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Global synchronization in lattices of coupled chaotic systems

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Based on the concept of matrix measures, we study global stability of synchronization in networks. Our results apply to quite general connectivity topology. In addition, a rigorous lower bound on the coupling strength for global synchronization of all oscillators is also obtained. Moreover, by merely checking the structure of the vector field of the single oscillator, we shall be able to determine if the system is globally synchronized. © *2007 American Institute of Physics*. [DOI: 10.1063/1.2754668]

Lattices of coupled chaotic oscillators model many systems of interest in physics, biology, and engineering. In particular, complete chaotic synchronization, all oscillators acquiring identical chaotic behavior, has received much attention analytically. There are, in general, two classes of results which give criteria for such synchronization. The first class of results linearizes around the synchronous manifold, and then computes the Lyapunov exponents/matrix measures or contraction factors of the variational equations to get local or global synchronization, respectively.[1–3] The second class of results uses the Lyapunov method by constructing a Lyapunov function to give analytical criteria for local or global synchronization.[4–14] This paper gives yet another approach by utilizing the concept of matrix measures to get global synchronization criteria. The coupling configuration of the networks is quite general, which includes asymmetric connections between nodes and/or some competitive ($g_{ij}<0$, $i\neq j$) couplings between cells $\mathbf{x}_i$ and $\mathbf{x}_j$, and partial-state coupling with nonzero off-diagonal connections. Moreover, by merely checking the structure of the vector field of the single oscillator, we shall be able to determine if the system is globally synchronized.

I. INTRODUCTION

During the past few decades the study of networks of dynamical systems has attracted increasing attention.[1–37] The purpose to connect dynamical systems in networks is to get them to solve problems cooperatively. For instance, such networks are needed for information processing in the brain.[21] The simplest mode of the coordinated motion between dynamical systems is their complete synchronization when all cells of the network acquire identical dynamical behavior. Consequently, one asks questions such as: What are the conditions for the stability of the synchronous state, especially with respect to coupling strengths and coupling configurations of the network? Typically, in networks of continuous time oscillators, the synchronous solution becomes stable when the coupling strength between oscillators exceeds a critical value. In this context, a central problem is to find the bounds on the coupling strength so that the stability of synchronization is guaranteed.

General approaches to local synchronization of coupled chaotic systems have been proposed, including the master stability function (MSF)-based criteria[1,16,32–35] originated by Pecora and Carroll,[1] and the matrix measures approach.[2] The former computes the Lyapunov exponent of the variational equations, while the latter uses the concept of matrix measures to give criteria on the variation equations. Recently, local synchronization in a complex network of asymmetrically coupled units was also obtained[18,25] via MSF-based criteria.

Global synchronization of coupled chaotic systems was also intensively studied. The methods include Lyapunov function-based criteria with symmetrical connections[4–8,10–14] or asymmetrical connections,[9,13] and the partial contraction approach.[3] For Lyapunov-based criteria, the partial-state coupling matrix, determining which state variables are coupled, is assumed to have the form satisfying Eq. (2.4c) while the partial contraction approach needs to verify the contraction of the system, depending on the state variables and time t, which is not a small task. In developing the theory of global synchronization of coupled chaotic systems, one needs to assume bounded dissipation of the coupled system; that is, all solutions of the coupled system are, in some sense, eventually bounded. Such assumption plays the role of an *a priori* estimate. However, in obtaining the theory of local synchronization, one does not need to know bounded dissipation of the coupled system. Thus, not surprisingly, the criteria in getting local synchronization are composed of a term that describes how chaotic the single system is and a term that depends on how the configuration of the networks is formed.

The purpose of this paper is yet to give another approach to study global synchronization of coupled chaotic systems. Our coupling rules are allowed to be asymmetric and/or some competitive ($g_{ij}<0$, $i\neq j$) couplings between cells $\mathbf{x}_i$ and $\mathbf{x}_j$, as long as the coupled system is bounded dissipative. In addition, the partial-state coupling in our approach is allowed to have the form satisfying (3.9a). Moreover, by merely checking the structure of the vector field of the single oscillator, we shall be able to determine if the system is globally synchronized. We also obtain a rigorous lower

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bound on the coupling strength for global synchronization of all oscillators with coupling configuration satisfying (2.4a) and (2.4b). Finally, the concept of matrix measures is introduced to obtain such global results.

We organize the paper as follows. Section II is to lay down the foundation of our paper. The main results are contained in Sec. III. Coupled Lorenz systems and coupled Duffing systems are used as illustrations. We also compare our results with those in Refs. 8 and 9.

II. BASIC FRAMEWORK AND PRELIMINARIES

In this paper, we will denote scalar variables in lower case, matrices in bold-type upper case, and vectors (or vector-valued functions) in bold-type lower case. We consider an array of m cells, coupled linearly together, with each cell being an n-dimensional system. The entire array is a system of nm ordinary differential equations. In particular, the state equations are

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{f}(\mathbf{x}_i,t) + d \cdot \sum_{j=1}^{m} g_{ij}\mathbf{D}\mathbf{x}_j, \quad i = 1,2,\ldots,m, \tag{2.1}$$

where $\mathbf{x}_i \in \mathbb{R}^n$, $\mathbf{f}:\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, and $\mathbf{D}$ is an $n \times n$ real matrix. Let

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_m \end{pmatrix}, \quad \mathbf{x}_i = \begin{pmatrix} x_{i,1} \\ \vdots \\ x_{i,n} \end{pmatrix}, \quad \text{and} \quad \mathbf{G} = (g_{ij})_{m\times m}. \tag{2.2}$$

Then (2.1) can be written as

$$\dot{\mathbf{x}} = \begin{pmatrix} \mathbf{f}(\mathbf{x}_1,t) \\ \vdots \\ \mathbf{f}(\mathbf{x}_m,t) \end{pmatrix} + d(\mathbf{G}\otimes\mathbf{D})\mathbf{x} =: \mathbf{F}(\mathbf{x},t) + d(\mathbf{G}\otimes\mathbf{D})\mathbf{x}, \tag{2.3a}$$

where $\otimes$ denotes the Kronecker product and

$$\mathbf{f}(\mathbf{x}_i,t) = \begin{pmatrix} f_1(\mathbf{x}_i,t) \\ \vdots \\ f_n(\mathbf{x}_i,t) \end{pmatrix}. \tag{2.3b}$$

We next impose conditions on coupling matrices $\mathbf{G}$ and $\mathbf{D}$. We assume that coupling matrix $\mathbf{G}$ satisfies the following:

(i) $\lambda = 0$ is a simple eigenvalue of $\mathbf{G}$ and $\mathbf{e} = [1,1,\ldots,1]^T_{1\times m}$ is its corresponding eigenvector; (2.4a)

(ii) All nonzero eigenvalues of $\mathbf{G}$ have negative real part. (2.4b)

We further assume that coupling matrix $\mathbf{D}$ is, without loss of generality, of the form

$$\mathbf{D} = \begin{pmatrix} \mathbf{I}_k & \mathbf{0} \\ \mathbf{0} & 0 \end{pmatrix}_{n\times n}. \tag{2.4c}$$

The index k, $1 \le k \le n$, means that the first k components of the individual system are coupled. If $k \ne n$, then the system is said to be partial-state coupled. Otherwise, it is said to be full-state coupled.

From time to time, we will refer to system (3) as the coupled system $(\mathbf{D},\mathbf{G},\mathbf{F}(\mathbf{x},t))$. To study synchronization of such a system, we permute the state variables in the following way:

$$\tilde{\mathbf{x}}_i = \begin{pmatrix} x_{1,i} \\ \vdots \\ x_{m,i} \end{pmatrix} \quad \text{and } \tilde{\mathbf{x}} = \begin{pmatrix} \tilde{\mathbf{x}}_1 \\ \vdots \\ \tilde{\mathbf{x}}_n \end{pmatrix}. \tag{2.5}$$

Then Eq. (2.3a) can be written as

$$\dot{\tilde{\mathbf{x}}} = \begin{pmatrix} \tilde{\mathbf{f}}_1(\tilde{\mathbf{x}},t) \\ \vdots \\ \tilde{\mathbf{f}}_n(\tilde{\mathbf{x}},t) \end{pmatrix} + d(\mathbf{D}\otimes\mathbf{G})\tilde{\mathbf{x}} =: \tilde{\mathbf{F}}(\tilde{\mathbf{x}},t) + d(\mathbf{D}\otimes\mathbf{G})\tilde{\mathbf{x}}, \tag{2.6a}$$

where

$$\tilde{\mathbf{f}}_i(\tilde{\mathbf{x}},t) = \begin{pmatrix} f_i(\mathbf{x}_1,t) \\ \vdots \\ f_i(\mathbf{x}_m,t) \end{pmatrix}. \tag{2.6b}$$

The purpose of such reformulation is twofold. First, a transformation of coordinates of $\tilde{\mathbf{x}}$ is to be applied to (2.6) so as to decompose the synchronous manifold. Second, once the synchronous manifold is decomposed, proving synchronization of Eq. (2.3a) is then equivalent to showing that the origin is asymptotically stable with respect to reduced system (3.3). From here on, we will treat $\tilde{}$ as a function that takes $\mathbf{x}$ into $\tilde{\mathbf{x}}$, or $\mathbf{x}_i$ into $\tilde{\mathbf{x}}_i$.

We next give the definition of the bounded dissipation of a system.

Definition 2.1. (i) A system of n ordinary differential equations is called bounded dissipative, provided that for any $r > 0$ and for any initial conditions $\mathbf{x}_0$ in $B_n(r)$, there exists a time $t^* \ge t_0$ such that $\|\mathbf{x}(t)\| \le \alpha_r$ for all $t \ge t^*$. (ii) If, in addition, α_r is independent of r, then the system is said to be uniformly bounded dissipative with respect to α_r.

To prove global synchronization of coupled chaotic systems, one needs to assume bounded dissipation, which plays the role of an *a priori* estimate. Without such an *a priori* estimate, as in the case of the Rössler system, global synchronization is much more difficult to obtain. Only local synchronization was reported numerically in literature (see, e.g., Ref. 5). We remark that in certain cases of the Rössler system, the trajectory of each oscillator grows unbounded, yet approaches each other (see, e.g., Ref. 5). An interesting question in this direction is how bounded dissipation of the coupled system is related to the uncoupled dynamics and its connectivity topology. Not many general theorems have been provided so far. In the case that $\mathbf{G}$ is diffusively coupled with periodic boundary conditions or zero flux and $\mathbf{D}$ satisfies (2.4c), it was shown in Ref. 6 that bounded dissipation of the single oscillator implies that of the coupled chaotic oscillators. Moreover, the absorbing domain of the coupled system is a topological product of the absorbing domain of each individual system.

In our derivation of synchronization of system (3), we need the concept of matrix measures. For completeness and ease of references, we also recall the following definition of matrix measures and their properties (see e.g., Ref. 38).

Definition 2.2. *Let* $\|\cdot\|_i$ *be an induced matrix norm on* $\mathbb{C}^{n\times n}$*. The matrix measure of matrix* $\mathbf{A}$ *on* $\mathbb{C}^{n\times n}$ *is defined to be* $\mu_i(\mathbf{A})=\lim_{\epsilon\to 0^+}\|I+\epsilon\mathbf{A}\|_i-1/\epsilon$.

Lemma 2.1. *Let* $\|\cdot\|_k$ *be an induced k norm on* $\mathbb{R}^{n\times n}$*, where* $k=1,2,\infty$*. Then each of matrix measure* $\mu_k(\mathbf{A})$*,* $k=1,2,\infty$*, of matrix* $\mathbf{A}=(a_{ij})$ *on* $\mathbb{R}^{n\times n}$ *is, respectively,*

$$\mu_\infty(\mathbf{A})=\max_i\left\{a_{ii}+\sum_{j\neq i}|a_{ij}|\right\}, \tag{2.7a}$$

$$\mu_1(\mathbf{A})=\max_j\left\{a_{jj}+\sum_{i\neq j}|a_{ij}|\right\}, \tag{2.7b}$$

and

$$\mu_2(\mathbf{A})=\lambda_{\max}(\mathbf{A}^H+\mathbf{A})/2. \tag{2.7c}$$

Here $\lambda_{\max}(\mathbf{A})$ *is the maximum eigenvalue of* $\mathbf{A}$.

Theorem 2.1. *(See, e.g., 3.5.32 of Ref. 38.) Consider the differential equation* $\dot{\mathbf{x}}(t)=\mathbf{A}(t)\mathbf{x}(t)+\mathbf{v}(t)$*,* $t\geq 0$*, where* $\mathbf{x}(t)\in\mathbb{R}^n$*,* $\mathbf{A}(t)\in\mathbb{R}^{n\times n}$*, and* $\mathbf{A}(t),\mathbf{v}(t)$ *are piecewise continuous. Let* $\|\cdot\|_i$ *be a norm on* $\mathbb{R}^n$*, and* $\|\cdot\|_i$*,* μ_i *denote, respectively, the corresponding induced norm and matrix measure on* $\mathbb{R}^{n\times n}$*. Then, whenever* $t\geq t_0\geq 0$*, we have*

$$\begin{aligned}&\|\mathbf{x}(t_0)\|\exp\left\{\int_{t_0}^{t}-\mu_i(-\mathbf{A}(s))ds\right\}\\&\quad-\int_{t_0}^{t}\exp\left\{\int_{s}^{t}-\mu_i(-\mathbf{A}(\tau))d\tau\right\}\|\mathbf{v}(s)\|ds\\&\quad\leq\|\mathbf{x}(t)\|\\&\quad\leq\|\mathbf{x}(t_0)\|\exp\left\{\int_{t_0}^{t}\mu_i(\mathbf{A}(s))ds\right\}\\&\qquad+\int_{t_0}^{t}\exp\left\{\int_{s}^{t}\mu_i(\mathbf{A}(\tau))d\tau\right\}\|\mathbf{v}(s)\|ds.\end{aligned} \tag{2.8}$$

To conclude this section, we define global synchronization as follows.

Definition 2.3. (i) System (3) is said to be globally synchronized if for any given initial values $\mathbf{x}_0$ there exists a $d=d_{\mathbf{x}_0}$ such that system (3) is synchronized for the initial conditions $\mathbf{x}_0$. Here $d_{\mathbf{x}_0}$ is a constant depending on $\mathbf{x}_0$. (ii) System (3) is said to be uniformly, globally synchronized if there exists a $d=d_1$ such that system (3) is synchronized for all initial values $\mathbf{x}_0$.

III. MAIN RESULTS

To study synchronization of (3), we first make a coordinate change to decompose the synchronous subspace. Let $\mathbf{A}$ be an $m\times m$ matrix of the form

$$\mathbf{A}=\begin{pmatrix}1&-1&0&\cdots&0\\0&\ddots&\ddots&\ddots&\vdots\\\vdots&\ddots&\ddots&\ddots&0\\0&\cdots&0&1&-1\\1&\cdots&\cdots&1&1\end{pmatrix}_{m\times m}=:\begin{pmatrix}\mathbf{C}\\\mathbf{e}^T\end{pmatrix}, \tag{3.1a}$$

where $\mathbf{e}$ is given as in (2.4a). It is then easy to see that $\mathbf{CC}^T$ is invertible and that

$$\mathbf{A}^{-1}=\left(\mathbf{C}^T(\mathbf{CC}^T)^{-1}\Big|\frac{\mathbf{e}}{m}\right). \tag{3.1b}$$

Setting

$$\mathbf{E}=\mathbf{I}_n\otimes\mathbf{A}, \tag{3.1c}$$

we see that

$$\begin{aligned}\mathbf{E}(\mathbf{D}\otimes\mathbf{G})\mathbf{E}^{-1}&=(\mathbf{I}_n\otimes\mathbf{A})(\mathbf{D}\otimes\mathbf{G})(\mathbf{I}_n\otimes\mathbf{A}^{-1})\\&=\mathbf{D}\otimes\mathbf{AGA}^{-1}\\&=\mathbf{D}\otimes\begin{pmatrix}\mathbf{CGC}^T(\mathbf{CC}^T)^{-1}&\mathbf{0}\\ *&0\end{pmatrix}=:\mathbf{D}\otimes\begin{pmatrix}\bar{\mathbf{G}}&\mathbf{0}\\ *&0\end{pmatrix}.\end{aligned} \tag{3.1d}$$

We remark, via (3.1d), that $\sigma(\mathbf{G})-\{0\}=\sigma(\bar{\mathbf{G}})$, where $\sigma(\mathbf{A})$ is the spectrum of matrix $\mathbf{A}$. Multiplying $\mathbf{E}$ to both sides of Eq. (2.6a), we get

$$\begin{aligned}\dot{\tilde{\mathbf{y}}}=:\mathbf{E}\dot{\tilde{\mathbf{x}}}&=\mathbf{E}\tilde{\mathbf{F}}(\tilde{\mathbf{x}},t)+d\mathbf{E}(\mathbf{D}\otimes\mathbf{G})\mathbf{E}^{-1}\tilde{\mathbf{y}}\\&=\mathbf{E}\tilde{\mathbf{F}}(\mathbf{E}^{-1}\tilde{\mathbf{y}},t)+d\left(\mathbf{D}\otimes\begin{pmatrix}\bar{\mathbf{G}}&\mathbf{0}\\ *&0\end{pmatrix}\right)\tilde{\mathbf{y}}.\end{aligned} \tag{3.2}$$

Let

$$\tilde{\mathbf{y}}=\begin{pmatrix}\tilde{\mathbf{y}}_1\\\vdots\\\tilde{\mathbf{y}}_n\end{pmatrix}.$$

Then

$$\tilde{\mathbf{y}}_i=\begin{pmatrix}x_{1,i}-x_{2,i}\\\vdots\\x_{m-1,i}-x_{m,i}\\\sum_{j=1}^m x_{j,i}\end{pmatrix}.$$

Setting

$$\tilde{\mathbf{y}}_i=\begin{pmatrix}\bar{\mathbf{y}}_i\\\sum_{j=1}^m x_{j,i}\end{pmatrix}\quad\text{and }\bar{\mathbf{y}}=\begin{pmatrix}\bar{\mathbf{y}}_1\\\vdots\\\bar{\mathbf{y}}_n\end{pmatrix},$$

we have that the dynamics of $\bar{\mathbf{y}}$ is satisfied by the following equation:

$$\dot{\bar{\mathbf{y}}}=d(\mathbf{D}\otimes\bar{\mathbf{G}})\bar{\mathbf{y}}+\bar{\mathbf{F}}(\bar{\mathbf{y}},t). \tag{3.3}$$

Here $\bar{\mathbf{F}}$ is obtained from $\mathbf{E}\tilde{\mathbf{F}}(\mathbf{E}^{-1}\tilde{\mathbf{y}},t)$ accordingly.

The task of obtaining global synchronization of system (3) is now reduced to showing that the origin is globally and asymptotically stable with respect to system (3.3). To this end, the space $\bar{\mathbf{y}}$ is broken into two parts: $\bar{\mathbf{y}}_c$, the coupled space, and $\bar{\mathbf{y}}_u$, the uncoupled space,

$$\bar{\mathbf{y}} = \begin{pmatrix} \bar{\mathbf{y}}_c \\ \bar{\mathbf{y}}_u \end{pmatrix} \quad \text{and } \bar{\mathbf{F}}(\bar{\mathbf{y}},t) = \begin{pmatrix} \bar{\mathbf{F}}_c(\bar{\mathbf{y}},t) \\ \bar{\mathbf{F}}_u(\bar{\mathbf{y}},t) \end{pmatrix}, \tag{3.4}$$

respectively. Here

$$\bar{\mathbf{y}}_c = \begin{pmatrix} \bar{\mathbf{y}}_1 \\ \vdots \\ \bar{\mathbf{y}}_k \end{pmatrix} \quad \text{and } \bar{\mathbf{y}}_u = \begin{pmatrix} \bar{\mathbf{y}}_{k+1} \\ \vdots \\ \bar{\mathbf{y}}_n \end{pmatrix}.$$

The dynamics on the coupled space with respect to the linear part is under the influence of $\bar{\mathbf{G}}$, which is asymptotically stable. The dynamics of the nonlinear part on coupled space can then be controlled by choosing a large coupling strength. As a matter of fact, it is easier to obtain synchronization of coupled chaotic systems with a larger coupled space. On the other hand, the uncoupled space has no stable matrix $\bar{\mathbf{G}}$ to play with. Thus, its corresponding vector field $\bar{\mathbf{F}}_u(\bar{\mathbf{y}},t)$ must have a certain structure to make the trajectory stay closer to the origin as time progresses, as we shall explain later.

Now, assume that $\bar{\mathbf{F}}_c(\bar{\mathbf{y}},t)$ satisfies a dual-Lipschitz condition with a dual-Lipschitz constant b_1. That is,

$$\|\bar{\mathbf{F}}_c(\bar{\mathbf{y}},t)\| \le b_1\|\bar{\mathbf{y}}\| \tag{3.5a}$$

whenever $\bar{\mathbf{y}}$ in the ball $B_{(m-1)n}(\alpha)$, and for all time t. Since the estimate in the right-hand side of (3.5a) depends on the whole space $\bar{\mathbf{y}}$, condition (3.5a) is a mild assumption provided that the coupled system is bounded dissipative. Write $\bar{\mathbf{F}}_u(\bar{\mathbf{y}},t)$ as

$$\bar{\mathbf{F}}_u(\bar{\mathbf{y}},t) = \mathbf{U}(t)\bar{\mathbf{y}}_u + (\bar{\mathbf{F}}_u(\bar{\mathbf{y}},t) - \mathbf{U}(t)\bar{\mathbf{y}}_u) =: \mathbf{U}(t)\bar{\mathbf{y}}_u + \bar{\mathbf{R}}_u(\bar{\mathbf{y}},t). \tag{3.5b}$$

Assume that $\mathbf{U}(t)$ is a block diagonal matrix of the form $\mathbf{U}(t)=$ diag $(\mathbf{U}_1(t),\ldots,\mathbf{U}_l(t))$, where $\mathbf{U}_j(t)$, $j=1,\ldots,l$, are matrices of size $(m-1)k_j\times(m-1)k_j$. Here $\Sigma_{j=1}^{l}k_j=n-k$, and $k_j\in\mathbb{N}$. We assume further that the following holds:

(i) The matrix measures $\mu_i(\mathbf{U}_j(t))$ are less than

$$-\gamma \text{ for all } t \text{ and all } j, \text{ where } \gamma>0; \tag{3.5c}$$

(ii) Let

$$\bar{\mathbf{R}}_u(\bar{\mathbf{y}},t) = \begin{pmatrix} \mathbf{R}_{u1}(\bar{\mathbf{y}},t) \\ \vdots \\ \mathbf{R}_{ul}(\bar{\mathbf{y}},t) \end{pmatrix}.$$

Then $\mathbf{R}_{uj}(\bar{\mathbf{y}},t)$, $j=1,\ldots,l$ satisfy a strong dual-Lipschitz condition with a strong dual-Lipschitz constant b_2. Specifically, let

$$\bar{\mathbf{y}}_u = \begin{pmatrix} \bar{\mathbf{y}}_{u1} \\ \vdots \\ \bar{\mathbf{y}}_{ul} \end{pmatrix},$$

written in accordance with the block structure of $\mathbf{U}(t)$. Then we assume that

$$\|\mathbf{R}_{uj}(\bar{\mathbf{y}},t)\| \le b_2 \left\| \begin{pmatrix} \bar{\mathbf{y}}_c \\ \bar{\mathbf{y}}_{u1} \\ \vdots \\ \bar{\mathbf{y}}_{uj-1} \end{pmatrix} \right\| \tag{3.5d}$$

whenever $\bar{\mathbf{y}}$ in the ball $B_{(m-1)n}(\alpha)$, and for all $j=1,\ldots,l$ and all time t.

Specifically, we break the vector field $\bar{\mathbf{F}}_u$ into (time-dependent) linear part $\mathbf{U}(t)\bar{\mathbf{y}}_u$ and nonlinear part $\bar{\mathbf{R}}_u(\bar{\mathbf{y}},t)$. We will further break $\mathbf{U}(t)$ into certain block diagonal forms if necessary. Note that form (3.5b) can always be achieved since the remaining term $\bar{\mathbf{R}}_u$ still depends on the whole space $\bar{\mathbf{y}}$. To take control of the dynamics on the linear part, we assume that the matrix measure of each diagonal block $\mathbf{U}_j(t)$ is negative. As to contain corresponding dynamics on the nonlinear part, we assume that (3.5d) holds. Note that though the nonlinear terms $\mathbf{R}_{uj}(\bar{\mathbf{y}},t)$ could possibly depend on the whole space, their norm estimates are required to depend only on the coupled space and uncoupled subspaces with their indexes proceeding j. In this setup, the nonlinear dynamics on uncoupled space can be iteratively controlled by choosing a large coupling strength. We also remark that if (3.5c) and (3.5d) are satisfied for l, the number of diagonal blocks, being one, then we do not need to further break $\mathbf{U}(t)$. Such further breaking is needed only if (3.5c) and (3.5d) are not satisfied. The proof in the following theorem gives exactly how the above strategy can be realized.

Theorem 3.1. *Let* $\mathbf{G}$ *and* $\mathbf{D}$ *be given as in (2.4). Assume that* $\bar{\mathbf{F}}$ *satisfies (3.5a), (3.5b), (3.5c), and (3.5d), and system (3.3) is uniformly bounded dissipative with respect to* α. *Let* $\lambda_1=\max\{\lambda_j|\lambda_j\in Re(\sigma(\bar{\mathbf{G}}))\}$. *If*

$$d > \frac{cb_1}{-\lambda_1+\epsilon}\left(1+\left(\frac{b_2}{\gamma}\right)^2\right)^{l/2} =: d_c, \tag{3.6}$$

where $\epsilon\ge 0$ *and* c *is some constant depending on* $\mathbf{G}$ *and* ϵ, *then* $\lim_{t\to\infty}\bar{\mathbf{y}}(t)=0$.

Proof. Since system (3.3) is uniformly bounded dissipative with respect to α, without loss of generality, we may assume that $\|\bar{\mathbf{y}}(t)\|\le\alpha$ for all time $t\ge t_0$. Using (3.5b), we write (3.3) as

$$\begin{pmatrix} \dot{\bar{\mathbf{y}}}_c \\ \dot{\bar{\mathbf{y}}}_u \end{pmatrix} = \begin{pmatrix} d(\mathbf{I}_k\otimes\bar{\mathbf{G}}) & \mathbf{0} \\ \mathbf{0} & \mathbf{U}(t) \end{pmatrix}\begin{pmatrix} \bar{\mathbf{y}}_c \\ \bar{\mathbf{y}}_u \end{pmatrix} + \begin{pmatrix} \bar{\mathbf{F}}_c(\bar{\mathbf{y}},t) \\ \bar{\mathbf{R}}_u(\bar{\mathbf{y}},t) \end{pmatrix}. \tag{3.7a}$$

Applying the variation of constant formula to (3.7a) on $\bar{\mathbf{y}}_c$, we get

$$\bar{\mathbf{y}}_c(t) = e^{(t-t_0)d(\mathbf{I}_k\otimes\bar{\mathbf{G}})}\bar{\mathbf{y}}_c(t_0) + \int_{t_0}^{t} e^{(t-s)d(\mathbf{I}_k\otimes\bar{\mathbf{G}})}\bar{\mathbf{F}}_c(\bar{\mathbf{y}}(s),s)ds.$$

Let $\lambda_1=\max\{\lambda_j|\lambda_j\in Re(\sigma(\mathbf{G})-\{0\})\}$. Then

$$\|e^{td(\mathbf{I}_k\otimes\bar{\mathbf{G}})}\| \le ce^{td\nu} \tag{3.7b}$$

for $\nu=\lambda_1+\epsilon$ and some constant c. Here $0<\epsilon<-\lambda_1$. Thus,

$$\|\bar{\mathbf{y}}_c(t)\| \le ce^{(t-t_0)d\nu}\|\bar{\mathbf{y}}_c(t_0)\| + cb_1\int_{t_0}^{t} e^{d(t-s)\nu}\|\bar{\mathbf{y}}(s)\|ds$$

$$\le ce^{(t-t_0)d\nu}\alpha + \frac{\alpha}{d}\frac{cb_1}{|\nu|} =: ce^{(t-t_0)d\nu}\alpha + \frac{\alpha}{d}c_0.$$

Let $\delta > 1$. We see that

$$\|\bar{\mathbf{y}}_c(t)\| \le \frac{\alpha}{d}c_0\delta, \tag{3.8a}$$

whenever $t \ge t_{0,1}$ for some $t_{0,1} > 0$. We then apply Theorem 2.1 on $\bar{\mathbf{y}}_{u1}$, and the resulting inequality is

$$\|\bar{\mathbf{y}}_{u1}(t)\| \le \|\bar{\mathbf{y}}_{u1}(t_{0,1})\|\exp\left\{\int_{t_{0,1}}^{t}\mu_i(\mathbf{U}_1(s))ds\right\} + \int_{t_{0,1}}^{t}\exp\left\{\int_{s}^{t}\mu_i(\mathbf{U}_1(\tau))d\tau\right\}\|\mathbf{R}_{u1}(\bar{\mathbf{y}}(s),s)\|ds.$$

It then follows from (3.5c), (3.5d), and (3.8a) that

$$\|\bar{\mathbf{y}}_{u1}(t)\| \le \alpha e^{-\gamma(t-t_{0,1})} + \frac{\alpha}{d}\frac{b_2}{\gamma}c_0\delta \le \frac{\alpha}{d}\frac{b_2}{\gamma}c_0\delta^2 =: \frac{\alpha}{d}c_1\delta^2, \tag{3.8b}$$

whenever $t \ge t_{1,1}$ for some $t_{1,1} \ge t_{0,1}$. Inductively, we get

$$\|\bar{\mathbf{y}}_{uj}(t)\| \le \frac{\alpha}{d}\left(\frac{b_2}{\gamma}\sqrt{\sum_{i=0}^{j-1}c_i^2}\right)\delta^{j+1} =: \frac{\alpha}{d}c_j\delta^{j+1}, \quad j = 2, \ldots, l, \tag{3.8c}$$

whenever $t \ge t_{j,1}(\ge t_{j-1,1})$. Letting $t_{l,1} = t_1$ and summing up (3.8a), (3.8b), and (3.8c), we get

$$\|\bar{\mathbf{y}}(t)\| = \sqrt{\sum_{j=1}^{l}\|\bar{\mathbf{y}}_{uj}(t)\|^2 + \|\bar{\mathbf{y}}_c(t)\|^2}$$

$$\le \frac{\alpha}{d}\left(1 + \left(\frac{b_2}{\gamma}\right)^2\right)^{l/2}\frac{cb_1}{|\nu|}\delta^{l+1} =: h\alpha,$$

whenever $t \ge t_1$. Choosing $d > (1 + (b_2/\gamma)^2)^{l/2}(cb_1/|\nu|\delta^{l+1})$, we see that the contraction factor h is strictly less than 1, and $\|\bar{\mathbf{y}}(t)\|$ contracts as time progresses. To complete the proof of the theorem, we note that $\delta > 1$ can be made arbitrary close to 1. Consequently, if $d > (1 + (b_2/\gamma)^2)^{l/2}(cb_1/|\nu|$, then h can still be made to be less than 1. □

Remark 3.1. (i) In case that $\bar{\mathbf{G}}$ is symmetric, then c and ϵ can be chosen to be one and zero, respectively. (ii) b_1 and b_2 could possibly depend on α. (iii) If system (3.3) is only bounded dissipative, then the estimate in (3.6) is still valid. However, in this case, b_1 and b_2 depend not only on α but also on $\mathbf{x}_0$.

Corollary 3.1. *Suppose $\bar{\mathbf{F}}$ and $\mathbf{G}$ are given as in Theorem 3.1. Let*

$$\mathbf{D} = \begin{pmatrix} \bar{\mathbf{D}}_{k\times k} & \mathbf{0} \\ \mathbf{0} & 0 \end{pmatrix}_{n\times n}, \quad \textit{where } \mathrm{Re}(\sigma(\bar{\mathbf{D}})) > 0. \tag{3.9a}$$

Assume, in addition, that either $\sigma(\mathbf{G})$ or $\sigma(\bar{\mathbf{D}})$ has no complex eigenvalue.

Then assertions in Theorem 3.1 still hold true, except d_c needs to be replaced by

$$d_c = \frac{cb_1}{|\nu|\min\{Re(\sigma(\bar{\mathbf{D}}))\}}\left(1 + \left(\frac{b_2}{\gamma}\right)^2\right)^{l/2}. \tag{3.9b}$$

Proof. The assumption on $\mathbf{D}$ is to ensure that (3.7b) is still valid. Other parts of the proof are similar to those in Theorem 3.1 and are thus omitted. □

We next turn our attention to finding conditions on the nonlinearities $f_i(\mathbf{u},t)$, $i = 1, \ldots, n$, $\mathbf{u} \in \mathbb{R}^n$, so that assumptions (3.5a), (3.5b), (3.5c), and (3.5d) are satisfied. To this end, we need the following notations: Let $\tilde{\mathbf{x}}_i$ and $\tilde{\mathbf{x}}$ be given as in (2.5). Define

$$[\tilde{\mathbf{x}}_i]^- = \begin{pmatrix} x_{1,i} \\ \vdots \\ x_{m-1,i} \end{pmatrix}, \quad \text{and } [\tilde{\mathbf{x}}]^- = \begin{pmatrix} [\tilde{\mathbf{x}}_1]^- \\ \vdots \\ [\tilde{\mathbf{x}}_n]^- \end{pmatrix}. \tag{3.10}$$

We then break $\tilde{\mathbf{F}}$ as given in (2.6a) into two parts so that the breaking is inconsistent with $\bar{\mathbf{y}}$ in (3.4). Specifically, we shall write

$$\tilde{\mathbf{F}}(\tilde{\mathbf{x}},t) = \begin{pmatrix} \tilde{\mathbf{F}}_c(\tilde{\mathbf{x}},t) \\ \tilde{\mathbf{F}}_u(\tilde{\mathbf{x}},t) \end{pmatrix}. \tag{3.11}$$

We are now in the position to state the following propositions.

Proposition 3.1. *Suppose that $f_i(\mathbf{x},t)$, $i = 1, 2, \ldots, k$ satisfy a Lipschitz condition in $B_n(\alpha/2)$ with a Lipschitz constant b_1. That is,*

$$|f_i(\mathbf{u},t) - f_i(\mathbf{v},t)| \le \frac{b_1}{k}\|\mathbf{u} - \mathbf{v}\|, \quad i = 1, 2, \ldots, k, \tag{3.12}$$

for all $\mathbf{u}$, $\mathbf{v}$ in $B_n(\alpha/2)$, and all time t. Then (3.5a) holds true.

Proof. Note that

$$\mathbf{E}\tilde{\mathbf{F}}(\tilde{\mathbf{x}},t) = \begin{pmatrix} \mathbf{A}\tilde{\mathbf{f}}_1(\tilde{\mathbf{x}},t) \\ \vdots \\ \mathbf{A}\tilde{\mathbf{f}}_n(\tilde{\mathbf{x}},t) \end{pmatrix},$$

where $\mathbf{A}$ is given as in (3.1a), and so

$$[\mathbf{A}\tilde{\mathbf{f}}_i(\tilde{\mathbf{x}},t)]^- = \begin{pmatrix} f_i(\mathbf{x}_1,t) - f_i(\mathbf{x}_2,t) \\ \vdots \\ f_i(\mathbf{x}_{m-1},t) - f_i(\mathbf{x}_m,t) \end{pmatrix}, \quad i = 1, 2, \ldots, n. \tag{3.13}$$

Since

$$\bar{\mathbf{F}}_c(\bar{\mathbf{y}},t) = \begin{pmatrix} [\mathbf{A}\tilde{\mathbf{f}}_1(\tilde{\mathbf{x}},t)]^- \\ \vdots \\ [\mathbf{A}\tilde{\mathbf{f}}_k(\tilde{\mathbf{x}},t)]^- \end{pmatrix},$$

we conclude that (3.5a) holds. □

From the above proposition, we see that the nonlinearities on the corresponding coupled space are only assumed to be Lipchitz. The following proposition is very useful in the sense that by checking how each component f_i of the nonlinearity $\mathbf{f}$ is formed, one would then be able to conclude whether (3.5c) and (3.5d) are satisfied.

Proposition 3.2. *Let* $\mathbf{u}=(u_1,\ldots,u_n)^T$ *and* $\mathbf{v}=(v_1,\ldots,v_n)^T$ *be vectors in* $B_n(\alpha/2)$. *Let* $w_p=\Sigma_{i=0}^{p}k_i$, $p=1,\ldots,l$, *where* $k_0=k$, *the dimension of coupled space, and* $k_1,\ldots,k_l$ *and* l *are given as in* (3.5c). *Write* $f_{w_{p-1}+i}(\mathbf{u},t)-f_{w_{p-1}+i}(\mathbf{v},t)$, $i=1,\ldots,k_p$, *as*

$$f_{w_{p-1}+i}(\mathbf{u},t)-f_{w_{p-1}+i}(\mathbf{v},t) = \sum_{j=1}^{k_p} q_{w_{p-1}+i,w_{p-1}+j}(\mathbf{u},\mathbf{v},t)(u_{w_{p-1}+j}-v_{w_{p-1}+j}) + r_{w_{p-1}+i}(\mathbf{u},\mathbf{v},t). \tag{3.14a}$$

We further assume that the following are true:

(i) *For*

$$p=1,\ldots,l,\ \text{let}\ \mathbf{Q}_{\mathbf{u},\mathbf{v},p}=(q_{w_{p-1}+i,w_{p-1}+j}(\mathbf{u},\mathbf{v},t)), \tag{3.14b}$$

where $1\le i,j\le k_p$. *Then* $\mu_*(\mathbf{V}_p)<-\gamma$ *for all* p, $\mathbf{u}$, $\mathbf{v}$, *in* $B_n(\alpha/2)$ *and all time* t, *where* $*=1,2,\infty$;

(ii) *Let* $\mathbf{r}_p=(r_{w_{p-1}+1}(\mathbf{u},\mathbf{v},t),\ldots,r_{w_p}(\mathbf{u},\mathbf{v},t))^T$. *We have that*

$$\|\mathbf{r}_p\| \le b_2 \left\| \begin{pmatrix} u_1-v_1 \\ \vdots \\ u_{w_{p-1}}-v_{w_{p-1}} \end{pmatrix} \right\| \tag{3.14c}$$

for all p, $\mathbf{u}$, $\mathbf{v}$ *in* $B_n(\alpha/2)$ *and all time* t. *Then* (3.5c) *and* (3.5d) *hold true for* $*=1,2,\infty$.

Proof. Since $r_i(\mathbf{u},\mathbf{v},t)$ depend on the whole space, $f_i(\mathbf{u},t)-f_i(\mathbf{v},t)$ can always be written as the form in (3.14a). Using (3.13) and (3.14a), we have that the matrices $\mathbf{U}_p(t)$ in the linear part of $\bar{\mathbf{F}}_u(\bar{\mathbf{y}},t)$ take the form

$$\mathbf{U}_p(t)=\sum_{w=1}^{m-1}\mathbf{Q}_{\mathbf{x}_w,\mathbf{x}_{w+1},p}(t)\otimes\mathbf{D}_w, \tag{3.15}$$

where $\mathbf{x}_w$ are given as in (2.2), and

$$(\mathbf{D}_w)_{ij}=\begin{cases}1 & i=j=w,\\ 0 & \text{otherwise,}\end{cases}\quad 1\le i,j\le m-1.$$

It then follows from (2.7a), (2.7b), and (3.15) that $\mu_*(\mathbf{U}_p(t))<-\gamma$ for $*=1$ or ∞. For $*=2$, we have that

$$\bigcup_{w=1}^{m-1}\sigma\{\mathbf{Q}_{\mathbf{x}_w,\mathbf{x}_{w+1},p}(t)+(\mathbf{Q}_{\mathbf{x}_w,\mathbf{x}_{w+1},p}(t))^T\} = \sigma\left\{\sum_{w=1}^{m-1}(\mathbf{Q}_{\mathbf{x}_w,\mathbf{x}_{w+1},p}(t)\otimes\mathbf{D}_w+(\mathbf{Q}_{\mathbf{x}_w,\mathbf{x}_{w+1},p}(t))^T\otimes\mathbf{D}_w)\right\} = \sigma(\mathbf{U}_p(t)+\mathbf{U}_p^T(t)),$$

where $\sigma(\mathbf{A})$ is the spectrum of $\mathbf{A}$. We remark that the first equality above can be verified by the definition of eigenvalues due to the structure of $\mathbf{U}_p(t)$. It then follows from (2.7c) that $\mu_2(\mathbf{U}_p(t))<-\gamma$. The remainder of the proof is similar to that of Proposition 3.1, and is thus omitted. □

Remark 3.2. The upshot of Proposition 3.2 is that, by only checking the "structure" of the vector field $\mathbf{f}$ of the single oscillator, one should be able to determine if our main result can be applied. To be precise, we begin with saving notations by setting $\mathbf{f}$ as $\mathbf{f}=\mathbf{f}(\mathbf{x},t)=(f_1(\mathbf{x},t),\ldots,f_n(\mathbf{x},t))^T$. We then check the form of the difference of the "uncoupled" part of dynamics. That is, we write $f_i(\mathbf{u},t)-f_i(\mathbf{v},t)$ in the form of (3.14a) with $i=k+1,\ldots,n$. If (3.14b) and (3.14c) can be satisfied, then $l=1$ gets the job done. Otherwise, we further break the uncoupled states into a set of smaller pieces to see if the resulting (3.14b) and (3.14c) are satisfied.

We are now ready to state the main theorems of the paper.

Theorem 3.2. *Assume that system* (3) *is (resp., uniformly) bounded dissipative. Let coupling matrices* $\mathbf{G}$ *and* $\mathbf{D}$ *satisfy* (2.4) *and the nonlinearities* $f_i(\mathbf{x},t)$, $i=1,2,\ldots,n$, *satisfy* (3.12) *and* (3.14). *Suppose* d *is greater than* d_c, *as given in* (3.6). *Then system* (3) *is (resp., uniformly) globally synchronized.*

Proof. The proof is a direct consequence of Propositions 3.1 and 3.2, and Theorem 3.1. □

Remark 3.3. From here on, we will refer to the assumptions in Theorem 3.2 as synchronization hypotheses.

Theorem 3.3. *Coupled system* $(\mathbf{D},\mathbf{G},\mathbf{F}(\mathbf{x},t))$, *given as in Corollary 3.1, is also (resp., uniformly) globally synchronized provided that its coupled system is (resp., uniformly) bounded dissipative and that* d *is greater than* d_c. *Here* d_c *is given in* (3.9b).

IV. APPLICATIONS

To see the effectiveness of our main results, we consider two examples in this section. These are coupled Lorenz equations,[8,26] and coupled Duffing oscillators.[37]

(i) We shall begin with Lorenz equations. Let $\mathbf{x}=(x_1,x_2,x_3)^T$,

$$\mathbf{f}(\mathbf{x},t)=\mathbf{f}(\mathbf{x})=(\sigma(x_2-x_1),rx_1-x_2-x_1x_3,-bx_3+x_1x_2)^T =:(f_1(\mathbf{x}),f_2(\mathbf{x}),f_3(\mathbf{x}))^T.$$

Here $\sigma=10$, $r=28$, and $b=8/3$. In the following cases (a)–(d), $\mathbf{G}$ denotes the diffusive coupling with zero flux and $\mathbf{D}$ is, respectively,

$$\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix},\quad \begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix},\quad \begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix},\quad \text{and}\quad \begin{pmatrix}0&0&0\\0&1&1\\0&0&1\end{pmatrix}.$$

For the first three cases, it was shown in Ref. 6 that all such coupled systems $(\mathbf{D},\mathbf{G},\mathbf{F}(\mathbf{x}))$ have the topological product of an absorbing domain

$$B=\left\{x_1^2+x_2^2+(x_3-r-\sigma)^2<\frac{b^2(r+\sigma)^2}{4(b-1)}=:\beta\right\}. \tag{4.1}$$

Hence, in each case, we will concentrate on the illustration of how our main results may or may not be applied.

(a) Let

$$\mathbf{D}=\mathbf{D}_1=\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

For "coupled" nonlinearity f_1, we get that

$$|f_1(\mathbf{u})-f_1(\mathbf{v})| = \sigma|(u_2-v_2)-(u_1-v_1)| \le \sqrt{2}\sigma\|\mathbf{u}-\mathbf{v}\|.$$

Hence, condition (3.5a) is satisfied. For "uncoupled" nonlinearities f_2 and f_3, we see that

$$\begin{aligned} f_2(\mathbf{u})-f_2(\mathbf{v}) &= (-u_2-u_1u_3+ru_1)-(-v_2-v_1v_3+rv_1)\\ &= [-(u_2-v_2)-u_1(u_3-v_3)]\\ &\quad +(r-v_3)(u_1-v_1) \end{aligned} \tag{4.2a}$$

and

$$\begin{aligned} f_3(\mathbf{u})-f_3(\mathbf{v}) &= (u_1u_2-bu_3)-(v_1v_2-bv_3)\\ &= [u_1(u_2-v_2)-b(u_3-v_3)]\\ &\quad +v_2(u_1-v_1). \end{aligned} \tag{4.2b}$$

Writing (4.2a) and (4.2b) in the vector form, we get

$$\begin{aligned} \begin{pmatrix} f_2(\mathbf{u})-f_2(\mathbf{v})\\ f_3(\mathbf{u})-f_3(\mathbf{v})\end{pmatrix} &= \begin{pmatrix}-1 & -u_1(t)\\ u_1(t)-b & \end{pmatrix}\begin{pmatrix}u_2-v_2\\ u_3-v_3\end{pmatrix}\\ &\quad + \begin{pmatrix}(r-v_3)(u_1-v_1)\\ v_2(u_1-v_1)\end{pmatrix}\\ &=: \mathbf{Q}_{\mathbf{u},\mathbf{v},1}(t)\begin{pmatrix}u_2-v_2\\ u_3-v_3\end{pmatrix}+\mathbf{r}_1. \end{aligned} \tag{4.2c}$$

Clearly, $\mu_2(\mathbf{Q}_{\mathbf{u},\mathbf{v},1}(t))=\max\{-1,-b\}=-1<0$, and $\|\mathbf{r}_1\| \le (\sigma+\sqrt{\beta})\cdot|u_1-v_1|$, where its estimate depends only on coupled space. Hence, conditions (3.14b) and (3.14c) are satisfied.

(b) Let

$$\mathbf{D}=\mathbf{D}_2=\begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

As in the case (a), the "coupled" nonlinearity f_2 is clearly Lipschitz on the absorbing domain. The difference of "uncoupled" nonlinearities f_1 and f_3 are given as follows:

$$f_1(\mathbf{u})-f_1(\mathbf{v}) = [-\sigma(u_1-v_1)]+\sigma(u_2-v_2),$$

$$\begin{aligned} f_3(\mathbf{u})-f_3(\mathbf{v}) &= [-b(u_3-v_3)]+u_1(u_2-v_2)\\ &\quad +v_2(u_1-v_1). \end{aligned}$$

If $l=1$ is chosen, then (3.14c) is violated. For in the case, the norm estimate in the right-hand side of (3.14c) can only depend on u_2-v_2. Now, if we choose $l=2$ and pick the space of the first diagonal block being the one associated with the nonlinearity f_1, then $\mathbf{Q}_{\mathbf{u},\mathbf{v},1}=(-\sigma)$ and $r_1=\sigma(u_2-v_2)$. Consequently, (3.14b) and (3.14c) are satisfied. Moreover, we have $\mathbf{Q}_{\mathbf{u},\mathbf{v},2}=(-b)$ and $r_2=u_1(u_2-v_2)+v_2(u_1-v_1)$, which depends only on the coupled space and the first uncoupled space. Thus, r_2 satisfies (3.14c).

(c) For illustration, we also consider

$$\mathbf{D}=\mathbf{D}_3=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

In this case, the uncoupled nonlinearities of f_1 and f_2 both contain the terms x_2 and x_1. The only feasible choice to break the uncoupled space is not to do any breaking. Consequently,

$$\mathbf{Q}_{\mathbf{u},\mathbf{v},1}=\begin{pmatrix}-\sigma & \sigma\\ r-u_3(t) & -1\end{pmatrix}.$$

For such $\mathbf{Q}_{\mathbf{u},\mathbf{v},1}$, its matrix measure cannot stay negative for all time. An indicated (see, e.g., Ref. 26), synchronization fails for this type of partial coupling.

(d) Let

$$\mathbf{D}=\mathbf{D}_4=\begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}.$$

To apply Theorem 3.3, we first note that for

$$\mathbf{D}=\mathbf{D}_5=\begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},$$

the corresponding coupled system $(\mathbf{D}_5,\mathbf{G},\mathbf{F}(\mathbf{x}))$ is indeed globally synchronized, and hence, so is the system $(\mathbf{D}_4,\mathbf{G},\mathbf{F}(\mathbf{x}))$. Note that bounded dissipation of the system $(\mathbf{D}_4,\mathbf{G},\mathbf{F}(\mathbf{x}))$ can be verified similarly as in Ref. 26.

(e) The works that are most related to ours are those in Refs. 8 and 9. While their estimates for d_c seem to be sharper than ours, which we shall illustrate in case (f), their connectivity topology requires that off-diagonal entries be non-negative. We only assume our connectivity topology satisfies (2.4a) and (2.4b). Consider, for instance, the following matrix:

$$\mathbf{G}=\begin{pmatrix}-1 & 2 & 0 & -1\\ -1 & -1 & 0 & 2\\ 2 & -1 & -3 & 2\\ 0 & 0 & 3 & -3\end{pmatrix}.$$

Such $\mathbf{G}$ has some negative off-diagonal entries and satisfies (2.4a) and (2.4b). In fact, the eigenvalues of $\mathbf{G}$ are 0, $-1\pm\sqrt{5}i$, and -6. Clearly, applying our results, we see immediately that the coupled system $(\mathbf{D}_i,\mathbf{G},\mathbf{F}(\mathbf{x}))$, $i=1,2,4$ is globally synchronized. Numerical results (see Fig. 1) indeed confirm synchronization of such connectivity topology. We remark that by constructing the Lyapunov function as given in Ref. 26, one would be able to show bounded dissipation of the coupled system with this particular connectivity topology.

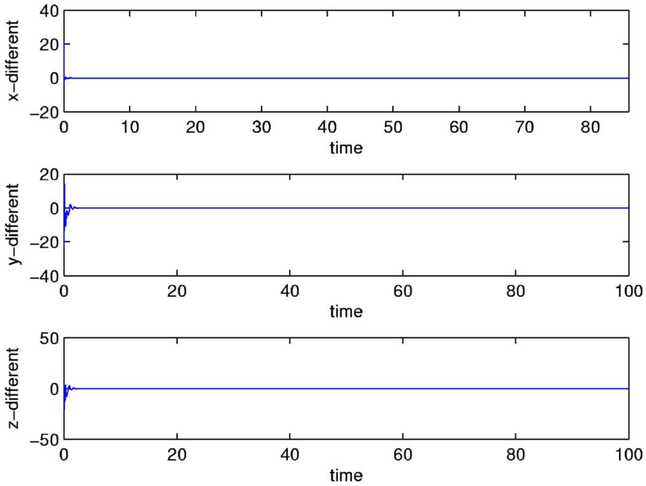

FIG. 1. (Color online) The difference of each component of two coupled oscillators in case (e).

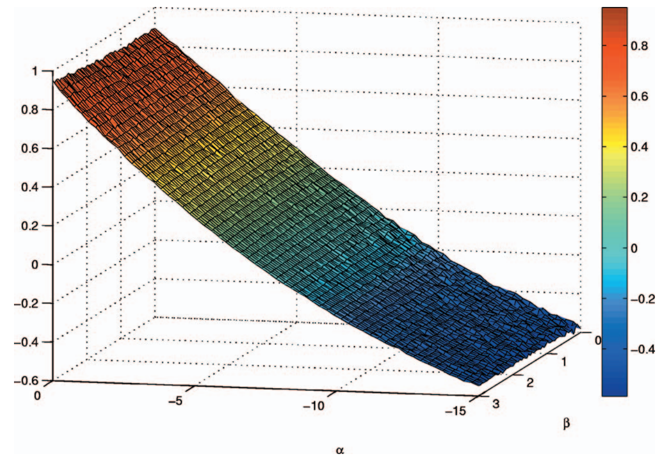

FIG. 2. (Color) The vertical axis denotes the maximum Lyapunov exponent of the variational equations, while the horizontal axis represents $\alpha = d\lambda$.

(f) In this part, we shall compute the lower bound for global synchronization for case (a) by using our method, those obtained in Ref. 8, and MSF, respectively. To compute d_c, given in (3.6), we note that $\bar{\mathbf{G}} = \mathbf{CGC}^T(\mathbf{CC}^T)^{-1} = \mathbf{C}(\mathbf{C}^T\mathbf{C})\mathbf{C}^T(\mathbf{CC}^T)^{-1} = \mathbf{CC}^T$. Since $\bar{\mathbf{G}}$ is symmetric, c and ϵ, given as in (3.7b), can be chosen to be 1 and 0, respectively. Consequently,

$$d_c = \frac{\sqrt{2}\sigma\sqrt{1+\beta+2\sigma\sqrt{\beta}+\sigma^2}}{4\sin^2\left(\frac{\pi}{2n}\right)}. \tag{4.3}$$

Here $4\sin^2(\pi/2n) = |\lambda_1|$. Applying Theorem 3.3, we see that the coupled system $(\mathbf{D},\mathbf{G},\mathbf{F}(\mathbf{x}))$ is uniformly, globally synchronized provided that the coupling strength d is greater than d_c. For $n=4$, $d_c \approx 1189$. In Ref. 8, the bound $\bar{d}_c$ for threshold of uniformly global synchronization is

$$\bar{d}_c = \begin{cases} \frac{a}{8}n^2 & \text{if } n \text{ is even} \\ \frac{a}{8}(n^2-1) & \text{if } n \text{ is odd.} \end{cases}$$

Here $a = [b(b+1)(r+\sigma)^2/16(b-1)] - \sigma$. For $n=4$, $\bar{d}_c \approx 1039$, which is slightly better than d_c.

Using the MSF criteria, we numerically (see Fig. 2) compute the maximum Lyapunov exponent of the variational equations with respect to the parameter α. We have, in this example, that if

$$\alpha = d\lambda_1 < -7.778, \tag{4.4}$$

then its maximum Lyapunov exponent is negative. Here $\lambda_1 = -4\sin^2(\pi/8)$ is the largest nonzero eigenvalue of $\mathbf{G}$. Hence, if $d > -7.778/\lambda_1 \approx 13.3$, then local synchronization of the coupled system $(\mathbf{D},\mathbf{G},\mathbf{F}(\mathbf{x}))$ can be realized.

(ii) Another formulation not considered in Refs. 7 and 8 is the Duffing oscillators. Specifically, the individual system considered is defined by

$$\dot{x}_1 = -\alpha x_1 - x_2^3 + a\cos wt, \tag{4.5a}$$

$$\dot{x}_2 = x_1, \tag{4.5b}$$

where α and a are positive constants. Letting $\mathbf{x} = (x_1, x_2)^T$, we have

$$\mathbf{f}(\mathbf{x},t) = (f_1(\mathbf{x},t), f_2(\mathbf{x})) = (-\alpha x_1 - x_2^3 + a\cos wt, x_1). \tag{4.6a}$$

Assume coupling matrices $\mathbf{D}$ and $\mathbf{G}$ are, respectively,

$$\mathbf{D}(c) = \begin{pmatrix} 1 & c \\ 0 & 0 \end{pmatrix} \tag{4.6b}$$

and

$$\mathbf{G}(\epsilon,r) = \begin{pmatrix} -2\epsilon & \epsilon-r & 0 & \cdots & 0 & \epsilon+r \\ \epsilon+r & -2\epsilon & \epsilon-r & \ddots & & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & -2\epsilon & \epsilon-r \\ \epsilon-r & 0 & \cdots & 0 & \epsilon+r & -2\epsilon \end{pmatrix}, \tag{4.6c}$$

where $\epsilon > 0$ and r are scalar diffusive and gradient coupling parameters, respectively. Note that

$$f_2(\mathbf{u}) - f_2(\mathbf{v}) = 0(u_2 - v_2) + (u_1 - v_1),$$

and so the matrix measure of the corresponding $\mathbf{Q}_{\mathbf{u},\mathbf{v},1}$ is zero. To apply our theorem, we need to make the following coordinate change.

Letting $y_2 = x_2$ and $y_1 = qx_1 + px_2$, we see that (4.5a) and (4.5b) become

$$\dot{y}_1 = \left(\frac{p}{q} - \alpha\right)y_1 + p\left(\alpha - \frac{p}{q}\right)y_2 - qy_2^3 + qa\cos wt =: \bar{\mathbf{f}}_1(\mathbf{y}), \tag{4.7a}$$

$$\dot{y}_2 = \frac{-p}{q}y_2 + \frac{1}{q}y_1 =: \bar{\mathbf{f}}_2(\mathbf{y}), \tag{4.7b}$$

and the corresponding coupled system (3.2) becomes

$$\dot{\tilde{\mathbf{y}}}_1 = \left(\frac{p}{q} - \alpha\right)\tilde{\mathbf{y}}_1 + p\left(\alpha - \frac{p}{q}\right)\tilde{\mathbf{y}}_2 - q\tilde{\mathbf{y}}_2^3 + \mathbf{g}(t) + d(qc-p)\mathbf{G}(\epsilon,r)\tilde{\mathbf{y}}_2 + d\mathbf{G}(\epsilon,r)\tilde{\mathbf{y}}_1, \quad (4.8a)$$

$$\dot{\tilde{\mathbf{y}}}_2 = -\frac{q}{p}\tilde{\mathbf{y}}_2 + \frac{1}{q}\tilde{\mathbf{y}}_1, \quad (4.8b)$$

where $\tilde{\mathbf{y}}_2^3 = (y_{1,2}^3, \dots, y_{m,2}^3)^T$ and $\mathbf{g}(t) = a\cos(wt)\,(1,\dots,1)^T$. In the following, we choose (p,q) to be $(1, c-1/d)$ as $c>0$, and to be $(-1,-1/d)$ as $c=0$, respectively. Then, in the case of $c>0$, Eq. (4.8) becomes

$$\dot{\tilde{\mathbf{y}}}_1 = d\mathbf{G}(\epsilon,r)\tilde{\mathbf{y}}_1 + \left(c - \alpha - \frac{1}{d}\right)\tilde{\mathbf{y}}_1 + \left(\alpha - c + \frac{1}{d}\right)\tilde{\mathbf{y}}_2 - \tilde{\mathbf{y}}_2^3 + \mathbf{g}(t) + \mathbf{G}(\epsilon,r)\tilde{\mathbf{y}}_2$$

$$=: d\mathbf{G}(\epsilon,r)\tilde{\mathbf{y}}_1 + \tilde{\mathbf{F}}_c(\tilde{\mathbf{y}},t),$$

$$\dot{\tilde{\mathbf{y}}}_2 = -\frac{1}{c-\frac{1}{d}}\tilde{\mathbf{y}}_2 + \tilde{\mathbf{y}}_1.$$

The purpose of the coordinate transformation is twofold. First is to make the dynamics of the linear part on the uncoupled space stable. In this case, the coefficient of $\tilde{\mathbf{y}}_2$ becomes negative when $d>2/c$. Second is to make sure the parameters in the nonlinear part of coupled space contain no bad influence of d, coupling strength. Otherwise, we may not be able to control its corresponding dynamics by choosing d large.

It is then easy to check that assumptions for Theorem 3.1 are all satisfied, and similar arguments can be followed for the case of $c=0$. Finally, in the Appendix, we will show that if $4\alpha/4+\alpha m^2 > c \geq 0$, $\epsilon>0$, and $r\in\mathbb{R}$, then the coupled system $(\mathbf{D}(c),\mathbf{G}(\epsilon,r),\mathbf{F}(\mathbf{x},t))$ is bounded dissipative. Thus, we can summarize the results as follows.

Theorem 4.1. *Let* $\mathbf{f}$, $\mathbf{D}(c)$, *and* $\mathbf{G}(\epsilon,r)$ *be given as in* (4.6a), (4.6b), and (4.6c), *respectively. Let* $0\leq c<4\alpha/4+\alpha^2 m$. *Then, the coupled system* $(\mathbf{D}(c),\mathbf{G}(\epsilon,r),\mathbf{F}(\mathbf{x},t))$ *is globally synchronized provided that d is chosen sufficiently large.*

Proof. It remains only to verify that $\mathbf{G}(\epsilon,r)$ satisfies assumptions (2.4a) and (2.4b). Indeed $\mathbf{G}(\epsilon,r)$ is a circulant matrix (see, e.g., Ref. 39); the eigenvalues λ_k of $\mathbf{G}(\epsilon,r)$ are

$$\lambda_k = -2\epsilon\left(1-\cos\frac{2k\pi}{n}\right) - i2r\sin\frac{2k\pi}{n}, \quad k=0,\dots,m-1.$$

□

Remark 4.1. (i) It was shown in Ref. 23 that there are positive constants d_0 and c_0 such that for $d\geq d_0$, $c\geq c_0$, the system $(\mathbf{D}(c),\mathbf{G}(\epsilon,0),\mathbf{F})$ given in (4.7) is synchronized. Our results also work for the case that c_0 is zero or small, or $\mathbf{G}(\epsilon,r)$, $r\neq 0$. (ii) It was shown in Ref. 15 that there are positive constants d_0 and c_0 such that for $d\geq d_0$, $c\geq c_0$, the system $(\mathbf{D}(c),\mathbf{G},\mathbf{F})$ is synchronized. Here $-\mathbf{G}$ is a positive definite matrix.

V. CONCLUSION

We have developed a theory to prove global synchronization in lattices of coupled chaotic systems. The results can be applied to quite general connectivity topology. In fact, it needs only to satisfy (2.4). In addition, a rigorous lower bound on the coupling strength to acquire global synchronization of the coupled system is obtained. Moreover, by merely checking the structure of the vector field of a single oscillator and verifying bounded dissipation of the coupled system, we shall be able to determine if the coupled system is synchronized or not. We conclude this paper by mentioning some possible future work. First, it is of great interest to extend our method to study the real world topology. Second, it is certainly worthwhile to study how bounded dissipation of the coupled system is related to the uncoupled dynamics and its connectivity topology. Third, it is interesting to study (global) synchronization of coupled systems, which lacks bounded dissipation, such as the Rössler system.

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APPENDIX: BOUNDED DISSIPATION

In this Appendix, we prove bounded dissipation of the systems considered in Sec. IV(ii). Setting $\tilde{\mathbf{x}}_2^3 = (x_{1,2}^3,\dots,x_{m,2}^3)^T$ and $\mathbf{g}(t)=a\cos(wt)\,(1,\dots,1)^T$, we see that (2.6) becomes

$$\dot{\tilde{\mathbf{x}}}_1 = -\alpha\tilde{\mathbf{x}}_1 - \tilde{\mathbf{x}}_2^3 + \mathbf{g}(t) + dc\mathbf{G}(\epsilon,r)\tilde{\mathbf{x}}_2 + d\mathbf{G}(\epsilon,r)\tilde{\mathbf{x}}_1, \quad (A1a)$$

$$\dot{\tilde{\mathbf{x}}}_2 = \tilde{\mathbf{x}}_1. \quad (A1b)$$

We consider the following scalar-valued function as the Lyapunov function of the coupled system $(\mathbf{D}(c),\mathbf{G}(\epsilon,r),\mathbf{F}(\mathbf{x},t))$:

$$U(\tilde{\mathbf{x}}_1,\tilde{\mathbf{x}}_2) = \frac{1}{2}\langle\tilde{\mathbf{x}}_1,\tilde{\mathbf{x}}_1\rangle + \sum_{i=1}^{m}\frac{x_{i,2}^4}{4} + c\langle\tilde{\mathbf{x}}_2,\tilde{\mathbf{x}}_1\rangle. \quad (A2)$$

Taking the time derivative of U along solutions of the coupled system $(\mathbf{D}(c),\mathbf{G}(\epsilon,r),\mathbf{F}(\mathbf{x},t))$, we have

$$\frac{dU}{dt} = \langle\tilde{\mathbf{x}}_1,\dot{\tilde{\mathbf{x}}}_1\rangle + \sum_{i=1}^{m}x_{i,2}^3 x_{i,1} + c\langle\tilde{\mathbf{x}}_1,\tilde{\mathbf{x}}_1\rangle + c\langle\tilde{\mathbf{x}}_2,\dot{\tilde{\mathbf{x}}}_1\rangle$$

$$= (c-\alpha)\langle\tilde{\mathbf{x}}_1,\tilde{\mathbf{x}}_1\rangle - c\alpha\langle\tilde{\mathbf{x}}_2,\tilde{\mathbf{x}}_1\rangle - c\langle\tilde{\mathbf{x}}_2,\tilde{\mathbf{x}}_2^3\rangle + \langle\tilde{\mathbf{x}}_1 + c\tilde{\mathbf{x}}_2,\mathbf{g}(t)\rangle + d\langle\tilde{\mathbf{x}}_1,\mathbf{G}(\epsilon,r)\tilde{\mathbf{x}}_1\rangle + 2dc\langle\tilde{\mathbf{x}}_1,\mathbf{G}(\epsilon,r)\tilde{\mathbf{x}}_2\rangle + dc^2\langle\tilde{\mathbf{x}}_2,\mathbf{G}(\epsilon,r)\tilde{\mathbf{x}}_2\rangle$$

$$= (c-\alpha)\langle\tilde{\mathbf{x}}_1,\tilde{\mathbf{x}}_1\rangle - c\alpha\langle\tilde{\mathbf{x}}_2,\tilde{\mathbf{x}}_1\rangle - c\langle\tilde{\mathbf{x}}_2,\tilde{\mathbf{x}}_2^3\rangle + \langle\tilde{\mathbf{x}}_1 + c\tilde{\mathbf{x}}_2,\mathbf{g}(t)\rangle + d(\tilde{\mathbf{x}}_1,\tilde{\mathbf{x}}_2)\left(\begin{pmatrix}1 & c\\ c & c^2\end{pmatrix}\otimes\mathbf{G}(\epsilon,r)\right)\begin{pmatrix}\tilde{\mathbf{x}}_1\\ \tilde{\mathbf{x}}_2\end{pmatrix}$$

$$\leq (c-\alpha)\langle\tilde{\mathbf{x}}_1,\tilde{\mathbf{x}}_1\rangle - c\alpha\langle\tilde{\mathbf{x}}_2,\tilde{\mathbf{x}}_1\rangle - c\langle\tilde{\mathbf{x}}_2,\tilde{\mathbf{x}}_2^3\rangle$$

$$+\langle \tilde{\mathbf{x}}_1 + c\tilde{\mathbf{x}}_2, \mathbf{g}(t)\rangle.$$

Note that the last inequality holds true since

$$\left(\begin{pmatrix} 1 & c \\ c & c^2 \end{pmatrix} \otimes \mathbf{G}(\epsilon,r)\right) + \left(\begin{pmatrix} 1 & c \\ c & c^2 \end{pmatrix} \otimes \mathbf{G}(\epsilon,r)\right)^T = \begin{pmatrix} 1 & c \\ c & c^2 \end{pmatrix} \otimes (\mathbf{G}(\epsilon,r)+\mathbf{G}(\epsilon,r)^T),$$

and $\mathbf{G}(\epsilon,r)+\mathbf{G}(\epsilon,r)^T$ is a nonpositive definite matrix. On the other hand, since

$$\langle \tilde{\mathbf{x}}_2, \tilde{\mathbf{x}}_2^3\rangle = \sum_{i=1}^{m} x_{2,i}^4 \geq \frac{1}{m}\left(\sum_{i=1}^{m} x_{i,2}^2\right)^2 \geq \frac{1}{m}\|\tilde{\mathbf{x}}_2\|_2^4,$$

we have

$$\frac{dU}{dt} \leq (c-\alpha)\|\tilde{\mathbf{x}}_1\|_2^2 + c\alpha\|\tilde{\mathbf{x}}_2\|_2\|\tilde{\mathbf{x}}_1\|_2 - \frac{c}{m}\|\tilde{\mathbf{x}}_2\|_2^4 + \sqrt{m}a(\|\tilde{\mathbf{x}}_1\|_2 + c\|\tilde{\mathbf{x}}_2\|_2) =: u(\|\tilde{\mathbf{x}}_2\|_1, \|\tilde{\mathbf{x}}_2\|_2).$$

We are now in a position to show bounded dissipation of the coupled system $(\mathbf{D}(c),\mathbf{G}(\epsilon,r),\mathbf{F}(\mathbf{x},t))$.

Proposition A.1.

(i) *If c satisfies the inequality*

$$0 < c < \min\left\{\frac{4\alpha}{4+\alpha^2 m}, \alpha\right\} = \frac{4\alpha}{4+\alpha^2 m}, \tag{A3}$$

then there exists a constant c_0 so that $dU/dt<0$ for $\|\tilde{\mathbf{x}}_2\|_1^2+\|\tilde{\mathbf{x}}_2\|_2^2 \geq c_0$;

(ii) *If $c=0$, then the first assertion of the proposition still holds true.*

Proof. Suppose $\|\tilde{\mathbf{x}}_2\|_2 \geq 1$. Then

$$u(\|\tilde{\mathbf{x}}_1\|_2, \|\tilde{\mathbf{x}}_2\|_2) \leq (c-\alpha)\|\tilde{\mathbf{x}}_1\|_2^2 + c\alpha\|\tilde{\mathbf{x}}_2\|_2\|\tilde{\mathbf{x}}_1\|_2 - \frac{c}{m}\|\tilde{\mathbf{x}}_2\|_2^2 + \sqrt{m}a(\|\tilde{\mathbf{x}}_1\|_2 + c\|\tilde{\mathbf{x}}_2\|_2) =: \bar{u}(\|\tilde{\mathbf{x}}_1\|_2, \|\tilde{\mathbf{x}}_2\|_2).$$

It then follows from (A3) that the level curve of $\bar{u}$ is a bounded closed curve. We shall call such an ellipse-like curve an elliptic in the plane. Thus, there exists a c_1 so that $dU/dt<0$ whenever $\|\tilde{\mathbf{x}}_2\|_1^2+\|\tilde{\mathbf{x}}_2\|_2^2 \geq c_1$ and $\|\tilde{\mathbf{x}}_2\|_2 \geq 1$. Let $\|\tilde{\mathbf{x}}_2\|_2 < 1$ and $\|\tilde{\mathbf{x}}_2\|_1^2+\|\tilde{\mathbf{x}}_2\|_2^2 \geq c_2$. Here c_2 is a constant to be determined. Then

$$u(\|\tilde{\mathbf{x}}_1\|_2, \|\tilde{\mathbf{x}}_2\|_2) \leq (c-\alpha)\|\tilde{\mathbf{x}}_1\|_2^2 + (c\alpha + \sqrt{m}a)\|\tilde{\mathbf{x}}_1\|_2 + \sqrt{m}ac =: h(\|\tilde{\mathbf{x}}_1\|_2).$$

Since $h(\|\tilde{\mathbf{x}}_1\|_2)$ is a parabola-like curve, which is open downward, there exists a $c_3>1$ such that $h(\|\tilde{\mathbf{x}}_1\|_2)<0$ whenever $\|\tilde{\mathbf{x}}_1\|_2 \geq c_3$. Thus, if $c_2 \geq c_3^2+1$, then $u(\|\tilde{\mathbf{x}}_1\|_2, \|\tilde{\mathbf{x}}_2\|_2)<0$ whenever $\|\tilde{\mathbf{x}}_2\|_2<1$ and $\|\tilde{\mathbf{x}}_1\|_2^2+\|\tilde{\mathbf{x}}_2\|_2^2 \geq c_2$. Picking $c_0=\max\{c_1,c_2\}$, we have that the assertion of the proposition holds true. □

Proposition A.2. *Assume (A3) holds true. Then* $\lim_{r\to\infty} U(\tilde{\mathbf{x}}_1,\tilde{\mathbf{x}}_2)=\infty$, *where* $r=\sqrt{\|\tilde{\mathbf{x}}_1\|^2+\|\tilde{\mathbf{x}}_2\|^2}$.

Proof. From Eq. (A2), we have that

$$U(\tilde{\mathbf{x}}_1,\tilde{\mathbf{x}}_2) = \frac{1}{2}\|\tilde{\mathbf{x}}_1\|^2 + \sum_{i=1}^{m}\frac{x_{i,2}^4}{4} + c\langle \tilde{\mathbf{x}}_2, \tilde{\mathbf{x}}_1\rangle \geq \frac{1}{2}\|\tilde{\mathbf{x}}_1\|^2 + \frac{1}{4m}\|\tilde{\mathbf{x}}_2\|^4 - c\|\tilde{\mathbf{x}}_2\|\cdot\|\tilde{\mathbf{x}}_1\|.$$

Let $(1/4m)b_1^2 > c^2$. Then suppose $\|\tilde{\mathbf{x}}_2\| > b_1$. We have

$$U(\tilde{\mathbf{x}}_1,\tilde{\mathbf{x}}_2) \geq \tfrac{1}{2}\|\tilde{\mathbf{x}}_1\|^2 + c^2\|\tilde{\mathbf{x}}_2\|^2 - c\|\tilde{\mathbf{x}}_2\|\|\tilde{\mathbf{x}}_1\| =: h_1(\|\tilde{\mathbf{x}}_1\|,\|\tilde{\mathbf{x}}_2\|),$$

since the level curve of $h_1(\|\tilde{\mathbf{x}}_1\|,\|\tilde{\mathbf{x}}_2\|)$ is elliptic-like in the plane. Thus, for any given $M>0$, there exists a $d_1>0$ such that $U(\tilde{\mathbf{x}}_1,\tilde{\mathbf{x}}_2)>M$ whenever $\|\tilde{\mathbf{x}}_1\|^2+\|\tilde{\mathbf{x}}_2\|^2 \geq d_1^2$ and $\|\tilde{\mathbf{x}}_2\| > b_1$.

Let $\|\tilde{\mathbf{x}}_2\| \leq b_1$. Then

$$U(\tilde{\mathbf{x}}_1,\tilde{\mathbf{x}}_2) \geq \tfrac{1}{2}\|\tilde{\mathbf{x}}_1\|^2 - cb_1\|\tilde{\mathbf{x}}_1\| =: h_2(\|\tilde{\mathbf{x}}_1\|,\|\tilde{\mathbf{x}}_2\|),$$

since $h_2(\|\tilde{\mathbf{x}}_1\|,\|\tilde{\mathbf{x}}_2\|)$ is a parabola-like curve, which is open upward in the plane. Thus, for any given $M>0$, there exists a $d_2>0$ such that $U(\tilde{\mathbf{x}}_1,\tilde{\mathbf{x}}_2)>M$ whenever $\|\tilde{\mathbf{x}}_1\|^2+\|\tilde{\mathbf{x}}_2\|^2 \geq d_2^2$ and $\|\tilde{\mathbf{x}}_2\| \leq b_1$. Picking $\delta=\max\{d_1,d_2\}$, we have that $U(\tilde{\mathbf{x}}_1,\tilde{\mathbf{x}}_2)>M$ for all $\|\tilde{\mathbf{x}}_1\|^2+\|\tilde{\mathbf{x}}_2\|^2 \geq \delta^2$. Thus, the assertion of the proposition holds true. □

Theorem A.1. *The coupled system* $(\mathbf{D}(c),\mathbf{G}(\epsilon,r),\mathbf{F}(\mathbf{x},t))$ *is bounded dissipative if condition (A3) holds true.*

Proof. The proof is a direct consequence of Propositions A.1 and A.2. □

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