Entanglement detection via the condition of quantum correlation

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(Received 18 May 2007; published 12 September 2007)

We develop a necessary condition of quantum correlation. It is utilized to construct a d-level bipartite Bell-type inequality which is strongly resistant to noise and requires only analyses of $O(d)$ measurement outcomes compared to the previous result $O(d^2)$. Remarkably, a connection between the arbitrary high-dimensional bipartite Bell-type inequality and entanglement witnesses is found. Through the necessary condition of quantum correlation, we propose that the witness operators to detect truly multipartite entanglement for a generalized Greenberger-Horne-Zeilinger (GHZ) state with two local measurement settings and a four-qubit singlet state with three settings. Moreover, we also propose a robust entanglement witness to detect a four-level tripartite GHZ state with only two local measurement settings.

DOI: 10.1103/PhysRevA.76.032313 PACS number(s): 03.67.Mn, 03.65.Ud

Entanglement is at the heart of quantum physics and a resource for quantum information processing [1]. Multipartite entanglement for two-level quantum systems (qubits) has attracted attention for its unusual features [2] and necessity in a large-scale realization of quantum computation and communication [3]. In particular, with the rapid development of technology for manipulating quantum states, multipartite entanglement has been created experimentally and then utilized for quantum information processing [4]. In addition, entangled qubits, entanglement for multilevel quantum systems (qudits) has been realized in a few physical systems [5]. Moreover, it has been proven that qudits have an advantage over qubits [6]. Thus, identifying whether an experiment's output is an entangled state for multipartite or multilevel systems is very important for further studies on quantum correlation and to perform reliable quantum protocols.

Bell-type inequalities (BIs) [7–9] and entanglement witnesses (EWs) [10–13] are widely used to verify quantum correlation. BIs are based on the local hidden variable theories whereas EWs rely on an utilization of the whole or partial knowledge of the entangled state to be created. However, a single systematic approach to construct EWs for entangled qudits and to connect BIs for arbitrary high-dimensional systems with EWs is still lacking. Investigations on how entangled qudits can be shown efficiently and what the fundamental feature is in entanglement verifications are both significant for a deeper understanding of quantum correlation of qudits [14] and for efficient manipulations to achieve quantum information processing [15].

In this work, we develop a necessary condition of quantum correlation. This enables d-level bipartite BIs to be tested with only analyses of $O(d)$ measurement outcomes for detection events which is much smaller than the previous result $O(d^2)$ [9,16]. In particular, a connection between arbitrary high-dimensional bipartite BIs and EWs is found. We then use the correlator operators involved in the necessary condition of quantum correlation to construct EWs for detecting genuine multipartite entanglement, which can only be generated with participation of all parties of a system, in the generalized Greenberger-Horne-Zeilinger (GHZ) state with two local measurement settings (LMSs) (which will be described in detail) and four-qubit singlet states [17] with only three LMSs. More recently, it has been shown that the four qubit singlet state is very useful for quantum secret sharing [18]. Through our method, the 15 LMSs required for the EW by Ref. [12] can be reduced greatly. In order to show the high generality of the condition of quantum correlation, we also describe an EW that can detect a four-level tripartite GHZ state [14] with only two LMSs. Moreover, the proposed EWs are resistant to noise. In what follows, an introduction to the necessary condition of quantum correlation will be given as a preliminary to further applications.

I. CORRELATION CONDITIONS FOR QUANTUM CORRELATION

In an experiment whose aim is to generate a multipartite entangled state $|\xi\rangle$, if the experimental conditions are imperfect, it is important to know whether an experimental output state still possesses multipartite quantum correlation which is close to the state $|\xi\rangle$. One EW for detecting genuine multipartite entanglement is given by Ref. [11] and formulated as

$$\mathcal{W}_{\xi}^{\mathrm{p}} = \alpha_{\xi}^{\mathrm{p}} \mathbb{1} - |\xi\rangle\langle\xi|, \tag{1}$$

where $\alpha_{\xi}^{p} = \max_{|\chi\rangle \in B} |\langle\chi|\xi\rangle|^2$ and B denotes the set of biseparable states. Although it is difficult to determine the overlap $\alpha_{\xi}^{\mathrm{p}}$, through the general method proposed by Bourennane *et al.* [12], one can perform this task. Thus, for some experimental output state, say ρ, if measured outcomes show that $\mathrm{Tr}(\mathcal{W}_{\xi}^{p}\rho) < 0$, the state ρ is identified as a genuine multipartite entanglement which is close to the state $|\xi\rangle$.

It is worth noting that complete knowledge of the state $|\xi\rangle$, i.e., all information about correlation characters, is uti-

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lized for the witness operator and in order to measure the operator $\mathcal{W}_{\xi}^{p}$ experimentally, the number of LMSs appears to increase with the number of qubits of the state $|\xi\rangle$ [12]. A LMS, denoted by $M:(\hat{V}_1,\ldots,\hat{V}_n)$ in this paper, means that single-qubit measurements of operator $\hat{V}_i$ for $i=1,\ldots,n$ are taken on the n remote parties in parallel. In addition, EWs with forms such as $\mathcal{W}_{\xi}^{p}$, the number of LMSs utilized to realize BIs typically increases exponentially with the number of parties of the state. Moreover, the analyses of measured outcomes for detection events also depend on the structures of BIs. A detection event means a set of measurement outcomes, denoted by $(v_1,\ldots,v_n)$, under some LMS. For example, the LMS $M_{2z}=(\sigma_z,\sigma_z)$ corresponds to four possible detection events: (0,0), (0,1), (1,0), and (1,1), where $v_i=0$ or 1 stands for the eigenvalue $(-1)^{v_i}$ of Pauli operator σ_z. The meaning of LMS and that of the detection event are strictly different.

The witness operators proposed in this paper to detect genuine multipartite entanglement have the following form:

$$\mathcal{W}_{\xi}=\alpha_{\xi}\mathbb{1}-\hat{C}_{\xi}, \qquad (2)$$

where α_{ξ} is some constant and $\hat{C}_{\xi}$ is the operator which is composed of several different kinds of correlator operators with necessary conditions of quantum correlations imbedded in the state $|\xi\rangle$. If outcomes of measurements show that $\mathrm{Tr}(\mathcal{W}_{\xi}\rho)<0$, the state ρ is identified as a truly multipartite entanglement. In what follows we will show that the operator $\hat{C}_{\xi}$ can be constructed systematically and measured with fewer LMSs for different kinds of pure multipartite entangled qubits or qudits.

Furthermore, through the same idea behind the method to construct correlator operators, a d-level bipartite BI is constructed and able to be tested experimentally with fewer analyses of detection events. We then consider the correlation conditions for quantum correlation involved in the approach to construct correlator operators utilized in EWs and BIs as a connection between them. We will see that the building blocks of the proposed EWs and BIs are all derived from the correlation conditions for quantum correlation.

In order to present the idea behind the correlation condition for quantum correlation clearly, let us first illustrate a derivation of correlation condition for the generalized four-qubit GHZ state

$$|\Phi(\theta,\phi)\rangle=\cos(\theta)|0000\rangle_z+e^{i\phi}\sin(\theta)|1111\rangle_z \qquad (3)$$

for $0<\theta<\pi/4$ and $0\leq\phi<\pi/2$, where $|v_1v_2v_3v_4\rangle_z=\otimes_{k=1}^{4}|v\rangle_{kz}$ for $v\in\{0,1\}$ and $|v\rangle_{kz}$ corresponds to an eigenstate of σ_z with eigenvalue $(-1)^v$ for the party k. For the four-qubit system, the kernel of our strategy for identifying correlation between a specific subsystem, say A, and another one, say B, under some LMS, M_l, relies on the sets of correlators with the following forms:

$$C_0^{(l)}=P(v_{A0},v_{B0})-P(v_{A1},v_{B0}), \qquad (4)$$

$$C_1^{(l)}=P(v_{A1},v_{B1})-P(v_{A0},v_{B1}), \qquad (5)$$

where $P(v_{Ai},v_{Bj})$ is the joint probability for obtaining the measured outcomes v_{Ai} for the A subsystem and v_{Bj} for the B one. By the values of the correlators for an experimental output state, we could identify correlations between outcomes of measurements for the subsystems.

Proposition 1. If the results of measurements reveal that $C_0^{(l)}$ and $C_1^{(l)}$ are all positive or all negative, i.e., $C_0^{(l)}C_1^{(l)}>0$, we are convinced that the outcomes of measurements performed on the A subsystem are correlated with the ones performed on the B subsystem.

Proof. If the A subsystem is independent of the B one, we recast $P(v_{Ai},v_{Bj})$ as $P(v_{Ai})P(v_{Bj})$, where $P(v_{Ai})$ and $P(v_{Bj})$ denote the marginal probabilities for obtaining results v_{Ai} and v_{Bj}, respectively. Then, we have

$$C_{0,n}^{(l)}=[P(v_{A0})-P(v_{A1})]P(v_{B0}), \qquad (6)$$

$$C_{1,n}^{(l)}=[P(v_{A1})-P(v_{A0})]P(v_{A1}). \qquad (7)$$

Since $P(v_{A1}),P(v_{B0})\geq 0$, we conclude that $C_0^{(l)}C_1^{(l)}\leq 0$. Therefore, $C_0^{(l)}C_1^{(l)}>0$ implies that the measured outcomes performed on the A subsystem are dependent on the one performed on the B subsystem. Q.E.D.

We start showing the strategy with the help of proposition 1. First, to describe the correlation between a specific party and others of the four-qubit system, we give four sets of correlator operators

$$\hat{C}_{0,nz}^{(z)}=(\hat{0}_{nz}-\hat{1}_{nz})\otimes\hat{0}_{mz}\otimes\hat{0}_{pz}\otimes\hat{0}_{qz}, \qquad (8)$$

$$\hat{C}_{1,nz}^{(z)}=(\hat{1}_{nz}-\hat{0}_{nz})\otimes\hat{1}_{mz}\otimes\hat{1}_{pz}\otimes\hat{1}_{qz}, \qquad (9)$$

for $n=1,\ldots,4$, where $\hat{v}_{nz}=|v\rangle_{nznz}\langle v|$ and n, m, p, and q denote four different parties under the LMS $M_{4z}=(\sigma_z,\sigma_z,\sigma_z,\sigma_z)$. In order to have compact forms, in what follows, symbols of tensor product will be omitted from correlator operators. Then, for some experimental output state, the expectation values of the Hermitian operators $\hat{C}_{0,n}^{(z)}$ and $\hat{C}_{1,n}^{(z)}$ are expressed in the following correlators in terms of joint probabilities:

$$C_{0,n}^{(z)}=P(v_n=0,v=0)-P(v_n=1,v=0), \qquad (10)$$

$$C_{1,n}^{(z)}=P(v_n=1,v=3)-P(v_n=0,v=3), \qquad (11)$$

where $v=\Sigma_{i=1,i\neq n}^{4}v_i$. By proposition 1, we know that if results of measurements reveal that $C_{0,n}^{(z)}C_{1,n}^{(z)}>0$, we are convinced that the outcomes of measurements performed on the nth party are correlated with the ones performed on the rest. If the nth party is independent of the rest, we have

$$C_{0,n}^{(z)}=[P(v_n=0)-P(v_n=1)]P(v=0),\quad C_{1,n}^{(z)}=[P(v_n=1)-P(v_n=0)]P(v=3),$$

and realize that $C_{0,n}^{(z)}C_{1,n}^{(z)}\leq 0$.

For the pure generalized four-qubit GHZ state $|\Phi(\theta,\phi)\rangle$ we have

$$C^{(z)}_{0,n,\Phi(\theta,\phi)} = \cos^2(\theta), \quad C^{(z)}_{1,n,\Phi(\theta,\phi)} = \sin^2(\theta) \tag{12}$$

and, hence, $C^{(z)}_{0,n,\Phi(\theta,\phi)}C^{(z)}_{1,n,\Phi(\theta,\phi)} > 0$, which describes the outcomes of measurements are correlated. Then the condition $C^{(z)}_{0,n}C^{(z)}_{1,n} > 0$ is a necessary condition of the pure generalized four-qubit GHZ state.

Further, we construct the following correlator operators to identify correlations between a specific group, which is composed of the nth and mth parties, and another:

$$\hat{C}^{(z)}_{0,nm} = (\hat{0}_{nz}\hat{0}_{mz} - \hat{1}_{nz}\hat{1}_{mz})\hat{0}_{pz}\hat{0}_{qz}, \tag{13}$$

$$\hat{C}^{(z)}_{1,nm} = (\hat{1}_{nz}\hat{1}_{mz} - \hat{0}_{nz}\hat{0}_{mz})\hat{1}_{pz}\hat{1}_{qz} \tag{14}$$

for $n,m=1,\ldots,4$ and $n\neq m$. Moreover, we can express the expectation values of the Hermitian operators $\hat{C}^{(z)}_{0,nm}$ and $\hat{C}^{(z)}_{1,nm}$ in terms of joint probabilities for some output state

$$C^{(z)}_{0,nm} = P(v_{nm}=0, v'=0) - P(v_{nm}=2, v'=0), \tag{15}$$

$$C^{(z)}_{1,nm} = P(v_{nm}=2, v'=2) - P(v_{nm}=0, v'=2), \tag{16}$$

where $v_{nm}=v_n+v_m$ and $v'=\Sigma^4_{i=1,i\neq n\neq m}v_i$. Proposition 1 shows that if the subsystem composed of the nth and the mth parties is uncorrelated with another one, the measured outcomes must satisfy $C^{(z)}_{0,nm}C^{(z)}_{1,nm} \leq 0$. On the other hand, $C^{(z)}_{0,nm}C^{(z)}_{1,nm} > 0$ indicates that they are dependent.

It is clear that, for a pure generalized four-qubit GHZ state, we have

$$C^{(z)}_{0,nm,\Phi(\theta,\phi)} = \cos^2(\theta), \quad C^{(z)}_{1,nm,\Phi(\theta,\phi)} = \sin^2(\theta), \tag{17}$$

and hence $C^{(z)}_{0,nm,\Phi(\theta,\phi)}C^{(z)}_{1,nm,\Phi(\theta,\phi)} > 0$. Thus we know that the subsystem composed of the nth and the mth parties are correlated with another. Therefore, the condition, $C^{(z)}_{0,nm}C^{(z)}_{1,nm} > 0$, is also a necessary condition of the state $|\Phi(\theta,\phi)\rangle$.

After introducing two correlation conditions for the pure generalized GHZ state under M_{4z}, let us progress toward the third one for correlation. Under the LMS, $M_{4x} = (\sigma_x,\sigma_x,\sigma_x,\sigma_x)$, we formulate four sets of correlators which correspond to the following operators for identifying correlations between the nth party and others:

$$\hat{C}^{(x)}_{0,n} = (\hat{0}_{nx} - \hat{1}_{nx}) \otimes \hat{\mathbf{E}}, \tag{18}$$

$$\hat{C}^{(x)}_{1,n} = (\hat{1}_{nx} - \hat{0}_{nx}) \otimes \hat{\mathbf{O}}, \tag{19}$$

where

$$\hat{\mathbf{E}} = (\hat{0}_{mx}\hat{0}_{px}\hat{0}_{qx} + \hat{0}_{mx}\hat{1}_{px}\hat{1}_{qx} + \hat{1}_{mx}\hat{0}_{px}\hat{1}_{qx} + \hat{1}_{mx}\hat{1}_{px}\hat{0}_{qx}), \tag{20}$$

$$\hat{\mathbf{O}} = (\hat{1}_{mx}\hat{1}_{px}\hat{1}_{qx} + \hat{1}_{mx}\hat{0}_{px}\hat{0}_{qx} + \hat{0}_{mx}\hat{1}_{px}\hat{0}_{qx} + \hat{0}_{mx}\hat{0}_{px}\hat{1}_{qx}). \tag{21}$$

From the expectation values of $\hat{C}^{(x)}_{0,n}$ and $\hat{C}^{(x)}_{1,n}$ for some state and proposition 1, we could know the correlation behavior of the system, i.e., for a system in which the nth party is uncorrelated with the rest under M_{4x}, the outcomes of measurements must satisfy the condition $C^{(x)}_{0,n}C^{(x)}_{1,n} \leq 0$.

TABLE I. Summaries of numerical results of $\alpha_\Phi(\theta,\phi)$ for $\mathcal{W}_\Phi(\theta,\phi)$, the parameters γ_Φ, which are utilized to prove $\mathcal{W}_\Phi(\theta,\phi)$ and $\delta_{\text{noise},\Phi}$ involved in robustness of the proposed witness operator for detecting truly multipartite entanglement. Three different cases for the state $|\Phi(\theta,\phi)\rangle$ corresponding to $\mathcal{W}_\Phi(\theta,\phi)$ have been demonstrated.

(θ,ϕ)	$(\frac{\pi}{4},\frac{\pi}{6})$	$(\frac{\pi}{4.9},0)$	$(\frac{\pi}{3.7},\frac{\pi}{9})$
α_Φ	9.01	9.21	8.92
γ_Φ	6.54	6.44	6.86
$\delta_{\text{noise},\Phi}$	0.139	0.150	0.169

For the pure state, $|\Phi(\theta,\phi)\rangle$, the expectation values of $\hat{C}^{(x)}_{k,n}$ is given by

$$C^{(x)}_{0,n,\Phi(\theta,\phi)} = C^{(x)}_{1,n,\Phi(\theta,\phi)} = \sin(2\theta)\cos(\phi)/2 \tag{22}$$

and ensure that there are correlations between measured outcomes under the LMS M_{4x}. Thus the condition $C^{(x)}_{0,n}C^{(x)}_{1,n} > 0$ is necessary for the pure generalized four-qubit GHZ state.

Entanglement imbedded in the pure generalized four-qubit GHZ state manifests itself via necessary conditions of correlations presented above under two LMSs. Therefore we combine all of the correlator operators involved in the necessary conditions

$$\hat{C}_\Phi = \hat{C}^{(z)} + \hat{C}^{(x)},$$

where

$$\hat{C}^{(z)} = \sum_{j=0}^{1}\Big(\sum_{n=1}^{4}\hat{C}^{(z)}_{j,n} + \sum_{m=2}^{4}\hat{C}^{(z)}_{j,1m}\Big) = 8(\hat{0}_{1z}\hat{0}_{2z}\hat{0}_{3z}\hat{0}_{4z} + \hat{1}_{1z}\hat{1}_{2z}\hat{1}_{3z}\hat{1}_{4z}) - \mathbb{1}, \tag{23}$$

$$\hat{C}^{(x)} = \sum_{n=1}^{4}\sum_{k=0}^{1}\hat{C}^{(x)}_{k,n} = 4\sigma_x\sigma_x\sigma_x\sigma_x, \tag{24}$$

and $\mathbb{1}$ is an identify operator, and then utilize the operator $\hat{C}_\Phi$ to construct a witness operator for detections of truly multipartite entanglement. Three example are shown as follows. The witness operator

$$\mathcal{W}_\Phi(\theta,\phi) = \alpha_\Phi(\theta,\phi)\mathbb{1} - \hat{C}_\Phi, \tag{25}$$

where $\alpha_\Phi(\theta,\phi)$ is some constant, detects genuine multipartite entanglement for the cases (θ,ϕ): $(\pi/4,\pi/6)$, $(\pi/4.9,0)$, and $(\pi/3.7,\pi/9)$. Table I gives a summary of $\alpha_\Phi(\theta,\phi)$ for these cases.

In order to prove that $\mathcal{W}_\Phi(\theta,\phi)$ is a EW for detecting genuine multipartite entanglement, we have to show the following comparison between

$$\mathcal{W}^p_\Phi(\theta,\phi) = \alpha^p_\Phi\mathbb{1} - |\Phi(\theta,\phi)\rangle\langle\Phi(\theta,\phi)| \tag{26}$$

and $\mathcal{W}_\Phi(\theta,\phi)$ [13]: if a state ρ satisfies $\text{Tr}[\mathcal{W}_\Phi(\theta,\phi)\rho] < 0$, it also satisfies $\text{Tr}[\mathcal{W}^p_\Phi(\theta,\phi)\rho] < 0$, i.e., $\mathcal{W}_\Phi(\theta,\phi) - \gamma_\Phi\mathcal{W}^p_\Phi(\theta,\phi) \geq 0$, where $\gamma_\Phi(\theta,\phi)$ is some positive constant.

TABLE II. Expectation values of three proposed EWs including $\mathcal{W}_{\Phi}(\frac{\pi}{4},\frac{\pi}{6})$, $\mathcal{W}_{\Phi}(\frac{\pi}{4.9},0)$, and $\mathcal{W}_{\Phi}(\frac{\pi}{3.7},\frac{\pi}{9})$ for the pure states $|\Phi\rangle$: $|\Phi(\frac{\pi}{4},\frac{\pi}{6})\rangle$, $|\Phi(\frac{\pi}{4.9},0)\rangle$, and $|\Phi(\frac{\pi}{3.7},\frac{\pi}{9})\rangle$.

$\|\Phi\rangle$	$\|\Phi(\frac{\pi}{4},\frac{\pi}{6})\rangle$	$\|\Phi(\frac{\pi}{4.9},0)\rangle$	$\|\Phi(\frac{\pi}{3.7},\frac{\pi}{9})\rangle$
$\mathrm{Tr}[\mathcal{W}_{\Phi}(\frac{\pi}{4},\frac{\pi}{6})\|\Phi\rangle\langle\Phi\|]$	−1.45	−1.83	−1.72
$\mathrm{Tr}[\mathcal{W}_{\Phi}(\frac{\pi}{4.9},0)\|\Phi\rangle\langle\Phi\|]$	−1.25	−1.63	−1.52
$\mathrm{Tr}[\mathcal{W}_{\Phi}(\frac{\pi}{3.7},\frac{\pi}{9})\|\Phi\rangle\langle\Phi\|]$	−1.55	−1.92	−1.81

Through the method given by Bourennane *et al.* [12], we derive the operator $\mathcal{W}_{\Phi}^{p}(\theta,\phi)$ and have $\alpha_{\Phi}^{p}=\cos^2(\theta)$ for $0<\theta\leq\pi/4$ and $\alpha_{\Phi}^{p}=\sin^2(\theta)$ for $\pi/4\leq\theta<\pi/2$. Table I summarizes the parameters γ_{Φ} utilized to prove that the proposed operators are indeed EWs for detecting truly multipartite entanglement.

In addition, we are concerned with the robustness to noise for the witness $\mathcal{W}_{\Phi}(\theta,\phi)$. The robustness of $\mathcal{W}_{\Phi}(\theta,\phi)$ depends on the noise tolerance $p_{\text{noise}}<\delta_{\text{noise}}$ is such that

$$\rho=\frac{p_{\text{noise}}}{2^N}\mathbb{1}+(1-p_{\text{noise}})|\Phi(\theta,\phi)\rangle\langle\Phi(\theta,\phi)|, \tag{27}$$

where p_{noise} describes the noise fraction, is identified as a genuine multipartite entanglement. Three cases for the robustness to noise for the witness $\mathcal{W}_{\Phi}(\theta,\phi)$ have been summarized in Table I.

Further, we show the expectation values of the proposed EWs for different pure states by Table II. From comparison with the results we know that a state, say $|\Phi(\theta',\phi')\rangle$, does not always give the smallest expectation value of the corresponding witness operator. One can identify with the operator $\mathcal{W}_{\Phi}(\theta',\phi')$ that an experimental output ρ is truly multipartite entanglement if $\mathrm{Tr}[\mathcal{W}_{\Phi}(\theta',\phi')\rho]<0$. Further, if $\mathrm{Tr}[\mathcal{W}_{\Phi}(\theta',\phi')\rho]<\mathrm{Tr}[\mathcal{W}_{\Phi}(\theta',\phi')|\Phi(\theta',\phi')\rangle\langle\Phi(\theta',\phi')|]$, the state ρ is not in the state $|\Phi(\theta',\phi')\rangle$ class.

An approach to derive $\hat{C}_{\Phi}$ shown above can be applied to the cases for arbitrary number of qubits straightforwardly. One can formulate sets of correlator operators to identify correlations between two subsystems under two LMSs and then construct the witness operators further. In particular, we have found that the proposed method also provides an analytical and systematic way to construct correlator operators for entangled states with local stabilizers and the corresponding EWs as the previous results [13,19].

Before proceeding further, let us give a brief summary and conclusion for this section. We have demonstrated a systematical method to derive correlator operators utilized to construct witness operators. The proposed correlator operators are based on necessary conditions of some pure multipartite entangled state to be created experimentally. Moreover, in the example, these witness operators can be measured with only two LMSs. In what follows, we will give two EWs in which the correlator operators can be constructed systematically. Through these cases for entanglement detection, one could realize that the proposed conditions of quantum correlations possess a wide generality.

II. EWs FOR MULTIPARTITE ENTANGLED STATES

A. Detection of genuine multipartite entanglement of the four-qubit singlet state

Very recently, four-party quantum secret sharing has been demonstrated via the resource of four photon entanglement [18], which is called the four-qubit singlet state [17]. Through the same method presented in the Introduction, we give an EW to detect the four-qubit singlet state.

The four-qubit singlet state is expressed as the following form:

$$|\Psi\rangle=\frac{1}{\sqrt{3}}\Big[|0011\rangle_z+|1100\rangle_z-\frac{1}{2}(|0110\rangle_z+|1001\rangle_z+|0101\rangle_z+|1010\rangle_z)\Big]. \tag{28}$$

Under the LMS M_{4z}, we formulate eight sets of criteria for identifying quantum correlation between a specific party and others: the first type of identifications include the following four sets of correlators:

$$\hat{C}_{0,m}^{(z)}=\hat{0}_{1z}\hat{0}_{2z}\hat{1}_{3z}\hat{1}_{4z}-X_m(\hat{0}_{1z}\hat{0}_{2z}\hat{1}_{3z}\hat{1}_{4z})X_m, \tag{29}$$

$$\hat{C}_{1,m}^{(z)}=\hat{1}_{1z}\hat{1}_{2z}\hat{0}_{3z}\hat{0}_{4z}-X_m(\hat{1}_{1z}\hat{1}_{2z}\hat{0}_{3z}\hat{0}_{4z})X_m, \tag{30}$$

where $X_m=\sigma_x$ is performed on the mth party for $m=1,\ldots,4$. Then, the second type of criteria are formulated as

$$\hat{C}_{0n,k}^{(z)}=[\hat{0}_{(2n+1)z}\hat{1}_{(2n+2)z}-X_k(\hat{0}_{(2n+1)z}\hat{1}_{(2n+2)z})X_k]\times(\hat{0}_{(2n\oplus3)z}\hat{1}_{(2n\oplus4)z}+\hat{1}_{(2n\oplus3)z}\hat{0}_{(2n\oplus4)z}), \tag{31}$$

$$\hat{C}_{1n,k}^{(z)}=[\hat{1}_{(2n+1)z}\hat{0}_{(2n+2)z}-X_k(\hat{1}_{(2n+1)z}\hat{0}_{(2n+2)z})X_k]\times(\hat{0}_{(2n\oplus3)z}\hat{1}_{(2n\oplus4)z}+\hat{1}_{(2n\oplus3)z}\hat{0}_{(2n\oplus4)z}), \tag{32}$$

where $k=(2n+1),(2n+2)$ for $n=0,1$; and the symbol "$\oplus$" behaves as the addition of modulo 4 for $n=1$ and as an ordinary addition for $n=0$. The expectation values of the operators $\hat{C}_{l,m}^{(z)}$ and $\hat{C}_{ln,k}^{(z)}$ for the pure four-qubit singlet state can be evaluated directly and are given by $C_{l,m,\Psi}^{(z)}=1/3$ and $C_{ln,k,\Psi}^{(z)}=1/6$ for $l=0,1$.

It is easy to see that the conditions involved in the expectation values of $\hat{C}_{l,m}^{(z)}$ and $\hat{C}_{ln,k}^{(z)}$:

$$C_{0,m}^{(z)}C_{1,m}^{(z)}>0 \quad\text{and}\quad C_{0n,k}^{(z)}C_{1n,k}^{(z)}>0, \tag{33}$$

are necessary for the pure four-qubit singlet state. The proof of this statement is similar to the one for proposition 1 presented in the first section.

For invariance of the wave function presented in the eigenbasis of σ_x (σ_y), in analogy, we can construct eight sets of Hermitian operators

$$(\hat{C}_{0,m}^{[x(y)]},\hat{C}_{1,m}^{[x(y)]}) \quad\text{and}\quad (\hat{C}_{0n,k}^{[x(y)]},\hat{C}_{1n,k}^{[x(y)]}),$$

via the replacement of the index z in above Hermitian operators by the index x (y) and constructing the operators in the eigenbasis of $\sigma_{x(y)}$. The expectation values of the above operators are all positive for the state $|\Psi\rangle$, and so we have the following necessary conditions of the state $|\Psi\rangle$:

$$C_{0,m}^{[x(y)]}C_{1,m}^{[x(y)]} > 0 \quad \text{and} \quad C_{0n,k}^{[x(y)]}C_{1n,k}^{[x(y)]} > 0. \tag{34}$$

Then, we combine all of the correlator operators proposed above:

$$\hat{C}_{\Psi} = \hat{C}_{\Psi}^{(x)} + \hat{C}_{\Psi}^{(y)} + \hat{C}_{\Psi}^{(z)}, \tag{35}$$

where

$$\hat{C}_{\Psi}^{(i)} = \sum_{l=0}^{1} (5\sum_{m=1}^{4} \hat{C}_{l,m}^{(i)} + \sum_{n=0}^{1} \sum_{k=2n+1}^{2n+2} \hat{C}_{ln,k}^{(i)}) \tag{36}$$

for $i=x,y,z$, and present a EW to detect the four-qubit singlet state. The following witness operator detects truly multipartite entanglement for states close to the state $|\Psi\rangle$:

$$\mathcal{W}_{\Psi} = \alpha_{\Psi}\mathbb{1} - \hat{C}_{\Psi}, \tag{37}$$

where $\alpha_{\Psi}=36.5$.

We use the method utilized for $\mathcal{W}_{\Phi}(\theta,\phi)$ to prove $\mathcal{W}_{\Psi}$ is a EW. First, we seek the witness operator $\mathcal{W}_{\Psi}^{p}$. Through Ref. [12], the operator is given by:

$$\mathcal{W}_{\Psi}^{p} = \frac{3}{4}\mathbb{1} - |\Psi\rangle\langle\Psi|. \tag{38}$$

Then, we have to show that if a state ρ satisfies $\mathrm{Tr}(\mathcal{W}_{\Psi}\rho) < 0$, it also satisfies $\mathrm{Tr}(\mathcal{W}_{\Psi}^{p}\rho) < 0$. We find that $\gamma_{\Psi}=30$ is such that $\mathcal{W}_{\Psi} - \gamma_{\Psi}\mathcal{W}_{\Psi}^{p} \geq 0$.

The sets of correlator operators $\hat{C}_{\Psi}^{(x)}$, $\hat{C}_{\Psi}^{(y)}$, and $\hat{C}_{\Psi}^{(z)}$ note that only three LMSs are used in the witness operator $\mathcal{W}_{\Psi}$. The number of LMSs is smaller than the required one, 15 LMSs, in Ref. [12]. Moreover, the robustness of the witness $\mathcal{W}_{\Psi}$ is specified by $\delta_{\text{noise},\Psi}=15/88 \simeq 0.170455$. This result satisfies the experimental requirement of robustness in Ref. [12].

B. Detection of genuine multipartite entanglement for a four-level tripartite system

In order to show further utilities of the proposed approach, we proceed to provide a witness to detect genuine multipartite entanglement close to a four-level tripartite GHZ state [14]:

$$|\mathrm{GHZ}_{4\times 3}\rangle = \frac{1}{2}\sum_{l=0}^{3} |l\rangle_{1z} \otimes |l\rangle_{2z} \otimes |l\rangle_{3z}. \tag{39}$$

First of all, with the knowledge of the wave function represented in the eigenbasis: $|l\rangle_{nz}$ for $n=1,2,3$, we have nine sets of correlator operators for identifying quantum correlation between the nth party and others, and are given by

$$\hat{C}_{nk,j}^{(z)} = (\hat{k} - \hat{s}_{kj})_{nz}\hat{k}_{pz}\hat{k}_{qz} \tag{40}$$

for $j=1,\ldots,9$; $\mathrm{k}=0,\ldots,3$; $n,p,q=1,2,3$, and $n \neq p \neq q$; where $\hat{s}_{kj}=\hat{0},\ldots,\hat{3}$; $\hat{k} \neq \hat{s}_{kj}$, and $\hat{s}_{kj} \neq \hat{s}_{k'j}$ for $k \neq k'$; and $\hat{C}_{nk,j}^{(z)} \neq \hat{C}_{nk,j'}^{(z)}$ for $j \neq j'$. To show $\hat{C}_{nk,j}^{(z)}$ explicitly, let us take the following set of operators numbered by $j=1$, for example,

$$\hat{C}_{n0,1}^{(z)} = (\hat{0}_{nz} - \hat{1}_{nz})\hat{0}_{pz}\hat{0}_{qz}, \quad \hat{C}_{n1,1}^{(z)} = (\hat{1}_{nz} - \hat{2}_{nz})\hat{1}_{pz}\hat{1}_{qz}, \quad \hat{C}_{n2,1}^{(z)} = (\hat{2}_{nz} - \hat{3}_{nz})\hat{2}_{pz}\hat{2}_{qz}, \quad \hat{C}_{n3,1}^{(z)} = (\hat{3}_{nz} - \hat{0}_{nz})\hat{3}_{pz}\hat{3}_{qz}.$$

Another example for the second set of operators $j=2$ could be the following one:

$$\hat{C}_{n0,2}^{(z)} = (\hat{0}_{nz} - \hat{2}_{nz})\hat{0}_{pz}\hat{0}_{qz}, \quad \hat{C}_{n1,2}^{(z)} = (\hat{1}_{nz} - \hat{3}_{nz})\hat{1}_{pz}\hat{1}_{qz}, \quad \hat{C}_{n2,2}^{(z)} = (\hat{2}_{nz} - \hat{0}_{nz})\hat{2}_{pz}\hat{2}_{qz}, \quad \hat{C}_{n3,2}^{(z)} = (\hat{3}_{nz} - \hat{1}_{nz})\hat{3}_{pz}\hat{3}_{qz}.$$

We progress to a correlation condition for the pure four-level tripartite GHZ state by the following proposition.

Proposition 2. If the expectation values of $\hat{C}_{nk,j}^{(z)}$ for some state are all positive for $k=1,\ldots,3$ under some j, the outcomes of measurements for the party n and the rest of the systems are correlated.

Proof. If the nth party is independent of the rest of the system, we can cast the expectation values of the operators $\hat{C}_{nk,j}^{(z)}$ as

$$C_{nk,j}^{(z)} = [P(v_n = k) - P(v_n = s_{kj})]P(v_p = k, v_q = k).$$

Since $P(v_p=k,v_q=k) \geq 0$, $C_{nk,j}^{(z)}$ should not be all positive. Thus $\hat{C}_{nk,j}^{(z)} > 0$ for all k's implies that the measured outcomes for the party n and the rest are correlated. Q.E.D.

All of the expectation values of the operators $\hat{C}_{nk,j}^{(z)}$ for the pure four-level tripartite GHZ state are given by $C_{nk,j,\mathrm{GHZ}_{4\times 3}}^{(z)} = 1/4$, which are greater than zero. We then consider that $\hat{C}_{nk,j}^{(z)} > 0$ as a necessary condition of the state. Second, if an observable with the eigenvector

$$|g\rangle_{nf} = \frac{1}{2}\sum_{h=0}^{3} \exp\left[-i\frac{2\pi h}{4}g\right]|h\rangle_{nz} \tag{41}$$

for $g=0,\ldots,3$, is measured for each party $n=1,2,3$, we give nine sets of correlator operators to identify quantum correlation between the nth party and others

$$\hat{C}_{nk,j}^{(f)} = (\hat{k} - \hat{s}_{kj})_{nf}\hat{V}_{klr}, \tag{42}$$

where

$$\hat{V}_{klr} = \sum_{l,r=0}^{3} \delta((k+l+r)\mathrm{mod}\ 4,0)\hat{l}_{pf}\hat{r}_{qf} \tag{43}$$

and definitions of $\hat{k}$, $\hat{s}_{kj}$, n, p, q, and j are the same as the ones mentioned for $\hat{C}_{nk,j}^{(z)}$. For $j=1$, the set of operators specified by the above equations could be:

$$\hat{C}_{n0,1}^{(f)} = (\hat{0} - \hat{1})(\hat{0}\hat{0} + \hat{1}\hat{3} + \hat{2}\hat{2} + \hat{3}\hat{1}), \quad \hat{C}_{n1,1}^{(f)} = (\hat{1} - \hat{2})(\hat{0}\hat{3} + \hat{1}\hat{2} + \hat{2}\hat{1} + \hat{3}\hat{0}), \quad \hat{C}_{n2,1}^{(f)} = (\hat{2} - \hat{3})(\hat{0}\hat{2} + \hat{1}\hat{1} + \hat{2}\hat{0} + \hat{3}\hat{3}), \quad \hat{C}_{n3,1}^{(f)} = (\hat{3} - \hat{0})(\hat{0}\hat{1} + \hat{1}\hat{0} + \hat{2}\hat{3} + \hat{3}\hat{2}).$$

For $j=2$, we could give the set of operators as follows:

$\hat{C}^{(f)}_{n0,2}=(\hat{0}-\hat{2})(\hat{0}\hat{0}+\hat{1}\hat{3}+\hat{2}\hat{2}+\hat{3}\hat{1})$, $\hat{C}^{(f)}_{n1,2}=(\hat{1}-\hat{3})(\hat{0}\hat{3}+\hat{1}\hat{2}+\hat{2}\hat{1}+\hat{3}\hat{0})$, $\hat{C}^{(f)}_{n2,2}=(\hat{2}-\hat{0})(\hat{0}\hat{2}+\hat{1}\hat{1}+\hat{2}\hat{0}+\hat{3}\hat{3})$, $\hat{C}^{(f)}_{n3,2}=(\hat{3}-\hat{1})(\hat{0}\hat{1}+\hat{1}\hat{0}+\hat{2}\hat{3}+\hat{3}\hat{2})$.

Please note that in order to have compact forms, we have omitted the subscripts nf, pf, and qf from the above examples. A correlation condition similar to the one discussed in proposition 2 is proposed by the statement, "if the expectation values of $\hat{C}^{(f)}_{nk,j}$ are all positive for $k=1,\ldots,3$ under some j, there are correlations between the measured outcomes for the party n and the rest of the systems." Since all of the expectation values of the operators $\hat{C}^{(f)}_{nk,j}$ for the pure four-level tripartite GHZ state are greater than zero, i.e., $C^{(f)}_{nk,j,\mathrm{GHZ}_{4\times3}}=1/4$, the correlation condition $\hat{C}^{(f)}_{nk,j}>0$ is then necessary for the state.

Therefore, through a linear combination of all of the correlator operators proposed above

$$\hat{C}_{\mathrm{GHZ}_{4\times3}}=\sum_{n=1}^{3}\sum_{j=1}^{9}\sum_{k=0}^{3}(1.5\hat{C}^{(z)}_{nk,j}+\hat{C}^{(f)}_{nk,j}), \quad (44)$$

the following witness operator detects genuine multipartite entanglement for states close to $|\mathrm{GHZ}_{4\times3}\rangle$:

$$\mathcal{W}_{\mathrm{GHZ}_{4\times3}}=\alpha_{\mathrm{GHZ}_{4\times3}}\mathbb{1}-\hat{C}_{\mathrm{GHZ}_{4\times3}}, \quad (45)$$

where $\alpha_{\mathrm{GHZ}_{4\times3}}=40.5$. We take an approach similar to the ones used in the previous proofs for EWs to prove that the above witness operator detects genuine multipartite entanglement. In order to show that if an experimental output state ρ satisfies $\mathrm{Tr}(\mathcal{W}_{\mathrm{GHZ}_{4\times3}}\rho)<0$, the state ρ also satisfies $\mathrm{Tr}(\mathcal{W}^{\mathrm{p}}_{\mathrm{GHZ}_{4\times3}}\rho)<0$, first, we deduce that

$$\mathcal{W}^{\mathrm{p}}_{\mathrm{GHZ}_{4\times3}}=\frac{1}{4}\mathbb{1}-|\mathrm{GHZ}_{4\times3}\rangle\langle\mathrm{GHZ}_{4\times3}|, \quad (46)$$

by the method proposed in Ref. [12]. Further, through the relation $\mathcal{W}_{\mathrm{GHZ}_{4\times3}}-36\mathcal{W}^{\mathrm{p}}_{\mathrm{GHZ}_{4\times3}}\geq0$ for the proposed witness operator, we then conclude that $\mathcal{W}_{\mathrm{GHZ}_{4\times3}}$ can be used to detect truly multipartite entanglement.

Furthermore, when a state mixes with white noise the proposed EW is very robust and it detects genuine multipartite entanglement if $p_{\mathrm{noise}}<0.4$. Thus, two local measurement settings are sufficient to detect genuine four-level tripartite entanglement around a pure four-level tripartite GHZ state.

III. BI FOR ARBITRARY HIGH-DIMENSIONAL BIPARTITE SYSTEMS

In order to derive a BI, we will begin with specifications of correlation conditions for quantum correlation of a two-qudit entangled state. Then, we will proceed to verify that any local hidden variable theory cannot reproduce the correlations embedded in the entangled state. This approach is opposite to the one presented in Ref. [9].

First, to specify the quantum correlation embedded in the two-qudit entangled state

$$|\psi_d\rangle=\frac{1}{\sqrt{d}}\sum_{l=0}^{d-1}|l\rangle_{1z}\otimes|l\rangle_{2z}, \quad (47)$$

we describe the wave function in the following eigenbasis of some observable $\hat{V}^{(q)}_k$:

$$|l\rangle_{kq}=\frac{1}{\sqrt{d}}\sum_{m=0}^{d-1}\exp\left[i\frac{2\pi m}{d}(l+n^{(q)}_k)\right]|m\rangle_{kz}, \quad (48)$$

for $k,q=1,2$, where $n^{(1)}_1=0$, $n^{(1)}_2=1/4$, $n^{(2)}_1=1/2$, and $n^{(2)}_2=-1/4$ correspond to four different LMSs $M_{ij}=(\hat{V}^{(i)}_1,\hat{V}^{(j)}_2)$ for $i,j=1,2$. From our knowledge of the four different representations of the state $|\psi_d\rangle$, we give four sets of correlators of quantum correlation

$$C^{(12)}_m=P[v^{(1)}_1=(-m)\mathrm{mod}\, d,v^{(2)}_2=m]-P[v^{(1)}_1=(1-m)\mathrm{mod}\, d,v^{(2)}_2=m], \quad (49)$$

$$C^{(21)}_m=P[v^{(2)}_1=(d-m-1)\mathrm{mod}\, d,v^{(1)}_2=m]-P[v^{(2)}_1=(-m)\mathrm{mod}\, d,v^{(1)}_2=m], \quad (50)$$

$$C^{(qq)}_m=P[v^{(q)}_1=(-m)\mathrm{mod}\, d,v^{(q)}_2=m]-P[v^{(q)}_1=(d-m-1)\mathrm{mod}\, d,v^{(q)}_2=m] \quad (51)$$

for $m=0,1,\ldots,d-1$ and $q=1,2$. The superscripts (ij), (i), and (j) indicate that some LMS M_{ij} has been selected. For the pure state $|\psi_d\rangle$ under M_{ij}, the correlator $C^{(ij)}_m$ can be evaluated analytically [9] and is given by

$$C^{(ij)}_{m,\psi_d}=\frac{1}{2d^3}[\csc^2(\pi/4d)-\csc^2(3\pi/4d)], \quad (52)$$

where $\csc(h)$ is the cosecant of h. Since $C^{(ij)}_{m,\psi_d}>0$ for all m's with any finite value of d, we ensure that there are correlations between outcomes of measurements performed on the state $|\psi_d\rangle$ under four different LMSs. The proof of this statement is similar to that for proposition 2. Hence the correlation conditions

$$C^{(ij)}_m>0 \quad (53)$$

are necessary for the pure two-qudit entangled state $|\psi_d\rangle$.

Thus, we take the summation of all $C^{(ij)}_m$'s

$$C_d=C^{(11)}+C^{(12)}+C^{(21)}+C^{(22)}, \quad (54)$$

where $C^{(ij)}=\sum_{m=0}^{d-1}C^{(ij)}_m$, as an identification of the state $|\psi_d\rangle$. We can evaluate the summation of all $C^{(ij)}_m$'s for the state $|\psi_d\rangle$, and then we have

$$C_{d,\psi_d}=\frac{2}{d^2}[\csc^2(\pi/4d)-\csc^2(3\pi/4d)]. \quad (55)$$

One can find that C_{d,ψ_d} is an increasing function of d. For instance, if $d=3$, one has $C_{3,\psi_3}\simeq2.87293$. In the limit of large d, we obtain $\lim_{d\to\infty}C_{d,\psi_d}=(16/3\pi)^2\simeq2.88202$.

We proceed to consider the maximum value of C_d for local hidden variable theories which is denoted by $C_{d,\mathrm{LHV}}$. The following proof is based on deterministic local models which are specified by fixing the outcome of all measurements. This consideration is general since any probabilistic model can be converted into a deterministic one [20]. Substituting a fixed set $(\tilde{v}_1^{(1)},\tilde{v}_2^{(1)},\tilde{v}_1^{(2)},\tilde{v}_2^{(2)})$ into

$$C_m^{(ij)} = P(v_1^{(i)} = \alpha_m^{(ij)}, v_2^{(2)} = m) - P(v_1^{(i)} = \beta_m^{(ij)}, v_2^{(j)} = m),$$

where $\alpha_m^{(ij)}$ and $\beta_m^{(ij)}$ denote the values involved in Eqs. (50)–(52), then we have the result

$$C_{m,\text{LHV}}^{(ij)} = \delta(\alpha_m^{(ij)},\tilde{v}_1^{(i)})\delta(m,\tilde{v}_2^{(j)}) - \delta(\beta_m^{(ij)},\tilde{v}_1^{(i)})\delta(m,\tilde{v}_2^{(j)}), \tag{56}$$

where $\delta(x,y)$ denotes the Kronecker delta symbol. Accordingly, C_d for local hidden variable theories turns into

$$\begin{aligned} C_{d,\text{LHV}} = {} & \delta((\tilde{v}_1^{(1)} + \tilde{v}_2^{(1)}) \bmod d, 0) - \delta(-(\tilde{v}_1^{(1)} + \tilde{v}_2^{(1)}) \bmod d, 1) \\ & + \delta(\tilde{v}_1^{(1)} + \tilde{v}_2^{(2)}) \bmod d, 0 - \delta((\tilde{v}_1^{(1)} + \tilde{v}_2^{(2)}) \bmod d, 1) \\ & + \delta((\tilde{v}_1^{(2)} + \tilde{v}_2^{(2)}) \bmod d, 0) - \delta(-(\tilde{v}_1^{(2)} \\ & + \tilde{v}_2^{(2)}) \bmod d, 1) + \delta(-(\tilde{v}_1^{(2)} + \tilde{v}_2^{(1)}) \bmod d, 1) \\ & - \delta((\tilde{v}_1^{(2)} + \tilde{v}_2^{(1)}) \bmod d, 0). \end{aligned} \tag{57}$$

There are three nonvanishing terms at most among the four positive δ functions and there exist four cases for it, for example, one is that if $\delta((\tilde{v}_1^{(1)}+\tilde{v}_2^{(1)}) \bmod d,0)=\delta((\tilde{v}_1^{(1)}+\tilde{v}_2^{(2)}) \bmod d,0)=\delta((\tilde{v}_1^{(2)}+\tilde{v}_2^{(2)}) \bmod d,0)=1$ is assigned, we obtain $\tilde{v}_2^{(1)}=\tilde{v}_2^{(2)}$ and then deduce that $\delta(-(\tilde{v}_1^{(2)}+\tilde{v}_2^{(1)}) \bmod d,1)=0$. We also know that there must exist one nonvanishing negative δ function and three vanishing negative ones in the $C_{d,\text{LHV}}$ under the same condition. In the example, the case is $\delta((\tilde{v}_1^{(2)}+\tilde{v}_2^{(1)}) \bmod d,0)=1$. With these facts, we conclude that $C_{d,\text{LHV}}\leq 2$. One can check other three cases for the four positive δ functions, and then they always result in the same bound. Thus, we realize that $C_{d,\psi_d} > C_{d,\text{LHV}}$ and the quantum correlations are stronger than the ones predicted by the local hidden variable theories.

For $d=2$, the proposed inequality $C_{2,\text{LHV}}\leq 2$ can be expressed explicitly in the form

$$\tilde{C}^{(11)} + \tilde{C}^{(12)} + \tilde{C}^{(22)} - \tilde{C}^{(21)} \leq 2, \tag{58}$$

where $\tilde{C}^{(ij)}=\Sigma_{k=0}^{1}(-1)^k\delta((\tilde{v}_1^{(i)}+\tilde{v}_2^{(j)}) \bmod d,k)$, and then we obtain the result which is known as the CHSH inequality after the discovery of Clauser, Horne, Shimony, and Holt [8]. On the other hand, from the quantum-mechanical point of view, we have a violation of the CHSH inequality by $C_{2,\psi_2}=2\sqrt{2}$.

A surprising feature of the inequality is that the total number of detection events required for analyses by each of the presented correlation functions $C^{(ij)}$ is only $2d$, which is much smaller than the result $O(d^2)$ shown in Ref. [16]. This implies that the proposed correlation functions contain only the dominant terms to identify correlations. However, the proposed BI is nontight from a geometric point of view [21]. Since the number of linear independent generators contained in the hyperplane $C_{d,\text{LHV}}=2$ is only $4d$ [19] which is smaller than $4d(d-1)$ involved in the condition of tightness [21], the BI is nontight.

Furthermore, if an experimental output state suffered from white noise and turned into a mixed one with the form

$$\rho = \frac{p_{\text{noise}}}{d^2}\mathbb{1} + (1 - p_{\text{noise}})|\psi_d\rangle\langle\psi_d|,$$

the value of C_d for the state ρ becomes $C_{d,\rho}=(1-p_{\text{noise}})C_{d,\psi_d}$. If the criterion $C_{d,\rho}>2$, i.e.,

$$p_{\text{noise}} < 1 - \frac{2}{C_{d,\psi_d}} \tag{59}$$

is imposed on the system, one ensures that the mixed state still exhibits quantum correlations in outcomes of measurements. For instance, to maintain the quantum correlation for the limit of large d, the system must have $p_{\text{noise}}<0.30604$.

On the other hand, it is worth comparing the noise tolerance of C_d with the one of the following EW:

$$\mathcal{W}_{\psi_d}^{p} = \frac{1}{d}\mathbb{1} - |\psi_d\rangle\langle\psi_d|. \tag{60}$$

Let the noise fraction be the form $p_{\text{noise}}=1-\epsilon$, where ϵ is a positive parameter. Then satisfying the condition of entanglement $\text{Tr}(\mathcal{W}_{\psi_d}^{p}\rho)<0$ implies that $\epsilon>1/(d+1)$. Therefore, in the case where $d\rightarrow\infty$, any state with $p_{\text{noise}}<1$ is detected as an entangled one. Hence, there is a significant difference between the noise tolerance of C_d and the one of $\mathcal{W}_{\psi_d}^{p}$ in the limit of large d.

IV. SUMMARY

Through the necessary condition of quantum correlation we develop a systematic approach to derive correlator operators for BIs and EWs. The d-level bipartite BI is strongly resistant to noise and can be tested with fewer analyses of measurement outcomes. The proposed EWs for the generalized GHZ, four-qubit singlet, and four-level tripartite GHZ states are robust to noise and require fewer experimental efforts to be realized. Therefore, the correlation conditions for quantum correlation involved in the approach to construct correlator operators utilized in EWs and BIs can be considered as a connection between them. The generality of the approach widely cover several (different) tasks of entanglement detections and pave the way for further studies on entangled qudits.

We thank J. W. Pan, Z. B. Chen, and Q. Zhang for useful discussions. This work was supported partially by the National Science Council of Taiwan under Grant No. NSC95-2119-M-009-030.

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