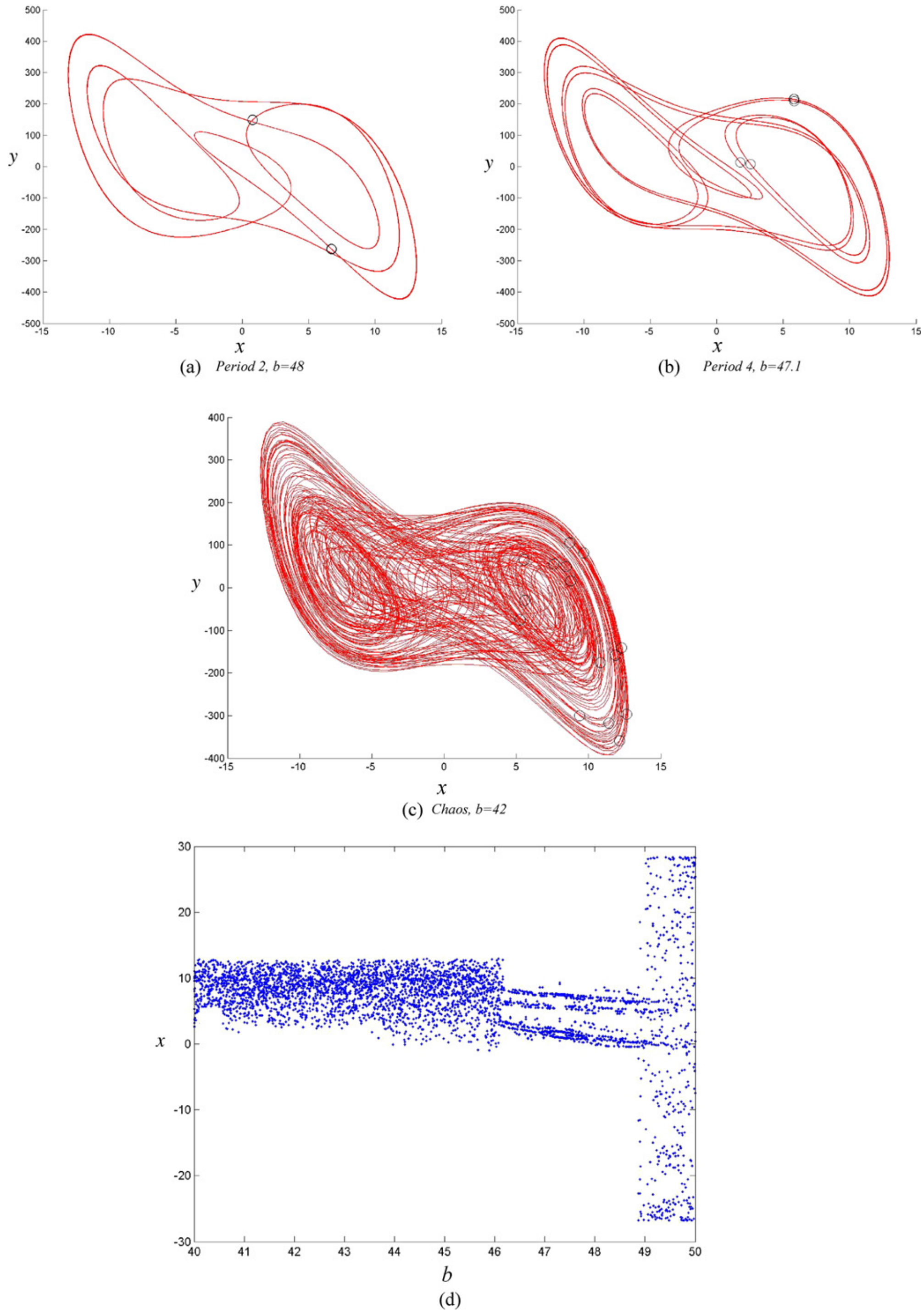


Fig. 8. The phase portraits, Poincaré maps and the bifurcation diagram for the fractional order modified Duffing system, x versus y and b versus x, $(q_1, q_2) = (1.4, 0.5)$.

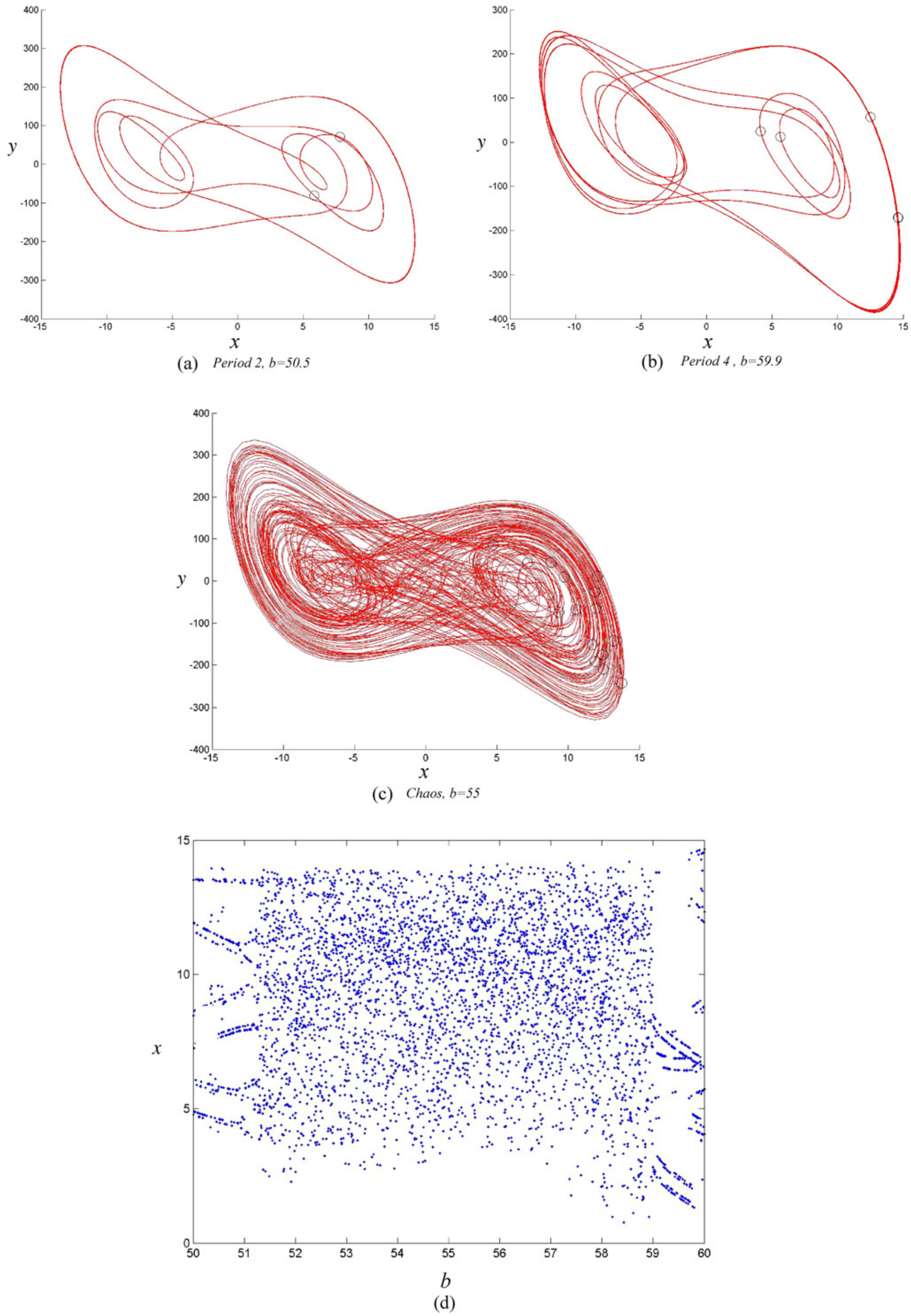
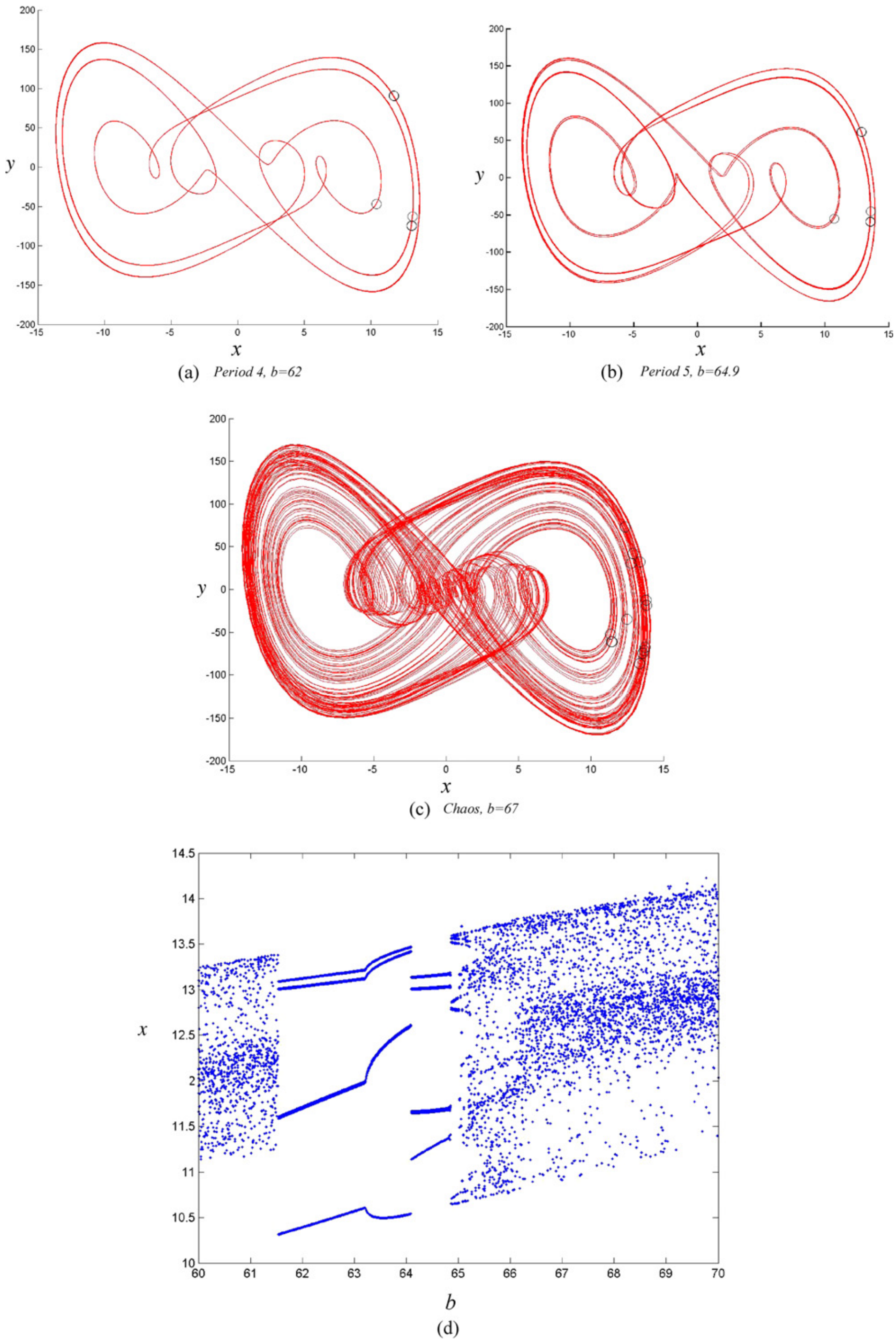


Fig. 9. The phase portraits, Poincaré maps and the bifurcation diagram for the fractional order modified Duffing system, x versus y and b versus x, $(q_1, q_2) = (1.3, 0.6)$.



(a) *Period 4, b=62*

(b) *Period 5, b=64.9*

(c) *Chaos, b=67*

(d)

Fig. 10. The phase portraits, Poincaré maps and the bifurcation diagram for the fractional order modified Duffing system, x versus y and b versus x, $(q_1, q_2) = (1.1, 0.8)$.

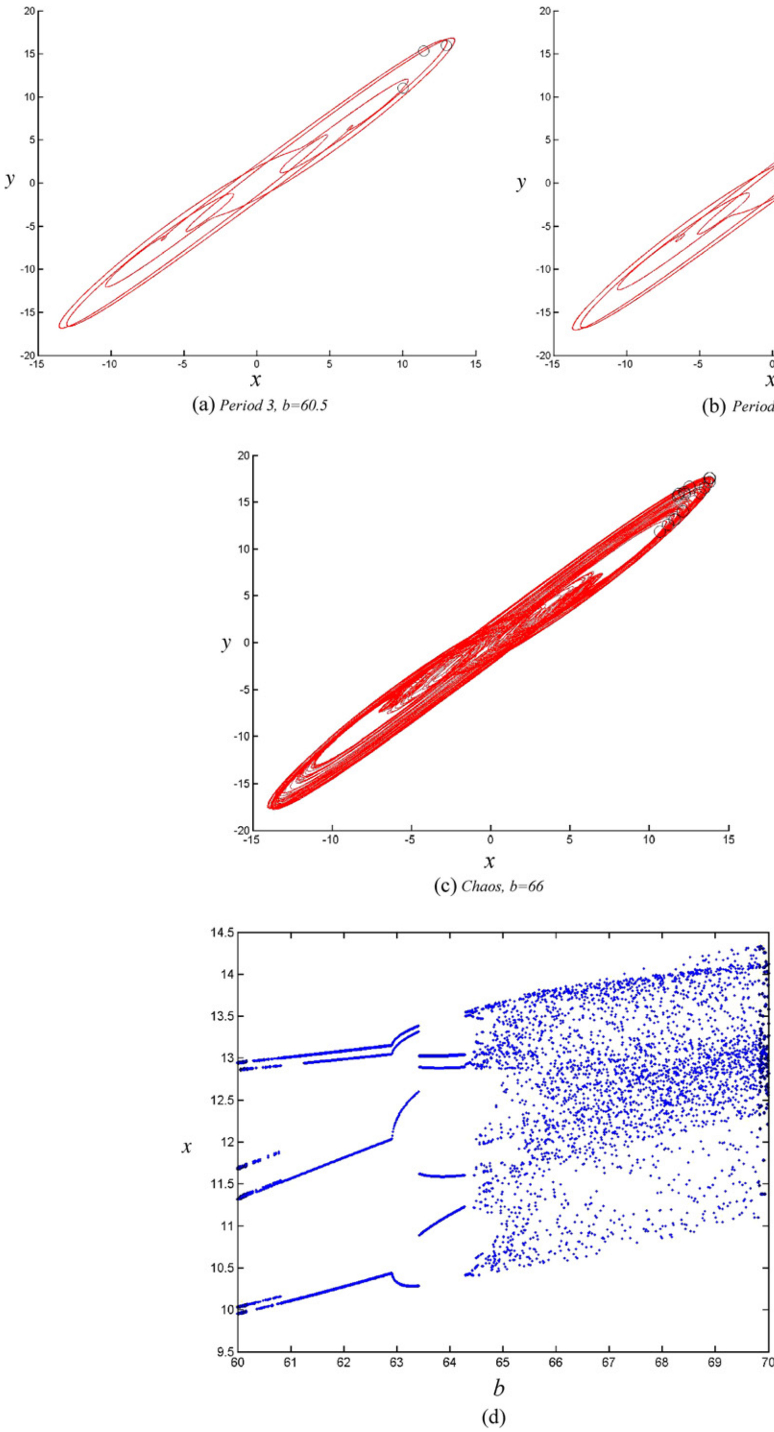


Fig. 11. The phase portraits, Poincaré maps and the bifurcation diagram for the fractional order modified Duffing system, x versus y and b versus x, $(q_1, q_2) = (0.1, 1.8)$.

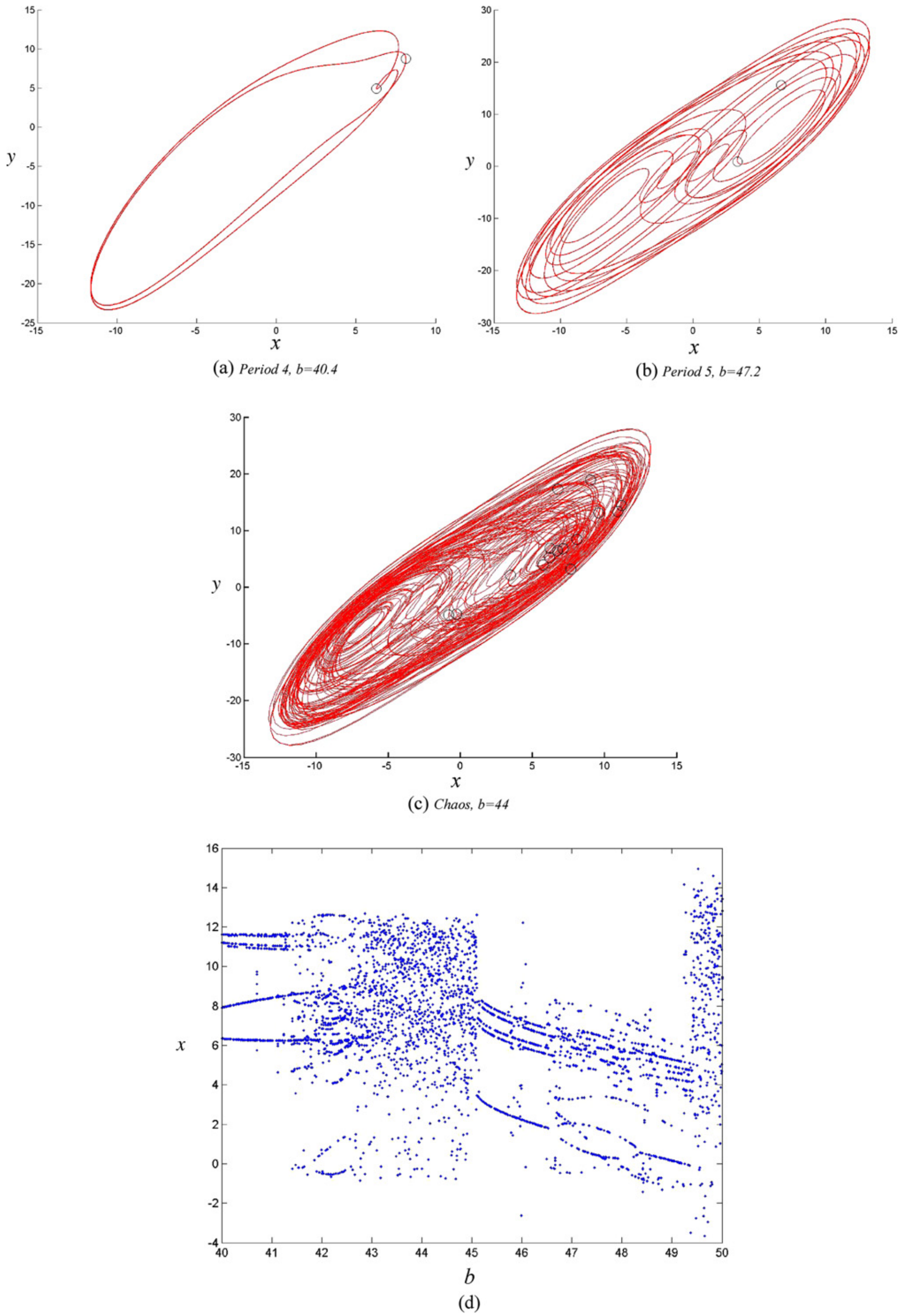


Fig. 12. The phase portraits, Poincaré maps and the bifurcation diagram for the fractional order modified Duffing system, x versus y and b versus x, $(q_1, q_2) = (0.3, 1.6)$.

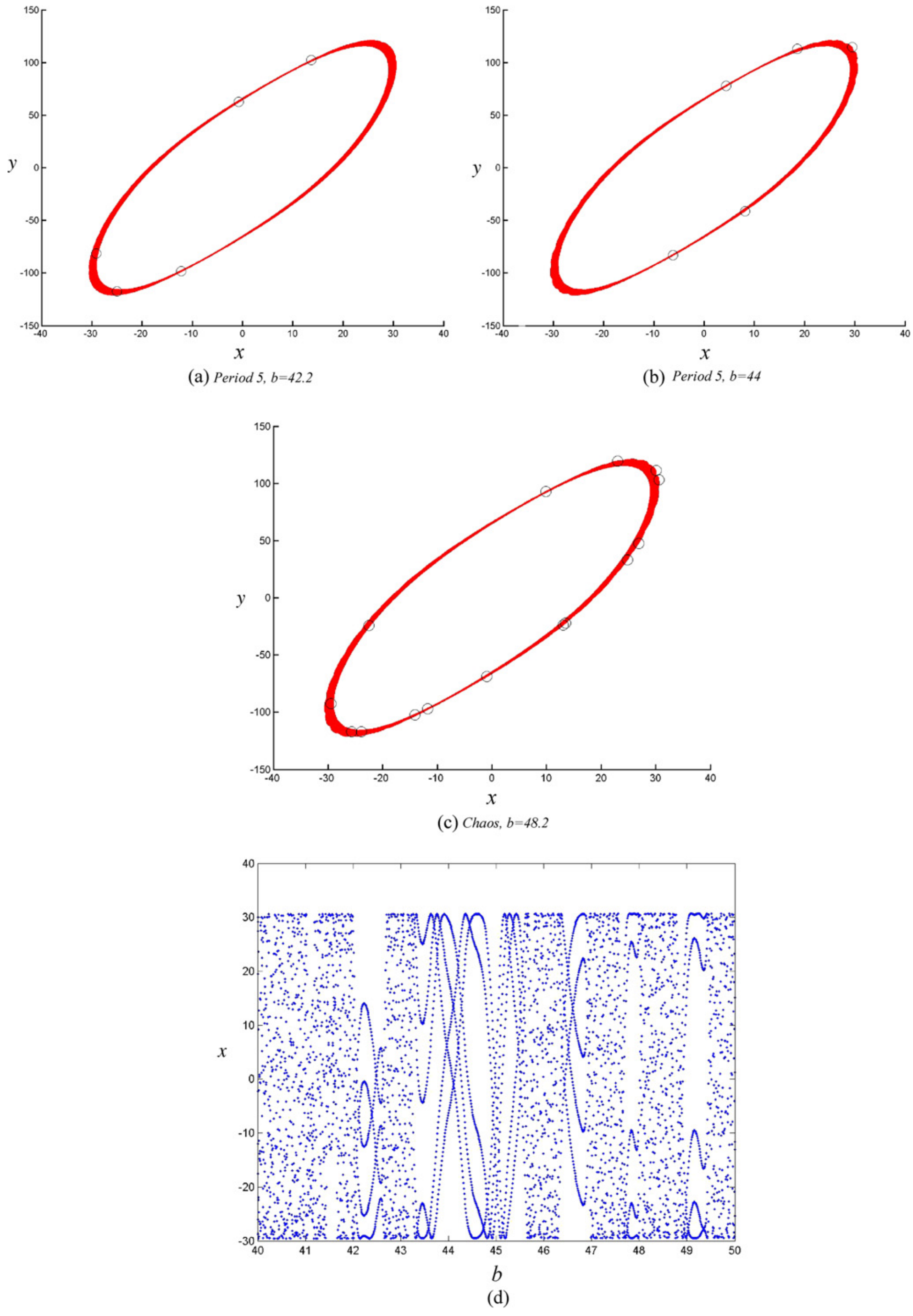


Fig. 13. The phase portraits, Poincaré maps and the bifurcation diagram for the fractional order modified Duffing system, x versus y and b versus x, $(q_1, q_2) = (0.4, 1.5)$.

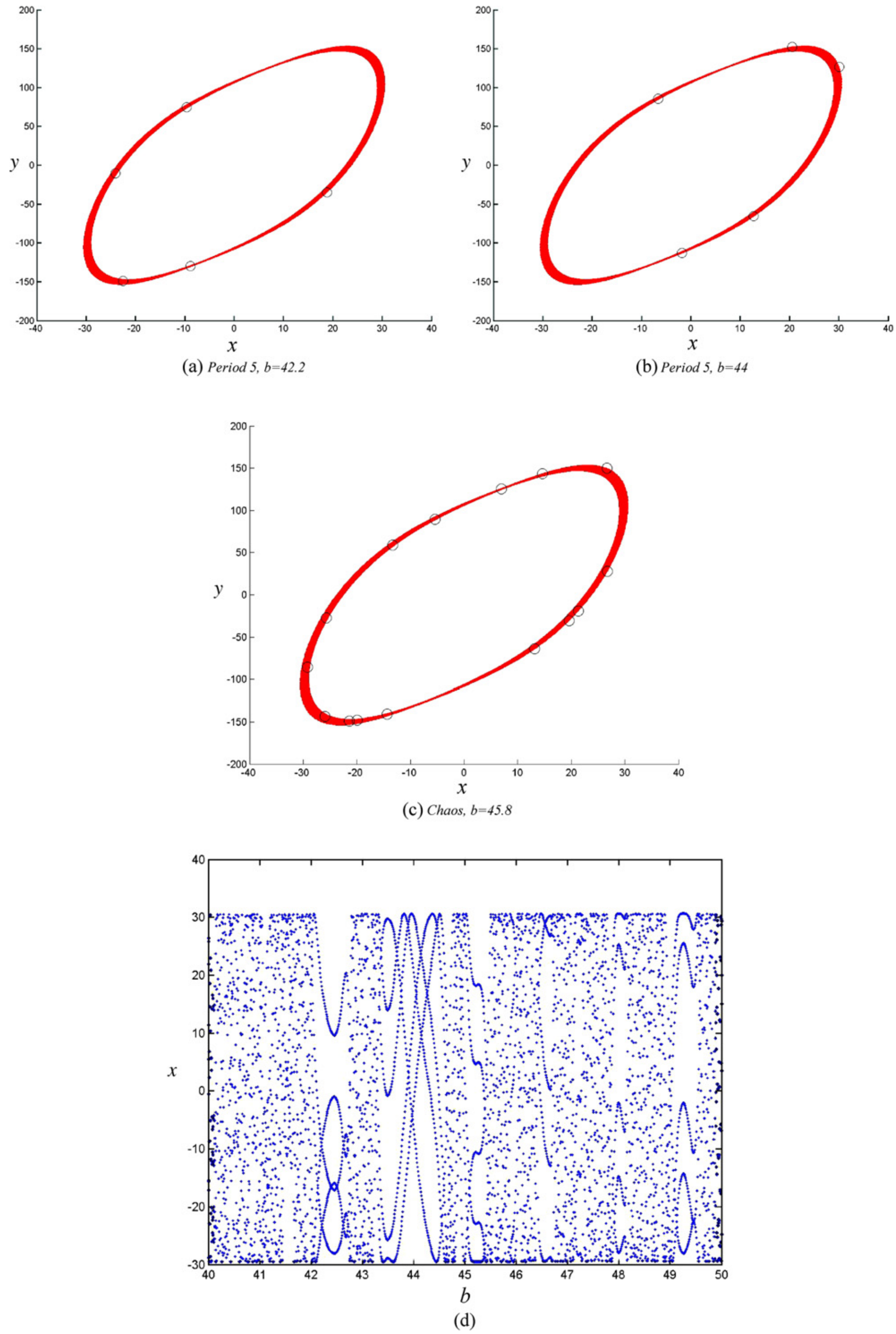


Fig. 14. The phase portraits, Poincaré maps and the bifurcation diagram for the fractional order modified Duffing system, x versus y and b versus x, $(q_1, q_2) = (0.5, 1.4)$.

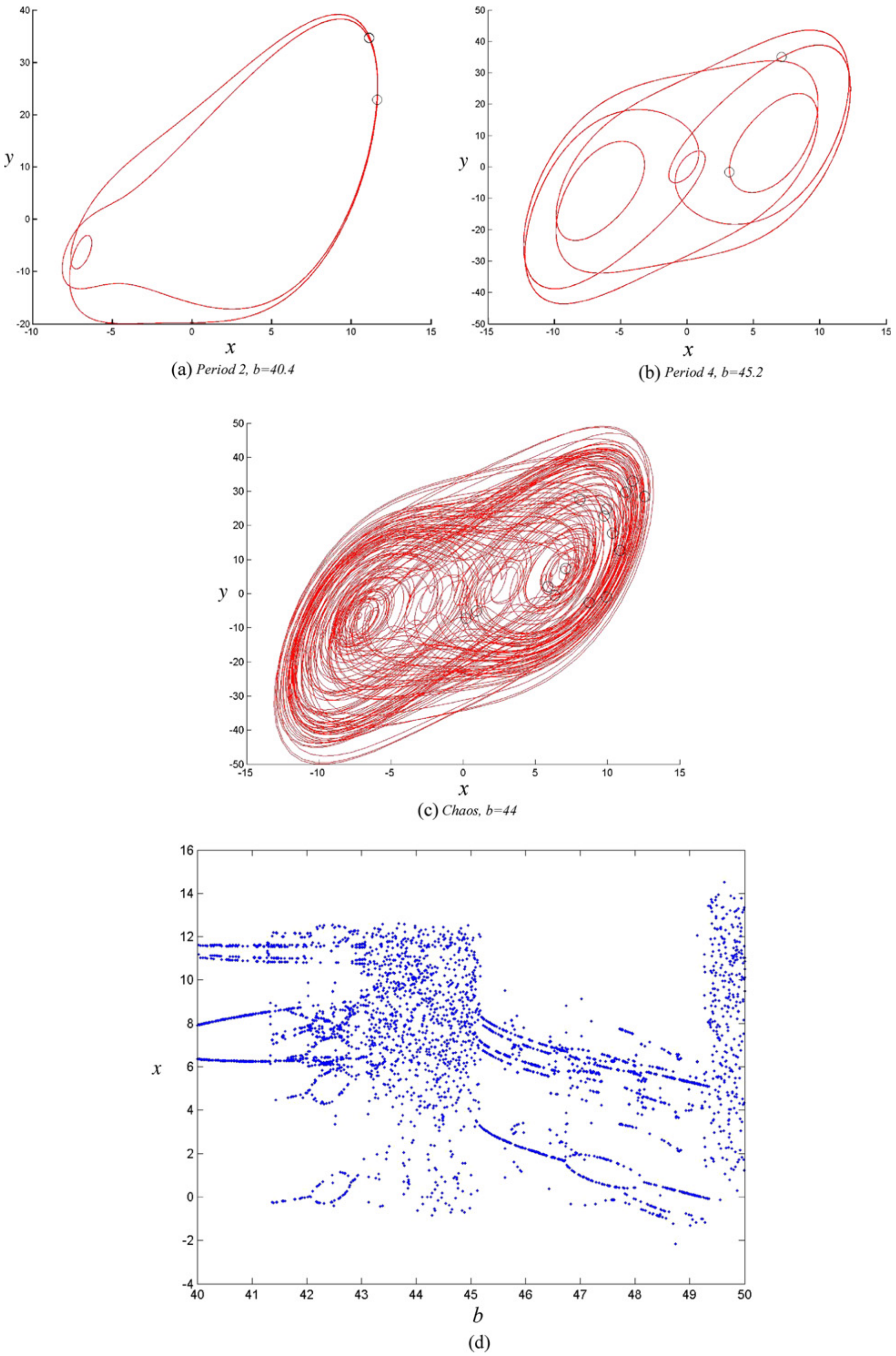
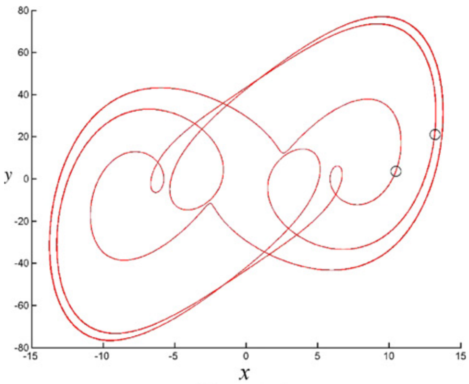
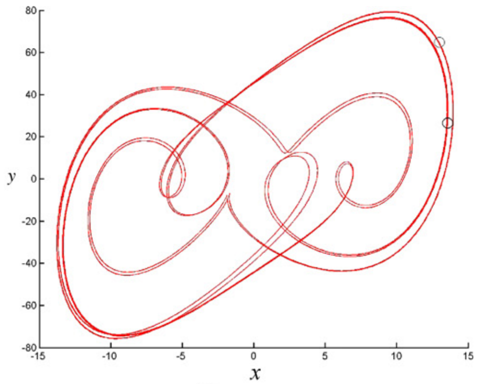


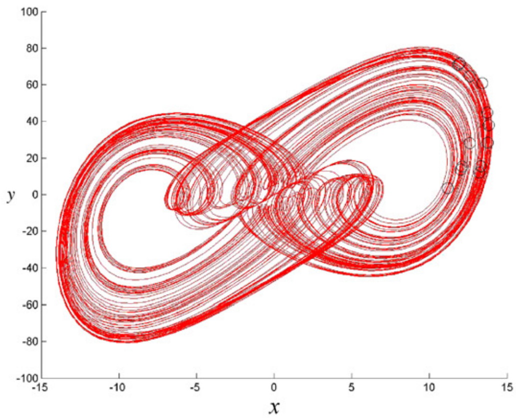
Fig. 15. The phase portraits, Poincaré maps and the bifurcation diagram for the fractional order modified Duffing system, x versus y and b versus x, $(q_1, q_2) = (0.6, 1.3)$.



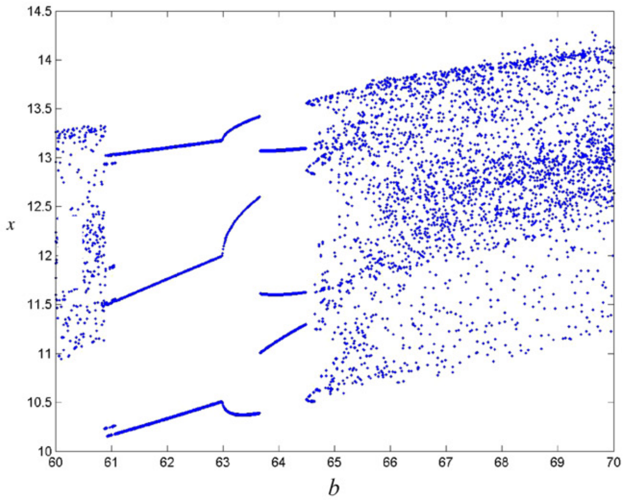
(a) *Period 3, b=63*



(b) *Period 4, b=64.7*

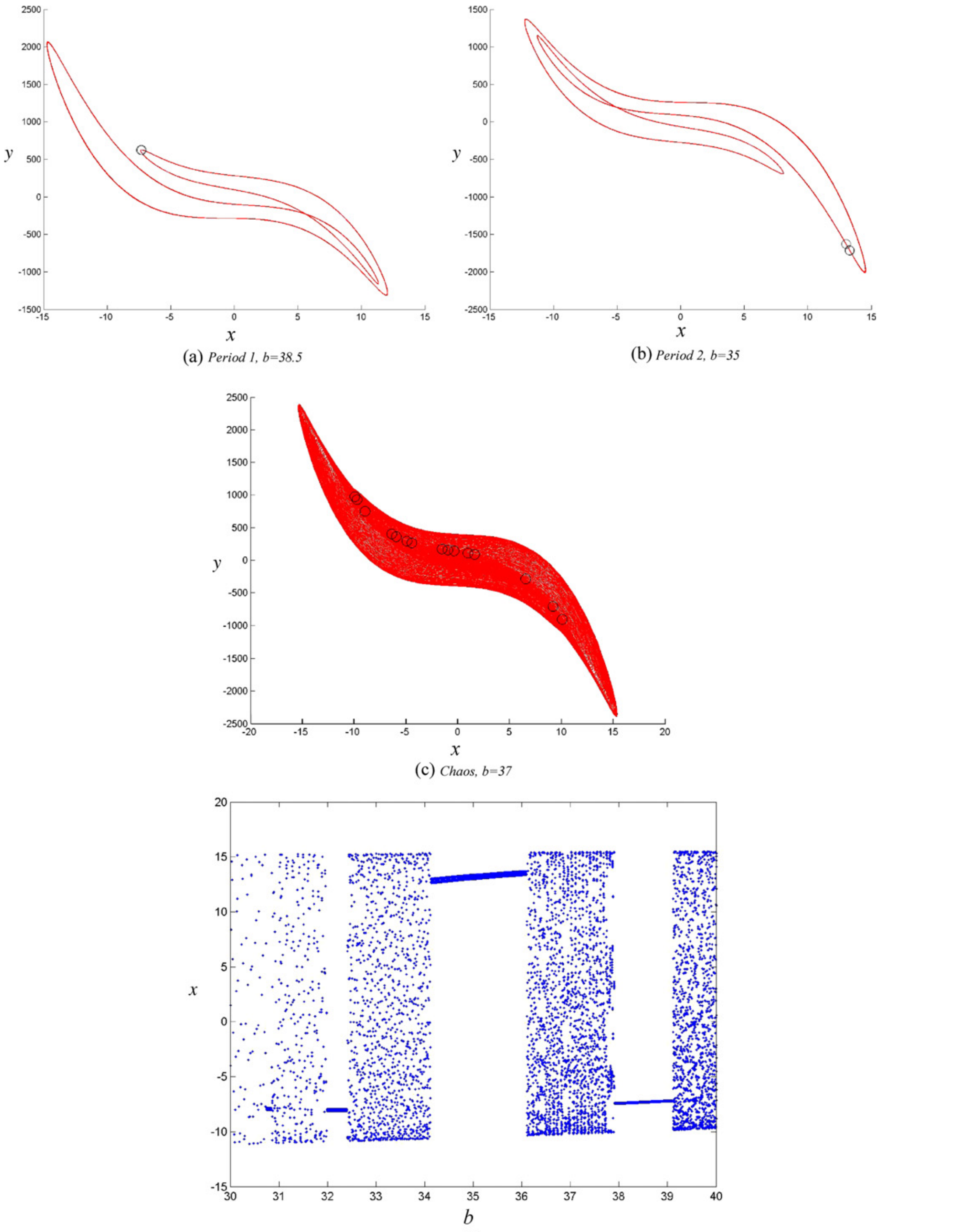


(c) *Chaos, b=66*



(d)

Fig. 16. The phase portraits, Poincaré maps and the bifurcation diagram for the fractional order modified Duffing system, x versus y and b versus x, $(q_1, q_2) = (0.8, 1.1)$.



(a) *Period 1, b=38.5*

(b) *Period 2, b=35*

(c) *Chaos, b=37*

(d)

Fig. 17. The phase portraits, Poincaré maps and the bifurcation diagram for the fractional order modified Duffing system, x versus y and b versus x, $(q_1, q_2) = (1.9, 0.1)$.

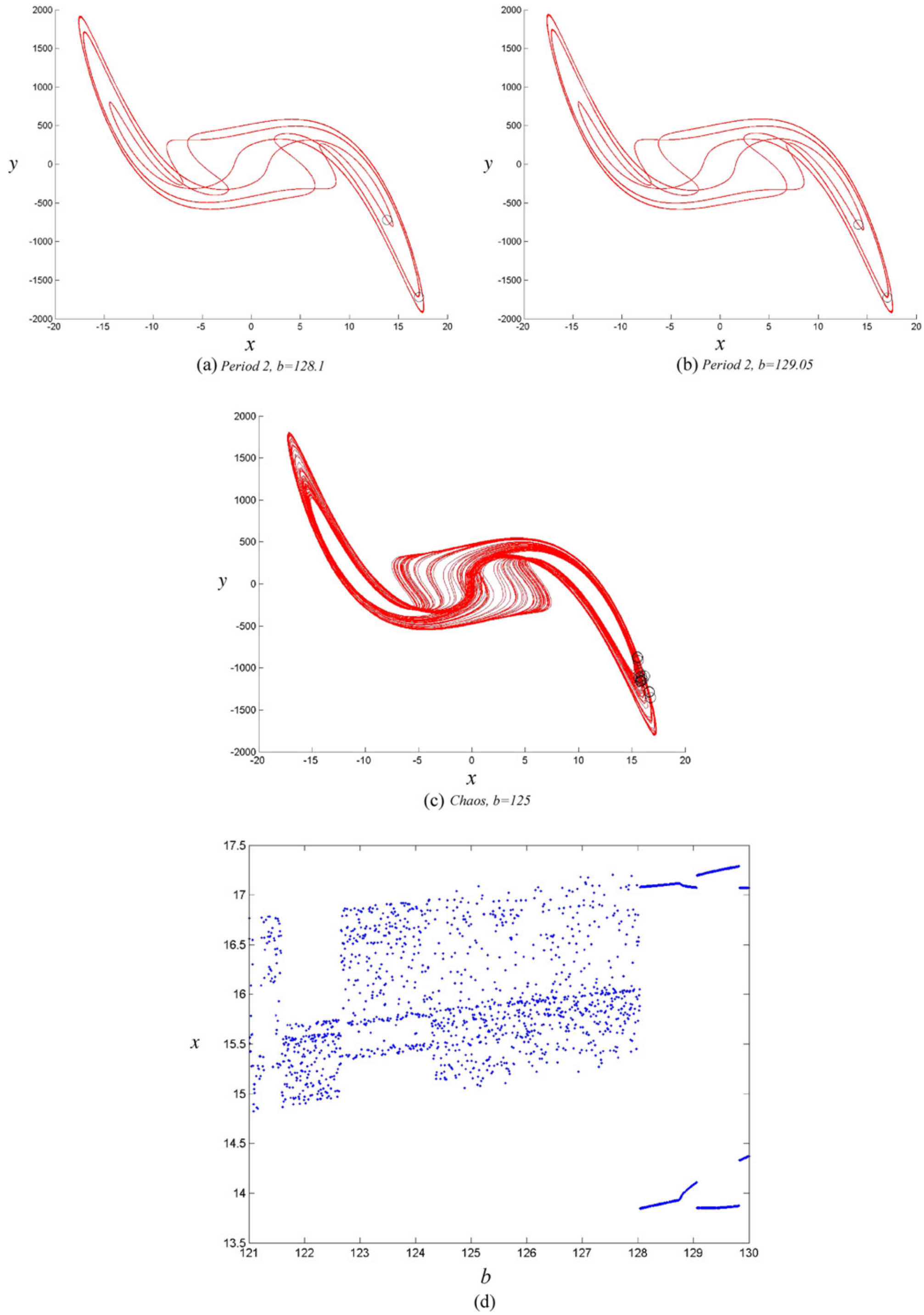


Fig. 18. The phase portraits, Poincaré maps and the bifurcation diagram for the fractional order modified Duffing system, x versus y and b versus x, $(q_1, q_2) = (1.8, 0.2)$.

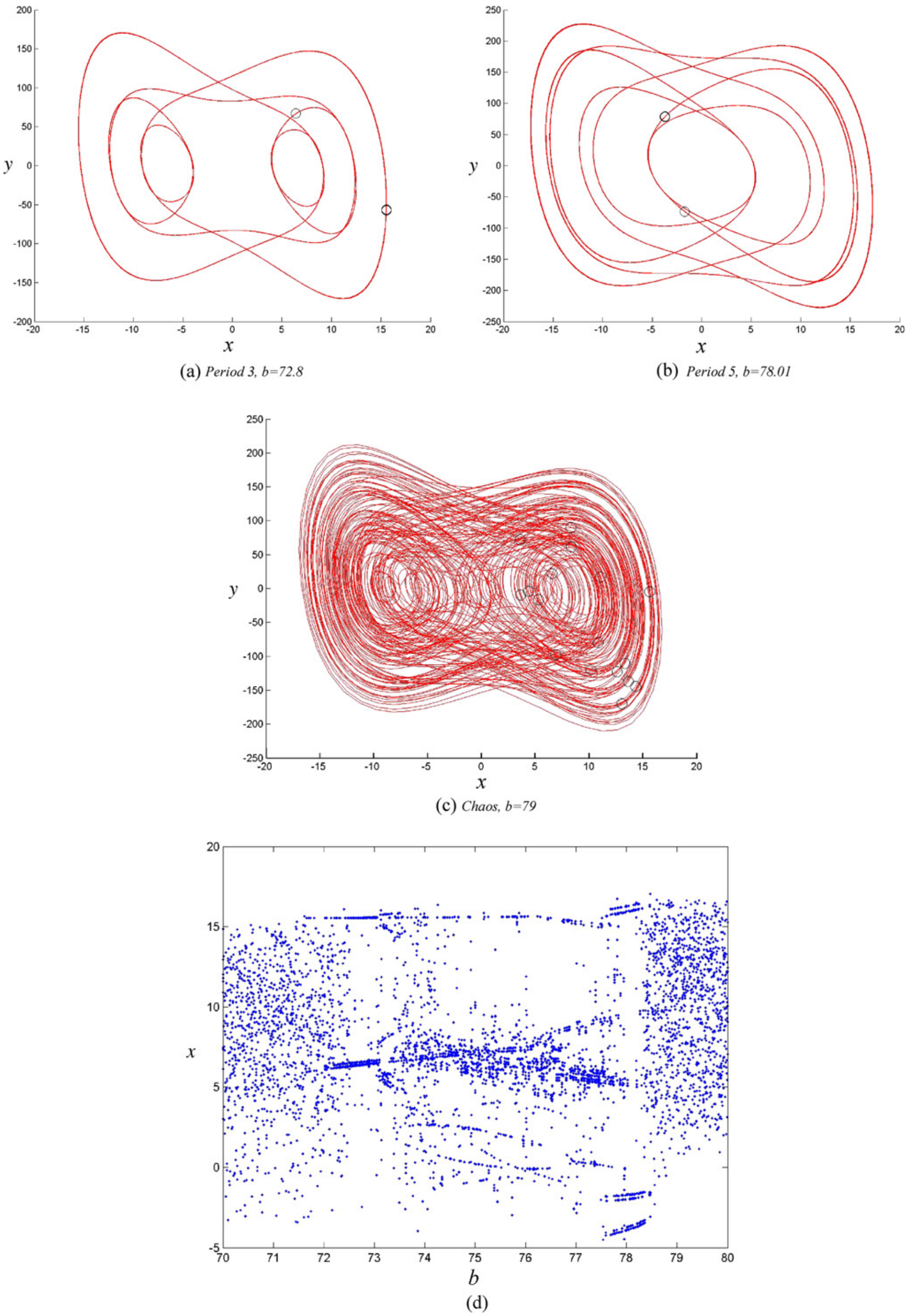


Fig. 19. The phase portraits, Poincaré maps and the bifurcation diagram for the fractional order modified Duffing system, x versus y and b versus x, $(q_1, q_2) = (1.2, 0.8)$.

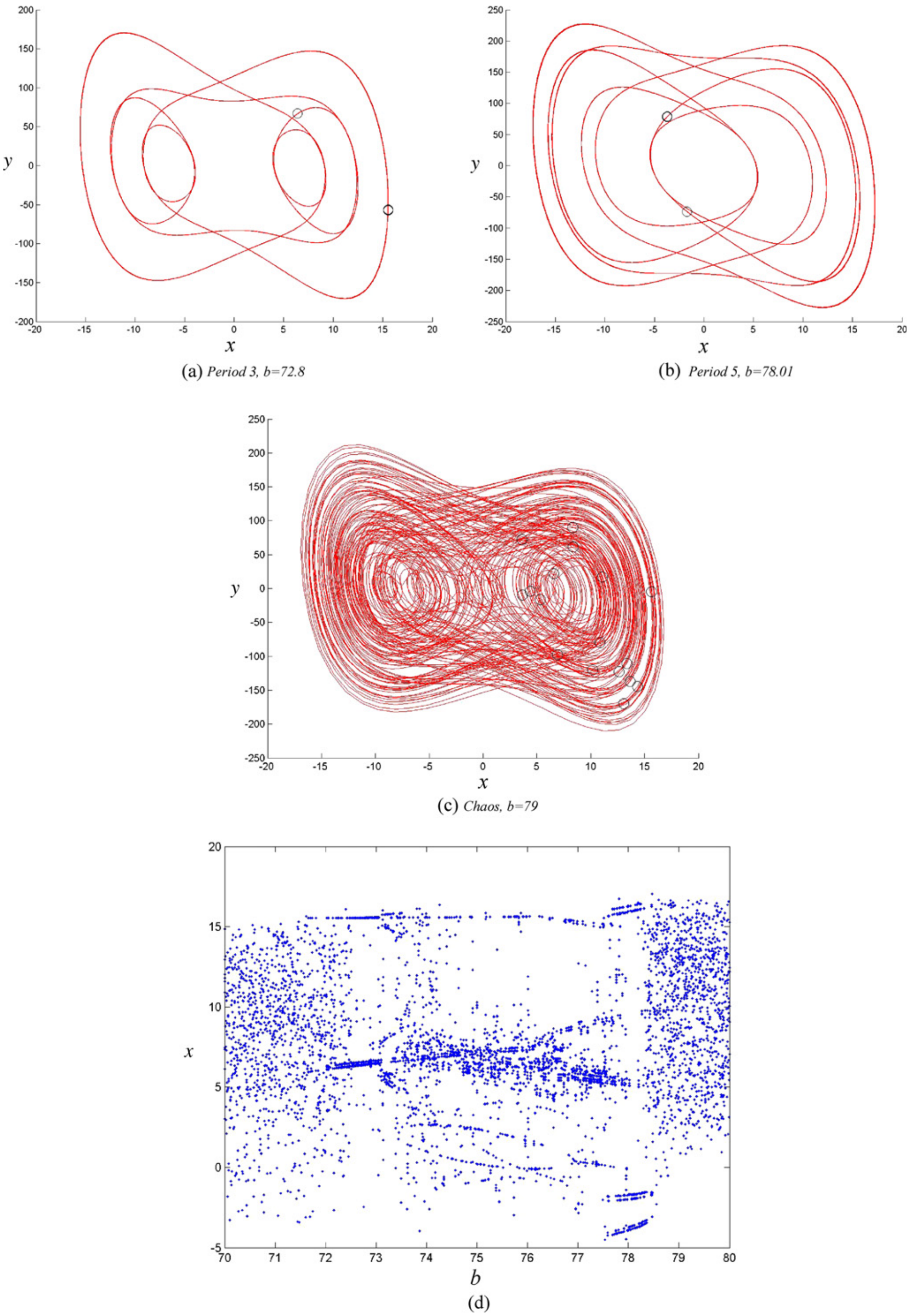
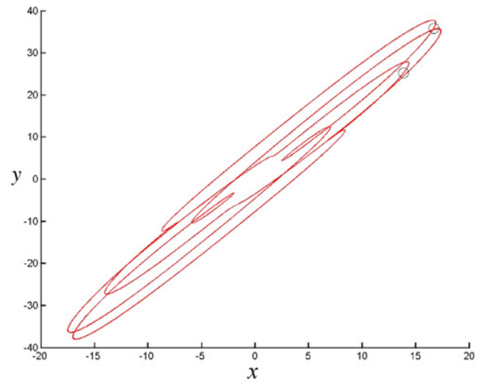
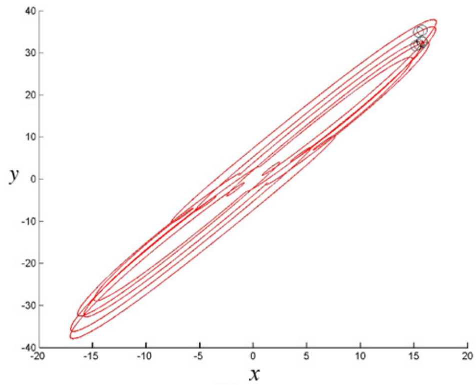


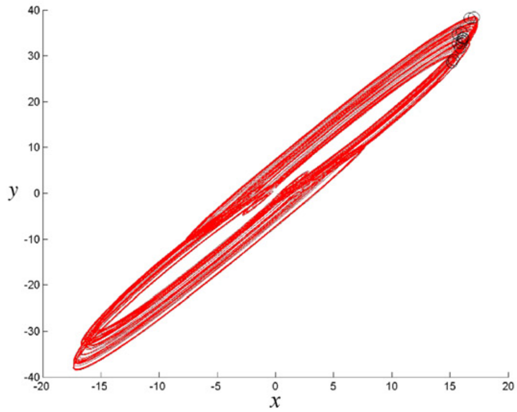
Fig. 20. The phase portraits, Poincaré maps and the bifurcation diagram for the fractional order modified Duffing system, x versus y and b versus x, $(q_1, q_2) = (1.1, 0.9)$.



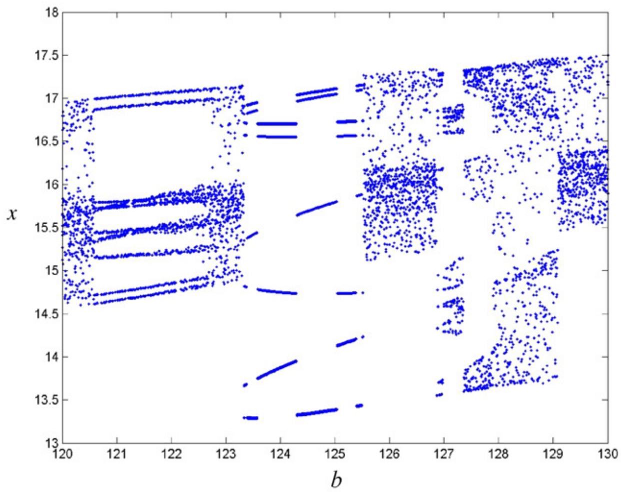
(a) *Period 4, b=124*



(b) *Period 6, b=121*



(c) *Chaos, b=126*



(d)

Fig. 21. The phase portraits, Poincaré maps and the bifurcation diagram for the fractional order modified Duffing system, x versus y and b versus x, $(q_1, q_2) = (0.2, 1.8)$.

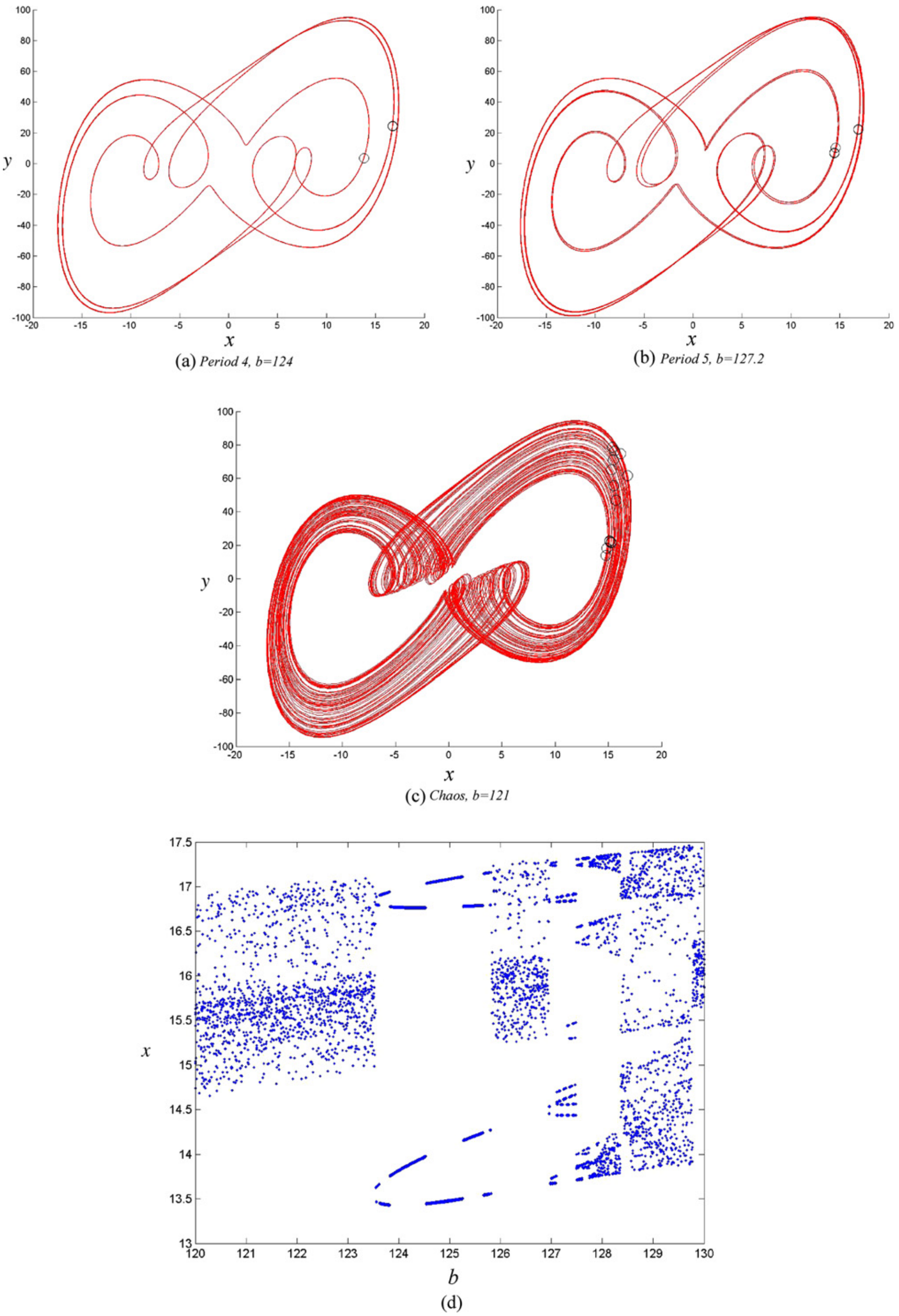


Fig. 22. The phase portraits, Poincaré maps and the bifurcation diagram for the fractional order modified Duffing system, x versus y and b versus x, $(q_1, q_2) = (0.8, 1.2)$.

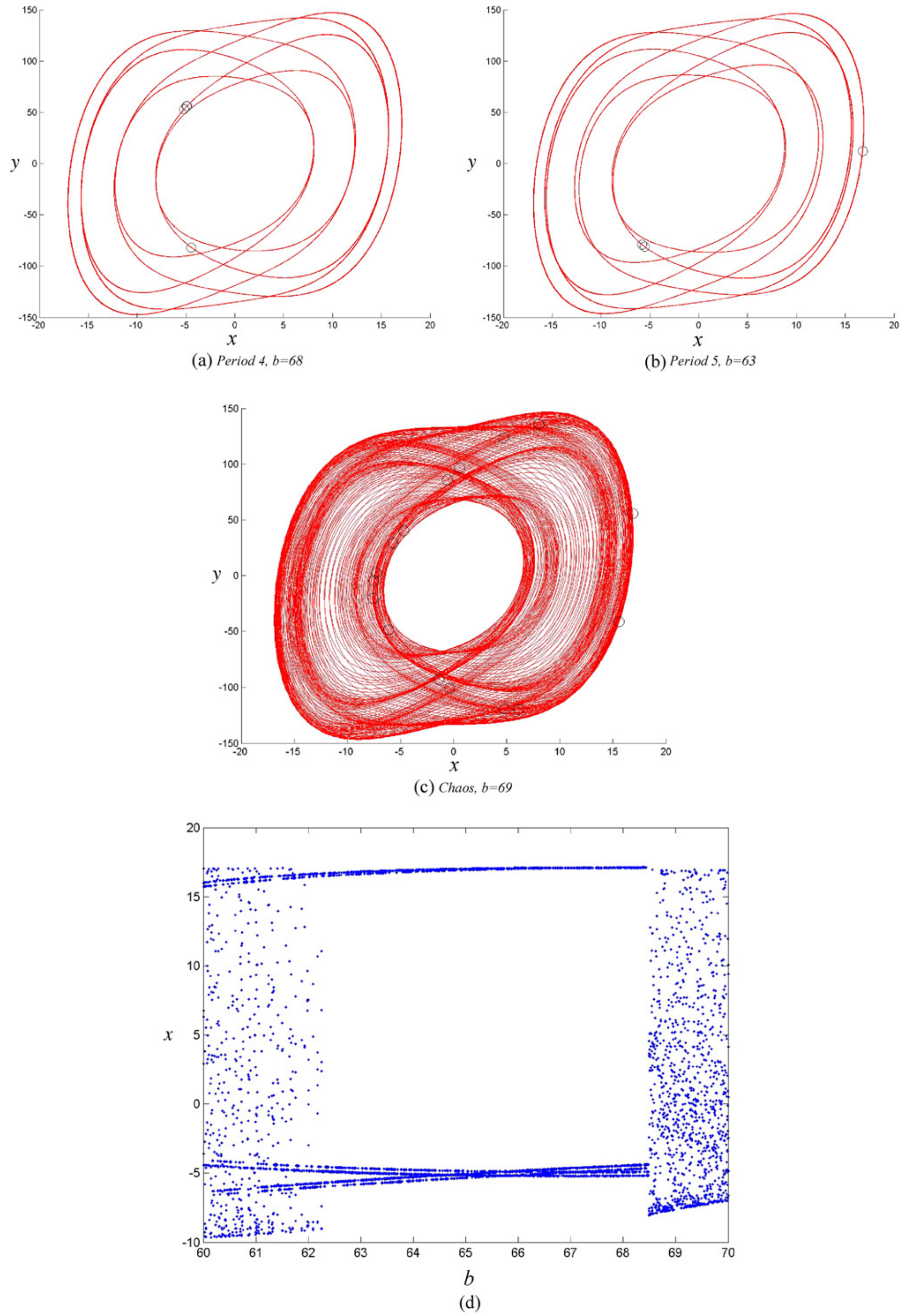


Fig. 23. The phase portraits, Poincaré maps and the bifurcation diagram for the fractional order modified Duffing system, x versus y and b versus x, $(q_1, q_2) = (0.9, 1.1)$.

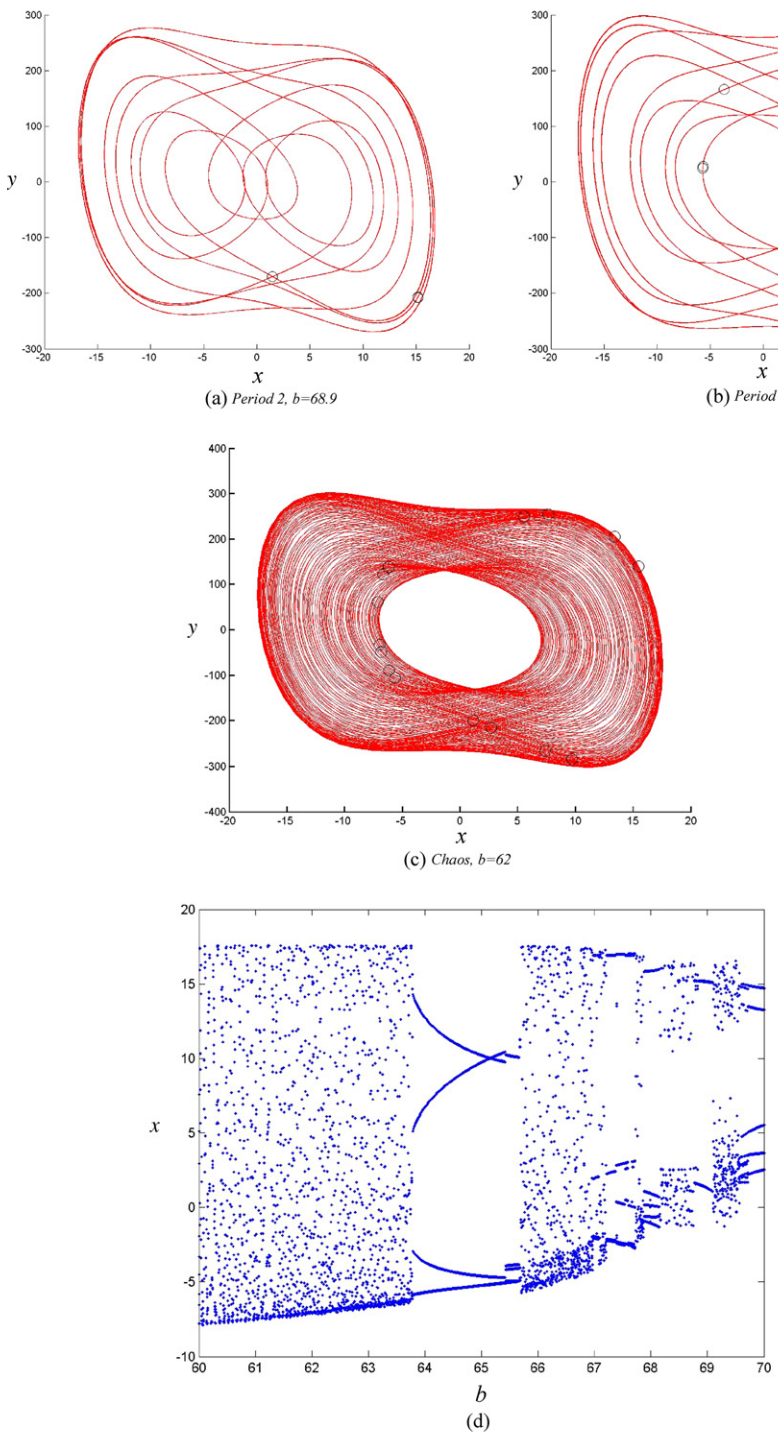


Fig. 24. The phase portraits, Poincaré maps and the bifurcation diagram for the fractional order modified Duffing system, x versus y and b versus x, $(q_1, q_2) = (1.2, 0.9)$.

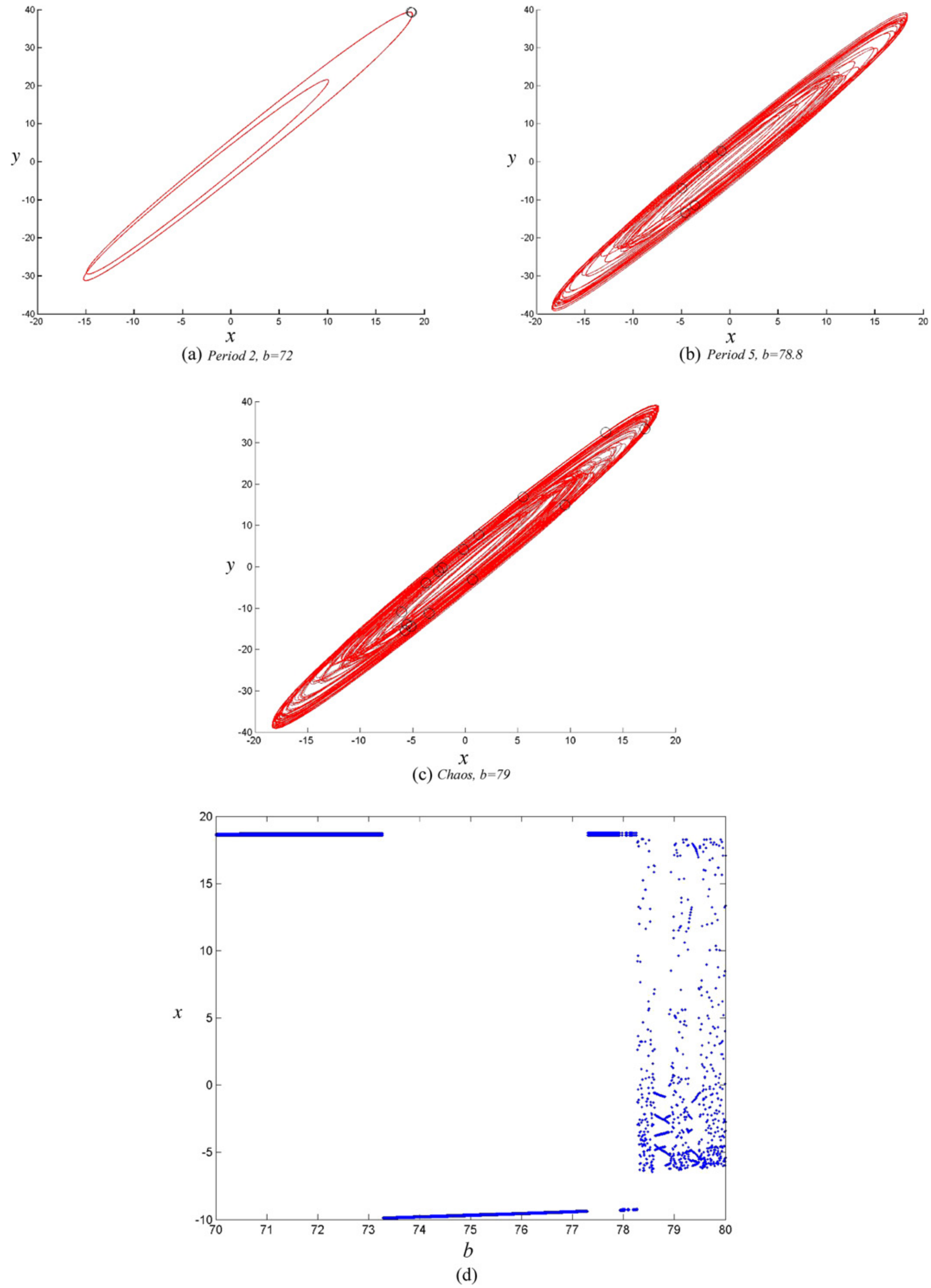


Fig. 25. The phase portraits, Poincaré maps and the bifurcation diagram for the fractional order modified Duffing system, x versus y and b versus x, $(q_1, q_2) = (0.2, 1.9)$.

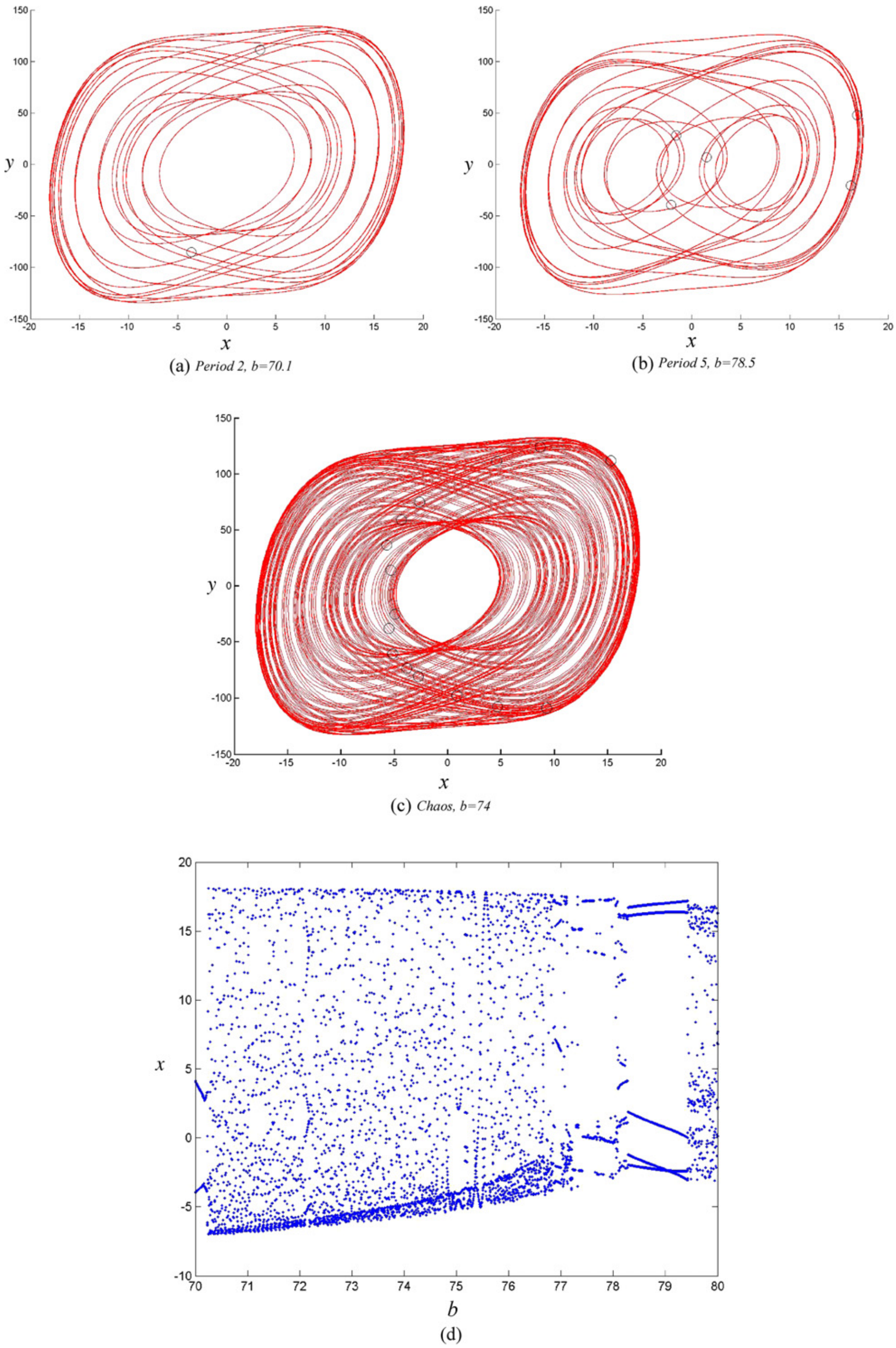


Fig. 26. The phase portraits, Poincaré maps and the bifurcation diagram for the fractional order modified Duffing system, x versus y and b versus x, $(q_1, q_2) = (0.9, 1.2)$.

5. Conclusions

In this paper we have studied the chaos in the fractional order modified Duffing system by phase portraits, Poincaré maps and bifurcation diagrams. The total orders of the system for the existence of chaos are 1.8, 1.9, 2.0 and 2.1.

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