圖-設計 計畫結案報告

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中文摘要：

圖-設計(Graph Design)主要是研究如何把一個圖 H 分割成很多個彼此互相同構的子圖 G，通常我們用 G|H 表示。由於 $K_k \mid \lambda K_v$ 對應於一個 $2-(v,k,\lambda)$ 設計， $K_k \mid \lambda K_{m(n)}$ 對應於 Group Divisible Design (可分組設計) $GD[n,m;k,\lambda]$，所以設計理論的研究自然在圖設計的研究中扮演非常重要的角色。

以上所提的兩種設計已經廣泛地應用在實驗設計中，所以，我們的研究除了增加更近代科技領域上的用途之外，也尋求不同的設計，例如 Grid-Block Design，它在 DNA Library Screening 方面有很重要的應用；另外，除了設計，也探討 packing 的應用，在同步光學網路及群試理論的平行式演算法上我們也得到很好的應用。

關鍵字：圖設計、方格圖設計、DNA 排序、同步光學網路、平行群試演算法。

英文摘要：

The main focus of the study of graph-design is to decompose a graph H into isomorphic copies of subgraphs G, denoted by G|H. It is well-known that $K_k \mid \lambda K_v$ is equivalent to the existence of a $2-(v,k,\lambda)$ design and $K_k \mid \lambda K_{m(n)}$ is equivalent to the existence of a group divisible design $GD[n,m;k,\lambda]$. Therefore, the study of combinatorial designs plays an important role in our study.

The use of combinatorial designs in experiment designs has been known for many occasions. Thus, we intend to add more which are related to modern technologies. Also, we expect to find more other types

of designs such as grid-block design. Note that this design has its application on DNA Library Screening (related to DNA sequencing). Besides, we also utilize the packing of graph to obtain well-constructed SONET and better disjunct matrices which are the main objectives in nonadaptive algorithms of group testing.

Keywords：Graph design, Grid-block design, DNA-sequencing, SONET, Non-adaptive algorithms.

報告內容：

在過去三年中，我們在圖設計的建構方面獲得不少成果，參見附錄。基本上我們的工作可以分成理論的建構與群試及網路的應用。前者，除了圖的圈分割[2,3,4,5,7,9,10]之外，我們也完成用配對[6,11]或較短路徑[8]來覆蓋一個圖。在應用方面，主要的工作之一是群試理論中建構 Non-adaptive algorithm 的 disjunct 矩陣，我們適當地利用圖設計的結果來完成工作[1,12]；另外，利用不同子圖來裝填完全圖也在同步光學網路上找到很好的應用[13]。

附錄：

1. A novel use of t-packings in constructing d-disjunct matrices (with F. K. Hwang), Discrete Applied Math., Vol. 154, Issue 12, 2006, 1759-1762.
2. Maximum Cyclic 4-cycle packings of the complete multipartite graph (with S. L. Wu), J. Combin. Optimization, Number 2-3, Vol. 14(2007), 365-382.
3. Maximal sets of hamiltonian cycles in $K_{2p}-F$ (with S. L. Logan and C. A.

Rodger), Discrete Math., 308(2008), 2822-2829.

4. All graphs with maximum degree three whose complements have 4-cycle decompositions (with C. M. Fu, C. A. Rodger and Todd Smith), Discrete Math. 308(2008), 2901-2909.
5. Maximal sets of Hamitonian cycles in D_n (with Liqun Pu and Hao Shen), Discrete Math., 308(2008), 3706-3710.
6. On the minimum sets of 1-factors covering a complete multipartite graph (with D. Cariolaro), J. Graph Theory, Vol. 58, Issue 3(2008), 239-250.
7. The Hamilton-Waterloo problem for two even cycle factors (with Kuo-Cjing Huang), Taiwanese J. Math., Vol. 12, No. 4, 2008, 933-940.
8. The linear 3-arboricity of $K_{n,n}$ and K_n (with K. C. Huang and C. H. Yen), Discrete Math., 308, Issue 17(2008), 3761-3769.
9. Multicolored parallelism of Hamiltonian cycles (with Y. H. Lo), Discrete Math., to appear.
10. $\overline{C_3}$-decompositions of D_t with quadratic leaves (with Liqun Pu and H. Sheu), Discrete Math., to appear.
11. Covering graphs with matchings of fixed size (with D. Cariolaro), Discrete Math., to appear.
12. A new construction of $\bar{3}$-separable matrices via improved decoding of Macula construction, Discrete Optim., to appear.
13. Minimizing SONET ADMs in indirectional WDM rings with grooming rate 7 (with C. Colbourn, G. Ge, A. Ling and Hui-Chuan Lu), SIAM J. Discrete Math., to appear.

Minimizing SONET ADMs in Unidirectional WDM Rings with Grooming Ratio 7

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Abstract

In order to reduce the number of add-drop multiplexers (ADMs) in SONET/WDM networks using wavelength add-drop multiplexing, certain graph decompositions can be used to form a 'grooming' that specifies the assignment of traffic to wavelengths. When traffic among nodes is all-to-all and uniform, the drop cost of such a decomposition is the sum, over all graphs in the decomposition, of the number of vertices of nonzero degree in the graph. The number of ADMs required is this drop cost. The existence of such decompositions with minimum cost, when every pair of sites employs no more than $\frac{1}{7}$ of the wavelength capacity, is determined within an additive constant. Indeed when the number n of sites satisfies $n \equiv 1 \pmod 3$ and $n \neq 19$, the determination is exact; when $n \equiv 0 \pmod 3$, $n \not\equiv 18 \pmod{24}$, and n is large enough, the determination is also exact; and when $n \equiv 2 \pmod 3$ and n is large enough, the gap between the cost of the best construction and the cost of the lower bound is independent of n and does not exceed 4.

1 Introduction

Traffic grooming in optical (SONET) rings arises from amalgamating C low rate signals onto a higher capacity wavelength [15, 25, 26]; C is the *grooming ratio*. Nodes initiate or terminate traffic on a wavelength using an *add-drop multiplexer* (ADM). Finding the minimum number of add-drop multiplexers (ADMs), $A(C, n)$, required in an n-node SONET ring with grooming ratio

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C, is equivalent to the following problem in graphs [4]: Given a number of nodes n and a grooming ratio C find a partition of the edges of K_n into subgraphs B_ℓ, $\ell = 1, \dots, s$ with $|E(B_\ell)| \le C$ such that $\sum_{1 \le \ell \le s} |V(B_\ell)|$ is minimum.

Optimal constructions for given grooming ratio C have been obtained using tools of graph and design theory [9]. Results are known for grooming ratio $C = 3$ [1], $C = 4$ [5, 23], $C = 5$ [3], $C = 6$ [2], $C \le \frac{1}{6}n(n-1)$ [5], and for large values of C [5]. Related problems have been studied for variable traffic requirements [8, 14, 22, 27, 29], for fixed traffic requirements [1, 3, 4, 5, 15, 21, 23, 24, 25, 28, 30], and in the case of bidirectional rings [10, 13]. The explicit correspondence between grooming and graph decomposition is developed in detail in [1, 11].

In this paper we consider grooming with grooming ratio 7. In Section 2 we employ linear programming duality to establish a general lower bound on $A(7, n)$. In Section 4 we determine $A(7, n)$ with the possible exception of $n = 19$ when $n \equiv 1 \pmod 3$. When $n \equiv 0 \pmod 3$ (Section 5) we determine $A(7, n)$ with finitely many possible exceptions except when $n \equiv 18 \pmod{24}$; in the latter case we establish a construction whose cost exceeds the lower bound by 1. When $n \equiv 2 \pmod 3$ (Section 6) we develop a set of constructions to establish that, with finitely many possible exceptions, the cost does not exceed the lower bound by more than 4, independent of n.

It is natural to ask why the case when $C = 7$ is of independent interest. Unlike all cases when $C \le 6$, the graph with the lowest ratio of number of vertices to number of edges does not have C edges; rather it is K_4, a 6-edge graph. This necessitates consideration of decompositions that do not use the minimum number of graphs, and hence determining the minimum number of wavelengths required is quite different than determining the minimum drop cost.

2 The Lower Bounds

We adapt a strategy using linear programming from [12] that was used in [11] to determine both the cost and the structure of certain optimal groomings. A grooming with ratio 7 is a decomposition of K_n into subgraphs each having at most 7 edges. Its *drop cost*, or just *cost*, is the sum of the numbers of vertices of nonzero degree over all graphs in the decomposition. $A(7, n)$ is the minimum drop cost of a grooming of K_n with grooming ratio 7. Figure 1 displays all connected graphs having at most 7 edges. The naming convention is as follows. For each number q of edges and p of vertices, suppose that there are $\gamma_{q,p}$ nonisomorphic graphs. These are named $G_{\ell,q,p}$ for $1 \le \ell \le \gamma_{q,p}$.

In a decomposition, let $\alpha_{\ell,q,p}$ be the number of occurrences of $G_{\ell,q,p}$ and let $\alpha_{q,p} = \sum_{\ell=1}^{\gamma_{q,p}} \alpha_{\ell,q,p}$. Then because every edge appears in exactly one of the chosen subgraphs,

$$\sum_{q=1}^{7} \sum_{p=1}^{8} \sum_{\ell=1}^{\gamma_{q,p}} q \cdot \alpha_{\ell,q,p} = \binom{n}{2} \tag{1}$$

In order to minimize drop cost, we must compute

$$\min \sum_{q=1}^{7} \sum_{p=1}^{8} \sum_{\ell=1}^{\gamma_{q,p}} p \cdot \alpha_{\ell,q,p} \tag{2}$$

Graph	$G_{\ell,q,p}$	deg. seq.	$\phi_{\ell,q,p}$	$\psi_{\ell,q,p}$
$G_{1,7,5}$		44222	4	3.5
$G_{2,7,5}$		43322	2.5	2
$G_{3,7,5}$		43331	1	2
$G_{4,7,5}$		33332	1	0.5
$G_{1,7,6}$		442211	4	5
$G_{2,7,6}$		522221	5.5	3.5
$G_{3,7,6}$		422222	5.5	3.5
$G_{4,7,6}$		532211	4	3.5
$G_{5,7,6}$		432221	4	3.5
$G_{6,7,6}$		433211	2.5	3.5
$G_{7,7,6}$		332222	4	2
$G_{1,7,7}$		4421111	4	6.5
$G_{2,7,7}$		5222111	5.5	5
$G_{3,7,7}$		4222211	5.5	5
$G_{4,7,7}$		5321111	4	5
$G_{5,7,7}$		4322111	4	5
$G_{6,7,7}$		4331111	2.5	5
$G_{7,7,7}$		2222222	7	3.5
$G_{8,7,7}$		3222221	5.5	3.5
$G_{9,7,7}$		3322211	4	3.5
$G_{10,7,7}$		3332111	2.5	3.5
$G_{1,7,8}$		71111111	4	8
$G_{2,7,8}$		44111111	4	8
$G_{3,7,8}$		52211111	5.5	6.5
$G_{4,7,8}$		42221111	5.5	6.5
$G_{5,7,8}$		62111111	4	6.5
$G_{6,7,8}$		53111111	4	6.5
$G_{7,7,8}$		43211111	4	6.5
$G_{8,7,8}$		22222211	7	5
$G_{9,7,8}$		32222111	5.5	5
$G_{10,7,8}$		33221111	4	5
$G_{11,7,8}$		33311111	2.5	5
$G_{1,6,4}$		3333	0	0
$G_{1,6,5}$		42222	4.5	3
$G_{2,6,5}$		43221	3	3
$G_{3,6,5}$		33222	3	1.5
$G_{4,6,5}$		33321	1.5	1.5
$G_{1,6,6}$		522111	4.5	4.5
$G_{2,6,6}$		422211	4.5	4.5
$G_{3,6,6}$		432111	3	4.5
$G_{4,6,6}$		222222	6	3
$G_{5,6,6}$		322221	4.5	3
$G_{6,6,6}$		332211	3	3
$G_{7,6,6}$		333111	1.5	3
$G_{1,6,7}$		5211111	4.5	6
$G_{2,6,7}$		4221111	4.5	6
$G_{3,6,7}$		4311111	3	6
$G_{4,6,7}$		6111111	3	6
$G_{5,6,7}$		2222211	6	4.5
$G_{6,6,7}$		3222111	4.5	4.5
$G_{7,6,7}$		3321111	3	4.5
$G_{1,5,4}$		3322	2	1
$G_{1,5,5}$		42211	3.5	4
$G_{2,5,5}$		22222	5	2.5
$G_{3,5,5}$		32221	3.5	2.5
$G_{4,5,5}$		33211	2	2.5
$G_{1,5,6}$		421111	3.5	5.5
$G_{2,5,6}$		511111	3.5	5.5
$G_{3,5,6}$		322111	3.5	4
$G_{4,5,6}$		331111	2	4
$G_{5,5,6}$		222211	5	3
$G_{1,4,4}$		2222	4	2
$G_{2,4,4}$		3221	2.5	2
$G_{1,4,5}$		41111	2.5	5
$G_{2,4,5}$		22211	4	3.5
$G_{3,4,5}$		32111	2.5	3.5
$G_{1,3,3}$		222	3	1.5
$G_{1,3,4}$		2211	3	3
$G_{2,3,4}$		3111	1.5	3
$G_{1,2,3}$		211	2	2.5
$G_{1,1,2}$		11	1	2

Figure 1: The Graphs

Figure 1 does not list disconnected graphs, but the cost of a disconnected graph is the sum of the costs of its components, so all feasible decompositions are accounted for. For every graph $G_{\ell,q,p}$, we find that $\frac{p}{q} \geq \frac{2}{3}$. Subtract $\frac{2}{3}\times(1)$ from (2) to restate the minimum drop cost $A(7, n)$ as

$$\frac{n(n-1)}{3} + \min \sum_{q=1}^{7} \sum_{p=1}^{8} \sum_{\ell=1}^{\gamma_{q,p}} (p - \frac{2}{3}q) \cdot \alpha_{\ell,q,p} \tag{3}$$

In (3) the triple summation is always nonnegative; it can be zero only when all graphs are isomorphic to K_4. However, structural restrictions can prohibit such a selection. In particular, considering the number $\binom{n}{2}$ of edges modulo 6,

$$\sum_{q=1}^{7} \sum_{p=1}^{8} \sum_{\ell=1}^{\gamma_{q,p}} (q \bmod 6) \cdot \alpha_{\ell,q,p} \equiv \begin{cases} 0 \pmod 6 \text{ if } n \equiv 0,1,4,9 \pmod{12} \\ 1 \pmod 6 \text{ if } n \equiv 2,11 \pmod{12} \\ 3 \pmod 6 \text{ if } n \equiv 3,6,7,10 \pmod{12} \\ 4 \pmod 6 \text{ if } n \equiv 5,8 \pmod{12} \end{cases} \tag{4}$$

We can relax this congruence to linear inequalities. For example, if $n \equiv 3, 6, 7, 10 \pmod{12}$,

$$\sum_{p=1}^{8} \left[\sum_{\ell=1}^{\gamma_{3,p}} \alpha_{\ell,3,p} + \frac{1}{3}\Big(\sum_{q\in\{1,4,7\}} \sum_{\ell=1}^{\gamma_{q,p}} \alpha_{\ell,q,p}\Big) + \frac{2}{3}\Big(\sum_{q\in\{2,5\}} \sum_{\ell=1}^{\gamma_{q,p}} \alpha_{\ell,q,p}\Big) \right] \geq 1 \tag{5}$$

because if there is no graph on three edges, there must be at least three graphs having $1 \pmod 3$ edges, or one having $1 \pmod 3$ edges and one having $2 \pmod 3$ edges.

Every vertex of K_n has degree congruent to $n-1 \bmod 3$; placing a K_4 in the decomposition does not change this congruence class at any vertex, and hence subgraphs other than K_4 may be needed to accommodate these vertex degrees. Let $\omega_{\ell,q,p}$ be the number of vertices whose degree is congruent to 1 modulo 3 in $G_{\ell,q,p}$, and let $\tau_{\ell,q,p}$ be the number of vertices whose degree is congruent to 2 modulo 3. Now if $n \equiv 0 \pmod 3$, every vertex has degree 2 modulo 3, and hence at every vertex there must either be a graph itself having degree 2 modulo 3, or two graphs each having degree 1 modulo 3 (there may be more). And if $n \equiv 2 \pmod 3$, every vertex has degree 1 modulo 3, and hence at every vertex there must either be a graph itself having degree 1 modulo 3, or two graphs each having degree 2 modulo 3. For convenience we write $\phi_{\ell,q,p} = \frac{1}{2}\omega_{\ell,q,p} + \tau_{\ell,q,p}$ and $\psi_{\ell,q,p} = \omega_{\ell,q,p} + \frac{1}{2}\tau_{\ell,q,p}$. These are tabulated for each graph in Figure 1. We conclude that

$$\begin{aligned} &\textstyle\sum_{q=1}^{7} \sum_{p=1}^{8} \sum_{\ell=1}^{\gamma_{q,p}} \phi_{\ell,q,p} \cdot \alpha_{\ell,q,p} \geq n \quad \text{if} \quad n \equiv 0 \pmod 3 \\ &\textstyle\sum_{q=1}^{7} \sum_{p=1}^{8} \sum_{\ell=1}^{\gamma_{q,p}} \psi_{\ell,q,p} \cdot \alpha_{\ell,q,p} \geq n \quad \text{if} \quad n \equiv 2 \pmod 3 \end{aligned} \tag{6}$$

Theorem 2.1 *The cost of an optimal grooming of K_n with grooming ratio 7, $A(7, n)$, is at least*

$$\begin{array}{lll} \frac{2}{3}\binom{n}{2} & \textit{if} & n \equiv 1,4 \pmod{12} \\ \frac{2}{3}\binom{n}{2} + 1 & \textit{if} & n \equiv 7,10 \pmod{12} \\ \frac{2}{3}\binom{n}{2} + \lceil\frac{n}{12}\rceil & \textit{if} & n \equiv 0,3,6,15,18,21 \pmod{24} \\ \frac{2}{3}\binom{n}{2} + \lceil\frac{n}{12}\rceil + 1 & \textit{if} & n \equiv 9,12 \pmod{24} \\ \lceil\frac{2}{3}\binom{n}{2} + \frac{2n}{21}\rceil & \textit{if} & n \equiv 5,8,17 \pmod{21} \\ & & \textit{or } n \equiv 2,23,32,53,56,77,62,83 \pmod{84} \\ \lceil\frac{2}{3}\binom{n}{2} + \frac{2n}{21}\rceil + 1 & \textit{if} & n \equiv 14,35,20,41,44,65,74,11 \pmod{84} \end{array}$$

Proof: We follow the strategy in [12]. Form a linear program whose variables are the $\{\alpha_{\ell,q,p}\}$s.

$$\begin{array}{c} \min \sum_{q=1}^{7} \sum_{p=1}^{8} \sum_{\ell=1}^{\gamma_{q,p}} (p - \frac{2}{3}q) \cdot \alpha_{\ell,q,p} \\ \text{subject to (4) suitably relaxed, (6), and nonnegativity of each variable} \end{array} \tag{7}$$

If $z^\star$ is the minimum, the cost of any grooming must be at least $\lceil \frac{2}{3}\binom{n}{2} + z^\star \rceil$, since the cost is integral. By forming the dual of (7), any feasible solution to the dual gives a lower bound on all primal feasible solutions, and hence a lower bound on $z^\star$.

Case 1: $n \equiv 1 \pmod 3$: When $n \equiv 1, 4 \pmod{12}$, the linear program is constrained only by nonnegativity, and the dual optimum is 0. When $n \equiv 7, 10 \pmod{12}$, (5) holds. Call its dual variable y_1. An assignment $y_1^\star$ is dual feasible if $y_1^\star \le p - 2$ for every graph $G_{\ell,3,p}$; $y_1^\star \le \frac{3}{2}(p - \frac{2}{3}q)$ for every graph $G_{\ell,q,p}$ with $q \in \{2,5\}$; and $y_1^\star \le 3(p - \frac{2}{3}q)$ for every graph $G_{\ell,q,p}$ with $q \in \{1,4,7\}$. By considering the graphs in Figure 1 the dual optimum of 1 occurs when $y_1^\star = 1$. This raises the lower bound by 1.

Case 2: $n \equiv 0 \pmod 3$: Consider the inequality from (6), and let y_2 be its dual variable. Each graph $G_{\ell,q,p}$ leads to the dual inequality $\phi_{\ell,q,p} y_2 \le p - \frac{2}{3}q$. The dual optimum of $\frac{n}{12}$ arises when $y_2^\star = \frac{1}{12}$; the only graph whose dual inequality is binding is $G_{1,7,5}$ with $\phi_{1,7,5} = 4$ and $5 - \frac{2}{3}7 = \frac{1}{3}$. We can compute the slackness of each variable; for $\alpha_{\ell,q,p}$, the slackness is $p - \frac{2}{3}q - \frac{1}{12}\phi_{\ell,q,p}$. A unit increase in the variable $\alpha_{\ell,q,p}$ increases the dual objective function value by the slackness. The only variables with slackness at most $\frac{1}{2}$ are $\alpha_{2,7,5}$ with slackness $\frac{1}{8}$, $\alpha_{3,7,5}$ and $\alpha_{4,7,5}$ with slackness $\frac{1}{4}$, and $\alpha_{1,5,4}$ with slackness $\frac{1}{2}$. Hence any decomposition of cost less than $\frac{n}{12} + \frac{1}{2}$ consists solely of graphs in $\{G_{\ell,7,5}\}$. To satisfy (6), $\alpha_{7,5} \ge \lceil \frac{n}{4} \rceil$. If $\alpha_{7,5} \ge \frac{n}{4} + \delta$, adjoining this inequality with dual variable y_3 yields a dual solution $\{y_2 = 0, y_3 = \frac{1}{3}\}$ of cost $\frac{n}{12} + \frac{\delta}{3}$, increasing the bound when $\delta \ge 3$. So $\lceil \frac{n}{4} \rceil \le \alpha_{7,5} < \frac{n}{4} + 3$. Because all graphs in the decomposition have six or seven edges, $\alpha_{7,5} \equiv 0 \pmod 3$. Thus when $n \equiv 9, 12 \pmod{24}$, $\alpha_{7,5} \equiv 3 \pmod 6$, violating (4). This increases the bound by 1 when $n \equiv 9, 12 \pmod{24}$.

Case 3: $n \equiv 2 \pmod 3$: Again consider the inequality from (6), and let y_2 be its dual variable. Each graph $G_{\ell,q,p}$ leads to the dual inequality $\psi_{\ell,q,p} y_2 \le p - \frac{2}{3}q$. The dual optimum of $\frac{2n}{21}$ arises when $y_2^\star = \frac{2}{21}$; the only graph whose dual inequality is binding is $G_{1,7,5}$ with $\psi_{1,7,5} = \frac{7}{2}$ and $5 - \frac{2}{3}7 = \frac{1}{3}$. We can compute the slackness of each variable; for $\alpha_{\ell,q,p}$, the slackness is $p - \frac{2}{3}q - \frac{2}{21}\psi_{\ell,q,p}$. The only variables with slackness at most $\frac{4}{7}$ are $\alpha_{2,7,5}$ and $\alpha_{3,7,5}$ with slackness $\frac{1}{7}$, $\alpha_{4,7,5}$ with slackness $\frac{2}{7}$, and $\alpha_{1,5,4}$ with slackness $\frac{4}{7}$. An increase of $\frac{4}{7}$ would result in an increase in the integer ceiling when $n \equiv 2, 11, 14, 20 \pmod{21}$, so in these cases we are restricted to K_4s and graphs in $\{G_{\ell,7,5}\}$ to meet the bound. To satisfy (6), $\alpha_{7,5} \ge \lceil \frac{2n}{7} \rceil$. If $\alpha_{7,5} \ge \frac{2n}{7} + \delta$, adjoining this inequality with dual variable y_3 yields a dual solution $\{y_2 = 0, y_3 = \frac{1}{3}\}$ of cost $\frac{2n}{21} + \frac{\delta}{3}$, increasing the bound when $\delta \ge 3$. So $\lceil \frac{2n}{7} \rceil \le \alpha_{7,5} < \frac{2n}{7} + 3$. Because all graphs in the decomposition have six or seven edges, $\alpha_{7,5} \equiv 1 \pmod 3$. Thus when $n = 21s + x$ for $x \in \{2, 11, 14, 20\}$, $\alpha_{7,5} = 6s+1, 6s+4, 6s+4, 6s+7$, respectively. This violates (4) precisely when $n \equiv 44, 65; 11, 74; 14, 35; 20, 41 \pmod{84}$, increasing the bound by 1 in these cases. □

We denote by $\mathcal{L}(7, n)$ the lower bound prescribed by Theorem 2.1.

3 Group Divisible Designs with Block Size Four

A *group divisible design* (GDD) is a triple $(X, \mathcal{G}, \mathcal{B})$ where X is a set of points, $\mathcal{G}$ is a partition of X into *groups*, and $\mathcal{B}$ is a collection of subsets of X called *blocks* such that any pair of distinct points from X occur together either in some group or in exactly one block, but not both. A K-GDD of type $g_1^{u_1} g_2^{u_2} \dots g_s^{u_s}$ is a GDD in which every block has size from the set K and in which there are u_i groups of size $g_i, i = 1, 2, \dots, s$.

A group divisible design $(X, \mathcal{G}, \mathcal{B})$ is *resolvable* if its block set $\mathcal{B}$ admits a partition into *parallel classes*, each parallel class being a partition of the point set X.

A *pairwise balanced design* (PBD) with parameters $(K; v)$ is a K-GDD of type 1^v.

The interested reader may refer to [6, 9] for the undefined terms as well as a general overview of design theory. The main recursive construction that we use is Wilson's Fundamental Construction (WFC) for GDDs (see, e.g. [9]).

Construction 3.1 *Let $(X, \mathcal{G}, \mathcal{B})$ be a GDD, and let $w : X \to Z^+ \cup \{0\}$ be a weight function on X. Suppose that for each block $B \in \mathcal{B}$, there exists a K-GDD of type $\{w(x) : x \in B\}$. Then there is a K-GDD of type $\{\sum_{x \in G} w(x) : G \in \mathcal{G}\}$.*

A *double group divisible design* (DGDD) is a quadruple $(X, \mathcal{H}, \mathcal{G}, \mathcal{B})$ where X is a set of points, $\mathcal{H}$ and $\mathcal{G}$ are partitions of X (into holes and groups, respectively) and $\mathcal{B}$ is a collection of subsets of X (blocks) such that

(i) for each block $B \in \mathcal{B}$ and each hole $H \in \mathcal{H}$, $|B \cap H| \leq 1$, and

(ii) any pair of distinct points from X which are not in the same hole occur either in some group or in exactly one block, but not both.

A K-DGDD of type $(g_1, h_1^v)^{u_1} (g_2, h_2^v)^{u_2} \dots (g_s, h_s^v)^{u_s}$ is a double group-divisible design in which every block has size from the set K and in which there are u_i groups of size g_i, each of which intersects each of the v holes in h_i points. (Thus, $g_i = h_i v$ for $i = 1, 2, \dots, s$. Not every DGDD can be expressed this way, of course, but this is the most general type that we require.) Thus, for example, a *modified group divisible design* K-MGDD of type g^u is a K-DGDD of type $(g, 1^g)^u$. A k-DGDD of type $(g, h^v)^k$ is an incomplete transversal design ITD $(k, g; h^v)$ and is equivalent to a set of $k-2$ holey MOLS of type h^v (see, e.g. [9]). A DGDD is *resolvable* if its block set admits a partition into parallel classes. We use the following existence result.

Theorem 3.2 [20] *There exists a 4-DGDD of type $(mt, m^t)^n$ if and only if $t, n \geq 4$ and $(t-1)(n-1)m \equiv 0 \pmod 3$ except for $(m, n, t) = (1, 4, 6)$ and except possibly for $m = 3$ and $(n, t) \in \{(6, 14), (6, 15), (6, 18), (6, 23)\}$.*

We also make use of the following simple construction for 4-GDDs:

Construction 3.3 [19] *Suppose that there is a 4-DGDD of type $(g_1, h_1^v)^{u_1}$ $(g_2, h_2^v)^{u_2} \dots (g_s, h_s^v)^{u_s}$ and that for each $i = 1, 2, \dots, s$ there is a 4-GDD of type $h_i^v a^1$ where a is a fixed non-negative integer. Then there is a 4-GDD of type $h^v a^1$ where $h = \sum_{i=1}^{s} u_i h_i$.*

The following results on TDs are known.

Theorem 3.4 *A TD(k, m) exists if:*

1. $k = 5$ *and* $m \geq 4$ *and* $m \notin \{6, 10\}$;
2. $k = 6$ *and* $m \geq 5$ *and* $m \notin \{6, 10, 14, 18, 22\}$;
3. $k = 7$ *and* $m \geq 7$ *and* $m \notin \{10, 14, 15, 18, 20, 22, 26, 30, 34, 38, 46, 60, 62\}$.

Finally, we employ the following results on 4-GDDs.

Theorem 3.5 ([9, III.1.3 Theorem 1.28]) *A* 4*-GDD of type* $3^u m^1$ *exists if and only if either* $u \equiv 0 \mod 4$ *and* $m \equiv 0 \mod 3, 0 \leq m \leq (3u-6)/2$*; or* $u \equiv 1 \mod 4$ *and* $m \equiv 0 \mod 6, 0 \leq m \leq (3u-3)/2$*; or* $u \equiv 3 \mod 4$ *and* $m \equiv 3 \mod 6$*,* $0 < m \leq (3u-3)/2$.

Theorem 3.6 ([17, Theorem 1.7]) *There exists a* 4*-GDD of type* $g^4 m^1$ *with* $m > 0$ *if and only if* $g \equiv m \equiv 0 \mod 3$ *and* $0 < m \leq \frac{3g}{2}$.

Theorem 3.7 ([18, Theorem 1.6]) *There exists a* 4*-GDD of type* $6^u m^1$ *for every* $u \geq 4$ *and* $m \equiv 0 \mod 3$ *with* $0 \leq m \leq 3u-3$ *except for* $(u, m) = (4, 0)$ *and except possibly for* $(u, m) \in \{(7, 15), (11, 21), (11, 24), (11, 27), (13, 27), (13, 33), (17, 39), (17, 42), (19, 45), (19, 48), (19, 51), (23, 60), (23, 63)\}$.

Theorem 3.8 ([16, Theorem 3.16]) *There exists a* 4*-GDD of type* $12^u m^1$ *for each* $u \geq 4$ *and* $m \equiv 0 \mod 3$ *with* $0 \leq m \leq 6(u-1)$.

Theorem 3.9 ([16, Theorem 5.21]) *There exists a* 4*-GDD of type* $2^u m^1$ *for each* $u \geq 6$*,* $u \equiv 0 \mod 3$ *and* $m \equiv 2 \mod 3$ *with* $2 \leq m \leq u-1$ *except for* $(u, m) = (6, 5)$ *and possibly excepting* $(u, m) \in \{(21, 17), (33, 23), (33, 29), (39, 35), (57, 44)\}$.

3.1 $g \in \{24, 84\}$

Lemma 3.10 *For each* $u \geq 4, u \notin \{7, 11, 13, 17, 19, 23\}$*, there exists a* 4*-GDD of type* $24^u m^1$ *with* $m \equiv 0 \mod 3$ *and* $0 \leq m \leq 12(u-1)$.

Proof: For $u = 4$, see Theorem 3.6. For each $u \geq 5, u \notin \{7, 11, 13, 17, 19, 23\}$, take a 4-GDD of type $6^u v^1$ with $v \equiv 0 \mod 3$ and $0 \leq v \leq 3(u-1)$, and remove the points on the last group of size v; apply weight 4, using 4-MGDDs of type 4^4 and resolvable $\{3\}$-MGDDs of type 4^3, to obtain a $\{3, 4\}$-DGDD of type $(24, 6^4)^u$ whose triples fall into $3v$ parallel classes. Adjoin $3v$ infinite points to complete the parallel classes and then fill in 4-GDDs of type $6^u t^1$ with $t \equiv 0 \mod 3$ and $0 \leq t \leq 3(u-1)$ to obtain a 4-GDD of type $24^u (3v+t)^1$, as desired. $\square$

Lemma 3.11 *For each* $u \in \{7, 11, 13, 17, 19, 23\}$*, there exists a* 4*-GDD of type* $24^u m^1$ *with* $m \equiv 0 \mod 3$ *and* $3(u-1) \leq m \leq 12(u-1)$.

Proof: For each u, start with a TD$(5, u)$ and adjoin an infinite point ∞ to the groups, then delete a finite point so as to form a $\{5, u+1\}$-GDD of type $4^u u^1$. Note that each block of size $u+1$ intersects the group of size u in the infinite point ∞ and each block of size 5 intersects the group of size u, but certainly not in ∞. Now, in the group of size u, we give ∞ weight 0 or $3(u-1)$ and give the remaining points weight 3, 6 or 9. Give all other points in the $\{5, u+1\}$-GDD weight 6. Replace the blocks in the $\{5, u+1\}$-GDD by 4-GDDs of types 6^u, $6^u(3(u-1))^1$, $6^4 3^1$, $6^4 6^1$, or $6^4 9^1$ to obtain the 4-GDDs as desired. □

Lemma 3.12 *For each* $u \in \{7, 11, 13, 17, 19, 23\}$*, there exists a* 4*-GDD of type* $24^u m^1$ *with* $m \equiv 0 \mod 3$ *and* $0 \leq m \leq 3(u-2)$.

Proof: Starting from a 4-DGDD of type $(24, 6^4)^u$ coming from Theorem 3.2 and applying Construction 3.3 with 4-GDDs of type $6^u m^1$ to fill in holes, we obtain most of the designs except for $(u, m) \in \{(7, 15), (11, 21), (11, 24), (11, 27), (13, 27), (13, 33), (17, 39), (17, 42), (19, 45), (19, 48), (19, 51), (23, 60), (23, 63)\}$.

For the remaining choices for (u, m), take a 4-GDD of type $6^u 3^1$ and remove the points of the last group of size 3; apply weight 4, using 4-MGDDs of type 4^4 and resolvable $\{3\}$-MGDDs of type 4^3, to obtain a $\{3, 4\}$-DGDD of type $(24, 6^4)^u$ whose triples fall into 9 parallel classes. Adjoin $m-9$ infinite points to complete the parallel classes and then fill in 4-GDDs of type $6^u(m-9)^1$. □

Combining Lemmas 3.10–3.12, we have the following theorem.

Theorem 3.13 *There exists a* 4*-GDD of type* $24^u m^1$ *for each* $u \geq 4$ *and* $m \equiv 0 \mod 3$ *with* $0 \leq m \leq 12(u-1)$.

Theorem 3.14 *There exists a* 4*-GDD of type* $84^u m^1$ *for each* $u \geq 4$ *and* $m \equiv 0 \mod 3$ *with* $0 \leq m \leq 42(u-1)$.

Proof: The proof is similar to that of Lemma 3.10. For each u, take a 4-GDD of type $12^u v^1$ with $v \equiv 0 \mod 3$ and $0 \leq v \leq 6(u-1)$, and remove the points on the last group of size v; apply weight 7, using 4-MGDDs of type 7^4 and resolvable $\{3\}$-MGDDs of type 7^3, to obtain a $\{3, 4\}$-DGDD of type $(84, 12^7)^u$ whose triples fall into $6v$ parallel classes. Adjoin $6v$ infinite points to complete the parallel classes and then fill in 4-GDDs of type $12^u t^1$ with $t \equiv 0 \mod 3$ and $0 \leq t \leq 6(u-1)$ to obtain a 4-GDD of type $84^u(6v+t)^1$, as desired. □

4 Constructions: $n \equiv 1 \pmod 3$

We settle small cases first.

Lemma 4.1 $A(7, n) = \mathcal{L}(7, n)$ *for* $n \in \{4, 7\}$.

Proof: The lower bound is met for $n = 4$ by a single K_4. The lower bound is realized when $n = 7$: Let $V = \{\infty\} \cup \{0, \ldots, 5\}$, and form the three $G_{1,7,5}$s $\{\{i, i+3\}, \{i, i+1\}, \{i, i+4\}, \{i+1, i+3\}, \{i+3, i+4\}, \{\infty, i\}, \{\infty, i+3\}\}$ for $i \in \{0, 1, 2\}$, arithmetic modulo 6. □

Lemma 4.2 $A(7,10) = \mathcal{L}(7,10) + 1 = 32.$

Proof: The lower bound of 31 is not met. To see this, the only primal variables with slackness at most $\frac{1}{3}$ are for $\{G_{\ell,7,5}\}$. But $6x + 7y = 45$ and $4x + 5y = 31$ admits only the solution $x = 4$ and $y = 3$, i.e. four K_4s and three graphs from $\{G_{\ell,7,5}\}$. There is a unique way to place four K_4s in a K_{10}, and its complement does not partition into three graphs from $\{G_{\ell,7,5}\}$. To produce a decomposition of cost 32, on the 10 points $\{0,\ldots,9\}$ form K_4s on $\{0,1,2,3\}$ and $\{0,4,5,6\}$, and the graphs

$$\begin{array}{ll}
G_{2,7,5} & \{\{2,4\},\{2,5\},\{2,7\},\{2,9\},\{4,7\},\{5,7\},\{4,9\}\} \\
G_{3,7,5} & \{\{3,9\},\{5,9\},\{6,9\},\{7,9\},\{3,6\},\{3,7\},\{6,7\}\} \\
G_{4,7,5} & \{\{3,4\},\{3,5\},\{3,8\},\{4,8\},\{5,8\},\{1,4\},\{1,5\}\} \\
G_{4,7,5} & \{\{0,7\},\{0,8\},\{0,9\},\{7,8\},\{8,9\},\{1,7\},\{1,9\}\} \\
G_{1,5,4} & \{\{1,8\},\{1,6\},\{2,8\},\{2,6\},\{6,8\}\}
\end{array}$$

□

Lemma 4.3 $\mathcal{L}(7,19) + 1 \leq A(7,19) \leq \mathcal{L}(7,19) + 2 = 117.$

Proof: The lower bound of 115 cannot be met. A maximum packing on 19 points has 25 K_4s [7]. Consider the linear program using (5). Using slackness, the only way to achieve a dual objective value of 1 in such a way that at least $21 = \binom{19}{2} - 25 \cdot 6$ edges do not appear in K_4s is to use three graphs in $\{G_{\ell,7,5}\}$. There are 249 nonisomorphic graphs that can be left by a maximum packing of 25 K_4s in K_{19} [2]. $G_{3,7,5}$ cannot be used because it contains a K_4, and the 25 K_4s form a maximum packing. Of the 249 graphs, 79 have degree sequence 3^{14}; 122 have degree sequence $6^1 3^{12}$ and 48 have degree sequence $6^2 3^{10}$. In order to use a $G_{1,7,5}$ there must be at least five vertices of degree 6 or larger; and for $G_{2,7,5}$ there must be at least three. Hence both are ruled out and the only possibiiity is three $G_{4,7,5}$s. This case can be eliminated by a simple computer search. Thus the drop cost cannot be 115. A solution with drop cost 117 follows:

24 K_4's: {0,1,2,4},{0,3,5,6},{0,7,8,9},{0,10,11,12},{0,13,14,15}
{0,16,17,18},{1,3,7,10},{1,5,8,11},{1,6,13,16},{1,9,14,17}
{1,12,15,18},{2,3,8,15},{2,5,9,18},{2,6,10,17},{2,7,12,13}
{2,11,14,16},{3,4,14,18},{3,9,12,16},{4,5,12,17},{4,6,9,15}
{5,10,15,16},{6,7,11,18},{6,8,12,14},{8,10,13,18}

one $G_{2,7,5}$:{{3,11},{3,13},{3,17},{11,15},{11,17},{13,17},{15,17}}

two $G_{4,7,5}$:{{4,7},{4,8},{4,16},{7,16},{7,17},{8,16},{8,17}} and
{{4,10},{4,11},{4,13},{9,10},{9,11},{9,13},{11,13}}

one $G_{7,6,6}$:{{5,7},{5,13},{5,14},{7,14},{7,15},{10,14}}

□

Theorem 4.4 *When* $n \equiv 1 \pmod 3$ *and* $n \notin \{10,19\}$, $A(7,n) = \mathcal{L}(7,n)$. *Moreover,* $A(7,10) = \mathcal{L}(7,10) + 1$ *and* $\mathcal{L}(7,19) + 1 \leq A(7,19) \leq \mathcal{L}(7,19) + 2$.

Proof: When $n \equiv 1, 4 \pmod{12}$, there is a 4-GDD of type 1^n with drop cost $\mathcal{L}(7, n)$. When $n \equiv 7, 10 \pmod{12}$ and $n \notin \{10, 19\}$, there is a 4-GDD of type $1^{n-7}7^1$ [7]; fill the hole with a solution from Lemma 4.1. □

5 Constructions: $n \equiv 0 \pmod 3$

The lower bound is met for $n = 3$ by a single K_3.

Lemma 5.1 $A(7, 6) = \mathcal{L}(7, 6) + 1 = 12$.

Proof: The lower bound of 11 is not met. A decomposition of cost 12 can be produced as follows:

$$\begin{array}{ll} G_{2,7,5} & \{\{0,1\},\{0,2\},\{0,4\},\{0,5\},\{1,4\},\{1,5\},\{2,4\}\} \\ G_{2,7,5} & \{\{1,2\},\{1,3\},\{2,3\},\{2,5\},\{3,4\},\{3,5\},\{4,5\}\} \\ G_{1,1,2} & \{\{0,3\}\} \end{array}$$

□

Lemma 5.2 $A(7, 9) = \mathcal{L}(7, 9) + 1 = 27$.

Proof: The lower bound of 26 is not met for $n = 9$ as follows. There can be at most three K_4s on nine points. If there are zero, at least six graphs are needed each having slackness at least $\frac{1}{3}$; because the total increase in the dual objective function is 2, all graphs must be from $\{G_{\ell,7,5}\}$ and cannot account for 36 edges. In the same manner, with one K_4, 30 edges must be accounted for by graphs in $\{G_{\ell,7,5}\}$, each with slackness $\frac{1}{3}$ and $G_{1,5,4}$ with slackness $\frac{2}{3}$; again this is not possible as 25 is not a multiple of 7. There remain cases with two or three K_4s; each can be eliminated by an exhaustive search.

A decomposition of cost 27 using graphs on at most six edges is given in [2]. We give a different solution here:

$$\begin{array}{ll} G_{1,7,5} & \{\{0,7\},\{0,8\},\{1,7\},\{1,8\},\{2,7\},\{2,8\},\{7,8\}\} \\ G_{4,7,5} & \{\{0,4\},\{0,5\},\{0,6\},\{1,4\},\{1,5\},\{1,6\},\{4,5\}\} \\ G_{4,7,5} & \{\{2,4\},\{2,5\},\{2,6\},\{3,4\},\{3,5\},\{3,6\},\{4,6\}\} \\ G_{4,7,5} & \{\{4,7\},\{4,8\},\{5,6\},\{5,7\},\{5,8\},\{6,7\},\{6,8\}\} \\ G_{1,6,4} & \{\{0,1\},\{0,2\},\{0,3\},\{1,2\},\{1,3\},\{2,3\}\} \\ G_{1,2,3} & \{\{3,7\},\{3,8\}\} \end{array}$$

□

Lemma 5.3 $A(7, 15) = \mathcal{L}(7, 15) = 72$.

Proof: Start with a Kirkman triple system of order 9 on $\{0, \ldots, 8\}$, in which the first parallel class is $\{B_0, B_1, B_2\}$. Then adjoin points $\{x_0, x_1, x_2, y_0, y_1, y_2\}$. Form nine K_4s by adding y_i to each block of the $(i+2)$nd parallel class. For $i \in \{0, 1, 2\}$ form a K_4 on $\{x_{i+2}\} \cup B_i$ and a $G_{1,7,5}$ in which the degree 4 vertices are x_i and x_{i+1} and the degree 2 vertices are the elements of B_i. Form a K_4 on $\{x_2, y_0, y_1, y_2\}$. What remains is a $G_{3,6,5}$. □

Lemma 5.4 $A(7,18) \leq \mathcal{L}(7,18) + 1 = 105$.

Proof: Form a 4-GDD of type 3^5 with groups $\{B_j : j = 0,1,2,3,4\}$. Then adjoin points $\{x_0, x_1, x_2\}$. For $i \in \{0,1,2\}$, form a $G_{1,7,5}$ by using the edge $\{x_i, x_{i+1 \bmod 3}\}$ and joining these vertices to each vertex in B_i and form a K_4 by adding $x_{i+2 \bmod 3}$ to B_i. For $i \in \{3,4\}$, form a $G_{3,6,5}$ by joining the vertices x_0 and x_1 to vertices in B_i and form a K_4 by adding x_2 to B_i. This decomposition is of cost 105. □

Lemma 5.5 $A(7,24) = \mathcal{L}(7,24) = 186$.

Proof: We give the solution on $\{0,1,2,3,4,5,6,7\} \times \mathbb{Z}_3$, writing element (i,j) as i_j.

$$\begin{array}{llll}
(0_0,0_1,1_0,4_2) & (0_0,1_1,5_0,6_1) & (0_0,2_0,3_1,3_2) & (0_0,2_1,5_1,5_2) \\
(0_0,2_2,7_0,7_2) & (0_0,6_0,6_2,7_1) & (1_0,1_1,2_1,7_0) & (1_0,2_2,5_1,6_1) \\
(1_0,3_1,5_0,7_1) & (1_0,3_2,4_1,6_2) & (2_0,2_1,4_2,6_1) & (3_0,5_0,6_2,7_2) \\
(4_0,4_1,5_2,7_2) & (3_0,4_0:0_0,1_0,2_0) & (3_0,4_1:5_1,6_1,7_1) &
\end{array}$$

The latter two orbits are graphs isomorphic to $G_{1,7,5}$. □

Theorem 5.6 $A(7,n) = \mathcal{L}(7,n)$ *when* $n \equiv 0 \pmod 3$, $n \not\equiv 18 \pmod{24}$ *and*

1. $n \geq 96$ *when* $n \equiv 0,3,6,9,15 \pmod{24}$;
2. $n \geq 276$ *when* $n \equiv 12 \pmod{24}$;
3. $n \geq 309$ *when* $n \equiv 21 \pmod{24}$.

$\mathcal{L}(7,n) \leq A(7,n) \leq \mathcal{L}(7,n) + 1$ *when* $n \equiv 18 \pmod{24}$ *and* $n \geq 114$.

Proof: If $m = n \bmod 24 \in \{0,3,6,9,15,18\}$ and $n \geq 96$, form a 4-GDD of type $24^{(n-m)/24}m^1$ from Theorem 3.13; place optimal groomings from Lemma 5.5 on each group of size 24, and an optimal grooming of size m on the exceptional group (from Lemmas 5.1, 5.2, or 5.3 when $m = 6, 9, 15$, respectively). When $m = 18$, use the grooming from Lemma 5.4, missing the lower bound by 1. When $m = 6$, reduce the drop cost by 1 by amalgamating the single edge from this grooming with a K_4 of the 4-GDD to form a $G_{3,7,5}$. When $m = 9$, reduce the drop cost by 1 by amalgamating both edges of the $G_{1,3,2}$ of this grooming with K_4s of the 4-GDD to form $G_{3,7,5}$s.

When $m = n \bmod 24 = 12$, form a 4-GDD of type 20^4, and add four infinite points. On each group, together with the four infinite points, place an optimal grooming from Lemma 5.5 aligning a K_4 on the four infinite points. Suppress the duplicate K_4s so produced. This establishes that $\mathcal{L}(7,84) = A(7,84)$. Then filling groups in a 4-GDD of type 24^t84^1 establishes that $A(7,24t+84) = \mathcal{L}(7,24t+84)$ when $t \geq 8$, i.e. for all $n \geq 276$.

When $m = n \bmod 24 = 21$, form a 4-GDD of type 23^4, and add one infinite point. On each group, together with the infinite point, place an optimal grooming from Lemma 5.5. This establishes that $\mathcal{L}(7,93) = A(7,93)$. Then filling groups in a 4-GDD of type 24^t93^1 establishes that $A(7,24t+93) = \mathcal{L}(7,24t+93)$ when $t \geq 9$, i.e. for all $n \geq 309$. □

6 Constructions: $n \equiv 2 \pmod 3$

Lemma 6.1 $A(7,n) = \mathcal{L}(7,n)$ *for* $n \in \{5,8\}$.

Proof: For K_5, note that $G_{1,7,5} \equiv K_5 \setminus K_3$. Partition K_8 as follows:

$G_{1,7,5}$	$\{\{0,1\},\{0,2\},\{0,3\},\{0,4\},\{1,2\},\{1,3\},\{1,4\}\}$
$G_{1,7,5}$	$\{\{6,7\},\{6,2\},\{6,3\},\{6,4\},\{7,2\},\{7,3\},\{7,4\}\}$
$G_{3,7,5}$	$\{\{1,5\},\{2,3\},\{2,4\},\{2,5\},\{3,4\},\{3,5\},\{4,5\}\}$
$G_{4,7,5}$	$\{\{1,6\},\{1,7\},\{0,5\},\{0,6\},\{0,7\},\{5,6\},\{5,7\}\}$

$\square$

Lemma 6.2 $A(7,11) = \mathcal{L}(7,11) = 39$.

Proof: Partition K_{11} on $\{\infty_1,\infty_2\} \cup (\mathbb{Z}_3 \times \mathbb{Z}_3)$ as follows. Include the K_4 $\{\infty_2, 0_2, 1_2, 2_2\}$. Form three $G_{2,7,5}$s as $\{\{i_0,(i+1)_1\},\{i_0,(i+2)_1\},\{i_0,(i+1)_2\},\{i_0,(i+2)_2\},\{(i+1)_1,(i+2)_1\},\{(i+1)_1,(i+2)_2\},\{(i+2)_1,(i+1)_2\}\}$ for $i \in \{0,1,2\}$. Then include three $G_{3,7,5}$s as $\{\{\infty_1,i_0\},\{\infty_1,i_1\},\{\infty_1,i_2\},\{i_0,i_1\},\{i_0,i_2\},\{i_1,i_2\},\{\infty_2,i_1\}\}$ for $i \in \{0,1,2\}$. Include one last $G_{3,7,5}$: $\{\{\infty_1,\infty_2\},\{\infty_2,0_0\},\{\infty_2,1_0\},\{\infty_2,2_0\},\{0_0,1_0\},\{0_0,2_0\},\{1_0,2_0\}\}$. $\square$

Lemma 6.3 $A(7,17) \le \mathcal{L}(7,17) + 1 = 94$.

Proof: Start with an $S(2,4,16)$ on $\mathbb{Z}_{15} \cup \{\infty\}$ with blocks $\{i,i+1,i+3,i+7\}$ for $i \in \mathbb{Z}_{15}$ and $\{\infty,i,i+5,i+10\}$ for $i \in \{0,1,2,3,4\}$. We adjoin a new point α and modify six of the blocks in the first orbit as follows:

Block	Remove	Add
$\{5,6,8,12\}$	$\{8,12\}$	$\{\alpha,5\},\{\alpha,8\}$
$\{7,8,10,14\}$	$\{8,14\}$	$\{\alpha,7\},\{\alpha,10\}$
$\{0,8,9,11\}$	$\{0,8\}$	$\{\alpha,0\},\{\alpha,9\}$
$\{3,11,12,14\}$	$\{12,14\}$	$\{\alpha,3\},\{\alpha,12\}$
$\{0,4,12,13\}$	$\{0,12\}$	$\{\alpha,4\},\{\alpha,13\}$
$\{0,2,6,14\}$	$\{0,14\}$	$\{\alpha,2\},\{\alpha,14\}$

Now add the K_4 on $\{0,8,12,14\}$. Then delete the K_4 on $\{\infty,1,6,11\}$; on $\{\alpha,\infty,1,6,11\}$, place a K_3 and a $G_{1,7,5}$. The result has 14 K_4s, one K_3, and seven graphs in $\{G_{\ell,7,5}\}$. $\square$

Lemma 6.4 *When* $n \equiv 2 \pmod 6$ *and* $n \ge 14$, $A(7,n) \le \frac{2}{3}\binom{n}{2} + \frac{n}{6} = \frac{2}{3}\binom{n}{2} + \frac{2n}{21} + \frac{n}{14}$.

Proof: Write $h = \frac{n}{2}$. When $h \equiv 1 \pmod 3$ and $h \ge 7$, a 4-GDD of type 2^h exists by Theorem 3.9. It has h groups and $\frac{h(h-1)}{3}$ blocks. For each group, choose a distinct block containing one point of the group (this is an easy exercise using systems of distinct representatives). Then adjoin the pair of each group to its corresponding block to obtain a $G_{3,7,5}$. $\square$

Lemma 6.5 *When* $n \equiv 5 \pmod 6$ *and* $n \ge 23$, $A(7,n) \le \frac{2}{3}\binom{n}{2} + \frac{2n}{21} + \frac{n+7}{14}$.

Proof: Write $h = \frac{n-5}{2}$. When $h \equiv 0 \pmod 3$ and $h \geq 9$, a 4-GDD of type $2^h 5^1$ exists by Theorem 3.9. For each group of size 2, choose a distinct block containing one point of the group and adjoin the pair of each group to its corresponding block to obtain a $G_{3,7,5}$. Then fill the group of size 5 using a solution from Lemma 6.1. □

In order to treat larger cases, we now develop a recursion.

Lemma 6.6 *There exists a decomposition of K_{21} into nine partial parallel classes of K_3s, and six $G_{1,7,5}$s.*

Proof: We present a solution on $\{0, 1, \ldots, 20\}$ with rows as partial parallel classes:

0 2 13	1 12 15	9 14 17	3 10 20	4 5 19	7 11 16	6 8 18
0 18 20	1 2 16	11 17 19	3 12 13	4 7 8	6 9 10	5 14 15
0 1 11	13 17 18	3 9 16	4 12 14	7 10 19	2 5 6	
0 3 5	1 8 17	4 13 16	7 9 20	6 11 15	2 10 14	
0 8 14	1 5 20	2 3 17	4 10 15	6 13 19	11 12 18	
0 9 15	1 13 14	3 18 19	4 6 20	2 7 12	5 8 16	
0 10 16	1 9 19	12 17 20	3 8 15	2 4 11	5 7 18	
0 12 19	1 10 18	15 16 17	6 7 14	2 8 9	11 13 20	
5 10 17	3 11 14	4 9 18	7 13 15	6 12 16	8 19 20	

The remaining edges partition into six $G_{1,7,5}$s: $\{\{7i+j, 7i+j+2\}, \{7i+j, 7i+4\}, \{7i+j, 7i+5\}, \{7i+j, 7i+6\}, \{7i+j+2, 7i+4\}, \{7i+j+2, 7i+5\}, \{7i+j+2, 7i+6\}\}$ for $j \in \{0,1\}$ and $i \in \{0,1,2\}$. □

We denote by $X(n)$ the excess over the lower bound, i.e. $X(n) = A(7,n) - \mathcal{L}(7,n)$.

Theorem 6.7 *Let $(V, \mathcal{G}, \mathcal{B})$ be a resolvable group-divisible design of type 7^n, in which the blocks of $\mathcal{B}$ are partitioned into parallel classes $\mathcal{P}_1, \ldots, \mathcal{P}_s$, and for $1 \leq i \leq s$ every block of $\mathcal{P}_i$ has size k_i. Suppose that, for $1 \leq i \leq s$, a* 4*-GDD of type $3^{k_i}\sigma_i^1$ exists, and that $\sum_{i=1}^s \sigma_i > 0$. Then*

$$A(7, 21n + 8 + \sum_{i=1}^{s} \sigma_i) \leq \mathcal{L}(7, 21n + 8 + \sum_{i=1}^{s} \sigma_i) + X(8 + \sum_{i=1}^{s} \sigma_i).$$

Proof: Suppose without loss of generality that $\sigma_1 > 0$. Give weight three to each point of the GDD $(V, \mathcal{G}, \mathcal{B})$. For $2 \leq i \leq s$, adjoin σ_i new infinite points, and place a 4-GDD of type $3^{k_i}\sigma_i^1$ on the inflation of each block of $\mathcal{P}_i$ together with these infinite points. Then proceed similarly for $\mathcal{P}_1$, but adding only $\sigma_1 - 1$ infinite points; in the 4-GDD, delete one point in the group of size σ_1 to form a $\{3,4\}$-GDD of type $3^{k_1}(\sigma_1 - 1)^1$ in which the blocks of size three form a (frame) parallel class on the $3k_i$ points. On each inflation of a group form a copy of the 21-point design from Lemma 6.6. The nine partial parallel classes of blocks of size 3 formed can be completed to nine parallel classes on the $21n$ points using the triples from the $\{3,4\}$-GDDs. Finally add nine further infinite points and extend each of the nine parallel classes to K_4s using these infinite points. The resulting design has a hole on the $8 + \sum_{i=1}^s \sigma_i$ infinite points added in total, which can be filled with a solution of cost $A(7, 8 + \sum_{i=1}^s \sigma_i)$. □

Corollary 6.8 *1.* $X(92) \leq X(29)$.

2. For $n \in \{11, 14, 17, 20, 23, 26, 29\}$*,* $X(84+n) \leq X(n)$.

3. For $n \in \{14, 20, 26, 32, 38, 44, 50\}$*,* $X(105+n) \leq X(n)$.

4. For $29 \leq n \leq 71$ *and* $n \equiv 2 \pmod 3$*,* $X(147+n) \leq X(n)$.

Proof: Apply Theorem 6.7 using an RTD$(k, 7)$ with $k = 3, 4, 5, 7$ as a resolvable GDD of type 7^k with $s = 7$ and $k_1 = \cdots = k_7 = k$. □

Corollary 6.9 *1. For* $29 \leq n \leq 80$ *and* $n \equiv 2 \pmod 3$*,* $X(168+n) \leq X(n)$.

2. For $32 \leq n \leq 92$ *and* $n \equiv 2 \pmod 6$*,* $X(189+n) \leq X(n)$.

3. For $41 \leq n \leq 107$ *and* $n \equiv 5 \pmod 6$*,* $X(231+n) \leq X(n)$.

4. For $44 \leq n \leq 134$ *and* $n \equiv 2 \pmod 6$*,* $X(273+n) \leq X(n)$.

5. For $53 \leq n \leq 164$ *and* $n \equiv 2 \pmod 3$*,* $X(336+n) \leq X(n)$.

Proof: Apply Theorem 6.7 using an RTD$(7, n)$ with $n = 8, 9, 11, 13, 16$ as a resolvable GDD of type 7^n with $s = n$ and $k_1 = \cdots = k_{n-1} = 7$ and $k_n = n$. □

Theorem 6.10 *For* $x \geq 4$*,* $0 \leq m \leq 42(x-1)$*,* $m \equiv 0 \pmod 3$*, and* $r \in \{11, 14, 17, 20, 23, 26, 29\}$*,*

$$A(7, 84x + m + r) \leq \mathcal{L}(7, 84x + m + r) + X(m + r).$$

Equivalently, $X(84x + m + r) \leq X(m + r)$.

Proof: Form a 4-GDD of type $84^x m^1$ from Theorem 3.14. Adjoin r infinite points, and place a solution on each group of size 84 together with the r points, leaving a hole on the r points (from Lemma 6.8(2)). On the $m + r$ points, place a solution with excess $X(m + r)$. □

Theorem 6.11 *For* $m \equiv 2 \pmod 3$ *and* $2 \leq m \leq 83$*,* $\mathcal{L}(7, 84x + m) \leq A(7, 84x + m) \leq \mathcal{L}(7, 84x + m) + X(84x + m)$*, where* $X(84x + m)$ *is given in Table 1 (using the final bold entry for* $X(84x + m)$ *in the row for* m *when the table does not provide a value). In particular,* $A(7, 84x + m) \leq \mathcal{L}(7, 84x + m) + 4$ *when* $84x + m > 1094$*.*

Proof: Apply Lemmas 6.1, 6.2, and 6.3 for $x = 0$ and $m \in \{5, 8, 11, 17\}$; then apply Lemmas 6.4 and 6.5 to provide an upper bound on $X(84x + m)$ in general. Now apply Corollary 6.8 and 6.9 to improve these upper bounds. Finally apply Theorem 6.10. □

	x													
m	0	1	2	3	4	5	6	7	8	9	10	11	12	13
2	1	6	12	18	2	6	6	6	6	6	6	6	6	**2**
5	0	6	12	4	4	6	6	6	6	6	**4**			
8	0	2	2	18	24	**2**								
11	0	0	12	4	**0**									
14	0	0	2	18	**0**									
17	1	1	13	5	**1**									
20	0	0	2	2	**0**									
23	2	2	14	6	**2**									
26	1	1	3	3	**1**									
29	**2**													
32	2	8	2	4	**2**									
35	2	0	2	20	**0**									
38	2	8	2	4	**2**									
41	2	0	2	20	**0**									
44	2	8	2	4	**2**									
47	3	1	3	21	**1**									
50	3	9	3	5	**3**									
53	4	2	2	22	4	**2**								
56	4	10	4	6	**4**									
59	4	2	2	22	4	**2**								
62	4	10	4	6	**4**									
65	4	2	2	2	4	**2**								
68	4	10	4	6	**4**									
71	5	3	3	3	5	**3**								
74	4	10	4	0	4	4	4	4	4	4	4	4	**0**	
77	6	12	4	4	6	6	6	6	6	**4**				
80	5	11	5	1	5	5	5	5	5	5	5	5	**1**	
83	6	12	4	4	6	6	6	6	6	**4**				

Table 1: Least Excesses for $84x + m$

7 Conclusions

Grooming with ratio 7 corresponds to the smallest ratio C for which optimal groomings do not consist primarily of C-edge graphs. Consequently, optimal grooming focusses on packings with K_4s in this case. Despite this, the structures of the edges not appearing in K_4s appear to exhibit patterns that repeat modulo 12, 24, and 84 when $n \equiv 1, 0, 2 \pmod 3$, respectively. In the latter case techniques for constructing optimal groomings in all cases would necessitate the direct construction of many 'small' groomings. Therefore in this paper, we have instead found near-optimal groomings in which the construction deviates from the lower bound by a fixed constant independent of n. When $n \equiv 0, 1 \pmod 3$, much more complete characterizations are given. Our conjecture is that, with few small exceptions, the lower bound proved here provides the correct cost of an optimal grooming.

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