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Decision Support

Rough set-based logics for multicriteria decision analysis

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Abstract

In this paper, we propose some decision logic languages for rule representation in rough set-based multicriteria analysis. The semantic models of these logics are data tables, each of which is comprised of a finite set of objects described by a finite set of criteria/attributes. The domains of the criteria may have ordinal properties expressing preference scales, while the domains of the attributes may not. The validity, support, and confidence of a rule are defined via its satisfaction in the data table.

Keywords: Rough sets; Fuzzy sets; Decision logic; Multicriteria decision analysis; Artificial intelligence

1. Introduction

The theory of knowledge has long been an important topic in many academic disciplines, such as philosophy, psychology, economics, and artificial intelligence, whereas the storage and retrieval of data is the main concern of information science. In modern experimental science, knowledge is usually acquired from observed data, which is a valuable resource for researchers and decision-makers. However, when the amount of data is large, it is difficult to analyze the data and extract knowledge from it. With the aid of computers, the vast amount of data stored in relational data tables can be transformed into symbolic knowledge automatically. Thus, intelligent data analysis has received a great deal of attention in recent years.

While data mining research concentrates on the design of efficient algorithms for extracting knowledge from data, how to bridge the semantic gap between structured data and human-comprehensible concepts has been a long-lasting challenge for the research community. Kruse et al. (1999) called this the *interpretability*

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problem of intelligent data analysis. Since discovered knowledge is only useful for a human user when he can understand its meaning, the knowledge representation formalism plays an important role in the utilization of the induced rules. A good representation formalism should have clear semantics so that a rule can be effectively validated with respect to the given data tables. In this regard, logic is one of the best choices. As noted by Zadeh (1996), humans usually compute with words instead of numbers, so if we can incorporate linguistically meaningful terms into the representation formalism, the induced rules may be more useful to human decision-makers.

The rough set theory proposed by Pawlak (1982) provides an effective tool for extracting knowledge from data tables. To represent and reason about the extracted knowledge, a decision logic (DL) was proposed in Pawlak (1991). The semantics of the logic is defined in a Tarskian style through the notions of models and satisfaction. While DL can be considered an example of classical logic in the context of data tables, different generalizations of DL corresponding to some non-classical logics are also desirable from the viewpoint of knowledge representation. For example, to deal with uncertain or incomplete information, some generalized decision logics have been proposed (Fan et al., 2001; Liau and Liu, 1999, 2001; Yao and Liau, 2002; Yao and Liu, 1999).

When rough set theory is applied to multi-criteria decision analysis (MCDA), it is crucial that preference-ordered attribute domains and decision classes be dealt with (Greco et al., 1997, 1998, 1999a, 2000, 2001a, 2002, 2004; Slowinski et al., 2002b). The original rough set theory cannot handle inconsistencies arising from violations of the dominance principle due to its use of the indiscernibility relation. Therefore, in the above-mentioned works, the indiscernibility relation is replaced by a dominance relation to solve the multi-criteria sorting problem, and the data table is replaced by a pairwise comparison table to solve multi-criteria choice and ranking problems. The approach is called the dominance-based rough set approach (DRSA). For MCDA problems, DRSA can induce a set of decision rules from sample decisions provided by decision-makers. The induced rules form a comprehensive preference model and can provide recommendations about a new decision-making environment.

A strong assumption about data tables is that each object takes exactly one value with respect to an attribute. However, in practice, we may only have incomplete information about the values of an object's attributes. Thus, more general data tables and decision logics are needed to represent and reason about incomplete information. For example, set-valued and interval set-valued data tables have been introduced to represent incomplete information (Kryszkiewicz, 1998; Kryszkiewicz and Rybiński, 1996a,b; Lipski, 1981; Yao and Liu, 1999). A generalized decision logic based on interval set-valued data tables is also proposed in Yao and Liu (1999). In these formalisms, the attribute values of an object may be a subset or an interval set in the domain. Since crisp subsets and interval sets are both special cases of fuzzy sets, further generalization of data tables is desirable to represent uncertain information. In data tables containing such information, an object can take a fuzzy subset of values for each attribute. To represent knowledge induced from uncertain data tables, the decision logic also needs to be generalized.

DRSA has also been extended to deal with missing values in MCDA problems (Greco et al., 2001a; Slowinski et al., 2002b). A data table with missing values is a special case of uncertain data tables. Therefore, we propose further extending DRSA to uncertain data tables and fuzzy data tables. In this paper, we present a logical treatment of DRSA in precise data tables, as well as uncertain and fuzzy data tables. Our approach is concerned with variants of DL for data tables.

The remainder of the paper is organized as follows. In Section 2, we review the decision logic proposed by Pawlak. In Sections 3–6, we respectively present generalized DL for preference-ordered data tables, preference-ordered uncertain data tables, preference-ordered fuzzy data tables, and pairwise comparison tables. For each logic, the syntax and semantics are described, and some quantitative measures for the rules of the logics are defined. Finally, in Section 7, we discuss the main contribution of this paper and indicate the direction of future research.

2. Classical data tables

In data mining problems, data is usually provided in the form of data tables (DT). A formal definition of a data table is given in Pawlak (1991).

Definition 1. A data table[1] is a tuple

$$T = (U, A, \{V_i | i \in A\},\ \{f_i | i \in A\}),$$

where U is a non-empty finite set, called the universe; A is a non-empty finite set of primitive attributes; for each $i \in A$, V_i is the domain of values for i; and for each $i \in A$, $f_i : U \to V_i$ is a total function.

Given a data table T, we denote its universe U and attribute set A by $Uni(T)$ and $Att(T)$, respectively.

In Pawlak (1991), a decision logic (DL) is proposed for the representation of knowledge discovered from data tables. It is called decision logic because it is particularly useful in a special kind of data table, called a *decision table*.[2] A decision table is a data table $T = (U, C \cup D, \{V_i | i \in A\}, \{f_i | i \in A\})$, where $Att(T)$ can be partitioned into two sets, C and D, called condition attributes and decision attributes respectively. Decision rules relating the condition and the decision attributes can be derived from the table by data analysis. A rule is then represented as an implication between the formulas of the logic.

The basic alphabet of a DL consists of a finite set of attribute symbols A, and a finite set of value symbols V_i for $i \in \mathsf{A}$. The syntax of DL is then defined as follows.

Definition 2

(1) An atomic formula of DL is a descriptor (i, v), where $i \in \mathsf{A}$ and $v \in \mathsf{V}_i$.
(2) The set of DL well-formed formulas (wff) is the smallest set containing the atomic formulas and closed under the Boolean connectives $\neg$, $\wedge$, and $\vee$.
(3) If φ and ψ are wffs of DL, then $\varphi \to \psi$ is a rule in DL, where φ is called the antecedent of the rule and ψ the consequent.

A data table $T = (U, A, \{V_i | i \in A\}, \{f_i | i \in A\})$ relates to a given DL if there is a bijection $\tau : \mathsf{A} \to A$ such that, for every $a \in \mathsf{A}$, $V_{\tau(a)} = \mathsf{V}_a$. Thus, by somewhat abusing the notation, we usually denote an atomic formula as (i, v), where $i \in A$ and $v \in V_i$ if the data tables are clear from the context. Intuitively, each element in the universe of a data table corresponds to a data record, and an atomic formula (which is in fact an attribute-value pair) describes the value of some attribute in the data record. Thus, the atomic formulas (and therefore the wffs) can be satisfied or not with respect to each data record. This generates a satisfaction relation between the universe and the set of wffs.

Definition 3. Given a DL and a data table $T = (U, A, \{V_i | i \in A\}, \{f_i | i \in A\})$ relating to it, the satisfaction relation $\models_T$ between U and the wffs of the DL is defined inductively as follows (the subscript T is omitted for brevity).

(1) $x \models (i, v)$ iff $f_i(x) = v$,
(2) $x \models \neg\varphi$ iff $x \not\models \varphi$,
(3) $x \models \varphi \wedge \psi$ iff $x \models \varphi$ and $x \models \psi$,
(4) $x \models \varphi \vee \psi$ iff $x \models \varphi$ or $x \models \psi$.

If φ is a DL wff, the set $m_T(\varphi)$ defined by

$$m_T(\varphi) = \{x \in U | x \models \varphi\}, \tag{1}$$

is called the meaning set of the formula φ in T. If T is understood, we simply write $m(\varphi)$.

A formula φ is said to be valid in a data table T (written as $\models_T \varphi$ or $\models \varphi$ for short when T is clear from the context) if and only if $m(\varphi) = U$. That is, φ is satisfied by all individuals in the universe.

A DL wff states the properties of individuals in the universe; therefore, it is satisfied by some individuals, but not by the others. However, the mined knowledge usually relates to the aggregated or statistical informa-

[1] Also called knowledge representation systems, information systems, or attribute-value systems.
[2] Note that for a general data table, the abbreviation DL can also be used to denote *data logic*.

tion of all individuals. Obviously, wffs that are valid in a data table represent a kind of knowledge that can be induced from the table, since they hold for all individuals. However, not all kinds of useful information are in the form of valid wffs. Sometimes, even probabilistic rules are very useful from the viewpoint of knowledge discovery. To quantify the usefulness of the mined rules, some measures have been proposed (e.g. Yao and Zhong, 1999; Yao and Liau, 2002). The most common measures are support and confidence.

Definition 4. Let Φ_1 be the set of all DL rules and $T = (U, A, \{V_i | i \in A\}, \{f_i | i \in A\})$ be a data table. Then

(1) the rule $\varphi \to \psi$ is valid in T iff $m_T(\varphi) \subseteq m_T(\psi)$;
(2) the absolute support function $asp_T : \Phi_1 \to \mathbb{N}$ is

$$asp_T(\varphi \to \psi) = |m_T(\varphi \wedge \psi)|;$$

(3) the relative support function $rsp_T : \Phi_1 \to [0, 1]$ is

$$rsp_T(\varphi \to \psi) = \frac{|m_T(\varphi \wedge \psi)|}{|U|}; \text{ and}$$

(4) the confidence function $cfd_T : \Phi_1 \to [0, 1]$ is

$$cfd_T(\varphi \to \psi) = \frac{|m_T(\varphi \wedge \psi)|}{|m_T(\varphi)|}.$$

Example 5. Let us use an example to illustrate the concept introduced in this section. Assume that Table 1 is a summary of reviewers' report for ten papers submitted to a journal. The table details ten papers evaluated by means of four attributes:

- o: originality,
- p: presentation,
- t: technical soundness, and
- d: overall evaluation (the decision attribute).

By Definition 1, the components of the data table are:

$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$,
$A = \{o, p, t, d\}$,
$V_o = V_p = V_t = \{1\ (\text{poor}), 2\ (\text{fair}), 3\ (\text{good}), 4\ (\text{excellent})\}$,
$V_d = \{1\ (\text{reject}), 2\ (\text{major revision}), 3\ (\text{minor revision}), 4\ (\text{accept})\}$,
$f_i(j) (i \in A,\ 1 \leqslant j \leqslant 10)$ denotes the jth element of column i of the table.

Table 1
A summary of reviewers' reports for 10 papers

$U \backslash A$	o	p	t	d
1	4	4	3	4
2	3	2	3	3
3	4	3	2	3
4	2	2	2	2
5	2	1	2	1
6	3	1	2	1
7	3	2	2	2
8	4	1	2	2
9	3	3	2	3
10	4	3	3	3

Thus, we have atomic formulas like $(o,4)$, $(p,1)$, and $(t,2)$; and formulas like $(o,4) \wedge (p,3)$ and $\neg(p,1) \vee \neg(t,1)$. The rule $r = (o,3) \wedge ((p,3) \vee (t,3)) \rightarrow (d,3)$ is valid, since $m((o,3) \wedge ((p,3) \vee (t,3))) = \{2,9\} \subseteq m((d,3)) = \{2,3,9,0\}$. Furthermore, we have $asp(r) = 2$, $rsp(r) = \frac{1}{5}$, and $cfd(r) = 1$.

3. Preference-ordered data tables

For MCDA problems, each object in a data table or decision table can be seen as a sample decision, and each condition attribute is a criterion for the decision. Since the domain of values of a criterion is usually ordered according to the decision-maker's preferences, we define a preference-ordered data table (PODT) as a tuple

$$T = (U, A, \{(V_i, \succeq_i) | i \in A\},\ \{f_i | i \in A\}),$$

where $T = (U, A, \{V_i | i \in A\}, \{f_i | i \in A\})$ is a classical data table; and for each $i \in A$, $\succeq_i \subseteq V_i \times V_i$ is a binary relation over V_i. The relation $\succeq_i$ is called a *weak preference relation* or *outranking* on V_i, and represents a preference over the set of objects with respect to the criterion i (Slowinski et al., 2002b). For $x, y \in U$, $f_i(x) \succeq_i f_i(y)$ means "x is at least as good as y with respect to criterion i.

To represent the rules induced from a PODT, we introduce preference-ordered decision logic (PODL). The syntax of PODL is the same as that of DL, except for the form of the atomic formulas. An atomic formula in PODL is a descriptor in the form of $(\geqslant_i, v)$ or $(\leqslant_i, v)$, where $i \in A$ and $v \in V_i$. The satisfaction relation between U and the set of PODL wffs is defined in the same way as for DL wffs, except that the satisfaction of an atomic formula is defined by $x \models (\geqslant_i, v)$ iff $f_i(x) \succeq v$, and by $x \models (\leqslant_i, v)$ iff $v \succeq f_i(x)$. Other semantic notions in DL, such as validity, support, and confidence, can all be used in the case of PODL without any modifications. The confidence function for PODL rules has also been defined by Greco et al. (2001b).

In Greco et al. (2001a), three types of rules are explicitly identified. We translate these rules into PODL rules as follows.

(1) $\bigwedge_{i \in C}(\geqslant_i, v_i) \rightarrow (\geqslant_d, v_d)$, where $C \subseteq A$ is a subset of condition attributes, $d \in A \backslash C$ is a decision attribute, $v_i \in V_i$ for all $i \in C$, and $v_d \in V_d$.
(2) $\bigwedge_{i \in C}(\leqslant_i, v_i) \rightarrow (\leqslant_d, v_d)$, where $C \subseteq A$ is a subset of condition attributes, $d \in A \backslash C$ is a decision attribute, $v_i \in V_i$ for all $i \in C$, and $v_d \in V_d$.
(3) $(\bigwedge_{i \in C_1}(\geqslant_i, v_i) \wedge \bigwedge_{i \in C_2}(\leqslant_i,\ v_i)) \rightarrow ((\geqslant_d, v_d) \wedge (\leqslant_d, v'_d))$, where $C_1 \cup C_2 \subseteq A$ is a subset of condition attributes, $d \in A \backslash (C_1 \cup C_2)$ is a decision attribute, $v_i \in V_i$ for all $i \in C_1 \cup C_2$, and $v_d, v'_d \in V_d$.

Example 6. Continuing with Example 5, let us assume that each V_i $(i = o, p, t, d)$ is now endowed with a weak preference relation $\succeq_i$ such that $4 \succeq_i 3 \succeq_i 2 \succeq_i 1$. Thus, we have atomic formulas like $(\geqslant_o, 4)$, $(\geqslant_p, 1)$, and $(\geqslant_t, 2)$. Let us now consider the following rules:

$$r_1 = (\geqslant_o, 3) \rightarrow (\geqslant_d, 3),$$

$$r_2 = (\leqslant_p, 2) \rightarrow (\leqslant_d, 2),$$

$$r_3 = (\geqslant_o, 4) \wedge (\leqslant_t, 2) \rightarrow (\geqslant_d, 2) \wedge (\leqslant_d, 3).$$

Then, we have

	asp	rsp	cfd
r_1	5	$\frac{1}{2}$	$\frac{5}{8}$
r_2	5	$\frac{1}{2}$	$\frac{5}{6}$
r_3	2	$\frac{1}{5}$	1

Among these rules, only r_3 is valid.

4. Preference-ordered uncertain data tables

PODL is suitable for the representation of rules induced from a PODT. However, the latter inherits the restriction of classical DT so that uncertain information cannot be represented. An uncertain data table is a generalization of DT such that the values of some or all attributes are imprecise (Fan et al., 2001; Dembczynski et al., 2002). An analogous generalization can be applied to PODT to define preference-ordered uncertain data tables (POUDT). Formally, a POUDT is a tuple

$$T = (U, A, \{(V_i, \succeq_i) | i \in A\}, \ \{f_i | i \in A\}),$$

where $U, A, \{(V_i, \succeq_i)|i \in A\}$ are defined as above, and for each $i \in A$, $f_i : U \to 2^{V_i} - \{\emptyset\}$. The intuition about POUDT is that the value of attribute i of an object x belongs to $f_i(x)$, though the value is not known exactly. When $f_i(x)$ is a singleton, we say that the value is precise. If all attribute values of T are precise, then T is said to be single-valued.

PODL is also generalized to preference-ordered uncertain decision logic (POUDL). The syntax of POUDL is same as that of PODL, except that its atomic formulas are of the form (i, s_i), where $i \in A$ and $s_i \subseteq V_i$. When $s_i = \{v \in V_i | v \succeq_i v_i\}$ (resp. $s_i = \{v \in V_i | v_i \succeq_i v\}$), we abbreviate (i, s_i) as $(\geqslant_i, v_i)$ (resp. $(\leqslant_i, v_i)$). To define the semantics of POUDL, we must first rewrite each wff into its normal form.

A wff is in a *conjunctive normal form* (CNF) if it is a conjunction of formulas of the form $\vee_{i \in B}(i, s_i)$, where $B \subseteq A$ is a subset of mutually distinct attributes. A wff is in a *disjunctive normal form* (DNF) if it is a disjunction of formulas of the form $\wedge_{i \in B}(i, s_i)$, where $B \subseteq A$ is a subset of mutually distinct attributes. Given a POUDL wff φ, its CNF and DNF are denoted by φ^c and φ^d, respectively. Any POUDL wff can be rewritten in both CNF and DNF by using Boolean algebra and the following rewriting rules:

$$\neg(i, s) = (i, V_i \setminus s),$$
$$(i, s_1) \vee (i, s_2) = (i, s_1 \cup s_2),$$
$$(i, s_1) \wedge (i, s_2) = (i, s_1 \cap s_2).$$

For the semantics of POUDL, we define the positive satisfaction relation $\models^+$ for CNF formulas and negative satisfaction relation $\models^-$ for DNF formulas. The definition is as follows:

(1) $x \models^+ (i, s)$ iff $f_i(x) \subseteq s$,
(2) $x \models^+ \varphi \vee \psi$ iff $x \models^+ \varphi$ or $x \models^+ \psi$,
(3) $x \models^+ \varphi \wedge \psi$ iff $x \models^+ \varphi$ and $x \models^+ \psi$,
(4) $x \models^- (i, s)$ iff $f_i(x) \cap s = \emptyset$,
(5) $x \models^- \varphi \wedge \psi$ iff $x \models^- \varphi$ or $x \models^- \psi$,
(6) $x \models^- \varphi \vee \psi$ iff $x \models^- \varphi$ and $x \models^- \psi$.

Then, for any POUDL wff φ, we define $x \models^+ \varphi$ iff $x \models^+ \varphi^c$, and $x \models^- \varphi$ iff $x \models^- \varphi^d$. According to the semantics of POUDL, $x \models^+ (\geqslant_i v_i)$ if for all $v \in f_i(x)$, v is preferred over v_i with respect to the criterion i. Therefore, we can be sure that, if $x \models^+ (\geqslant_i v_i)$ holds, then the value of criterion i of x will at least reach the level of v_i no matter what the actual value is. Analogously, if $x \models^- (\geqslant_i v_i)$ holds, we can be sure that the value of criterion i of x will not be above the level of v_i no matter what the actual value is.

For each POUDL wff φ and a given POUDT T, we define two meaning sets:

$$m_T^+(\varphi) = \{x \in U | x \models^+ \varphi\},$$
$$m_T^-(\varphi) = \{x \in U | x \models^- \varphi\};$$

$m_T^+(\varphi)$ is the set of objects that are known to satisfy φ, and $m_T^-(\varphi)$ is the set of objects that are known not to satisfy φ. The indeterminate region of φ with respect to T is defined as

$$m_T^*(\varphi) = U \setminus (m^+(\varphi) \cup m_T^-(\varphi)).$$

As usual, the subscript T can be omitted if it is clear from the context. Using the notations from rough set theory, we also define

$$\underline{m}(\varphi) = m^+(\varphi) \text{ and } \overline{m}(\varphi) = U \setminus m^-(\varphi).$$

Note that the three types of rules mentioned in Section 3 can also be represented in POUDL, though the semantics is quite different.

The quantitative measures of the rules' usefulness can be defined by the notion of completion of a POUDT. Let $T = (U, A, \{(V_i, \succeq_i) | i \in A\}, \{f_i | i \in A\})$ be a POUDT. Then, a PODT $S = (U, A, \{(V_i, \succeq_i) | i \in A\}, \{f'_i | i \in A\})$ is a *completion* of T if $f'_i(x) \in f_i(x)$ for all $i \in A$ and $x \in U$. The number of completions of T is equal to $\prod_{i\in A, x\in U} |f_i(x)|$. Let $CL(T)$ denote the set of all completions of T. If we identify a singleton set with its element by slightly abusing the notation, then a completion of a POUDT (i.e., a PODT) can be considered as a special case of POUDT. This yields the following definition.

Definition 7. Let $T = (U, A, \{(V_i, \succeq_i) | i \in A\}, \{f_i | i \in A\})$ be a POUDT and $\varphi \to \psi$ be a POUDL rule, then

(1) $\varphi \to \psi$ is strongly valid in T if $\overline{m}(\varphi) \subseteq m^+(\psi)$ and weakly valid in T if $m^+(\varphi) \subseteq m^+(\psi)$;
(2) the absolute support interval of $\varphi \to \psi$ is

$$asi_T(\varphi \to \psi) = \left[\min_{S\in CL(T)} asp_S(\varphi \to \psi), \max_{S\in CL(T)} asp_S(\varphi \to \psi)\right];$$

(3) the relative support interval of $\varphi \to \psi$ is

$$rsi_T(\varphi \to \psi) = \left[\min_{S\in CL(T)} rsp_S(\varphi \to \psi), \max_{S\in CL(T)} rsp_S(\varphi \to \psi)\right] \text{ and}$$

(4) the confidence interval of $\varphi \to \psi$ is

$$cfi_T(\varphi \to \psi) = \left[\min_{S\in CL(T)} cfd_S(\varphi \to \psi), \max_{S\in CL(T)} cfd_S(\varphi \to \psi)\right].$$

The next two propositions show how these measures are calculated.

Proposition 8. *Let φ be a POUDL wff and T be a POUDT. Then, for all $x \in Uni(T)$, we have*

(1) $x \models^+_T \varphi$ *iff* $x \models_S \varphi$ *for all* $S \in CL(T)$,
(2) $x \models^-_T \varphi$ *iff* $x \not\models \varphi$ *for all* $S \in CL(T)$, *and*
(3) $x \in m^*_T(\varphi)$ *iff there exist* $S_1, S_2 \in CL(T)$ *such that* $x \models_{S_1} \varphi$ *and* $x \not\models_{S_2} \varphi$.

Proof. We first note that if S is a PODT, then for any POUDL wff φ, $x \models_S \varphi$ iff $x \models_S \varphi^c$ iff $x \models_S \varphi^d$. Thus, without loss of generality, we only need to consider wffs in CNF or DNF. Let us now prove the first equivalence. The second equivalence can be proved analogously, and the third follows from the first two.

($\Rightarrow$): If φ is in CNF, then we have $x \models^+_T \varphi$ iff $x \models^+_T \vee_{i\in B}(i, s_i)$ for each conjunct $\vee_{i\in B}(i, s_i)$ of φ. Now, $x \models^+_T \vee_{i\in B}(i, s_i)$ implies that there exists $i \in B$ such that $f_i(x) \subseteq s_i$. This, in turn, implies that $x \models_S \vee_{i\in B}(i, s_i)$ for any $S \in CL(T)$. Thus, $x \models^+_T \varphi$ implies that $x \models_S \varphi$ for all $S \in CL(T)$.
($\Leftarrow$): If φ is in CNF and $x \models_S \varphi$ for all $S \in CL(T)$, then for any conjunct $\vee_{i\in B}(i, s_i)$ of φ, we have $x \models_S \vee_{i\in B}(i, s_i)$ for any $S \in CL(T)$. Assume $x \not\models^+_T \vee_{i\in B}(i, s_i)$ for some conjunct $\vee_{i\in B}(i, s_i)$ of φ; then $f_i(x) \not\subseteq s_i$ holds for all $i \in B$. Thus, since the attributes in B are mutually distinct, we can have an $S = (U, A, \{(V_i, \succeq_i) | i \in A\}, \{f'_i | i \in A\}) \in CL(T)$ such that $f'_i(x) \in f_i(x) \setminus s_i$ for all $i \in B$. Obviously, this implies that $x \not\models_S \vee_{i\in B}(i, s_i)$ and contradicts the fact that $x \models_S \vee_{i\in B}(i, s_i)$ for any $S \in CL(T)$. Therefore, we can derive $x \models^+_T \vee_{i\in B}(i, s_i)$ for any conjunct $\vee_{i\in B}(i, s_i)$ of φ, and consequently, $x \models^+_T \varphi$. $\square$

Proposition 9. *Let $\varphi \to \psi$ be a POUDL rule and T be a POUDT, then we have*

(1) $|\underline{m}(\varphi \wedge \psi)| = \min_{S \in CL(T)} asp_S(\varphi \to \psi)$,

$|\overline{m}(\varphi \wedge \psi)| = \max_{S \in CL(T)} asp_S(\varphi \to \psi)$;

(2) $\frac{|\underline{m}(\varphi \wedge \psi)|}{|U|} = \min_{S \in CL(T)} rsp_S(\varphi \to \psi)$,

$\frac{|\overline{m}(\varphi \wedge \psi)|}{|U|} = \max_{S \in CL(T)} rsp_S(\varphi \to \psi)$;

(3) $\frac{|\underline{m}(\varphi \wedge \psi)|}{|\overline{m}(\varphi) \setminus (m^*(\varphi) \cap m^-(\varphi \wedge \neg\psi))|} = \min_{S \in CL(T)} cfd_S(\varphi \to \psi)$,

$\frac{|\overline{m}(\varphi \wedge \psi)|}{|\overline{m}(\varphi) \setminus (m^*(\varphi) \cap m^-(\varphi \wedge \psi))|} = \max_{S \in CL(T)} cfd_S(\varphi \to \psi)$.

Proof. The only non-trivial case to be proved is the third one. We first note that

$$\min_{S \in CL(T)} cfd_S(\varphi \to \psi) = \min_{S \in CL(T)} \frac{|m_S(\varphi \wedge \psi)|}{|m_S(\varphi)|}.$$

Let S^* be a completion of T such that

$$S^* \in \arg\min_{S \in CL(T)} \frac{|m_S(\varphi \wedge \psi)|}{|m_S(\varphi)|}.$$

Then, for any object x, we can consider the following cases:

Case 1: If $x \in \overline{m}(\varphi) \setminus (m^*(\varphi) \cap m^-(\varphi \wedge \neg\psi))$ and $x \notin m_{S^*}(\varphi)$, then $x \notin \underline{m}(\varphi)$ since $\underline{m}(\varphi) \subseteq m_S(\varphi)$ for any $S \in CL(T)$. This means $x \in m^*(\varphi)$ and $x \notin m^-(\varphi \wedge \neg\psi)$, so there exists $S \in CL(T)$ such that $x \in m_S(\varphi \wedge \neg\psi)$ by Proposition 8. Since the attribute values of different objects in a completion can be independently determined, we can define $S' \in CL(T)$ such that for each attribute i, $f_i^{S'}(x) = f_i^S(x)$ and $f_i^{S'}(y) = f_i^{S^*}(y)$ for all $y \neq x$, where $f_i^{S'}, f_i^S$, and $f_i^{S^*}$ correspond to the attribute functions of S', S, and S^*, respectively. Then $|m_{S'}(\varphi)| = |m_{S^*}(\varphi)| + 1$ and $|m_{S'}(\varphi \wedge \psi)| = |m_{S^*}(\varphi \wedge \psi)|$. This contradicts the minimality assumption for S^*.
Case 2: If $x \in m_{S^*}(\varphi)$ and $x \notin \overline{m}(\varphi) \setminus (m^*(\varphi) \cap m^-(\varphi \wedge \neg\psi))$, then $x \in m^*(\varphi)$ and $x \in m^-(\varphi \wedge \neg\psi)$, since $m_S(\varphi) \subseteq \overline{m}(\varphi)$ for any $S \in CL(T)$. From $x \in m_{S^*}(\varphi)$ and $x \in m^-(\varphi \wedge \neg\psi)$, we can derive $x \in m_{S^*}(\varphi \wedge \psi)$ by Proposition 8. From $x \in m^*(\varphi)$, we can find an $S \in CL(T)$ such that $x \nvDash_S \varphi$ (and, of course, $x \nvDash_S \varphi \wedge \psi$). Thus, we can also define $S' \in CL(T)$ such that for each attribute i, $f_i^{S'}(x) = f_i^S(x)$ and $f_i^{S'}(y) = f_i^{S^*}(y)$ for all $y \neq x$. Then, $|m_{S'}(\varphi)| = |m_{S^*}(\varphi)| - 1$ and $|m_{S'}(\varphi \wedge \psi)| = |m_{S^*}(\varphi \wedge \psi)| - 1$, which implies that

$$\frac{|m_{S'}(\varphi \wedge \psi)|}{|m_{S'}(\varphi)|} \leqslant \frac{|m_{S^*}(\varphi \wedge \psi)|}{|m_{S^*}(\varphi)|}.$$

Therefore, without loss of generality, we can assume that $m_{S^*}(\varphi) = \overline{m}(\varphi) \setminus (m^*(\varphi) \cap m^-(\varphi \wedge \neg\psi))$.

Now, if there exists $x \in m_{S^*}(\varphi \wedge \psi)$ and $x \notin \underline{m}(\varphi \wedge \psi)$, then $x \in \overline{m}(\varphi) \setminus (m^*(\varphi) \cap m^-(\varphi \wedge \neg\psi))$, so we can consider two possibilities.

Case 1: $x \in m^+(\varphi)$. Then, from the assumption, we can derive $x \in m^*(\psi)$.
Case 2: $x \in m^*(\varphi)$ and $x \in m^*(\varphi \wedge \neg\psi)$.

In both cases, we can find an $S \in CL(T)$ such that $x \models_S \varphi \wedge \neg\psi$. Let S' be a completion of T that has the same attribute values as S for x and the same attribute values as S^* for objects that are not x. Then $|m_{S'}(\varphi)| = |m_{S^*}(\varphi)|$ and $|m_{S'}(\varphi \wedge \psi)| = |m_{S^*}(\varphi \wedge \psi)| - 1$. This contradicts the minimality assumption for S^*. Therefore, $m_{S^*}(\varphi \wedge \psi) = \underline{m}(\varphi \wedge \psi)$, which means that there exists an

$$S^* \in \arg\min_{S \in CL(T)} \frac{|m_S(\varphi \wedge \psi)|}{|m_S(\varphi)|},$$

such that

$$\frac{|\underline{m}(\varphi \wedge \psi)|}{|\overline{m}(\varphi) \setminus (m^*(\varphi) \cap m^-(\varphi \wedge \neg\psi))|} = cfd_{S^*}(\varphi \to \psi).$$

Therefore, the result for the lower bound is proved. The result for the upper bound can be proved in an analogous way. □

In general, the attribute symbols in the antecedent of a rule and those in the consequent are disjoint. For this special type of rule, the confidence interval can be simplified slightly.

Corollary 10. *Let A_φ denote the set of attributes appearing in a wff φ, and $\varphi \to \psi$ be a POUDL rule such that $A_\varphi \cap A_\psi = \emptyset$. Then, $cfi(\varphi \to \psi) = [low, up]$, where*

$$low = \frac{|\underline{m}(\varphi \wedge \psi)|}{|\overline{m}(\varphi) \setminus (m^*(\varphi) \cap m^+(\psi))|}, \text{ and}$$

$$up = \frac{|\overline{m}(\varphi \wedge \psi)|}{|\overline{m}(\varphi) \setminus (m^*(\varphi) \cap m^-(\psi))|}.$$

Proof. It suffices to show that $m^*(\varphi) \cap m^-(\varphi \wedge \neg\psi) = m^*(\varphi) \cap m^+(\psi)$ and $m^*(\varphi) \cap m^-(\varphi \wedge \psi) = m^*(\varphi) \cap m^-(\psi)$. We only prove the former, as the latter can be proved analogously.

($\subseteq$): If both $x \in m^*(\varphi) \cap m^-(\varphi \wedge \neg\psi)$ and $x \notin m^*(\varphi) \cap m^+(\psi)$ hold, then there exists $S_1 \in CL(T)$ such that $x \models_{S_1} \varphi$ (from $x \in m^*(\varphi)$); and there also exists $S_2 \in CL(T)$ such that $x \models_{S_2} \neg\psi$ (from $x \notin m^+(\psi)$). Since A_φ and A_ψ are disjoint, we can construct an $S \in CL(T)$ such that

$$f'_i(x) = \begin{cases} f_i^1(x) & \text{if } i \in A_\varphi, \\ f_i^2(x) & \text{if } i \in A_\psi, \end{cases}$$

where f_i^1, f_i^2, and f'_i are attribute functions of S_1, S_2, and S, respectively. Hence, we have $x \models_S \varphi \wedge \neg\psi$, which contradicts $x \in m^-(\varphi \wedge \neg\psi)$ by Proposition 8.

($\supseteq$): If $x \in m^*(\varphi) \cap m^+(\psi)$, then for all $S \in CL(T)$, $x \models_S \psi$ holds, which implies that $x \not\models_S \varphi \wedge \neg\psi$. We then have $x \in m^*(\varphi) \cap m^-(\varphi \wedge \neg\psi)$ by Proposition 8. □

Example 11. We use a modified example for route selection from (Warren et al., 2004) to illustrate POUDT and POUDL. The example is concerned with a route selection problem for solid waste management. In Table 2, six routes are evaluated by means of 2 attributes w (weight capacity) and s (surface condition), and the evaluation results are described by the decision attribute d. The domain of values for w is $V_w = \{l \text{ (low)}, m \text{ (medium)}, h \text{ (high)}\}$ and the domain of values for s is $V_s = \{v \text{ (very good)}, g \text{ (good)}, b \text{ (bad)}\}$. The evaluation results are then divided into three levels $V_d = \{1, 2, 3\}$. We assume that each domain is endowed with a preference relation such that $h \succ_w m \succ_w l$, $v \succ_s g \succ_s b$, and $3 \succ_d 2 \succ_d 1$, where $u \succ_i v$ means $u \succeq_i v \wedge v \not\succeq_i u$ for $i \in A$ and $u, v \in V_i$. Due to the incompleteness of the information, some attribute values appearing in the table are non-singleton. Note that this POUDT has 64 completions.

Let us now consider a POUDL wff $\varphi_1 = (\geqslant_w m) \vee ((\leqslant_w l) \wedge (\geqslant_s g))$. The CNF of φ_1 is $\varphi_1^c = (w, V_w) \wedge ((\geqslant_w m) \vee (\geqslant_s g))$. We can see that $x_2 \models^+ \varphi_1$, since both $x_2 \models^+ (w, V_w)$ and $x_2 \models^+ (\geqslant_s g)$ hold. Indeed, in any completion of T, we have either $f'_w(x_2) = l$ and $f'_s(x_2) = g$ or $f'_w(x_2) = m$ and $f'_s(x_2) = g$; therefore, $x_2 \models \varphi_1$ always holds. Note that $x_2 \models^+ \varphi_1$ cannot be verified if we do not transform φ_1 into its CNF. Analogously, let φ_2 denote the POUDL wff $(\leqslant_w l) \wedge ((\geqslant_w m) \vee (\leqslant_s b))$, then its DNF is $\varphi_2^d =$

Table 2
A POUDT for route selection

$U\backslash A$	w	s	d
x_1	$\{m,h\}$	$\{v\}$	3
x_2	$\{l,m\}$	$\{g\}$	2
x_3	$\{m\}$	$\{g,v\}$	2
x_4	$\{l,m\}$	$\{b,g\}$	2
x_5	$\{l\}$	$\{b\}$	1
x_6	$\{h\}$	$\{g,v\}$	3

$(w,\emptyset) \vee ((\leqslant_w l) \wedge (\leqslant_s b))$. It is easy to verify $x_2 \models^- \varphi_2$ by the semantics. Again, this cannot be done if φ_2 is not transformed into its CNF.

Let $\varphi = (\geqslant_w, m)$, $\psi_1 = (\geqslant_d, 2)$, and $\psi_2 = (\geqslant_d, 3)$ be three wffs of POUDL, then we consider the rules $\varphi \rightarrow \psi_1$ and $\varphi \rightarrow \psi_2$. We have

$$\begin{aligned}
&\underline{m}(\varphi) = \{x_1, x_3, x_6\}, \quad \overline{m}(\varphi) = \{x_1, x_2, x_3, x_4, x_6\}, \quad m^*(\varphi) = \{x_2, x_4\},\\
&\underline{m}(\varphi \wedge \psi_1) = \{x_1, x_3, x_6\}, \quad \overline{m}(\varphi \wedge \psi_1) = \{x_1, x_2, x_3, x_4, x_6\}, \quad m^+(\psi_1) = \{x_1, x_2, x_3, x_4, x_6\},\\
&m^-(\psi_1) = \{x_5\}, \quad \underline{m}(\varphi \wedge \psi_2) = \{x_1, x_6\}, \quad \overline{m}(\varphi \wedge \psi_2) = \{x_1, x_6\},\\
&m^+(\psi_2) = \{x_1, x_6\}, \quad \text{and} \quad m^-(\psi_2) = \{x_2, x_3, x_4, x_5\}.
\end{aligned}$$

Thus, we obtain the *asi*, *rsi*, and *cfi* of these two rules as follows:

	asi	*rsi*	*cfi*
$\varphi \rightarrow \psi_1$	$[3,5]$	$[\frac{1}{2}, \frac{5}{6}]$	$[1,1]$
$\varphi \rightarrow \psi_2$	$[2,2]$	$[\frac{1}{3}, \frac{1}{3}]$	$[\frac{2}{5}, \frac{2}{3}]$

It can be verified that these values indeed satisfy equalities in Proposition 9. Note that the rule $\varphi \rightarrow \psi_1$ is strongly valid, whereas the rule $\varphi \rightarrow \psi_2$ is neither weakly valid, nor strongly valid.

5. Preference-ordered fuzzy data tables

The preference-ordered fuzzy data table (POFDT) is a further generalization of POUDT. An approach for dealing with fuzzy information in PODT has been proposed in Greco et al. (1999b). In this section, we propose an alternative based on our logical formalism. For any domain V, let $\mathbb{NF}(V)$ denote the set of all normalized fuzzy subsets of V. Recall that a fuzzy subset of domain V is normalized if $\sup_{x \in V} \mu(x) = 1$, where μ is the membership function of the fuzzy subset. A POFDT is a tuple

$$T = (U, A, \{(V_i, \succeq_i) | i \in A\}, \{f_i | i \in A\}),$$

where $U, A, \{(V_i, \succeq_i) | i \in A\}$ are defined as above, and for each $i \in A$, $f_i : U \rightarrow \mathbb{NF}(V_i)$.

For the representation of rules induced from POFDT, we can imagine several generalized decision languages, such as those introduced in Liau and Liu (1999, 2001) and Fan et al. (2001). However, for simplicity, we use the syntax of PODL and interpret the wffs of PODL with respect to POFDT. Thus, the language of preference-ordered fuzzy decision logic (POFDL) is simply the language of PODL.

For the semantics of POFDL, we define the valuation function with respect to a POFDT over the wffs of POFDL. The function is denoted by E_T and defined by

(1) $E_T(x, (\geqslant_i, v)) = \inf\{1 - \mu_i^x(v_i) | v_i \in V_i, v_i \not\succeq_i v\}$, where μ_i^x is the membership function of $f_i(x)$;
(2) $E_T(x, (\leqslant_i, v)) = \inf\{1 - \mu_i^x(v_i) | v_i \in V_i, v_i \not\preceq_i v\}$, where μ_i^x is the membership function of $f_i(x)$;
(3) $E_T(x, \neg\varphi) = 1 - E_T(x, \varphi)$;
(4) $E_T(x, \varphi \wedge \psi) = E_T(x, \varphi) \otimes E_T(x, \psi)$, where $\otimes$: $[0,1] \times [0,1] \rightarrow [0,1]$ is a t-norm[3]; and
(5) $E_T(x, \varphi \vee \psi) = E_T(x, \varphi) \oplus E_T(x, \psi)$, where $\oplus$ is the t-conorm defined by $a \oplus b = 1 - (1-a) \otimes (1-b)$.

[3] A binary operation $\otimes$ is a t-norm iff it is associative, commutative, and increasing in both places, and $1 \otimes a = a$ and $0 \otimes a = 0$ for all $a \in [0,1]$.

Note that, if we consider the membership function μ_i^x as a possibility distribution on the domain V_i, $E_T(x,(\geqslant_i,v))$ corresponds to the necessity measure (Zadeh, 1978) of the subset $\{v_i \in V_i | v_i \succeq_i v\}$. The same remark holds for $E_T(x,(\leqslant_i,v))$.

The valuation function can be extended to cover the POFDL rules by the following equation:

$$E_T(x,\varphi \to \psi) = E_T(x,\varphi) \to_{\otimes} E_T(x,\psi),$$

where $\to_{\otimes}$: $[0,1]\times[0,1]\to[0,1]$ is the residuated implication function for $\otimes$, defined as $a \to_{\otimes} b = \sup\{x | x\otimes a \leqslant b\}$. As usual, we can omit the subscript T from E_T if this does not cause confusion.

The notions of validity, support, and confidence can be modified as follows.

Definition 12. Let Φ_2 be the set of all POFDL rules and T be a POFDT. Then

(1) the validity function val_T: $\Phi_2 \to [0,1]$ is
$$val_T(\varphi \to \psi) = \otimes_{x\in U} E_T(x,\varphi \to \psi);$$
(2) the absolute support function asp_T: $\Phi_2 \to [0,|U|]$ is
$$asp_T(\varphi \to \psi) = \sum_{x\in U} E_T(x,\varphi \wedge \psi);$$
(3) the relative support function rsp_T: $\Phi_2 \to [0,1]$ is
$$rsp_T(\varphi \to \psi) = \frac{\sum_{x\in U} E_T(x,\varphi \wedge \psi)}{|U|};\ \text{and}$$
(4) the confidence function cfd_T: $\Phi_2 \to [0,1]$ is
$$cfd_T(\varphi \to \psi) = \frac{\sum_{x\in U} E_T(x,\varphi \wedge \psi)}{\sum_{x\in U} E_T(x,\varphi)}.$$

Example 13. We use a project evaluation system to illustrate POFDT and POFDL. Assume some projects are evaluated with respect to originality, presentation, and technical feasibility. The set of attributes is the same as in Example 5. The domains of values of these attributes are $[0,10]$ endowed with ordinary ordering of real numbers. However, due to the difficulty of precise evaluation, the attribute values for these projects are fuzzy subsets represented by linguistic labels $\{a\,(\text{excellent}), b\,(\text{good}), c\,(\text{fair}), d\,(\text{poor})\}$. The membership functions of these subsets are given in Fig. 1, and the evaluation results are presented in Table 3.

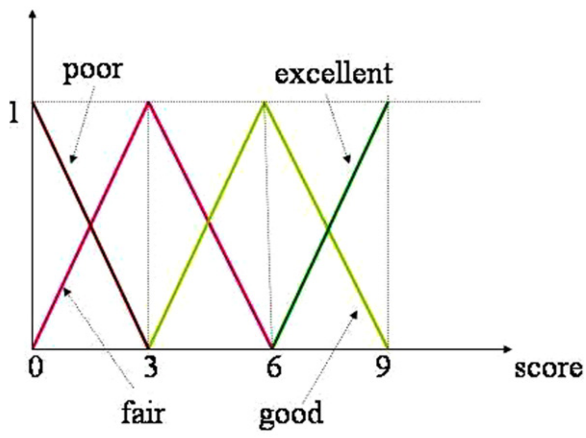

Fig. 1. The membership functions of fuzzy sets.

Table 3
A data table for project evaluation

$U\backslash A$	o	p	t	d
x_1	a	a	b	a
x_2	b	c	b	b
x_3	a	b	c	b
x_4	c	c	c	c
x_5	c	d	c	d
x_6	b	d	c	d
x_7	b	c	c	c
x_8	a	d	c	c

The membership functions are defined by

$$\mu_a(v) = \begin{cases} \frac{v-7}{3} & \text{if } 7 \leqslant v \leqslant 10, \\ 0 & \text{elsewhere,} \end{cases}$$
$$\mu_b(v) = \begin{cases} \frac{2v-10}{3} & \text{if } 5 \leqslant v \leqslant \frac{13}{2}, \\ \frac{16-2v}{3} & \text{if } \frac{13}{2} \leqslant v \leqslant 8, \\ 0 & \text{elsewhere,} \end{cases}$$
$$\mu_c(v) = \begin{cases} \frac{2v-4}{3} & \text{if } 2 \leqslant v \leqslant \frac{7}{2}, \\ \frac{10-2v}{3} & \text{if } \frac{7}{2} \leqslant v \leqslant 5, \\ 0 & \text{elsewhere,} \end{cases}$$
$$\mu_d(v) = \begin{cases} \frac{3-v}{3} & \text{if } 0 \leqslant v \leqslant 3, \\ 0 & \text{elsewhere.} \end{cases}$$

Let us now consider three POFDL atomic formulas: $\varphi_1 = (\geqslant_o, 8)$, $\varphi_2 = (\geqslant_t, 4)$, and $\psi = (\geqslant_d, 6)$. The truth value of an atomic formula for an object depends on the attribute values of the object as follows:

	$E_T(x, (\geqslant_i, 8))$	$E_T(x, (\geqslant_i, 4))$	$E_T(x, (\geqslant_i, 6))$
$f_i(x) = a$	$\frac{2}{3}$	1	1
$f_i(x) = b$	0	1	$\frac{1}{3}$
$f_i(x) = c$	0	0	0
$f_i(x) = d$	0	0	0

If we use the Łukasiewicz t-norm and implication (Hájek, 1998) defined by

$$a \otimes b = \max(a + b - 1, 0), \text{ and}$$
$$a \rightarrow_{\otimes} b = \min(1, 1 - a + b),$$

we can obtain the truth values of the wffs of the objects as follows:

	φ_1	φ_2	ψ	$\varphi_1 \wedge \varphi_2$	$\varphi_1 \wedge \varphi_2 \wedge \psi$	$\varphi_1 \wedge \varphi_2 \rightarrow \psi$
x_1	$\frac{2}{3}$	1	1	$\frac{2}{3}$	$\frac{2}{3}$	1
x_2	0	1	$\frac{1}{3}$	0	0	1
x_3	$\frac{2}{3}$	0	$\frac{1}{3}$	0	0	1
x_4	0	0	0	0	0	1
x_5	0	0	0	0	0	1
x_6	0	0	0	0	0	1
x_7	0	0	0	0	0	1
x_8	$\frac{2}{3}$	0	0	0	0	1

Therefore, if r is the rule $\varphi_1 \wedge \varphi_2 \rightarrow \psi$, we have $val_T(r) = 1$, $asp_T(r) = \frac{2}{3}$, $rsp_T(r) = \frac{1}{12}$, and $cfd_T(r) = 1$.

Note that, though POFDT is a generalization of POUDT, the quantitative measures of POFDL are not calculated through the completions of POFDT as in the case of POUDT. In fact, it is unclear how the notion of completion can be generalized to POFDT. We can envision at least two possibilities. One way is to consider a completion of a POFDT as a pair (T, c), where T is a classical DT and $c \in [0,1]$; the other way is to define it as a pair (T, μ), where T is a classical DT and $\mu : Uni(T) \to [0,1]$. Based on these definitions, we can derive the bounds of quantitative measures of POFDL rules as in Proposition 9. However, the detailed definitions and derivations of such results will be addressed in our future research.

6. Pairwise comparison decision logic

Greco et al. (1997, 1998, 1999a) proposed the pairwise comparison table (PCT) to handle multicriteria choice or ranking problems. In a PCT, the strength of preferences between objects, instead of the evaluation scores of objects, are stored with respect to each criterion. Formally, a PCT is a tuple

$$T = (U, A, \{H_i | i \in A\}, \{f_i | i \in A\}),$$

where U and A are defined as above; and for each $i \in A$, H_i is a finite set of integers, and $f_i : U \times U \to H_i$ encodes the preferential information[4]. Each H_i denotes a different grade of preference (such as "very weak preference", "weak preference", "strong preference", etc.) with respect to the criterion i. If $f_i(x,y) = h > 0$, then x is preferred to y by degree h with respect to the criterion i. If $f_i(x,y) = h < 0$, then x is not preferred to y by degree h with respect to the criterion i. If $f_i(x,y) = 0$, then x is similar to y with respect to the criterion i. A PCT is *coherent* if for each $i \in A$ and $x, y \in U$, $f_i(x,y) > 0$ implies $f_i(y,x) \leqslant 0$ and $f_i(x,y) < 0$ implies $f_i(y,x) \geqslant 0$. In this paper, we only consider a coherent PCT.

To represent rules induced from a PCT, we propose pairwise comparison decision logic (PCDL). An atomic formula of PCDL is a descriptor of the form $(i, \geqslant_h)$ or $(i, \leqslant_h)$, where $i \in A$ and $h \in H_i$, and the wffs and rules of PCDL are defined in the same way as those for the other decision logic languages discussed in this paper. However, unlike other logics, where wffs are evaluated with respect to an object, the wffs of PCDL are evaluated with respect to a pair of objects. More precisely, the satisfaction of a wff with respect to a pair of objects (x,y) is defined as follows:

(1) $(x,y) \models (i, \geqslant_h)$ iff $f_i(x,y) \geqslant h$,
(2) $(x,y) \models (i, \leqslant_h)$ iff $f_i(x,y) \leqslant h$,
(3) $(x, y) \models \neg\varphi$ iff $(x,y) \not\models \varphi$,
(4) $(x,y) \models \varphi \wedge \psi$ iff $(x,y) \models \varphi$ and $(x,y) \models \psi$,
(5) $(x,y) \models \varphi \vee \psi$ iff $(x,y) \models \varphi$ or $(x,y) \models \psi$.

If φ is a PCDL wff and T is a PCT, the set $m_T(\varphi)$ defined by

$$m_T(\varphi) = \{(x,y) \in U \times U | (x,y) \models \varphi\}, \tag{2}$$

is called the meaning set of the formula φ in T. If T is understood, we simply write $m(\varphi)$.

Definition 14. Let Φ_3 be the set of all PCDL rules and $T = (U, A, \{H_i | i \in A\}, \{f_i | i \in A\})$ be a PCT. Then,

(1) the rule $\varphi \to \psi$ is valid in T iff $m_T(\varphi) \subseteq m_T(\psi)$;
(2) the absolute support function $asp_T : \Phi_3 \to \mathbb{N}$ is

$$asp_T(\varphi \to \psi) = |m_T(\varphi \wedge \psi)|;$$

(3) the relative support function $rsp_T : \Phi_3 \to [0,1]$ is

$$rsp_T(\varphi \to \psi) = \frac{|m_T(\varphi \wedge \psi)|}{|U|^2}; \text{ and}$$

[4] Without loss of generality, we slightly change the original definition in Greco et al. (1997, 1998, 1999a).

(4) the confidence function $cfd_T : \Phi_3 \rightarrow [0,1]$ is

$$cfd_T(\varphi \rightarrow \psi) = \frac{|m_T(\varphi \wedge \psi)|}{|m_T(\varphi)|}.$$

Note that the confidence function for PCDL rules was previously used to define the variable consistency model of a PCT (Slowinski et al., 2002a).

Without loss of generality, we can rename the elements of U as natural numbers from 0 to $|U| - 1$. Then, each f_i can be seen as a $|U| \times |U|$ matrix M_i over domain H_i. Thus, we can employ matrix algebra to test the validity of a rule and calculate its support and confidence in an analogous way to that proposed in Liau (2004a,b).

By using PCDL, the three types of decision rules mentioned in Greco et al. (2001a) can be represented as follows:

(1) $D_{\geqslant}$-decision rules:

$$\bigwedge_{i\in C}(i, \geqslant_{h_i}) \rightarrow (d, \geqslant_1),$$

(2) $D_{\leqslant}$-decision rules:

$$\bigwedge_{i\in C}(i, \leqslant_{h_i}) \rightarrow (d, \leqslant_{-1}),$$

(3) $D_{\geqslant\leqslant}$-decision rules:

$$\bigwedge_{i\in C_1}(i, \geqslant_{h_i}) \wedge \bigwedge_{i\in C_2}(i, \leqslant_{h_i}) \rightarrow (d, \geqslant_1) \vee (d, \leqslant_{-1}),$$

where C, C_1, and $C_2 \subseteq A$ are sets of criteria, and $d \in A$ is the decision attribute. We assume that $\{-1, 1\} \subseteq H_d$ so that $f_d(x, y) = 1$ means that x outranks y, and $f_d(x, y) = -1$ means that y outranks x.

Example 15. Let us define a PCT from the PODT introduced in Example 6. The PCT is defined as

$$(U, A, \{H_i | i \in A\}, \{f_i' | i \in A\}),$$

where U and A are defined as in Example 5, $H_i = \{-3, -2, -1, 0, 1, 2, 3\}$, and f_i' is defined as $f_i'(x, y) = f_i(x) - f_i(y)$ for all $x, y \in U$ and $i \in A$, where f_i is also defined as in Example 5. Let us consider the following two rules:

$$r_1 = (o, \geqslant_2) \rightarrow (d, \geqslant_1),$$
$$r_2 = (p, \leqslant_{-2}) \rightarrow (d, \leqslant_{-1}).$$

Then, we have

	asp	rsp	cfd
r_1	7	7%	$\frac{7}{8}$
r_2	15	15%	1

Note that the rule r_2 is valid, even though it only has a support value of 0.15. Furthermore, since in this example, $m((d, \geqslant_1) \vee (d, \leqslant_0)) = U \times U$ holds, the $D_{\geqslant\leqslant}$-decision rules are always valid and have a confidence value equal to 1.

7. Conclusions

In this paper, we present some logics that are useful in the representation of rules induced from preference-ordered data tables. Such data tables are commonly used in MCDA. The main advantage of using logic is its syntax and semantics are precise. As DL is a precise way to represent decision rules induced from classical data tables, we use PODL and PCDL to reformulate the decision rules induced from PODT and PCDT in DRSA, respectively. Though this seems a trivial step, it maps the decision rules induced from PODT and PCDT into precise logical formulas and gives them a formal semantics. The less trivial task is to generalize PODL to POUDL and POFDL. While the issue of missing values has been addressed in classical DT or PODT, we deal with uncertain values or fuzzy values. In particular, we derive the closed form for the lower and upper bounds of the confidence and support values of POUDL rules in each precise completion of the POUDT. We also present the semantics of POFDL rules based on possibility theory.

While this paper is primarily concerned with the syntax and declarative semantics of the logics, efficient algorithms for data mining based on the logical representations are also urgently needed. Developing such algorithms is an important research direction.

In addition to decision logic, information logic is another kind of logic arising from data tables (Demri and Orlowska, 2002). The semantics of information logic is the same as the Kripke semantics for modal logics. We believe that it would also be interesting to explore information logics with respect to dominance relations.

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