Panpositionable Hamiltonicity of the Alternating Group Graphs

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The alternating group graph AG_n is an interconnection network topology based on the Cayley graph of the alternating group. There are some interesting results concerning the hamiltonicity and the fault tolerant hamiltonicity of the alternating group graphs. In this article, we propose a new concept called panpositionable hamiltonicity. A hamiltonian graph G is panpositionable if for any two different vertices x and y of G and for any integer l satisfying $d(x,y) \leq l \leq |V(G)| - d(x,y)$, there exists a hamiltonian cycle C of G such that the relative distance between x, y on C is l. We show that AG_n is panpositionable hamiltonian if $n \geq 3$. NETWORKS, Vol. 50(2), 146–156 2007

Keywords: alternating group graph; hamiltonian; hamiltonian connected; panpositionable hamiltonian

1. INTRODUCTION

Network topology is usually represented by a graph where the vertices represent processors and the edges represent the links between processors. For graph definitions and notation we follow Ref. [6]. Let $G = (V, E)$ be a graph, where V is a finite set and E is a subset of $\{(u, v) \mid (u, v)$ is an unordered pair of $V\}$. We say that V is the vertex set and E is the edge set of G. Two vertices u and v are adjacent if $(u, v) \in E$. A path P is represented by $\langle v_0, v_1, v_2, \dots, v_k \rangle$. The length of a path P is the number of edges in P, denoted by $L(P)$. We sometimes write the path $\langle v_0, v_1, v_2, \dots, v_k \rangle$ as $\langle v_0, P_1, v_i, v_{i+1}, \dots, v_j, P_2, v_t, \dots, v_k \rangle$, where P_1 is the path $\langle v_0, v_1, \dots, v_i \rangle$ and P_2 is the path $\langle v_j, v_{j+1}, \dots, v_t \rangle$. It is possible to write a path $\langle v_0, v_1, P, v_1, v_2, \dots, v_k \rangle$ if $L(P) = 0$. We use $d_G(u, v)$, or simply $d(u, v)$ if there is no ambiguity, to denote the distance between u and v in a graph G, i.e., the length of a shortest path joining u and v in G. A cycle is a path with at least three vertices such that the first vertex is the same as the last one. We use $d_C(u, v)$ and $D_C(u, v)$ to denote the shorter and the longer distance between u and v on a cycle C of G, respectively. It is possible that $D_C(u, v) = d_C(u, v)$ if the lengths of the two disjoint paths joining u and v in C are equal. A path is a hamiltonian path if its vertices are distinct and span V. A graph G is hamiltonian connected if there exists a hamiltonian path joining any two vertices of G. A hamiltonian cycle of G is a cycle that traverses every vertex of G exactly once. A graph G is hamiltonian if there exists a hamiltonian cycle in G. The hamiltonian properties are important aspects of designing an interconnection network. Many related works have appeared in the literature [1, 5, 7].

We propose a new concept called panpositionable hamiltonicity. A hamiltonian graph G is panpositionable if for any two different vertices x and y of G and for any integer l satisfying $d(x, y) \leq l \leq |V(G)| - d(x, y)$, there exists a hamiltonian cycle C of G such that the relative distance between x, y on C is l; more precisely, $d_C(x, y) = l$ if $l \leq \lfloor \frac{|V(G)|}{2} \rfloor$ or $D_C(x, y) = l$ if $l > \frac{|V(G)|}{2}$. Given a hamiltonian cycle C, if $d_C(x, y) = l$, we have $D_C(x, y) = |V(G)| - d_C(x, y)$. Therefore, a graph is panpositionable hamiltonian if for any integer l with $d(x, y) \leq l \leq \frac{|V(G)|}{2}$, there exists a hamiltonian cycle C of G with $d_C(x, y) = l$. One trivial example, the complete graph K_n with $n \geq 3$, is panpositionable.

There are several requirements in designing a good topology for an interconnection network, such as connectivity and hamiltonicity. The hamiltonian property is one of the major requirements in designing an interconnection network. The hamiltonian property is fundamental to the deadlock-free routing algorithms of distributed systems [8, 9]. A high-reliability network design can be based on constructing a hamiltonian cycle in an interconnection network. Similar

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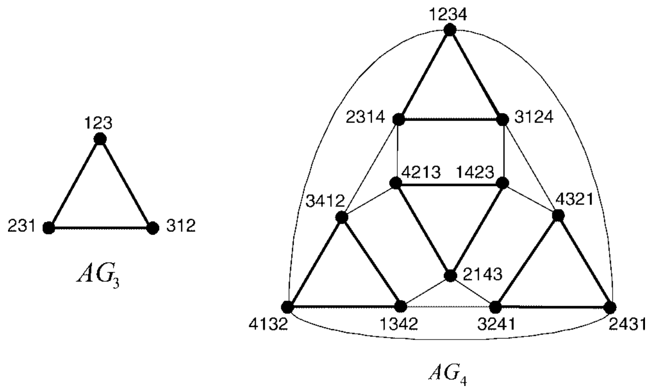


FIG. 1. AG_3 and AG_4.

to the importance of hamiltonicity for the communication between processors in an interconnection network, panpositionable hamiltonicity allows more flexible communication in a hamiltonian network. The panpositionable hamiltonian property inherits the hamiltonian property and advances it further. The concept is interesting and useful in the study of interconnection networks.

The alternating group graph [7] was proposed by Jwo et al. The interconnection scheme is based on the Cayley graph of the set of all even permutations. Cheng and Lipman investigated some properties of this family of graphs in Refs. [2–4]. Let $\langle n\rangle = \{1, 2, \ldots, n\}$. Then $p = p_1p_2 \ldots p_n$ is a permutation of elements in $\langle n\rangle$ if $p_i \neq p_j$ for all $i \neq j$. Suppose that p_i and p_j are two different symbols in p with $i < j$; then the pair (i,j) is an inversion if $p_i > p_j$. A permutation is even if the number of inversions is even. Let g_i^+ and g_i^- be two operations such that pg_i^+ and pg_i^- are permutations obtained from permutation p by rotating the symbols in position 1, 2, and i from left to right and from right to left, respectively. For example, we have $1234g_4^+ = 4132$ and $1234g_4^- = 2431$. The n-dimensional alternating group graph AG_n is the graph (V_n, E_n), where V_n is the set of all even permutations, and $E_n = \{(p, q) \mid p, q \in V_n$, and $q = pg_i^+$ or $q = pg_i^-$ for $i = 3, 4, \ldots, n\}$. Figure 1 illustrates AG_3 and AG_4. An alternating group graph AG_n is a regular graph of degree $2(n-2)$ with $\frac{n!}{2}$ vertices, and AG_n is vertex symmetric and edge symmetric [7]. The diameter of AG_n is $\lfloor \frac{3(n-2)}{2} \rfloor$. There are some studies concerning hamiltonicity of the alternating group graph. Jwo et al. [7] showed that the alternating group graph is hamiltonian and hamiltonian connected. Chang et al. [1] showed that AG_n is $(n-2)$-vertex fault tolerant hamiltonian and $(n-3)$-vertex fault tolerant hamiltonian connected if $n \geq 4$. A graph G is panconnected if there exists a path of length l joining any two different vertices x and y with $d(x, y) \leq l \leq |V(G)| - 1$. A graph G is pancyclic if it contains a cycle of length l for each l satisfying $3 \leq l \leq |V(G)|$. Chang et al. also proved that AG_n is panconnected and pancyclic for all $n \geq 3$ [1].

In this article, we study the panpositionable hamiltonicity of the alternating group graph AG_n. We show that the alternating group graph AG_n is panpositionable hamiltonian for all $n \geq 3$. In the following section, we discuss some basic properties of the alternating group graph. In Section 3, we prove our main theorem. In the final section, we present our conclusion and explain some relationship between the panpositionable hamiltonian property and the panconnected property.

2. SOME PROPERTIES OF THE ALTERNATING GROUP GRAPHS

For each $n \geq 3$, let $V_n[i] = \{p \mid p = p_1p_2 \ldots p_n$ and $p_n = i\}$. It is the set of all vertices with the rightmost position being i. Let $AG_n[i]$ denote the subgraph of AG_n induced by $V_n[i]$. It is easy to see that each $AG_n[i]$ is isomorphic to AG_{n-1}. Thus, AG_n can be recursively constructed from n copies of AG_{n-1}. Each $AG_n[i]$ represents a subcomponent of AG_n. Let I be a subset of $\{1, 2, \ldots, n\}$. We use $AG_n(I)$ to denote the subgraph of AG_n induced by $\bigcup_{i\in I} V_n[i]$. We call $AG_n(I)$ an incomplete alternating group graph if $|I| < n$. We observe that each $AG_n[i]$ can be recursively decomposed into its smaller subcomponents. We use $E_n^{i,j}$ to denote the set of edges between $AG_n[i]$ and $AG_n[j]$. Let F be a faulty set, which may include faulty edges, faulty vertices, or both. The good edge set $GE_n^{i,j}(F)$ is the set of edges $(u, v) \in E_n^{i,j}$ such that $\{u, v, (u, v)\} \cap F = \emptyset$. We need some basic properties of the alternating group graphs. The following proposition follows directly from the definition of the alternating group graphs.

Proposition 1. *Let n be an integer with $n \geq 5$, and let i and j be two distinct elements of $\langle n\rangle$. Suppose that H is one subcomponent of $AG_n[j]$ with the $(n-1)$th position being h and the nth position being j for some $h \in \langle n\rangle - \{i,j\}$. Then $|E_n^{i,j}| = (n-2)!$, and the number of edges between $AG_n[i]$ and H is $(n-3)!$. Moreover, if (u, v) and (u', v') are distinct edges in $E_n^{i,j}$, then $\{u, v\} \bigcap \{u', v'\} = \emptyset$, and $(u, u') \in E(AG_n[i])$ if and only if $(v, v') \in E(AG_n[j])$.*

Let $i \in \langle n\rangle$, and let u be a vertex in $AG_n[i]$. We say that v is a neighbor of u if v is adjacent to u. We use $N^I(u)$ to denote the set of all neighbors of u, which are in $AG_n(I)$. Particularly, we use $N^*(u)$ as an abbreviation of $N^{\langle n\rangle - \{i\}}(u)$. We call vertices in $N^*(u)$ the outer neighbors of u. Obviously, $|N^i(u)| = 2(n-3)$ and $|N^*(u)| = 2$. We say that vertex u is adjacent to subcomponent $AG_n[j]$ if u has an outer neighbor in $AG_n[j]$. Then, we define the adjacent subcomponent $AS(u)$ of u as $\{j \mid u$ is adjacent to $AG_n[j]\}$. We have the following proposition:

Proposition 2. *Suppose that $n \geq 4$ and $i \in \langle n\rangle$. Let u and v be two distinct vertices in $AG_n[i]$.*

(a) If $d(u, v) = 1$, then $|AS(u) \cap AS(v)| = 1$ and $AS(u) \neq AS(v)$.
(b) If $d(u, v) = 2$, then $AS(u) \neq AS(v)$.

Proof. Let $u = u_1u_2 \ldots u_n$, $v = v_1v_2 \ldots v_n$, and $u_n = v_n = i$. If $d(u, v) = 1$, we have $v = ug_k^c$ for some $3 \leq k \leq n-1$ and $c \in \{+, -\}$. Without loss of generality, we may assume that $c = +$. This means that $u_1 = v_2$, $u_2 = v_k$, and $u_k = v_1$ if $v = ug_k^+$. Let ug_n^+ and ug_n^- be the two outer neighbors of u. Let vg_n^+ and vg_n^- be the two outer neighbors of v. Then, $AS(u) = \{u_1, u_2\}$ and $AS(v) = \{v_1, v_2\}$. Thus $AS(u) \cap AS(v) = \{u_1\} = \{v_2\}$ and $|AS(u) \cap AS(v)| = 1$. It is obvious that $AS(u) \neq AS(v)$.

If $d(u, v) = 2$, there exists a vertex $w \in V(AG_n[i])$ such that $d(u, w) = d(w, v) = 1$, because $u, v \in AG_n[i]$. Let $w = w_1w_2 \dots w_n$. If $u = wg_k^+$ and $v = wg_k^-$ for $3 \le k \le n-1$, then $AS(u) = \{w_1, w_k\}$ and $AS(v) = \{w_2, w_k\}$. If $u = wg_x^c$ and $v = wg_y^c$ for some $x \neq y$, $3 \le x, y \le n-1$, and $c \in \{+, -\}$, then $w_x \in AS(u)$ and $w_x \notin AS(v)$. Thus $AS(u) \neq AS(v)$. The statement follows. ∎

Chang et al. studied the fault hamiltonicity and fault hamiltonian connectivity of the alternating group graphs in Ref. [1]. The result is listed as follows.

Theorem 1. [1] *Alternating group graphs AG_n, $n \ge 4$, are $n-2$ vertex-fault tolerant hamiltonian and $n-3$ vertex-fault tolerant hamiltonian connected.*

The above theorem states that with up to $n-2$ faulty vertices AG_n still has a hamiltonian cycle, and with up to $n-3$ faulty vertices AG_n is still hamiltonian connected. The following lemmas consider the hamiltonian connectivity of the subgraphs $AG_n(I)$ of the alternating group graphs AG_n.

Lemma 1. *Suppose that $I \subseteq \{1, 2, 3, 4\}$ with $|I| \ge 2$. If $x \in V(AG_4[i])$, $y \in V(AG_4[j])$, and (x, y) is an edge between $AG_4[i]$ and $AG_4[j]$ with $i \neq j \in I$, then there is a hamiltonian path of $AG_4(I)$ joining x and y.*

Proof. The alternating group graph AG_4 is known to be edge symmetric. Without loss of generality, we may consider that $x = 3241$ and $y = 1342$, which are two adjacent vertices of AG_4 in Figure 1. If $I = \{1, 2\}$, then $\langle 3241, 4321, 2431, 4132, 3412, 1342\rangle$ forms a hamiltonian path of $AG_4(I)$ from x to y. If $I = \{1, 2, 3\}$, then $\langle 3241, 2431, 4321, 1423, 2143, 4213, 3412, 4132, 1342\rangle$ forms a hamiltonian path of $AG_4(I)$ from x to y. If $I = \{1, 2, 3, 4\}$, then by Theorem 1, AG_4 is hamiltonian connected. Hence the lemma follows. ∎

Lemma 2. *Suppose that*

1. *$n \ge 5$,*
2. *$I \subseteq \langle n\rangle$ with $|I| \ge 2$,*
3. *$F \subseteq V(AG_n) \cup E(AG_n)$, and*
4. *$AG_n[l] - F$ is hamiltonian connected for each $l \in I$ and $|F| \le 2n - 7$.*

Then, for any $x \in V(AG_n[i])$ and $y \in V(AG_n[j])$ with $i \neq j \in I$, there is a hamiltonian path of $AG_n(I) - F$ joining x and y.

Proof. Consider that $|F| \le 2n - 7$. Suppose that $|GE_n^{i_1,i_2}(F)| < 3$ for some $i_1, i_2 \in \langle n\rangle$. Since $|E_n^{i_1,i_2}| = (n-2)! \ge 2(n-2)$, this implies that $|F| > 2n - 7$. We get a contradiction. Hence we have $|GE_n^{i_1,i_2}(F)| \ge 3$. We prove this lemma by induction on $|I|$. Suppose that $|I| = 2$, and $I = \{i, j\}$ for some i, j. Since $|GE_n^{i,j}(F)| \ge 3$, there exists an edge $(u, v) \in GE_n^{i,j}(F)$ such that $u \neq x \in V(AG_n[i])$ and $v \neq y \in V(AG_n[j])$. By the assumption that each $AG_n[l] - F$ is hamiltonian connected, there is a hamiltonian path P_1 of $AG_n[i] - F$ from x to u and a hamiltonian path P_2 of $AG_n[j] - F$ from v to y. Thus $\langle x, P_1, u, v, P_2, y\rangle$ forms a hamiltonian path of $AG_n(I) - F$ from x to y.

Assume that the result is true for all I' with $2 \le |I'| < |I|$. Thus there is an $i' \in I$ with $i' \neq i, j$. Since $|GE_n^{i',j}(F)| \ge 3$, there exists an edge $(u, v) \in GE_n^{i',j}(F)$ such that $u \in V(AG_n[i'])$ and $v \neq y \in V(AG_n[j])$. Then, there is a hamiltonian path P_1 of $AG_n(I - \{j\}) - F$ from x to u and a hamiltonian path P_2 of $AG_n[j] - F$ from v to y. Thus $\langle x, P_1, u, v, P_2, y\rangle$ forms a hamiltonian path of $AG_n(I) - F$ from x to y. Hence the lemma follows. ∎

Jwo et al. [7] presented a shortest path routing algorithm for the alternating group graph AG_n, and gave some characterizations of the minimum length path between two arbitrary vertices in AG_n. With this algorithm, we can find a minimum length path between any two distinct vertices of AG_n as stated in the following proposition.

Proposition 3. [7] *Let $i, j \in \langle n\rangle$ and $i \neq j$. Suppose that $u = u_1u_2 \cdots u_n$ and $v = v_1v_2 \cdots v_n$ are two vertices in AG_n.*

(a) If $u_n = v_n = i$, then u and v belong to the same subcomponent $AG_n[i]$. A shortest path from u to v in AG_n is completely contained in $AG_n[i]$. That is, $d(u, v) = d_{AG_n[i]}(u, v)$.

(b) If $u_n = i$, $v_n = j$, and $v_x = i$ for some $x \in \{1, 2\}$, there exists a vertex $s \in V(AG_n[i])$ adjacent to v such that $\langle u, P, s, v\rangle$ is the minimum length path between u and v in AG_n, where P is a path completely contained in $AG_n[i]$. That is, $d(u, v) = d_{AG_n[i]}(u, s) + 1 = d(u, s) + 1$.

(c) If $u_n = i$, $v_n = j$, and $v_x = i$ for some $x \in \{3, 4, \dots, n-1\}$, there exist vertices $s \in V(AG_n[i])$ and $t \in V(AG_n[j])$, where t is adjacent to v, and $(s, t) \in E_n^{i,j}$, such that $\langle u, P, s, t, v\rangle$ is the minimum length path between u and v in AG_n, where P is a path completely contained in $AG_n[i]$. That is, $d(u, v) = d_{AG_n[i]}(u, s) + 2 = d(u, s) + 2$.

Example. *Suppose that u and v are two vertices in AG_5. If $u = 12345$ and $v = 21435$, then $u \in V(AG_5[5])$ and $v \in V(AG_5[5])$. A shortest path from u to v is* $12345 \xrightarrow{g_3^+} 31245 \xrightarrow{g_4^-} 14235 \xrightarrow{g_3^+} 21435$, *and case (a) holds. If $u = 12345$ and $v = 15432$, then $u \in V(AG_5[5])$ and $v \in V(AG_5[2])$. A shortest path from u to v is* $12345 \xrightarrow{g_3^+} 31245 \xrightarrow{g_4^-} 14235 \xrightarrow{g_3^+} 21435 \xrightarrow{g_5^-} 15432$, *and case (b) holds. If $u = 12345$ and $v = 34512$, then $u \in V(AG_5[5])$ and $v \in V(AG_5[2])$. A shortest path from u to v is* $12345 \xrightarrow{g_4^-} 24315 \xrightarrow{g_5^-} 45312 \xrightarrow{g_3^+} 34512$, *and case (c) holds.*

3. PANPOSITIONABLE HAMILTONICITY OF THE ALTERNATING GROUP GRAPHS

In this section, we prove that the alternating group graph AG_n is panpositionable hamiltonian for all $n \geq 3$. The basic idea is to study AG_3 and AG_4 first, and then to prove the result for $n \geq 5$ by induction on n.

Lemma 3. *Alternating group graphs AG_n are panpositionable hamiltonian for $n = 3, 4$.*

Proof. For $n = 3$, since AG_3 is isomorphic to the complete graph K_3, the result holds for $n = 3$ trivially. Now, we consider THE case $n = 4$. Let u and v be any two vertices of AG_4 in Figure 1.

The alternating group graph is known to be vertex symmetric and edge symmetric, and to have diameter $\lfloor \frac{3(n-2)}{2} \rfloor$ [7]. The diameter of AG_4 is 3. Without loss of generality, to prove this lemma it is enough to consider $u = 1234$ and $v = 3124$ for $d(u, v) = 1$; $u = 1234$ and $v = 4321$ for $d(u, v) = 2$; $u = 1234$ and $v = 2143$ for $d(u, v) = 3$. For each $l \in \{d(u, v), d(u, v) + 1, \ldots, \frac{|V(AG_4)|}{2}\}$, we construct a hamiltonian cycle HC of AG_4 such that $d_{HC}(u, v) = l$. These hamiltonian cycles HC are listed below.

$d(u,v)$	$d_{HC}(u,v)$	The cycle HC
1	1	⟨1234, 3124, 4321, 2431, 3241, 2143, 1423, 4213, 2314, 3412, 1342, 4132, 1234⟩
1	2	⟨1234, 2314, 3124, 4321, 1423, 4213, 3412, 4132, 1342, 2143, 3241, 2431, 1234⟩
1	3	⟨1234, 2431, 4321, 3124, 1423, 4213, 2143, 3241, 1342, 4132, 3412, 2314, 1234⟩
1	4	⟨1234, 2431, 4321, 1423, 3124, 2314, 3412, 4213, 2143, 3241, 1342, 4132, 1234⟩
1	5	⟨1234, 2314, 4213, 2143, 1423, 3124, 4321, 2431, 3241, 1342, 3412, 4132, 1234⟩
1	6	⟨1234, 2431, 4132, 1342, 3241, 4321, 3124, 1423, 2143, 4213, 3412, 2314, 1234⟩
2	2	⟨1234, 3124, 4321, 3241, 2431, 4132, 3412, 1342, 2143, 1423, 4213, 2314, 1234⟩
2	3	⟨1234, 3124, 1423, 4321, 2431, 3241, 2143, 1342, 4132, 3412, 4213, 2314, 1234⟩
2	4	⟨1234, 2314, 3124, 1423, 4321, 2431, 3241, 1342, 2143, 4213, 3412, 4132, 1234⟩
2	5	⟨1234, 3124, 2314, 4213, 1423, 4321, 2431, 3241, 2143, 1342, 3412, 4132, 1234⟩
2	6	⟨1234, 2314, 3412, 4213, 1423, 3124, 4321, 2431, 3241, 2143, 1342, 4132, 1234⟩
3	3	⟨1234, 3124, 1423, 2143, 3241, 4321, 2431, 4132, 1342, 3412, 4213, 2314, 1234⟩
3	4	⟨1234, 3124, 4321, 3241, 2143, 1423, 4213, 2314, 3412, 1342, 4132, 2431, 1234⟩
3	5	⟨1234, 3124, 4321, 2431, 3241, 2143, 1423, 4213, 2314, 3412, 1342, 4132, 1234⟩
3	6	⟨1234, 2431, 3241, 4321, 3124, 1423, 2143, 1342, 4132, 3412, 4213, 2314, 1234⟩

Thus the lemma holds. ∎

In the following lemma, we show that there exist two vertex disjoint paths spanning all the vertices in an incomplete alternating group graph. We need the lemma later in our main theorem. One may skip the proof temporarily, and come back to it later.

Lemma 4. *Suppose that $n \geq 5$, $I \subseteq \langle n \rangle$ with $|I| \geq 2$, $x_1 \in V(AG_n[i_1])$ and $x_2 \in V(AG_n[i_2])$ with $i_1 \neq i_2 \in I$. Then, for any pair of distinct vertices (y_1, y_2) in $V(AG_n(I))$, there exist two disjoint paths, one joining x_1 and y_i for some $i \in \{1, 2\}$, and the other joining x_2 and y_j with $i \neq j$, such that these two paths span all the vertices in $AG_n(I)$.*

Proof. Let $i_1, i_2, \ldots, i_{|I|}$ be $|I|$ distinct indices of $\langle n \rangle$. We prove this lemma by finding two disjoint paths P_1 and P_2 in $AG_n(I)$ such that P_1 joins x_1 and y_i, and P_2 joins x_2 and y_j with $i \neq j$. Moreover, P_1 and P_2 span all the vertices in $AG_n(I)$. According to the location of y_1 and y_2, we have the following cases:

CASE 1. Suppose that y_1 and y_2 are located in different subcomponents, and x_1 and x_2 are not both adjacent to y_i for every $i \in \{1, 2\}$.

SUBCASE 1.1. Suppose that x_1, x_2, y_i, and y_j are located in four different subcomponents. We assume that $y_i \in V(AG_n[i_3])$ and $y_j \in V(AG_n[i_4])$ with $|I| \geq 4$. See Figure 2a for an illustration. By Lemma 2, we can find a hamiltonian path P_1 from x_1 to y_i in $AG_n(\{i_1, i_3\})$. Similarly, we can find a hamiltonian path P_2 from x_2 to y_j in $AG_n(I - \{i_1, i_3\})$. Therefore, P_1 and P_2 are two disjoint paths spanning all the vertices in $AG_n(I)$.

SUBCASE 1.2. Suppose that one of y_1, y_2 and one of x_1, x_2 are located in the same subcomponent. Without loss of generality, we may assume that x_1 and y_i are located in the same subcomponent, x_2 and y_j are located in different subcomponents, $y_i \in V(AG_n[i_1])$, and $y_j \in V(AG_n[i_3])$ with $|I| \geq 3$. See Figure 2b for an illustration. By Theorem 1, since $AG_n[i_1]$ is hamiltonian connected, we can find a hamiltonian path P_1 from x_1 to y_i in $AG_n[i_1]$. By Lemma 2, we can

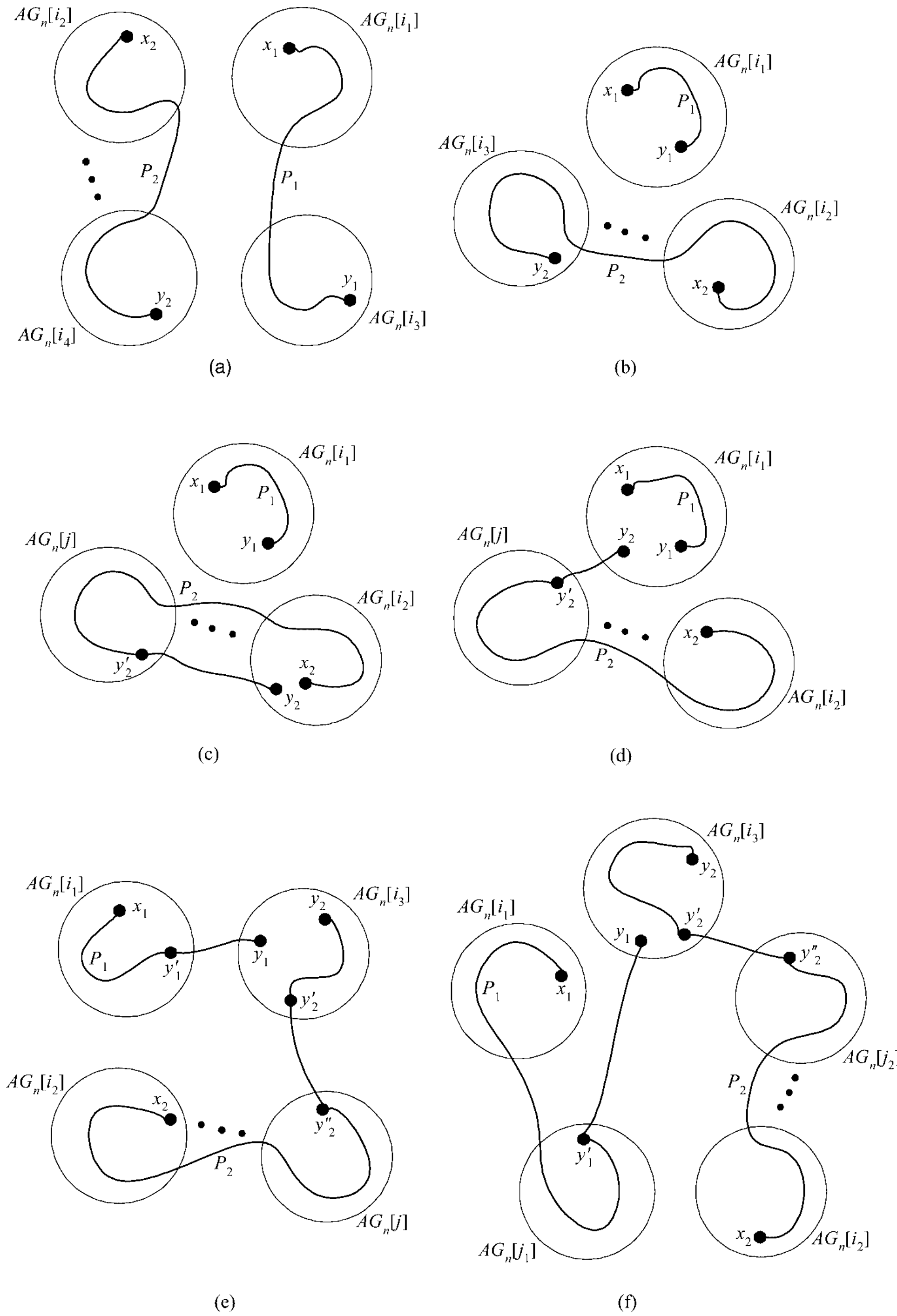


FIG. 2. Illustrations for Lemma 4.

find a hamiltonian path P_2 from x_2 to y_j in $AG_n(I - \{i_1\})$. Therefore, P_1 and P_2 are two disjoint paths spanning all the vertices in $AG_n(I)$.

SUBCASE 1.3. Suppose that x_1 and y_i are located in the same subcomponent for some $i \in \{1, 2\}$, and x_2 and y_j are located in the same subcomponent with $i \neq j$. We assume that $y_i \in V(AG_n[i_1])$ and $y_j \in V(AG_n[i_2])$ with $|I| \geq 2$. See Figure 2c for an illustration. Without loss of generality, we may assume that $i = 1$ and $j = 2$. By Theorem 1, since $AG_n[i_1]$ is hamiltonian connected, we can find a hamiltonian path P_1 from y_1 to x_1 in $AG_n[i_1]$. If $|I| \geq 3$, since $|N^*(y_2)| = 2$, we can find an edge $(y_2, y_2') \in E_n^{i_2,j}$ such that $y_2' \in V(AG_n[j])$ for some $j \in I - \{i_1, i_2\}$. By Lemma 2, we can find a hamiltonian path P_2' from y_2' to x_2 in $AG_n(I - \{i_1\}) - \{y_2\}$. Let $P_2 = \langle y_2, y_2', P_2', x_2 \rangle$. If $|I| = 2$, by Theorem 1, there is a hamiltonian path P_2' from

y_2 to x_2 in $AG_n[i_2]$. Let $P_2 = \langle y_2, P'_2, x_2\rangle$. Therefore, P_1 and P_2 are two disjoint paths spanning all the vertices in $AG_n(I)$.

CASE 2. Suppose that y_i and y_j are located in the same subcomponent, and x_1 and x_2 are not both adjacent to y_i for every $i \in \{1, 2\}$.

SUBCASE 2.1. Suppose that $y_1, y_2 \in V(AG_n[i_1])$ or $y_1, y_2 \in V(AG_n[i_2])$ with $|I| \geq 2$. See Figure 2d for an illustration. Without loss of generality, we consider the former case and assume that $i = 1$ and $j = 2$. By Theorem 1, $AG_n[i_1] - \{y_2\}$ is hamiltonian connected, hence we can find a hamiltonian path P_1 from y_1 to x_1 in $AG_n[i_1] - \{y_2\}$. If $|I| \geq 3$, since $|N^*(y_2)| = 2$, we can find an edge $(y_2, y'_2) \in E_n^{i_1,j}$ such that $y'_2 \in V(AG_n[j])$ for some $j \in I - \{i_1, i_2\}$. By Lemma 2, we can find a hamiltonian path P'_2 from y'_2 to x_2 in $AG_n(I - \{i_1\})$. If $|I| = 2$, there exists an edge $(y_2, y'_2) \in E_n^{i_1,i_2}$ such that $y'_2 \in V(AG_n[i_2])$. By Theorem 1, there is a hamiltonian path P'_2 from y'_2 to x_2 in $AG_n[i_2]$. Let $P_2 = \langle y_2, y'_2, P'_2, x_2\rangle$. Therefore, P_1 and P_2 are two disjoint paths spanning all the vertices in $AG_n(I)$.

SUBCASE 2.2. Suppose that $y_1, y_2 \in V(AG_n[i_3])$. Without loss of generality, we consider two subcases:

SUBCASE 2.2.1. Suppose there exists some $x_i \in AS(y_1)$ for $i \in \{1, 2\}$ with $|I| \geq 3$. See Figure 2e for an illustration. Without loss of generality, we may assume that $i = 1$. Since $x_1 \in AS(y_1)$, we can find an edge $(y_1, y'_1) \in E_n^{i_1,i_3}$ such that $y'_1 \in V(AG_n[i_1])$ and $x_1 \neq y'_1$. By Theorem 1, we can find a hamiltonian path P'_1 from y'_1 to x_1 in $AG_n[i_1]$. Let $P_1 = \langle y_1, y'_1, P'_1, x_1\rangle$. Let $y'_2 \neq y_1 \in V(AG_n[i_3])$. By Theorem 1, since $AG_n[i_3] - \{y_1\}$ is hamiltonian connected, we can find a hamiltonian path P''_2 from y_2 to y'_2 in $AG_n[i_3] - \{y_1\}$. If $|I| \geq 4$, since $|N^*(y'_2)| = 2$, we can find an edge $(y'_2, y''_2) \in E_n^{i_3,j}$ such that $y''_2 \in V(AG_n[j])$ for some $j \in I - \{i_1, i_2, i_3\}$. By Lemma 2, we can find a hamiltonian path P'_2 from y''_2 to x_2 in $AG_n(I - \{i_1, i_3\})$. If $|I| = 3$, there exists an edge $(y'_2, y''_2) \in E_n^{i_3,i_2}$ such that $y''_2 \in V(AG_n[i_2])$. By Theorem 1, there is a hamiltonian path P'_2 from y''_2 to x_2 in $AG_n[i_2]$. Let $P_2 = \langle y_2, P''_2, y'_2, y''_2, P'_2, x_2\rangle$. Therefore, P_1 and P_2 are two disjoint paths spanning all the vertices in $AG_n(I)$.

SUBCASE 2.2.2. Suppose that $x_1, x_2 \notin AS(y_1) \cup AS(y_2)$ with $|I| \geq 4$. See Figure 2f for an illustration. Since $|N^*(y_1)| = 2$, we can find an edge $(y_1, y'_1) \in E_n^{i_1,j_1}$ such that $y'_1 \in V(AG_n[j_1])$ for some $j_1 \in I - \{i_1, i_2, i_3\}$. By Lemma 2, we can find a hamiltonian path P'_1 from y'_1 to x_1 in $AG_n(\{i_1, j_1\})$. Let $P_1 = \langle y_1, y'_1, P'_1, x_1\rangle$. Let $y'_2 \in V(AG_n[i_3])$ and $y'_2 \in N^{i_3}(y_1)$. By Proposition 2, we have $AS(y_1) \neq AS(y'_2)$. By Theorem 1, since $AG_n[i_3] - \{y_1\}$ is hamiltonian connected, we can find a hamiltonian path P''_2 from y_2 to y'_2 in $AG_n[i_3] - \{y_1\}$. If $|I| \geq 5$, since $|N^*(y'_2)| = 2$, we can find an edge $(y'_2, y''_2) \in E_n^{i_3,j_2}$ such that $y''_2 \in V(AG_n[j_2])$ for some $j_2 \in I - \{i_1, i_2, i_3, j_1\}$. By Lemma 2, we can find a hamiltonian path P'_2 from y''_2 to x_2 in $AG_n(I - \{i_1, i_3, j_1\})$. If $|I| = 4$, since $|N^*(y'_2)| = 2$, we can find an edge $(y'_2, y''_2) \in E_n^{i_3,i_2}$ such that $y''_2 \in V(AG_n[i_2])$. Since $AG_n[i_2]$ is hamiltonian connected, there is a hamiltonian path P'_2 from y''_2 to x_2 in $AG_n[i_2]$. Let $P_2 = \langle y_2, P''_2, y'_2, y''_2, P'_2, x_2\rangle$. Therefore, P_1 and P_2 are two disjoint paths spanning all the vertices in $AG_n(I)$.

CASE 3. Suppose that x_1 and x_2 are adjacent to y_i for some $i \in \{1, 2\}$. Without loss of generality, we assume that $i = 1$. If $y_2 \in V(AG_n[i_1])$, let $P_1 = \langle x_1, y_1\rangle$. Suppose that $F = \{x_1, y_1\}$. By Lemma 2, we can find a hamiltonian path P_2 from x_2 to y_2 in $AG_n(I) - F$. If $y_2 \notin V(AG_n[i_1])$, let $P_1 = \langle x_2, y_1\rangle$. Suppose that $F = \{x_2, y_1\}$. By Lemma 2, we can find a hamiltonian path P_2 from x_1 to y_2 in $AG_n(I) - F$. Therefore, P_1 and P_2 are two disjoint paths spanning all the vertices in $AG_n(I)$.

Thus the lemma follows. ■

We now prove our main result.

Theorem 2. *Alternating group graphs AG_n are panpositionable hamiltonian if $n \geq 3$.*

Proof. We prove this theorem by induction on n. By Lemma 3, AG_3 and AG_4 are panpositionable hamiltonian. Suppose that the result holds for AG_{n-1} for some $n \geq 5$. We observe that AG_n can be recursively constructed from n copies of AG_{n-1}, and each AG_{n-1} is panpositionable hamiltonian by the inductive hypothesis, for all $n \geq 5$. Let s and t be two distinct vertices of AG_n. Then for each $l \in \{d(s,t), d(s,t) + 1, d(s,t) + 2, \ldots, \frac{|V(AG_n)|}{2}\}$, we shall find a hamiltonian cycle of AG_n such that the distance between s and t on the cycle is l. The idea of the proof is to expand the path between s and t to various lengths by inserting one or more subcomponents of AG_{n-1}. We achieve this purpose by our expanding algorithm described below, and we can construct a path connecting s and t with the length of the path being l for any integer l with $d(s,t) \leq l \leq \frac{|V(AG_n)|}{2}$.

CASE 1. Suppose that s and t belong to the same subcomponent $AG_n[i]$. There will be two subcases in Case 1; Figure 3a and b illustrate Subcase 1.1 and Subcase 1.2, respectively. See Figure 3a first. Suppose that $s, t \in V(AG_n[i])$ for some $i \in \langle n\rangle$. By Proposition 3, $d(s,t) = d_{AG_n[i]}(s,t)$. Since $AG_n[i]$ is isomorphic to AG_{n-1}, by the inductive hypothesis, for each $l_0 \in \{d(s,t), d(s,t)+1, d(s,t)+2, \ldots, |V(AG_n[i])| - d(s,t)\}$, we can construct a hamiltonian cycle HC_i of $AG_n[i]$ such that the distance between s and t on the cycle is l_0. Let u and v be the two neighbors of t on HC_i. Let $HC_i = \langle s, LP, u, t, v, RP, s\rangle$, and let $P_0 = \langle s, LP, u, t\rangle$. Without loss of generality, let $L(P_0) = l_0$. By Proposition 2, $d(t,u) = 1$, so we have $|AS(t) \cap AS(u)| = 1$. This means that we can find a subcomponent $AG_n[j_1]$ for which $j_1 \in \langle n\rangle - \{i\}$, such that there exist two disjoint edges (u, p_1) and (t, q_1) in E_n^{i,j_1}. By Proposition 1, $(p_1, q_1) \in E(AG_n[j_1])$. Since $|N^*(t)| = 2$, we can find a subcomponent $AG_n[h_t]$ different from $AG_n[i]$ and $AG_n[j_1]$, and a vertex $t' \in V(AG_n[h_t])$ such that $(t, t') \in E_n^{i,h_t}$ for some $h_t \in \langle n\rangle - \{i, j_1\}$. By Proposition 2,

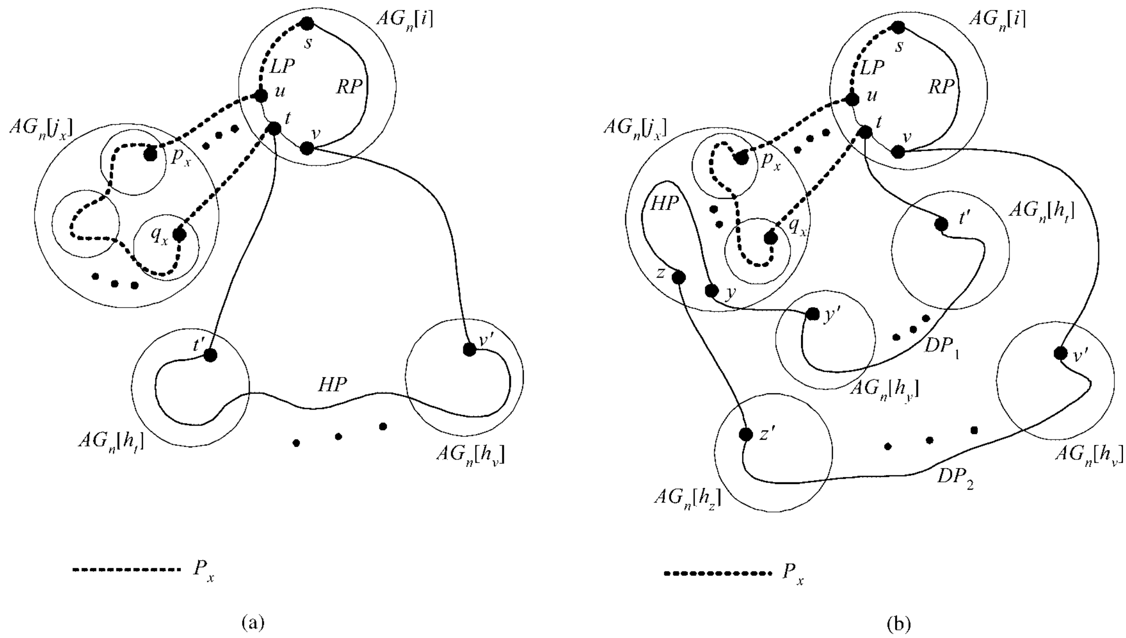


FIG. 3. Theorem 2, case 1.

$d(t,v) \le 2$ and hence $AS(t) \supseteq \{j_1, h_t\}$, $AS(t) \ne AS(v)$, and $|N^*(v)| = 2$. So we can find another subcomponent $AG_n[h_v]$ and a vertex $v' \in V(AG_n[h_v])$ such that $(v, v') \in E_n^{i,h_v}$ for some $h_v \in \langle n \rangle - \{i, j_1, h_t\}$. By Lemma 2, there exists a hamiltonian path HP of $AG_n(\langle n \rangle - \{i\})$ joining t' and v'. Thus $\langle s, P_0, t, t', HP, v', v, RP, s \rangle$ forms a hamiltonian cycle, and for each $l_0 \in \{d(s,t), d(s,t)+1, d(s,t)+2, \ldots, |V(AG_n[i])| - d(s,t)\}$, the distance between s and t on the cycle is l_0.

Now we present an algorithm called st-expansion to expand the path P_0 between s and t to various lengths. We describe the details as follows.

We can insert one subcomponent of $AG_n[j_1]$ into P_0 as follows. See Figure 4a for an illustration. Because p_1 and q_1 are adjacent, we may regard them as in the same subcomponent of $AG_n[j_1]$, say C. The subcomponent C is isomorphic to AG_{n-2}. By Theorem 1, there is a hamiltonian path HP_1 of C joining p_1 and q_1 with $L(HP_1) = |V(AG_{n-2})| - 1$. We can insert more than one subcomponent of $AG_n[j_1]$ into P_0 as follows. See Figure 4b for an illustration. We regard p_1 and q_1 as in different subcomponents of $AG_n[j_1]$. By Lemma 1 if $n = 5$ and by Lemma 2 if $n > 5$, there is a hamiltonian path HP_1 joining p_1 and q_1 with $L(HP_1) = m|V(AG_{n-2})| - 1$, where m is the number of subcomponents of $AG_n[j_1]$ we wanted to insert. Thus we can construct a path HP_1 between p_1 and q_1 such that $L(HP_1) = I_1|V(AG_{n-2})| - 1$ for each integer I_1 with $1 \le I_1 \le n-1$. Let $P_1 = \langle s, LP, u, p_1, HP_1, q_1, t \rangle$. Thus we have $L(P_1) = l_0 + I_1|V(AG_{n-2})| = l_0 + \frac{I_1(n-2)!}{2}$. Since $d(s,t) \le l_0 \le |V(AG_n[i])| - d(s,t)$, we have $\frac{I_1(n-2)!}{2} + d(s,t) \le L(P_1) \le \frac{I_1(n-2)!}{2} + \frac{(n-1)!}{2} - d(s,t)$. For each $1 \le I_1 \le n-1$, $\frac{(I_1-1)(n-2)!}{2} + \frac{(n-1)!}{2} - d(s,t) \ge \frac{I_1(n-2)!}{2} + d(s,t)$ if $n \ge 5$. Therefore, for each $l_1 \in \{d(s,t), d(s,t)+1, d(s,t)+2, \ldots, (n-1)! - d(s,t)\}$, we can construct a path P_1 from s to t such that the distance between s and t on the path is l_1.

Similar as above, we can expand the path between s and t more. For each $2 \le x \le \lfloor \frac{n}{2} \rfloor$, let u_{x-1} and t_{x-1} be two adjacent vertices on HP_{x-1}, where HP_{x-1} is a hamiltonian path of $AG_n[j_{x-1}]$ joining p_{x-1} and q_{x-1}. By Propositions 1 and 2, there exist two distinct edges (u_{x-1}, p_x) and (t_{x-1}, q_x) in $E_n^{j_{x-1}, j_x}$ for some $j_x \in \langle n \rangle - \{i, h_t, h_v, j_1, \ldots, j_{x-1}\}$ such that $(p_x, q_x) \in E(AG_n[j_x])$. See Figure 4c for an illustration. We can insert one subcomponent of $AG_n[j_x]$ into P_0 as follows. Because p_x and q_x are adjacent, we may regard them as in the same subcomponent of $AG_n[j_x]$, say C. The subcomponent C is isomorphic to AG_{n-2}. By Theorem 1, there is a hamiltonian path HP_x of C joining p_x and q_x with $L(HP_x) = |V(AG_{n-2})| - 1$. We can insert more than one subcomponent of $AG_n[j_x]$ into P_0 as follows. We regard p_x and q_x as in different subcomponents of $AG_n[j_x]$. By Lemma 1 if $n = 5$ and by Lemma 2 if $n > 5$, there is a hamiltonian path HP_x joining p_x and q_x with $L(HP_x) = m|V(AG_{n-2})| - 1$, where m is the number of subcomponents of $AG_n[j_x]$ we wanted to insert. Thus, we can construct a path HP_x between p_x and q_x such that $L(HP_x) = I_x|V(AG_{n-2})| - 1$ for each integer I_x with $1 \le I_x \le n-1$. Let $P_x = \langle s, LP, u, p_1, \ldots, p_x, HP_x, q_x, \ldots, q_1, t \rangle$. Thus we have $L(P_x) = l_0 + (x-1)|V(AG_{n-1})| + I_x|V(AG_{n-2})| = l_0 + \frac{(x-1)(n-1)!}{2} + \frac{I_x(n-2)!}{2}$. Since $d(s,t) \le l_0 \le |V(AG_n[i])| - d(s,t)$, we have $\frac{(x-1)(n-1)!}{2} + \frac{I_x(n-2)!}{2} + d(s,t) \le L(P_x) \le \frac{I_x(n-2)!}{2} + \frac{x(n-1)!}{2} - d(s,t)$. For each $1 \le I_x \le n-1$, $\frac{(I_x-1)(n-2)!}{2} + \frac{x(n-1)!}{2} - d(s,t) \ge \frac{I_x(n-2)!}{2} + \frac{(x-1)(n-1)!}{2} + d(s,t)$ if $n \ge 5$. Therefore, for each $l_x \in \{d(s,t), d(s,t)+1, d(s,t)+2, \ldots, \frac{(x+1)(n-1)!}{2} - d(s,t)\}$, we can construct a path P_x from s to t such that the distance between s and t on the path is l_x by using st-expansion. Notice that the maximal value of l_x is $\frac{(\lfloor \frac{n}{2} \rfloor + 1)(n-1)!}{2} - d(s,t)$, which is greater than $\frac{n!}{4}$, and $\frac{|V(AG_n)|}{2} = \frac{n!}{4}$. Hence for any integer l with $d(s,t) \le l \le \frac{|V(AG_n)|}{2}$, we

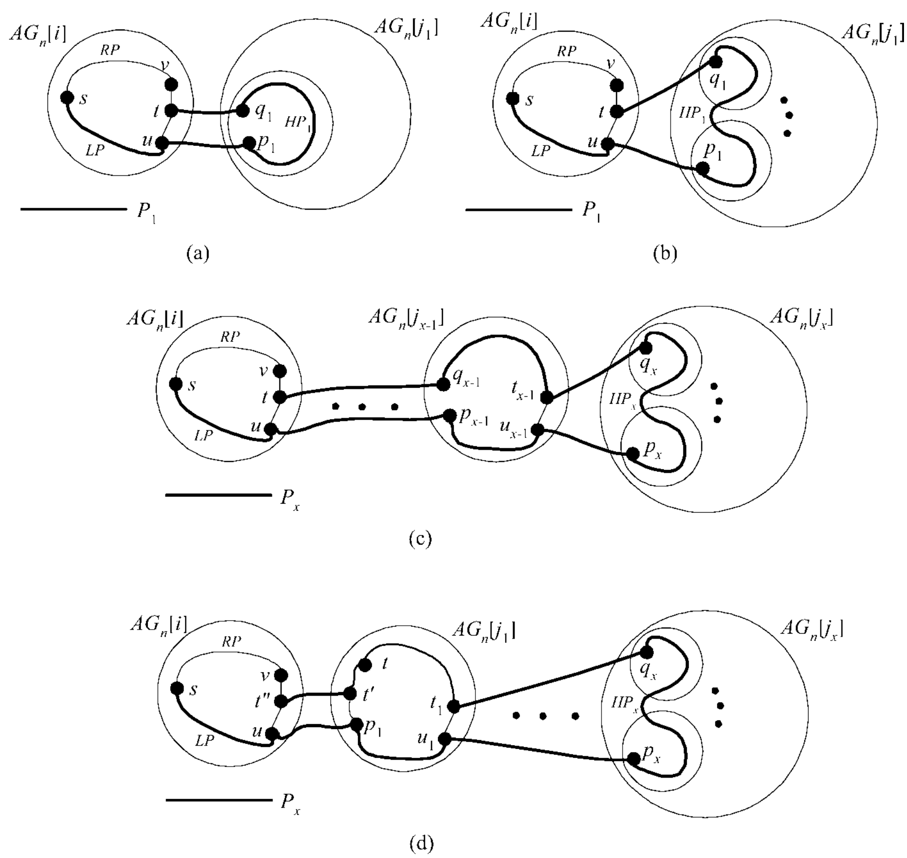


FIG. 4. An illustration of *st*-expansion.

can construct a path joining s and t with the length of the path being l. We will use *st*-expansion for the remaining cases of the proof.

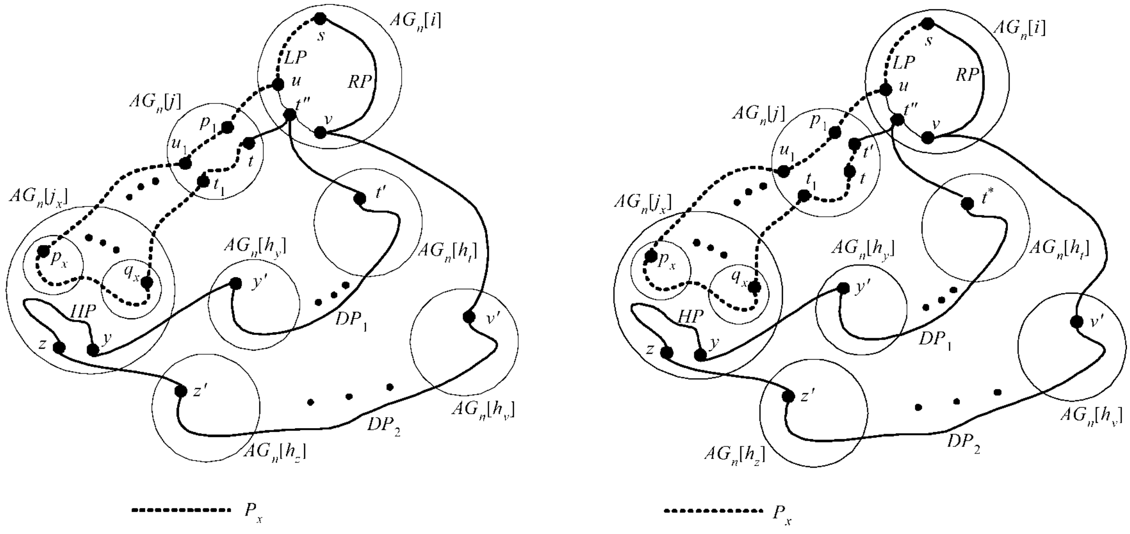
To construct a hamiltonian cycle, we consider the following two subcases:

SUBCASE 1.1. All the vertices of $AG_n(\{j_1,\ldots,j_x\})$ are on the path P_x for some $1 \le x \le \lfloor \frac{n}{2} \rfloor$. See Figure 3a for an illustration. By Lemma 2, there is a hamiltonian path HP of $AG_n(\langle n\rangle - \{i,j_1,\ldots,j_x\})$ joining t' and v' in which $t' \in V(AG_n[h_t])$ and $v' \in V(AG_n[h_v])$. Thus $\langle s, P_x, t, t', HP, v', v, RP, s\rangle$ forms a hamiltonian cycle, and for each $l \in \{d(s,t), d(s,t)+1, d(s,t)+2,\ldots,\frac{|V(AG_n)|}{2}\}$, the distance between s and t on the cycle is l.

SUBCASE 1.2. Not all the vertices of $AG_n(\{j_1,\ldots,j_x\})$ are on the path P_x for some $1 \le x \le \lfloor \frac{n}{2} \rfloor$. See Figure 3b for an illustration. Then we can find two adjacent vertices y and z in $AG_n[j_x]$, which are not on the path P_x. Let $F \subseteq V(P_x)$. By Proposition 1 and Proposition 2, there exist two distinct edges $(y,y') \in E_n^{j_x,h_y}$ and $(z,z') \in E_n^{j_x,h_z}$ such that $y' \ne t' \in V(AG_n[h_y])$ and $z' \ne v' \in V(AG_n[h_z])$, respectively. If $AG_n[j_x] - F$ is isomorphic to AG_{n-2}, by Theorem 1, there is a hamiltonian path HP from y to z in $AG_n[j_x] - F$. If $AG_n[j_x] - F$ contains more than one subcomponent of $AG_n[j_x]$, by Lemma 1 if $n = 5$, and by Lemma 2 if $n > 5$, there is a hamiltonian path HP from y to z in $AG_n[j_x] - F$. By Lemma 4, there exist two disjoint paths DP_1 and DP_2, such that DP_1 joins t' and y', and DP_2 joins v' and z'. Moreover, the two paths span all of the vertices in $AG_n(\langle n\rangle - \{i,j_1,\ldots,j_x\})$. Thus $\langle s, P_x, t, t', DP_1, y', y, HP, z, z', DP_2, v', v, RP, s\rangle$ forms a hamiltonian cycle, and for each $l \in \{d(s,t), d(s,t)+1, d(s,t)+2,\ldots,\frac{|V(AG_n)|}{2}\}$, the distance between s and t on the cycle is l.

Now, we consider the case in which s and t belong to different subcomponents of AG_n. Suppose that $s \in V(AG_n[i])$ and $t \in V(AG_n[j])$ for any $i \ne j \in \langle n\rangle$. By Proposition 3, there exists a minimum length path connecting s and t with the form $\langle s, MP, t'', t\rangle$ or $\langle s, MP, t'', t', t\rangle$, where MP is a path in $AG_n[i]$, $t'' \in V(AG_n[i])$, and $t' \in V(AG_n[j])$. That is, $d(s,t) = d_{AG_n[i]}(s,t'')+1 = d(s,t'')+1$ or $d(s,t) = d_{AG_n[i]}(s,t'')+2 = d(s,t'')+2$. Thus we have the following cases:

CASE 2. Suppose that s and t belong to different subcomponents of AG_n, and the minimum length path connecting s and t has the form $\langle s, MP, t'', t\rangle$. Then $d(s,t) = d(s,t'')+1$. See Figure 5a for an illustration. Since $AG_n[i]$ is isomorphic to AG_{n-1}, by the inductive hypothesis, for each $l_0 \in \{d(s,t''), d(s,t'')+1, d(s,t'')+2,\ldots,|V(AG_n[i])| - d(s,t'')\}$, we can construct a hamiltonian cycle HC_i of $AG_n[i]$ such that the distance between s and t'' on the cycle is l_0. Let



(a) (b)

FIG. 5. Theorem 2, Case 2 and Case 3.

u and v be the two neighbors of t'' on HC_i. Let $HC_i = \langle s, LP, u, t'', v, RP, s\rangle$, and let $P_0 = \langle s, LP, u, t'', t\rangle$. Without loss of generality, let $L(P_0) = l_0 + 1$. By Proposition 2, $d(t'', u) = 1$, so we have $|AS(t'') \cap AS(u)| = 1$. This means that we can find a subcomponent $AG_n[j_1]$ in which $j_1 \in \langle n\rangle - \{i\}$. If $t \notin V(AG_n[j_1])$, by using st''-expansion, the proof is the same as Case 1 but we replace vertex t in Case 1 with vertex t'' in this case. So we consider the case in which $t \in V(AG_n[j_1])$, that is, $j_1 = j$. Let $q_1 = t$. There exist two disjoint edges (u, p_1) and (t'', q_1) in E_n^{i,j_1}. By Proposition 1, $(p_1, q_1) \in E(AG_n[j_1])$. By Proposition 2, $d(t'', v) \leq 2$ hence $AS(t'') = \{j_1\}$, and $AS(t'') \neq AS(v)$. Since $|N^*(t'')| = 2$, we can find a subcomponent $AG_n[h_t]$, and a vertex $t' \in V(AG_n[h_t])$ such that $(t'', t') \in E_n^{i,h_t}$ for some $h_t \in \langle n\rangle - \{i, j_1\}$. Since $|N^*(v)| = 2$ and $AS(t'') \neq AS(v)$, we can find a subcomponent $AG_n[h_v]$, and a vertex $v' \in V(AG_n[h_v])$ such that $(v, v') \in E_n^{i,h_v}$ for some $h_v \in \langle n\rangle - \{i, j_1, h_t\}$. By Lemma 2, there exists a hamiltonian path HP of $AG_n(\langle n\rangle - \{i\})$ joining t and v'. Thus $\langle s, P_0, t, HP, v', v, RP, s\rangle$ forms a hamiltonian cycle, and for each $l_0 \in \{d(s,t), d(s,t)+1, d(s,t)+2, \ldots, |V(AG_n[i])| - d(s,t) + 1\}$, the distance between s and t on the cycle is l_0.

By using st''-expansion, for any integer l'' with $d(s, t'') \leq l'' \leq \frac{|V(AG_n)|}{2}$, we can construct a path joining s and t'' with the length of the path being l''. Therefore, for any integer l with $d(s,t) \leq l \leq \frac{|V(AG_n)|}{2}$, we can construct a path joining s and t with the length of the path being l.

To construct a hamiltonian cycle, we consider the following two subcases:

SUBCASE 2.1. All the vertices of $AG_n(\{j_1, \ldots, j_x\})$ are on the path P_x for some $1 \leq x \leq \lfloor \frac{n}{2} \rfloor$. By Lemma 2, there is a hamiltonian path HP of $AG_n(\langle n\rangle - \{i, j_1, \ldots, j_x\})$ joining t' and v' where $t' \in V(AG_n[h_t])$ and $v' \in V(AG_n[h_v])$. Thus $\langle s, P_x, t, t'', t', HP, v', v, RP, s\rangle$ forms a hamiltonian cycle, and for each $l \in \{d(s,t), d(s,t)+1, d(s,t)+2, \ldots, \frac{|V(AG_n)|}{2}\}$, the distance between s and t on the cycle is l.

SUBCASE 2.2. Not all the vertices of $AG_n(\{j_1, \ldots, j_x\})$ are on the path P_x for some $1 \leq x \leq \lfloor \frac{n}{2} \rfloor$. See Figure 5a for an illustration. Then, we can find two adjacent vertices y and z in $AG_n(j_x)$, which are not on the path P_x. Let $F \subseteq V(P_x)$. By Proposition 1 and Proposition 2, there exist two distinct edges $(y, y') \in E_n^{j_x,h_y}$ and $(z, z') \in E_n^{j_x,h_z}$ such that $y' \neq t' \in V(AG_n[h_y])$ and $z' \neq v' \in V(AG_n[h_z])$, respectively. If $AG_n[j_x] - F$ is isomorphic to AG_{n-2}, by Theorem 1, there is a hamiltonian path HP from y to z in $AG_n[j_x] - F$. If $AG_n[j_x] - F$ contains more than one subcomponent of $AG_n[j_x]$, by Lemma 1 if $n = 5$, and by Lemma 2 if $n > 5$, there is a hamiltonian path HP from y to z in $AG_n[j_x] - F$. By Lemma 4, there exist two disjoint paths DP_1 and DP_2, such that DP_1 joins t' and y', and DP_2 joins v' and z'. Moreover, the two paths span all of the vertices in $AG_n(\langle n\rangle - \{i, j_1, \ldots, j_x\})$. Thus $\langle s, P_x, t, t'', t', DP_1, y', y, HP, z, z', DP_2, v', v, RP, s\rangle$ forms a hamiltonian cycle, and for each $l \in \{d(s,t), d(s,t)+1, d(s,t)+2, \ldots, \frac{|V(AG_n)|}{2}\}$, the distance between s and t on the cycle is l.

CASE 3. Suppose that s and t belong to different subcomponents of AG_n, and the minimum length path connecting s and t has the form $\langle s, MP, t'', t', t\rangle$. Then $d(s,t) = d(s,t'') + 2$. See Figure 5b for an illustration. Since $AG_n[i]$ is isomorphic to AG_{n-1}, by the inductive hypothesis, for each $l_0 \in \{d(s,t''), d(s,t'')+1, d(s,t'')+2, \ldots, |V(AG_n[i])| - d(s,t'')\}$, we can construct a hamiltonian cycle HC_i of $AG_n[i]$ such that the distance between s and t'' on the cycle is l_0. Let u and v be the two neighbors of t'' on HC_i. Let $HC_i =$

$\langle s, LP, u, t'', v, RP, s\rangle$, and let $P_0 = \langle s, LP, u, t'', t', t\rangle$. Without loss of generality, let $L(P_0) = l_0 + 2$. By Proposition 2, $d(t'', u) = 1$, so we have $|AS(t'') \cap AS(u)| = 1$. This means that we can find a subcomponent $AG_n[j_1]$ for which $j_1 \in \langle n\rangle - \{i\}$. If $t, t' \notin V(AG_n[j_1])$, by using st''-expansion, the proof is the same as Case 1 but we replace vertex t in Case 1 with vertex t'' in this case. So we consider the case in which $t, t' \in V(AG_n[j_1])$, that is, $j_1 = j$. There exist two disjoint edges (u, p_1) and (t'', t') in E_n^{i,j_1}. By Proposition 1, $(p_1, t') \in E(AG_n[j_1])$. By Proposition 2, $d(t'', v) \leq 2$ hence $AS(t'') = \{j_1\}$, and $AS(t'') \neq AS(v)$. Since $|N^*(t'')| = 2$, we can find a subcomponent $AG_n[h_t]$, and a vertex $t^* \in V(AG_n[h_t])$ such that $(t'', t^*) \in E_n^{i,h_t}$ for some $h_t \in \langle n\rangle - \{i, j_1\}$. Since $|N^*(v)| = 2$ and $AS(t'') \neq AS(v)$, we can find a subcomponent $AG_n[h_v]$, and a vertex $v' \in V(AG_n[h_v])$ such that $(v, v') \in E_n^{i,h_v}$ for some $h_v \in \langle n\rangle - \{i, j_1, h_t\}$. Let $F \subseteq V(AG_n)$ and $F' = \{t^*\}$. By Lemma 2, there exists a hamiltonian path HP of $AG_n(\langle n\rangle - \{i\})$ joining t and v'. Thus $\langle s, P_0, t, HP, v', v, RP, s\rangle$ forms a hamiltonian cycle, and for each $l_0 \in \{d(s,t), d(s,t)+1, d(s,t)+2, \ldots, |V(AG_n[i])| - d(s,t) + 1\}$, the distance between s and t on the cycle is l_0.

Now we modify the st-expansion slightly to expand the path P_0 between s and t to various lengths. We describe the details as follows.

Suppose that $n \geq 6$. See Figure 4d for an illustration. We can insert one subcomponent of $AG_n[j_1]$, which is isomorphic to AG_{n-2}, into P_0 as follows. Because $d(p_1, t) = 2$, which is less than the diameter of AG_{n-2}, and by the symmetric property of the alternating group graph, we may regard p_1 and t as in the same subcomponent of $AG_n[j_1]$, say C. By Lemma 2, there is a hamiltonian path HP_1 of $C - F_j$ joining p_1 and t with $L(HP_1) = |V(AG_{n-2})| - 2$. Let C^* be the m subcomponents of $AG_n[j_1]$ we wanted to insert into P_0, where m is the number of subcomponents of $AG_n[j_1]$. We regard p_1 and t as in different subcomponents of $AG_n[j_1]$. By Lemma 2, there is a hamiltonian path HP_1 of $C^* - F_j$ joining p_1 and t with $L(HP_1) = m|V(AG_{n-2})| - 2$. Thus, we can construct a path HP_1 between p_1 and t such that $L(HP_1) = I_1|V(AG_{n-2})| - 2$ for each integer I_1 with $1 \leq I_1 \leq n-1$. Let $P_1 = \langle s, LP, u, p_1, HP_1, t\rangle$. Thus, we have $L(P_1) = l_0 + I_1|V(AG_{n-2})| - 2 = l_0 + \frac{I_1(n-2)!}{2} - 2$. Since $d(s,t) - 2 \leq l_0 \leq |V(AG_n[i])| - d(s,t) + 2$, we have $\frac{I_1(n-2)!}{2} + d(s,t) - 4 \leq L(P_1) \leq \frac{I_1(n-2)!}{2} + \frac{(n-1)!}{2} - d(s,t)$. For each $1 \leq I_1 \leq n-1$, $\frac{(I_1-1)(n-2)!}{2} + \frac{(n-1)!}{2} - d(s,t) \geq \frac{I_1(n-2)!}{2} + d(s,t) - 4$ if $n \geq 6$. Therefore, for each $l_1 \in \{d(s,t), d(s,t)+1, d(s,t)+2, \ldots, (n-1)! - d(s,t)\}$, we can construct a path P_1 from s to t such that the distance between s and t on the path is l_1. Then, similar to the st-expansion described in Case 1, we can expand the path between s and t such that for each $l_x \in \{d(s,t), d(s,t)+1, d(s,t)+2, \ldots, \frac{(x+1)(n-1)!}{2} - d(s,t)\}$, we can construct a path P_x from s to t such that the distance between s and t on the path is l_x. Hence for any integer l with $d(s,t) \leq l \leq \frac{|V(AG_n)|}{2}$, we can construct a path joining s and t with the length of the path being l.

For $n = 5$, that is, AG_5, we have $d(s,t) = 4$ in this case. As described above, $\langle s, LP, u, t'', t, HP, v', v, RP, s\rangle$ forms a hamiltonian cycle, and for each $l_0 \in \{4, 5, 6, \ldots, 12\}$, the distance between s and t on the cycle is l_0. Let $F_j \subseteq V(AG_n[j_1])$ and $F_j = \{t'\}$. By Theorem 1, we can find a hamiltonian path HP_1 of $AG_n[j_1] - F_j$ joining p_1 and t. Let $P_1 = \langle s, LP, u, p_1, HP_1, t\rangle$. We have $11 \leq L(P_1) \leq 19$. Therefore, for each $l_1 \in \{4, 5, 6, \ldots, 19\}$, we can construct a path P_1 from s to t such that the distance between s and t on the path is l_1 in AG_5. Let u_1 and t_1 be two adjacent vertices on HP_1. Then, for each $l_2 \in \{4, 5, 6, \ldots, \frac{|V(AG_5)|}{2}\}$, we can construct a path P_2 from s to t such that the distance between s and t on the path is l_2 by u_1t_1-expansion.

To construct a hamiltonian cycle, the proof is the same as that given in Subcase 2.1 and Subcase 2.2 by replacing vertex t' in Case 2 with vertex t^* in this case.

Hence the theorem is proved. ∎

4. CONCLUDING REMARKS

In this article, we have proposed a new concept called panpositionable hamiltonicity. We now explain some relationship between panpositionable hamiltonicity and panconnectivity. A hamiltonian graph G is panpositionable if for any two different vertices x and y of G and for any integer l satisfying $d(x,y) \leq l \leq |V(G)| - d(x,y)$, there exists a hamiltonian cycle C of G such that the relative distance between x, y on C is l. A graph G is panconnected if there exists a path of length l joining any two different vertices x and y with $d(x,y) \leq l \leq |V(G)| - 1$. If G is panpositionable hamiltonian, it is clear that there exists a path of length l joining any two different vertices x and y with $d(x,y) \leq l \leq |V(G)| - d(x,y)$. If we can find that G contains a path of length l joining any two different vertices x and y with $|V(G)| - d(x,y) + 1 \leq l \leq |V(G)| - 1$, then we can conclude that G is panconnected. By Theorem 1, the fault tolerant hamiltonian properties of the alternating group graph AG_n, there exists a path of length l joining any two different vertices x and y with $\frac{n!}{2} - 4 \leq l \leq \frac{n!}{2} - 1$ in AG_n if $n \geq 4$ [1]. Therefore, we can obtain the following known result as a corollary.

Corollary 1. [1] *Alternating group graphs AG_n are panconnected for all $n \geq 4$.*

We give an example to show that a panconnected graph G is not necessarily panpositionable hamiltonian. Let $n, s_1, s_2, \ldots, s_r$ be integers with $1 \leq s_1 < s_2 < \cdots < s_r$. The circulant graph $C(n; s_1, s_2, \ldots, s_r)$ is a graph with vertex set $\{0, 1, \ldots, n-1\}$. Two vertices i and j are adjacent if and only if $i - j = \pm s_k \pmod{n}$ for some k where $1 \leq k \leq r$. We can check that $C(n; 1, 2)$ is panconnected by brute force for $n \in \{5, 6, 7, 8, 9, 10\}$. However, $C(10; 1, 2)$ is not panpositionable hamiltonian. Figure 6 shows the structure of $C(10; 1, 2)$. Consider vertex 0 and vertex 2, with $d(0, 2) = 1$. We prove by contradiction that $C(10; 1, 2)$ does not contain a hamiltonian cycle HC with $d_{HC}(0, 2) = 5$. Suppose to the contrary that HC is a hamiltonian cycle

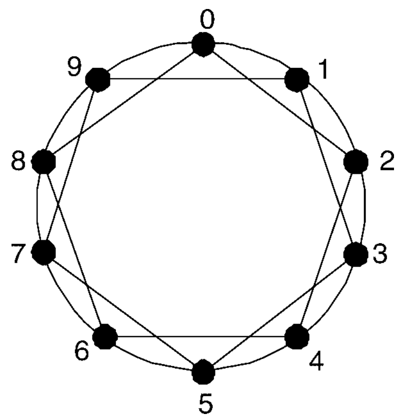


FIG. 6. The circulant graph $C(10; 1, 2)$.

of $C(10; 1, 2)$ with $d_{HC}(0, 2) = 5$. There are three possible paths, $P_1 = \langle 0, 8, 9, 1, 3, 2 \rangle$, $P_2 = \langle 0, 9, 1, 3, 4, 2 \rangle$, and $P_3 = \langle 0, 1, 3, 5, 4, 2 \rangle$, of length 5 joining vertex 0 and vertex 2. If HC contains P_1, then the edges $(0, 1), (0, 2), (0, 9)$ cannot belong to HC. If HC contains P_2 or P_3, then the edges $(2, 0), (2, 1), (2, 3)$ cannot belong to HC. Hence for $n = 10$, there does not exist any hamiltonian cycle in $C(10; 1, 2)$ such that the distance on the cycle between vertex 0 and vertex 2 is 5. So $C(10; 1, 2)$ is not panpositionable hamiltonian. In fact, the circulant graph $C(n; 1, 2)$ is panconnected for every $n \geq 5$, but it is not panpositionable hamiltonian for some values of n. Therefore, the panpositionable hamiltonian property is a stronger property for an interconnection network. Future work will try to find the panpositionable hamiltonicity of other interconnection networks and some relationships between these hamiltonian-like concepts.

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