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Pragmatical generalized synchronization of chaotic systems with uncertain parameters by adaptive control

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Abstract

A new kind of generalized synchronization of two chaotic systems with uncertain parameters is proposed. Based on a pragmatical asymptotical stability theorem and an assumption of equal probability for ergodic initial conditions, an adaptive control law is derived so that it can be proved strictly that the common null solution of error dynamics and of parameter dynamics is actually asymptotically stable, i.e. these two identical systems are in generalized synchronization and the estimated parameters approach the uncertain values. It is called pragmatical generalized synchronization. Finally, two numerical examples are studied for two Quantum-CNN oscillator chaotic systems to show the effectiveness of the proposed generalized synchronization strategy with a double Duffing chaotic system as a goal system.

Keywords: Quantum Cellular Neural Network (Quantum-CNN); Chaos; Pragmatical generalized synchronization; Pragmatic asymptotical stability theorem; Adaptive control

1. Introduction

The synchronization phenomenon has the following feature: the trajectories of the drive and response systems are identical notwithstanding starting from different initial conditions. However, slight errors of initial conditions, for chaotic dynamical systems, will lead to completely different trajectories [1–4]. Therefore, how controlling two chaotic systems to be synchronized is an attractive objective [5–8]. Many approaches have been presented for the synchronization of chaotic systems such as linear and nonlinear feedback control [9,10]. Most of them are based on the exact knowledge of the system structure and parameters. But in practice, some or all of the system parameters are uncertain. Moreover, these parameters change from time to time. A lot of works have proceeded to solve this problem by adaptive synchronization [11,12]. In the current scheme of adaptive synchronization [13–15], the traditional Lyapunov stability theorem and Babalat lemma are used to prove that the error vector approaches zero as time approaches infinity. But the question of why the estimated parameters also approach uncertain values has remained without answer. Based on a pragmatical asymptotical stability theorem and an assumption of equal probability for ergodic initial conditions [16,17], the question is answered.

Among many kinds of synchronizations [18–24], the generalized synchronization is investigated [25–30]. This means that there exists a given functional relationship between the states of the master and that of the slave $y = G(x)$. In this paper, a special kind of generalized synchronizations

$$y = G(x) = x + F(t) \tag{1}$$

is studied, where x, y are the state vectors of the master and of the slave, respectively. $F(t)$ is a given vector function of time which may take various forms, either regular or chaotic functions of time. When $F(t) = 0$, it reduces to a complete synchronization [31,32].

As two numerical examples, two identical Quantum Cellular Neural Network (Quantum-CNN) chaotic systems [33] and a double Duffing chaotic system are used as the master system, slave system, and goal system, respectively. The goal system

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gives a chaotic $F(t)$. Quantum-CNN oscillator equations are derived from a Schrödinger equation taking into account quantum dot cellular automata structures to which in the last decade a wide interest has been devoted, with particular attention towards quantum computing.

This paper is organized as follows. In Section 2, by the pragmatical asymptotical stability theorem, a new pragmatical generalized synchronization scheme by adaptive control is given. In Section 3, adaptive controllers are designed for the pragmatical generalized synchronization of two Quantum-CNN chaotic oscillators with a double Duffing chaotic system as a goal system in two examples. Numerical simulations are also given in Section 3. Finally, conclusions are drawn in Section 4.

2. Pragmatical generalized synchronization scheme, by adaptive control

There are two identical nonlinear dynamical systems, and the master system controls the slave system. The master system is given by

$$\dot{x} = Ax + f(x, B) \tag{2}$$

where $x = [x_1, x_2, \ldots, x_n]^{\mathrm{T}} \in R^n$ denotes a state vector, A is an $n \times n$ uncertain constant coefficient matrix, f is a nonlinear vector function, and B is a vector of uncertain constant coefficients in f.

The slave system is given by

$$\dot{y} = \hat{A}y + f(y, \hat{B}) + u(t) \tag{3}$$

where $y = [y_1, y_2, \ldots, y_n]^{\mathrm{T}} \in R^n$ denotes a state vector, $\hat{A}$ is an $n \times n$ estimated coefficient matrix, $\hat{B}$ is a vector of estimated coefficients in f, and $u(t) = [u_1(t), u_2(t), \ldots, u_n(t)]^{\mathrm{T}} \in R^n$ is a control input vector.

Our goal is to design a controller $u(t)$ so that the state vector of the slave system (3) asymptotically approaches the state vector of the master system (2) plus a given chaotic vector function $F(t) = [F_1(t), F_2(t), \ldots, F_n(t)]^{\mathrm{T}}$. This is a special kind of generalized synchronization; y is a given function of x:

$$y = G(x) = x + F(t). \tag{4}$$

The synchronization can be accomplished when $t \to \infty$; the limit of the error vector $e(t) = [e_1, e_2, \ldots, e_n]^{\mathrm{T}}$ approaches zero:

$$\lim_{t\to\infty} e = 0 \tag{5}$$

where

$$e = x - y + F(t). \tag{6}$$

From Eq. (6) we have

$$\dot{e} = \dot{x} - \dot{y} + \dot{F}(t) \tag{7}$$

$$\dot{e} = Ax - \hat{A}y + f(x, B) - f(y, \hat{B}) + \dot{F}(t) - u(t). \tag{8}$$

A Lyapunov function $V(e, \tilde{A}_c, \tilde{B}_c)$ is chosen as a positive definite function

$$V(e, \tilde{A}_c, \tilde{B}_c) = \frac{1}{2}e^{\mathrm{T}}e + \frac{1}{2}\tilde{A}_c^{\mathrm{T}}\tilde{A}_c + \frac{1}{2}\tilde{B}_c^{\mathrm{T}}\tilde{B}_c \tag{9}$$

where $\tilde{A} = A - \hat{A}$, $\tilde{B} = B - \hat{B}$, $\tilde{A}_c$ and $\tilde{B}_c$ are two column matrices whose elements are all the elements of matrix $\tilde{A}$ and of matrix $\tilde{B}$, respectively.

Its derivative along any solution of the differential equation system consisting of Eq. (8) and update parameter differential equations for $\tilde{A}_c$ and $\tilde{B}_c$ is

$$\dot{V}(e, \tilde{A}_c, \tilde{B}_c) = e^{\mathrm{T}}[Ax - \hat{A}y + Bf(x) - \hat{B}f(y) + \dot{F}(t) - u(t)] + \tilde{A}_c\dot{\tilde{A}}_c + \tilde{B}_c\dot{\tilde{B}}_c \tag{10}$$

where $u(t)$, $\dot{\tilde{A}}_c$, and $\dot{\tilde{B}}_c$ are chosen so that $\dot{V} = e^{\mathrm{T}}Ce, C$ is a diagonal negative definite matrix, and $\dot{V}$ is a negative semi-definite function of e and parameter differences $\tilde{A}_c$ and $\tilde{B}_c$. In the current scheme of adaptive synchronization [13–15], the traditional Lyapunov stability theorem and Babalat lemma are used to prove that the error vector approaches zero, as time approaches infinity. But the question of why the estimated parameters also approach uncertain parameters remains unanswered. By the pragmatical asymptotical stability theorem, the question can be answered strictly.

The stability for many problems in real dynamical systems is actual asymptotical stability, although it may not be mathematical asymptotical stability. The mathematical asymptotical stability demands that trajectories from all initial states in the neighborhood of the zero solution must approach the origin as $t \to \infty$. If there are only a small part or even a few of the initial states from which the trajectories do not approach the origin as $t \to \infty$, the zero solution is not mathematically asymptotically stable. However, when the probability of occurrence of an event is zero, it means the event does not occur actually. If the probability of occurrence of the event that the trajectories from the initial states are such that they do not approach zero when $t \to \infty$, is zero, the stability of the zero solution is actual asymptotical stability though it is not mathematical asymptotical stability. In order to analyze the asymptotical stability of the equilibrium point of such systems, the pragmatical asymptotical stability theorem is used.

Let X and Y be two manifolds of dimensions m and n ($m < n$), respectively, and φ be a differentiable map from X to Y; then $\varphi(X)$ is a subset of Lebesque measure 0 of Y [34]. For an autonomous system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x_1, \ldots, x_n) \tag{11}$$

where $x = [x_1, \ldots, x_n]^{\mathrm{T}}$ is a state vector, the function $f = [f_1, \ldots, f_n]^{\mathrm{T}}$ is defined on $D \subset R^n$ and $\|x\| \le H > 0$. Let $x = 0$ be an equilibrium point for the system (11). Then

$$f(0) = 0. \tag{12}$$

Definition. The equilibrium point for the system (11) is pragmatically asymptotically stable provided that with initial points on C which is a subset of Lebesque measure 0 of D, the behaviors of the corresponding trajectories cannot be determined, while with initial points on $D - C$, the corresponding trajectories behave as if they agree with traditional asymptotical stability [16,17].

Theorem. *Let $V = [x_1, \ldots, x_n]^{\mathrm{T}} : D \to R_+$ be positive definite and analytic on D, such that the derivative of V through Eq. (11), $\dot{V}$, is negative semi-definite.*

Let X be the m-manifold consisting of the point set for which $\forall x \neq 0$, $\dot{V}(x) = 0$ and D is an n-manifold. If $m + 1 < n$, then the equilibrium point of the system is pragmatically asymptotically stable.

Proof. Since every point of X can be passed by a trajectory of Eq. (11), which is one dimensional, the collection of these trajectories, C, is an $(m + 1)$-manifold [16,17].

If $m + 1 < n$, then the collection C is a subset of Lebesque measure 0 of D. By the above definition, the equilibrium point of the system is pragmatically asymptotically stable. □

If an initial point is ergodicly chosen in D, *the probability of the initial point falling on the collection C is zero. Here, equal probability is assumed for every point chosen as an initial point in the neighborhood of the equilibrium point.* Hence, the event that the initial point is chosen from collection C *does not occur actually*. Therefore, under the equal probability assumption, pragmatical asymptotical stability becomes actual asymptotical stability. When the initial point falls on $D - C$, $\dot{V}(x) < 0$, the corresponding trajectories behave as if they agree with traditional asymptotical stability because by the existence and uniqueness of the solution of the initial-value problem, these trajectories never meet C.

In Eq. (9) V is a positive definite function of n variables, i.e. p error state variables and $n - p = m$ differences between unknown and estimated parameters, while $\dot{V} = e^{\mathrm{T}}Ce$ is a negative semi-definite function of n variables. Since the number of error state variables is always more than one, $p > 1$, $m + 1 < n$ is always satisfied; by the pragmatical asymptotical stability theorem we have

$$\lim_{t\to\infty} e = 0 \tag{13}$$

and the estimated parameters approach the uncertain parameters. The pragmatical generalized synchronization is obtained. Therefore, the equilibrium point of the system is *pragmatically asymptotically stable. Under the equal probability assumption, it is actually asymptotically stable for both error state variables and parameter variables*.

3. Numerical results of pragmatical generalized chaos synchronization of two Quantum-CNN oscillators by adaptive control

Case I. The chaotic states of a goal system, a double Duffing chaotic system, used as $F(t)$.

For a two-cell Quantum-CNN, the following differential equations are used [33] as the master system:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}x_1 = -2a_1\sqrt{1-x_1^2}\sin x_2 \\ \dfrac{\mathrm{d}}{\mathrm{d}t}x_2 = -\omega_1(x_1 - x_3) + 2a_1\dfrac{x_1}{\sqrt{1-x_1^2}}\cos x_2 \\ \dfrac{\mathrm{d}}{\mathrm{d}t}x_3 = -2a_2\sqrt{1-x_3^2}\sin x_4 \\ \dfrac{\mathrm{d}}{\mathrm{d}t}x_4 = -\omega_2(x_3 - x_1) + 2a_2\dfrac{x_3}{\sqrt{1-x_3^2}}\cos x_4 \end{cases} \tag{14}$$

where x_1, x_3 are polarizations, x_2, x_4 are quantum phase displacements, a_1 and a_2 are proportional to the inter-dot energy inside each cell and ω_1 and ω_2 are parameters that weigh effects on the cell of the difference of the polarization of neighboring cells, like the cloning templates in traditional CNNs. Let $a_1 = 4.9$, $a_2 = 4.9$, $\omega_1 = 3.03$, $\omega_2 = 1.83$.

A slave system is described by

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}y_1 = -2\hat{a}_1\sqrt{1-y_1^2}\sin y_2 \\ \dfrac{\mathrm{d}}{\mathrm{d}t}y_2 = -\hat{\omega}_1(y_1 - y_3) + 2\hat{a}_1\dfrac{y_1}{\sqrt{1-y_1^2}}\cos y_2 \\ \dfrac{\mathrm{d}}{\mathrm{d}t}y_3 = -2\hat{a}_2\sqrt{1-y_3^2}\sin y_4 \\ \dfrac{\mathrm{d}}{\mathrm{d}t}y_4 = -\hat{\omega}_2(y_3 - y_1) + 2\hat{a}_2\dfrac{y_3}{\sqrt{1-y_3^2}}\cos y_4. \end{cases} \tag{15}$$

In order to lead (y_1, y_2, y_3, y_4) to $(x_1 + F_1(t), x_2 + F_2(t), x_3 + F_3(t), x_4 + F_4(t))$, we add u_1, u_2, u_3, and u_4 to each equation of Eq. (15), respectively:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}y_1 = -2\hat{a}_1\sqrt{1-y_1^2}\sin y_2 + u_1 \\ \dfrac{\mathrm{d}}{\mathrm{d}t}y_2 = -\hat{\omega}_1(y_1 - y_3) + 2\hat{a}_1\dfrac{y_1}{\sqrt{1-y_1^2}}\cos y_2 + u_2 \\ \dfrac{\mathrm{d}}{\mathrm{d}t}y_3 = -2\hat{a}_2\sqrt{1-y_3^2}\sin y_4 + u_3 \\ \dfrac{\mathrm{d}}{\mathrm{d}t}y_4 = -\hat{\omega}_2(y_3 - y_1) + 2\hat{a}_2\dfrac{y_3}{\sqrt{1-y_3^2}}\cos y_4 + u_4. \end{cases} \tag{16}$$

Subtracting Eq. (16) from Eq. (14), we obtain an error dynamics. The initial values of the master system and the slave system are taken as $x_1(0) = 0.8$, $x_2(0) = -0.77$, $x_3(0) = -0.72$, $x_4(0) = 0.57$, $y_1(0) = 0.1$, $y_2(0) = 0.28$, $y_3(0) = 0.42$, and $y_4(0) = -0.72$, respectively.

The goal system for generalized synchronization is a double Duffing chaotic system

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = z_1 - z_1^3 - \delta_1 z_2 + f_1\cos\psi_1 t \\ \dot{z}_3 = z_4 \\ \dot{z}_4 = z_3 - z_3^3 - \delta_2 z_4 + f_2\cos\psi_2 t \end{cases} \tag{17}$$

where $\delta_1 = 13.5$, $\delta_2 = 12.5$, $f_1 = -24.9$, $f_2 = -33.1$, $\Psi_1 = 10.9$, $\Psi_2 = 19.9$, $z_1(0) = 0.75$, $z_2(0) = -0.3$, $z_3(0) = -0.4$, and $z_4(0) = 0.5$.

We have

$$\lim_{t\to\infty} e_i = \lim_{t\to\infty}(x_i - y_i + z_i) = 0, \quad i = 1, 2, 3, 4 \tag{18}$$

where $\dot{e} = \dot{x} - \dot{y} + \dot{z}$, and

$$\begin{aligned}
\dot{e}_1 &= -2a_1\sqrt{1-x_1^2}\sin x_2 + 2\hat{a}_1\sqrt{1-y_1^2}\sin y_2 - u_1 + \dot{z}_1\\
\dot{e}_2 &= -\omega_1(x_1 - x_3) + \hat{\omega}_1(y_1 - y_3)\\
&\quad + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 - 2\hat{a}_1\frac{y_1}{\sqrt{1-y_1^2}}\cos y_2 - u_2 + \dot{z}_2\\
\dot{e}_3 &= -2a_2\sqrt{1-x_3^2}\sin x_4 + 2\hat{a}_2\sqrt{1-y_3^2}\sin y_4 - u_3 + \dot{z}_3\\
\dot{e}_4 &= -\omega_2(x_3 - x_1) + \hat{\omega}_2(y_3 - y_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4\\
&\quad - 2\hat{a}_2\frac{y_3}{\sqrt{1-y_3^2}}\cos y_4 - u_4 + \dot{z}_4
\end{aligned} \tag{19}$$

where $e_1 = x_1 - y_1 + z_1$, $e_2 = x_2 - y_2 + z_2$, $e_3 = x_3 - y_3 + z_3$, and $e_4 = x_4 - y_4 + z_4$.

Choose a Lyapunov function in the form of a positive definite function:

$$\begin{aligned}
&V(e_1, e_2, e_3, e_4, \tilde{a}_1, \tilde{a}_2, \tilde{\omega}_1, \tilde{\omega}_2)\\
&\quad = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + e_4^2 + \tilde{a}_1^2 + \tilde{a}_2^2 + \tilde{\omega}_1^2 + \tilde{\omega}_2^2)
\end{aligned} \tag{20}$$

where $\tilde{a}_1 = a_1 - \hat{a}_1$, $\tilde{a}_2 = a_2 - \hat{a}_2$, $\tilde{\omega}_1 = \omega_1 - \hat{\omega}_1$, $\tilde{\omega}_2 = \omega_2 - \hat{\omega}_2$ and $\hat{a}_1, \hat{a}_2, \hat{\omega}_1, \hat{\omega}_2$ are estimates of uncertain parameters a_1, a_2, ω_1, and ω_2, respectively.

Its time derivative is

$$\begin{aligned}
\dot{V} &= e_1[-2a_1\sqrt{1-x_1^2}\sin x_2 + 2\hat{a}_1\sqrt{1-y_1^2}\sin y_2 - u_1 + z_2]\\
&\quad + e_2\Big[-\omega_1(x_1 - x_3) + \hat{\omega}_1(y_1 - y_3)\\
&\quad + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 - 2\hat{a}_1\frac{y_1}{\sqrt{1-y_1^2}}\cos y_2 - u_2\\
&\quad + z_1 - z_1^3 - \delta_1 z_2 + f_1\cos\psi_1 t\Big]\\
&\quad + e_3[-2a_2\sqrt{1-x_3^2}\sin x_4 + 2\hat{a}_2\sqrt{1-y_3^2}\sin y_4\\
&\quad - u_3 + z_4] + e_4\Big[-\omega_2(x_3 - x_1) + \hat{\omega}_2(y_3 - y_1)\\
&\quad + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4 - 2\hat{a}_2\frac{y_3}{\sqrt{1-y_3^2}}\cos y_4\\
&\quad - u_4 + z_3 - z_3^3 - \delta_2 z_4\\
&\quad + f_2\cos\psi_2 t\Big] + \tilde{a}_1(-\dot{\hat{a}}_1) + \tilde{a}_2(-\dot{\hat{a}}_2)\\
&\quad + \tilde{\omega}_1(-\dot{\hat{\omega}}_1) + \tilde{\omega}_2(-\dot{\hat{\omega}}_2).
\end{aligned} \tag{21}$$

Choose

$$\begin{aligned}
u_1 &= -2a_1\sqrt{1-x_1^2}\sin x_2 + 2\hat{a}_1\sqrt{1-y_1^2}\sin y_2 + \hat{a}_1 e_1\\
&\quad + \frac{\hat{a}_1 z_2}{a_1} + \tilde{a}_1^2\\
u_2 &= 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 - 2\hat{a}_1\frac{y_1}{\sqrt{1-y_1^2}}\cos y_2 + z_1 - z_1^3\\
&\quad + f_1\cos\psi_1 t + \hat{\omega}_1 e_2 - \omega_1(x_1 - x_3) + \hat{\omega}_1(y_1 - y_3)\\
&\quad - \frac{\hat{\omega}_1\delta_1}{\omega_1}z_2 + \tilde{\omega}_1^2\\
u_3 &= -2a_2\sqrt{1-x_3^2}\sin x_4 + 2\hat{a}_2\sqrt{1-y_3^2}\sin y_4\\
&\quad + \hat{a}_2 e_3 + \frac{\hat{a}_2 z_4}{a_2} + \tilde{a}_2^2\\
u_4 &= 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4 - 2\hat{a}_2\frac{y_3}{\sqrt{1-y_3^2}}\cos y_4 + z_3 - z_3^3\\
&\quad + f_2\cos\psi_2 t + \hat{\omega}_2 e_4 - \omega_2(x_3 - x_1) + \hat{\omega}_2(y_3 - y_1)\\
&\quad - \frac{\hat{\omega}_2\delta_2}{\omega_2}z_4 + \tilde{\omega}_2^2
\end{aligned} \tag{22}$$

$$\begin{aligned}
\dot{\tilde{a}}_1 &= -\dot{\hat{a}}_1 = -\frac{e_1 z_2}{a_1} + \tilde{a}_1 e_1 - e_1^2\\
\dot{\tilde{\omega}}_1 &= -\dot{\hat{\omega}}_1 = \frac{\delta_1}{\omega_1}e_2 z_2 + \tilde{\omega}_1 e_2 - e_2^2\\
\dot{\tilde{a}}_2 &= -\dot{\hat{a}}_2 = -\frac{e_3 z_4}{a_2} + \tilde{a}_2 e_3 - e_3^2\\
\dot{\tilde{\omega}}_2 &= -\dot{\hat{\omega}}_2 = \frac{\delta_2}{\omega_2}e_4 z_4 + \tilde{\omega}_2 e_4 - e_4^2.
\end{aligned} \tag{23}$$

The initial values of estimates for uncertain parameters are $\hat{a}_1(0) = \hat{a}_2(0) = \hat{\omega}_1(0) = \hat{\omega}_2(0) = 0$.

Substituting Eqs. (22) and (23) into Eq. (21), we obtain

$$\dot{V} = -a_1 e_1^2 - \omega_1 e_2^2 - a_2 e_3^2 - \omega_2 e_4^2 \le 0 \tag{24}$$

which is a negative semi-definite function of e_1, e_2, e_3, e_4, $\tilde{a}_1$, $\tilde{a}_2$, $\tilde{\omega}_1$, and $\tilde{\omega}_2$. The Lyapunov asymptotical stability theorem is not satisfied. We cannot obtain that the common origin of error dynamics (19) and parameter dynamics (23) is asymptotically stable. Now, D is an 8-manifold, $n = 8$ and the number of error state variables $p = 4$. When $e_1 = e_2 = e_3 = e_4 = 0$ and $\tilde{a}_1, \tilde{a}_2, \tilde{\omega}_1, \tilde{\omega}_2$ take arbitrary values, $\dot{V} = 0$, so X is a 4-manifold, $m = n - p = 8 - 4 = 4$. $m + 1 < n$ is satisfied. By the pragmatical asymptotical stability theorem, error vector e approaches zero and the estimated parameters also approach the uncertain parameters. The pragmatical generalized synchronization is obtained. The equilibrium point $e_1 = e_2 = e_3 = e_4 = \tilde{a}_1 = \tilde{a}_2 = \tilde{\omega}_1 = \tilde{\omega}_2 = 0$ is pragmatically asymptotically stable. Under the assumption of equal probability, it is actually asymptotically stable. The numerical results are shown in Fig. 1. After 10 s, the generalized synchronization is accomplished.

Case II. The cubics of chaotic states of the goal system, a double Duffing chaotic system, used as $F(t)$.

We demand

$$\lim_{t\to\infty} e_i = \lim_{t\to\infty}(x_i - y_i + z_i^3) = 0, \quad i = 1, 2, 3, 4 \tag{25}$$

and then

$$\dot{e} = \dot{x} - \dot{y} + 3z^2\dot{z}. \tag{26}$$

$$\begin{aligned}
\dot{e}_1 &= -2a_1\sqrt{1-x_1^2}\sin x_2 + 2\hat{a}_1\sqrt{1-y_1^2}\sin y_2 - u_1 + 3z_1^2\dot{z}_1\\
\dot{e}_2 &= -\omega_1(x_1-x_3) - \hat{\omega}_1(y_1-y_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2\\
&\quad - 2\hat{a}_1\frac{y_1}{\sqrt{1-y_1^2}}\cos y_2 - u_2 + 3z_2^2\dot{z}_2 \qquad (27)\\
\dot{e}_3 &= -2a_2\sqrt{1-x_3^2}\sin x_4 + 2\hat{a}_2\sqrt{1-y_3^2}\sin y_4 - u_3 + 3z_3^2\dot{z}_3\\
\dot{e}_4 &= -\omega_2(x_3-x_1) + \hat{\omega}_2(y_3-y_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4\\
&\quad - 2\hat{a}_2\frac{y_3}{\sqrt{1-y_3^2}}\cos y_4 - u_4 + 3z_4^2\dot{z}_4
\end{aligned}$$

where $e_1 = x_1 - y_1 + z_1^3$, $e_2 = x_2 - y_2 + z_2^3$, $e_3 = x_3 - y_3 + z_3^3$, and $e_4 = x_4 - y_4 + z_4^3$.

Choose a Lyapunov function in the form of a positive definite function:

$$\begin{aligned}
&V(e_1, e_2, e_3, e_4, \tilde{a}_1, \tilde{a}_2, \tilde{\omega}_1, \tilde{\omega}_2)\\
&\quad = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + e_4^2 + \tilde{a}_1^2 + \tilde{a}_2^2 + \tilde{\omega}_1^2 + \tilde{\omega}_2^2) \qquad (28)
\end{aligned}$$

where $\tilde{a}_1 = a_1 - \hat{a}_1$, $\tilde{a}_2 = a_2 - \hat{a}_2$, $\tilde{\omega}_1 = \omega_1 - \hat{\omega}_1$, $\tilde{\omega}_2 = \omega_2 - \hat{\omega}_2$ and $\hat{a}_1, \hat{a}_2, \hat{\omega}_1, \hat{\omega}_2$ are estimates of uncertain parameters a_1, a_2, ω_1, and ω_2, respectively.

Its time derivative is

$$\begin{aligned}
\dot{V} &= e_1[-2a_1\sqrt{1-x_1^2}\sin x_2 + 2\hat{a}_1\sqrt{1-y_1^2}\sin y_2\\
&\quad - u_1 + 3z_1^2 z_2] + e_2\Bigg[-\omega_1(x_1-x_3) + \hat{\omega}_1(y_1-y_3)\\
&\quad + 3z_2^2(z_1 - z_1^3 - \delta_1 z_2 + f_1\cos\psi_1 t)\\
&\quad + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 - 2\hat{a}_1\frac{y_1}{\sqrt{1-y_1^2}}\cos y_2 - u_2\Bigg]\\
&\quad + e_3[-2a_2\sqrt{1-x_3^2}\sin x_4 + 2\hat{a}_2\sqrt{1-y_3^2}\sin y_4\\
&\quad - u_3 + 3z_3^2 z_4] + e_4\Bigg[-\omega_2(x_3-x_1) + \hat{\omega}_2(y_3-y_1)\\
&\quad + 3z_4^2(z_3 - z_3^3 - \delta_2 z_4 + f_2\cos\psi_2 t)\\
&\quad + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4 - 2\hat{a}_2\frac{y_3}{\sqrt{1-y_3^2}}\cos y_4 - u_4\Bigg]\\
&\quad + \tilde{a}_1(-\dot{\hat{a}}_1) + \tilde{a}_2(-\dot{\hat{a}}_2) + \tilde{\omega}_1(-\dot{\hat{\omega}}_1) + \tilde{\omega}_2(-\dot{\hat{\omega}}_2). \qquad (29)
\end{aligned}$$

Choose

$$\begin{aligned}
u_1 &= -2a_1\sqrt{1-x_1^2}\sin x_2 + 2\hat{a}_1\sqrt{1-y_1^2}\sin y_2\\
&\quad + \hat{a}_1 e_1 + \frac{3\hat{a}_1 z_1^2 z_2}{a_1} + \tilde{a}_1^2\\
u_2 &= 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 - 2\hat{a}_1\frac{y_1}{\sqrt{1-y_1^2}}\cos y_2\\
&\quad + 3z_2^2(z_1 - z_1^3 + f_1\cos\psi_1 t)\\
&\quad + \hat{\omega}_1 e_2 - \omega_1(x_1-x_3) + \hat{\omega}_1(y_1-y_3)\\
&\quad - \frac{3\delta_1}{\omega_1}\hat{\omega}_1 z_2^3 + \tilde{\omega}_1^2 \qquad (30)\\
u_3 &= -2a_2\sqrt{1-x_3^2}\sin x_4 + 2\hat{a}_2\sqrt{1-y_3^2}\sin y_4\\
&\quad + \hat{a}_2 e_3 + \frac{3\hat{a}_2 z_3^2 z_4}{a_2} + \tilde{a}_2^2\\
u_4 &= 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4 - 2\hat{a}_2\frac{y_3}{\sqrt{1-y_3^2}}\cos y_4\\
&\quad + 3z_4^2(z_3 - z_3^3 + f_2\cos\psi_2 t) + \hat{\omega}_2 e_4 - \omega_2(x_3-x_1)\\
&\quad + \hat{\omega}_2(y_3-y_1) - \frac{3\delta_2}{\omega_2}\hat{\omega}_2 z_4^3 + \tilde{\omega}_2^2
\end{aligned}$$

$$\begin{aligned}
\dot{\tilde{a}}_1 &= -\dot{\hat{a}}_1 = -\frac{3e_1 z_1^2 z_2}{a_1} + \tilde{a}_1 e_1 - e_1^2\\
\dot{\tilde{\omega}}_1 &= -\dot{\hat{\omega}}_1 = \frac{3\delta_1}{\omega_1}e_2 z_2^3 + \tilde{\omega}_1 e_2 - e_2^2\\
\dot{\tilde{a}}_2 &= -\dot{\hat{a}}_2 = -\frac{3e_3 z_3^2 z_4}{a_2} + \tilde{a}_2 e_3 - e_3^2\\
\dot{\tilde{\omega}}_2 &= -\dot{\hat{\omega}}_2 = \frac{3\delta_2}{\omega_2}e_4 z_4^3 + \tilde{\omega}_2 e_4 - e_4^2.
\end{aligned} \tag{31}$$

The initial values of estimates for uncertain parameters are $\hat{a}_1(0) = \hat{a}_2(0) = \hat{\omega}_1(0) = \hat{\omega}_2(0) = 0$. Substituting Eqs. (30) and (31) into Eq. (29), it can be rewritten as

$$\dot{V} = -a_1 e_1^2 - \omega_1 e_2^2 - a_2 e_3^2 - \omega_2 e_4^2 \le 0 \tag{32}$$

which is a negative semi-definite function of e_1, e_2, e_3, e_4, $\tilde{a}_1, \tilde{a}_2, \tilde{\omega}_1$, and $\tilde{\omega}_2$. The Lyapunov asymptotical stability theorem is not satisfied. We cannot obtain that the common origin of error dynamics (27) and parameter dynamics (31) is asymptotically stable. In our case, $\dot{V} = 0$ when $e_1 = e_2 = e_3 = e_4 = 0$, and $\tilde{a}_1$, $\tilde{a}_2$, $\tilde{\omega}_1$, and $\tilde{\omega}_2$ take arbitrary values. $n = 8$, $m = 4$, $m + 1 < n$ is satisfied. By the pragmatical asymptotical stability theorem, the equilibrium point $e_1 = e_2 = e_3 = e_4 = \tilde{a}_1 = \tilde{a}_2 = \tilde{\omega}_1 = \tilde{\omega}_2 = 0$ is pragmatically asymptotically stable. Under the assumption of equal probability, it is actually asymptotically stable. The error vector e approaches zero and the estimated parameters approach the uncertain parameters. The numerical results are shown in Fig. 2. After 10 s, the generalized synchronization is accomplished.

4. Conclusions

In this paper pragmatical generalized synchronization of adaptive control is studied. The pragmatical asymptotical stability theorem fills the vacancy between the actual asymptotical stability and mathematical asymptotical stability; the conditions of the Lyapunov function for pragmatical asymptotical stability are lower than those for traditional asymptotical stability. By using this theorem, with the same conditions for the Lyapunov function, $V > 0$, $\dot{V} \le 0$, as in

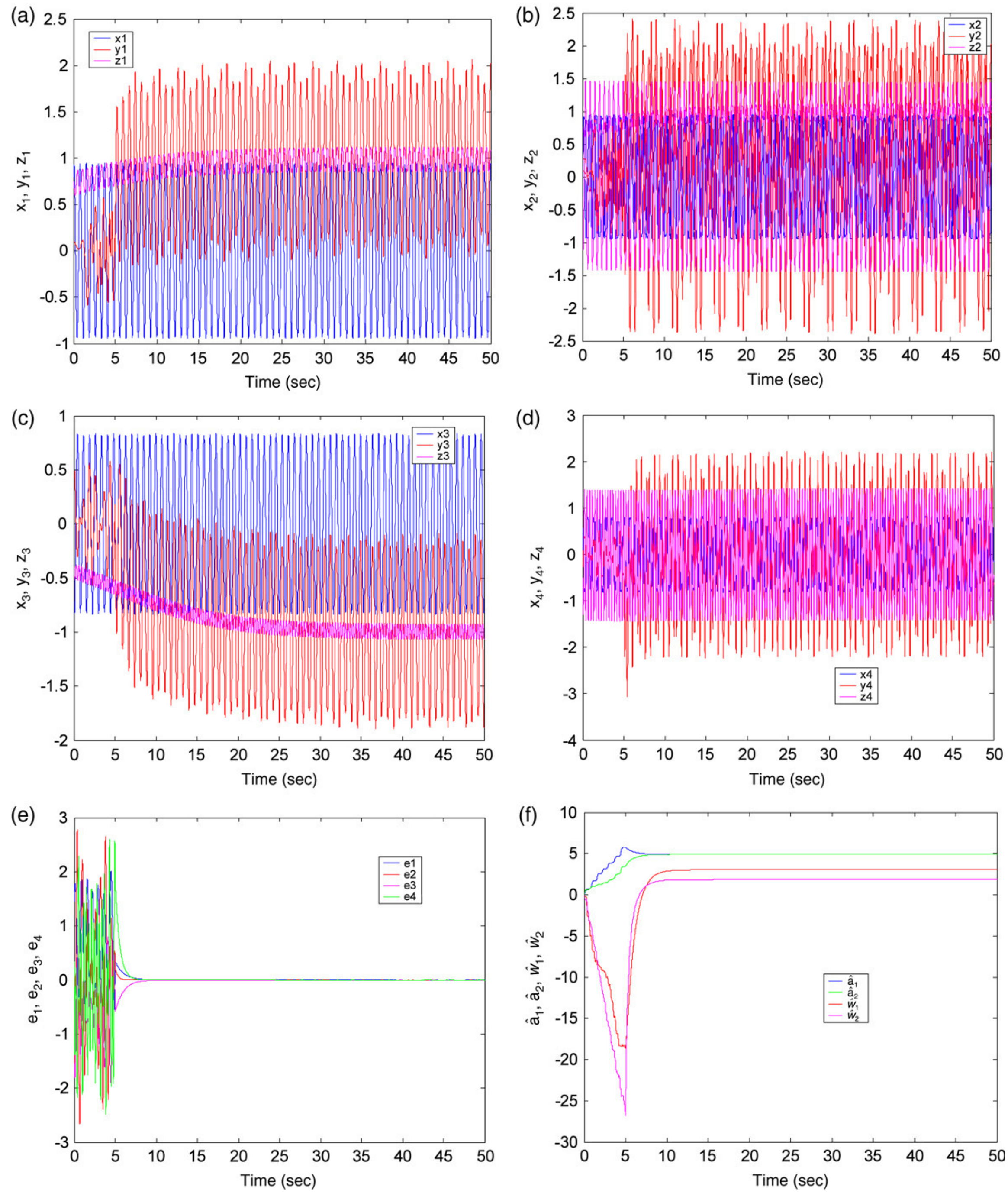

Fig. 1. Time histories of states, state errors, z_1, z_2, z_3, z_4, $\hat{a}_1$, $\hat{a}_2$, $\hat{w}_1$, and $\hat{w}_2$ for Case I with $a_1 = 4.9$, $a_2 = 4.9$, $\omega_1 = 3.03$, $\omega_2 = 1.83$.

the current scheme of adaptive synchronization, we not only obtain the generalized synchronization of chaotic systems but also prove that the estimated parameters approach the uncertain values. Two Quantum-CNN chaotic systems and a double Duffing chaotic system are used as the master system, slave system, and goal system, respectively, in two cases: the chaotic states of a goal system, a double Duffing chaotic system, used as $F(t)$ and the cubics of chaotic states of the same goal system used as $F(t)$. These generalized synchronizations of chaotic systems by adaptive control can be used to increase the security of communication.

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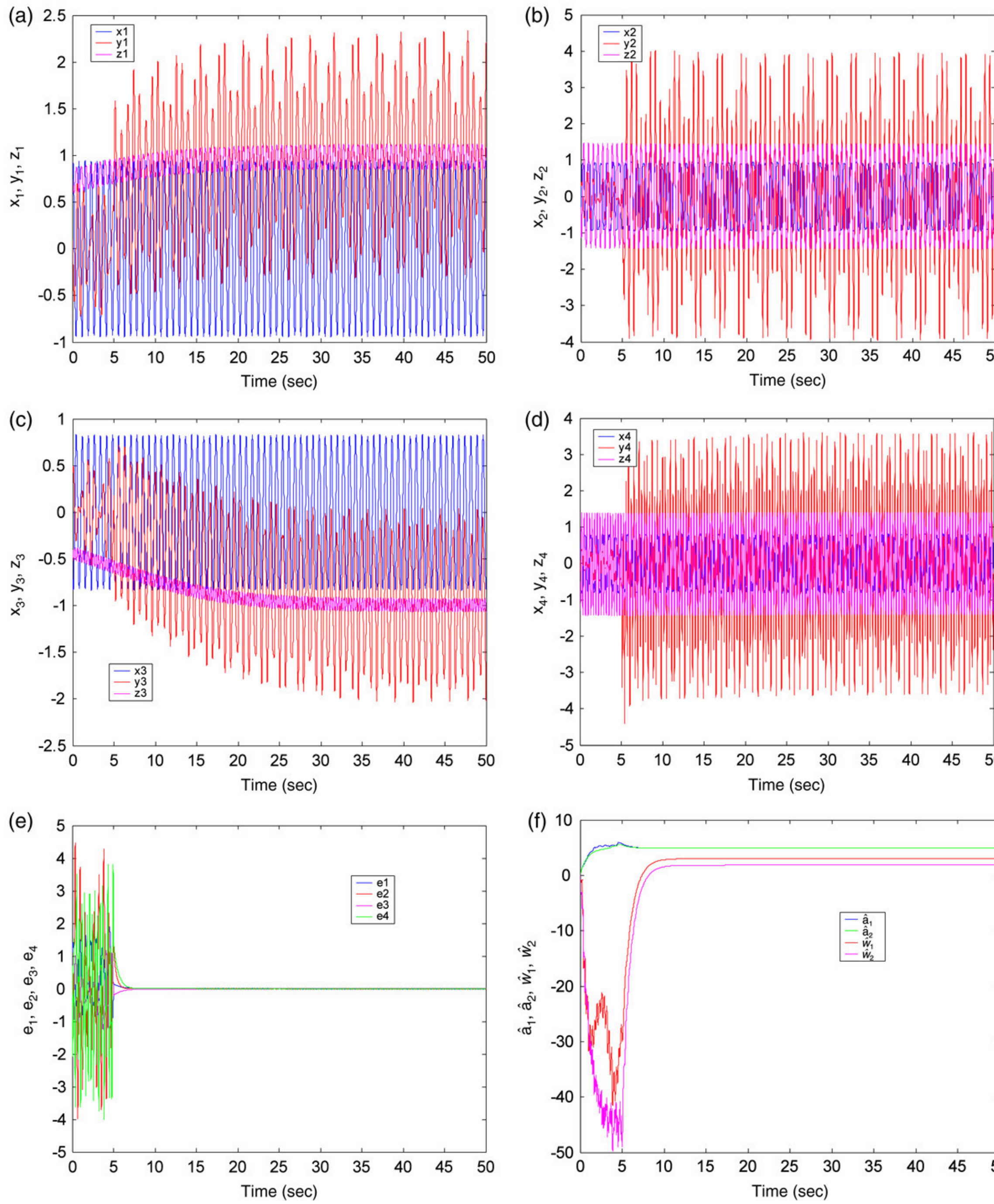

Fig. 2. Time histories of states, state errors, z_1, z_2, z_3, z_4, $\hat{a}_1$, $\hat{a}_2$, $\hat{w}_1$, and $\hat{w}_2$ for Case II $a_1 = 4.9$, $a_2 = 4.9$, $\omega_1 = 3.03$, $\omega_2 = 1.83$.

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