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Chaos, control and synchronization of a fractional order rotational mechanical system with a centrifugal governor

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Abstract

Chaos, its control and synchronization for a fractional order rotational mechanical system with a centrifugal governor are studied for both the autonomous and the nonautonomous cases. It is found that chaos exists in the fractional order systems with order less than and more than the number of states of the system. Controlling the chaotic motion of a fractional order system to its equilibrium point is obtained for both the autonomous and the nonautonomous cases. The rotational mechanical systems with the same fractional order and with the different fractional orders are synchronized by linear coupling for both the autonomous and the nonautonomous cases.

1. Introduction

Fractional calculus is 300 years old. Although it has a long history, the applications of fractional calculus to physics and engineering have just started [1]. Many systems are known to display fractional order dynamics, such as viscoelastic systems [2], dielectric polarization, and electromagnetic waves. More recently, there is a new trend to investigate the control and dynamics of fractional order dynamical systems [3–9]. In [3] it has been shown that nonlinear chaotic systems can still behave chaotically when their models become fractional. In [4], chaos control was investigated for fractional chaotic systems, where controllers have been designed using the ‘‘backstepping’’ method of nonlinear control design. It was demonstrated that nonlinear controllers designed to stabilize the integral chaotic model would still stabilize the fractional order model in [5]. In [6], it is found that chaos exists in the fractional order Chen system with order less than 3. Linear feedback control of chaos in this system is studied. In [7], chaos synchronization of fractional order chaotic systems are studied. In [8], author presents a conjecture that fourth-order hyperchaotic nonlinear systems can still produce hyperchaotic behavior with a total system order of $3+\varepsilon$, where $1>\varepsilon>0$. In this paper, we study the chaotic behaviors in the fractional order autonomous and nonautonomous nonlinear systems of rotational mechanical system with a centrifugal governor [9]. By utilizing approximation approach of fractional operator, it is shown that systems with total order less than or more than three, the number of the states of the systems, exhibit chaos. Chaos control and chaos synchronization have been widely studied for the integral order systems [13–27], they also are studied in this paper for the fractional order systems.

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This paper is organized as follows. In Section 2, the review and the approximation of a fractional operator are given. In Section 3, the chaos in the autonomous fractional order rotational mechanical system with a centrifugal governor is shown by phase portraits and bifurcation diagrams. In Section 4, the chaos control of the chaotic motion of the autonomous system to its equilibrium point is obtained by a linear state feedback controller, and the chaos synchronizations of the two identical systems are accomplished by linear coupling for both the same order cases and the different order cases. In Section 5, the chaos in the nonautonomous fractional order system is shown by phase portraits and bifurcation diagrams. In Section 6, the chaos control and the chaos synchronization of the nonautonomous fractional order system are studied by the similar methods as that used for autonomous system. In Section 7, conclusions are drawn.

2. The review and the approximation of a fractional operator

The commonly used definition for a general fractional derivative is the Riemann–Liouville definition [10]. The Riemann–Liouville definition is given here:

$$\frac{\mathrm{d}^q f(t)}{\mathrm{d}t^q} = \frac{1}{\Gamma(n-q)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t \frac{f(\tau)}{(t-\tau)^{q-n+1}} \mathrm{d}\tau,$$

where $\Gamma(\cdot)$ is a Gamma function and n is an integer such that $n-1 \leqslant q < n$. This definition is different from the usual intuitive definition of derivative. Thus, it is necessary to develop approximations to the fractional operators using the standard integer order operators. Fortunately, the Laplace transform which is the basic engineering tool for analyzing linear systems is still applicable and works:

$$L\left\{\frac{\mathrm{d}^q f(t)}{\mathrm{d}t^q}\right\} = s^q L\{f(t)\} - \sum_{k=0}^{n-1} s^k \left[\frac{\mathrm{d}^{q-1-k} f(t)}{\mathrm{d}t^{q-1-k}}\right]_{t=0}, \quad \text{for all } q,$$

where n is an integer such that $n-1 \leqslant q < n$. Upon considering the initial conditions to be zero, this formula reduces to the more expected form

$$L\left\{\frac{\mathrm{d}^q f(t)}{\mathrm{d}t^q}\right\} = s^q L\{f(t)\}.$$

Linear transfer function approximations of the fractional integrator [11] is adopted. Basically the idea is to approximate the system behavior based on frequency domain arguments. [12] gives approximations for $1/s^q$ with $q = 0.1$–0.9 in steps of 0.1. These approximations will be used in this paper.

3. Chaos in an autonomous fractional order rotational mechanical system with a centrifugal governor

The autonomous system is studied in this section. The standard derivatives are replaced by the fractional derivatives as follows:

$$\begin{cases} \dfrac{\mathrm{d}^q x}{\mathrm{d}t^q} = y, \\ \dfrac{\mathrm{d}^q y}{\mathrm{d}t^q} = r(z+\omega_0)^2 \sin(x-\varphi_0)\cos(x-\varphi_0) - \sin(x-\varphi_0) - Cy, \\ \dfrac{\mathrm{d}^q z}{\mathrm{d}t^q} = k\cos(x-\varphi_0) - F, \end{cases} \tag{1}$$

where q is the fractional order, $C = 0.7$, $r = 0.25$, $k = 3.5$, $F = 1.942$, $\cos\varphi_0 = \frac{F}{k}$ and $\omega_0^2 = \frac{k}{rF}$. Simulations are performed for $q = 0.8$, 0.9, 1.1, 1.2. The simulation results indicate that chaos indeed exists in the fractional order autonomous system with order both less than 3 and more than 3. When $q = 0.9$ and 1.1, chaotic attractors are found and the phase portraits are shown in Figs. 1 and 2, respectively. Bifurcation diagrams which assure the existence of chaos are shown in Figs. 3 and 4. When $q = 0.8$ and 1.2, no chaotic behavior is found. This indicates that the lowest and the highest limits of the fractional order for this system to be chaotic may be in the ranges $0.8 < q < 0.9$ and $1.1 < q < 1.2$, respectively. Thus, the lowest total order and the highest total order found for the existence of chaos in this system are 2.7 and 3.3, respectively.

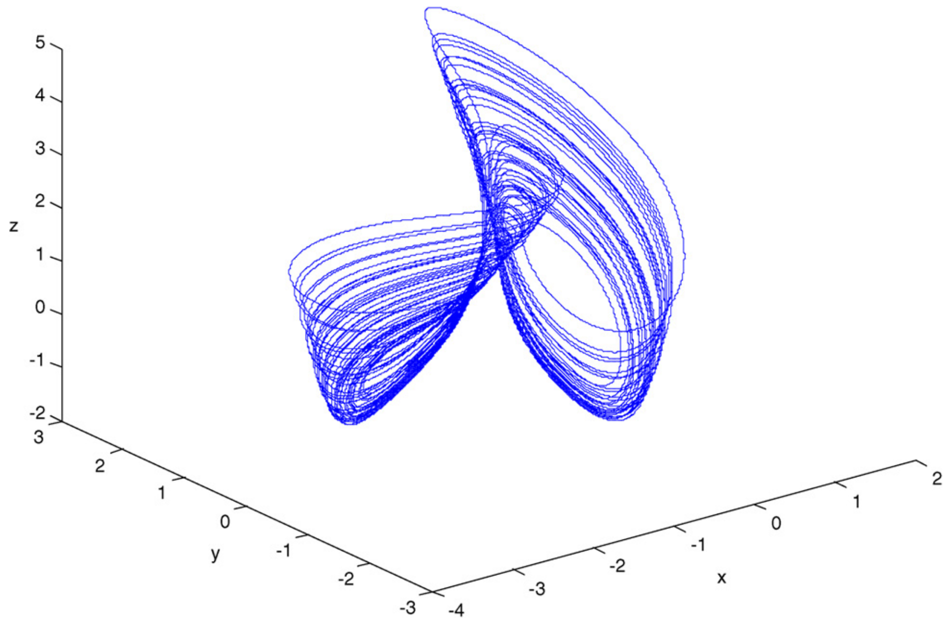

Fig. 1. The phase portrait of the autonomous fractional order system with order $q = 0.9$ and $k = 3.5$.

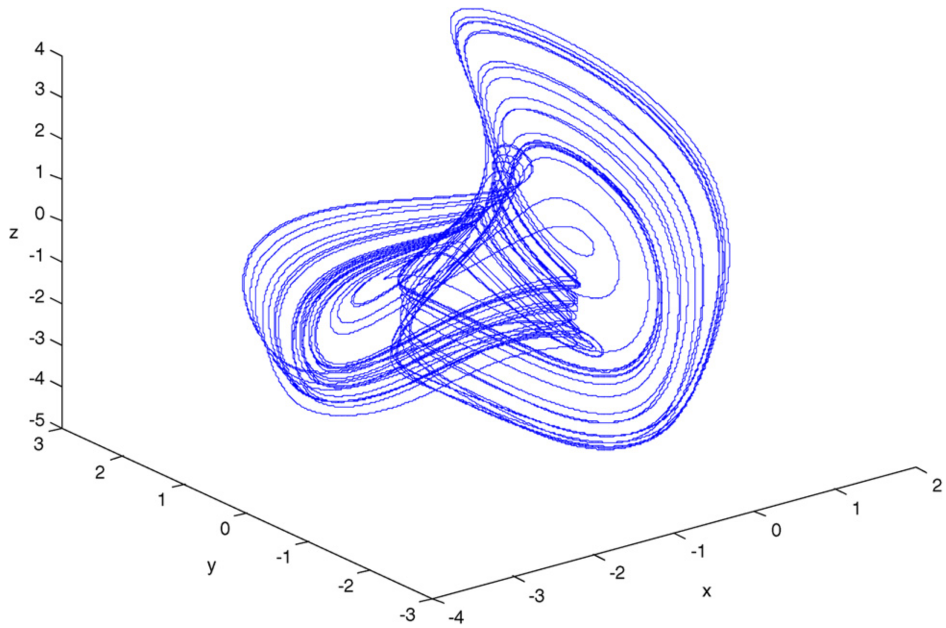

Fig. 2. The phase portrait of the autonomous fractional order system with order $q = 1.1$ and $k = 3.5$.

4. Chaos control and chaos synchronization of autonomous fractional order system

4.1. Chaos control of the autonomous fractional order system

Here the issue of controlling the chaotic motion of the autonomous fractional order chaotic system to its equilibrium point is discussed. Simplifying the fractional order autonomous chaotic system (1) in a compact vector form, we have

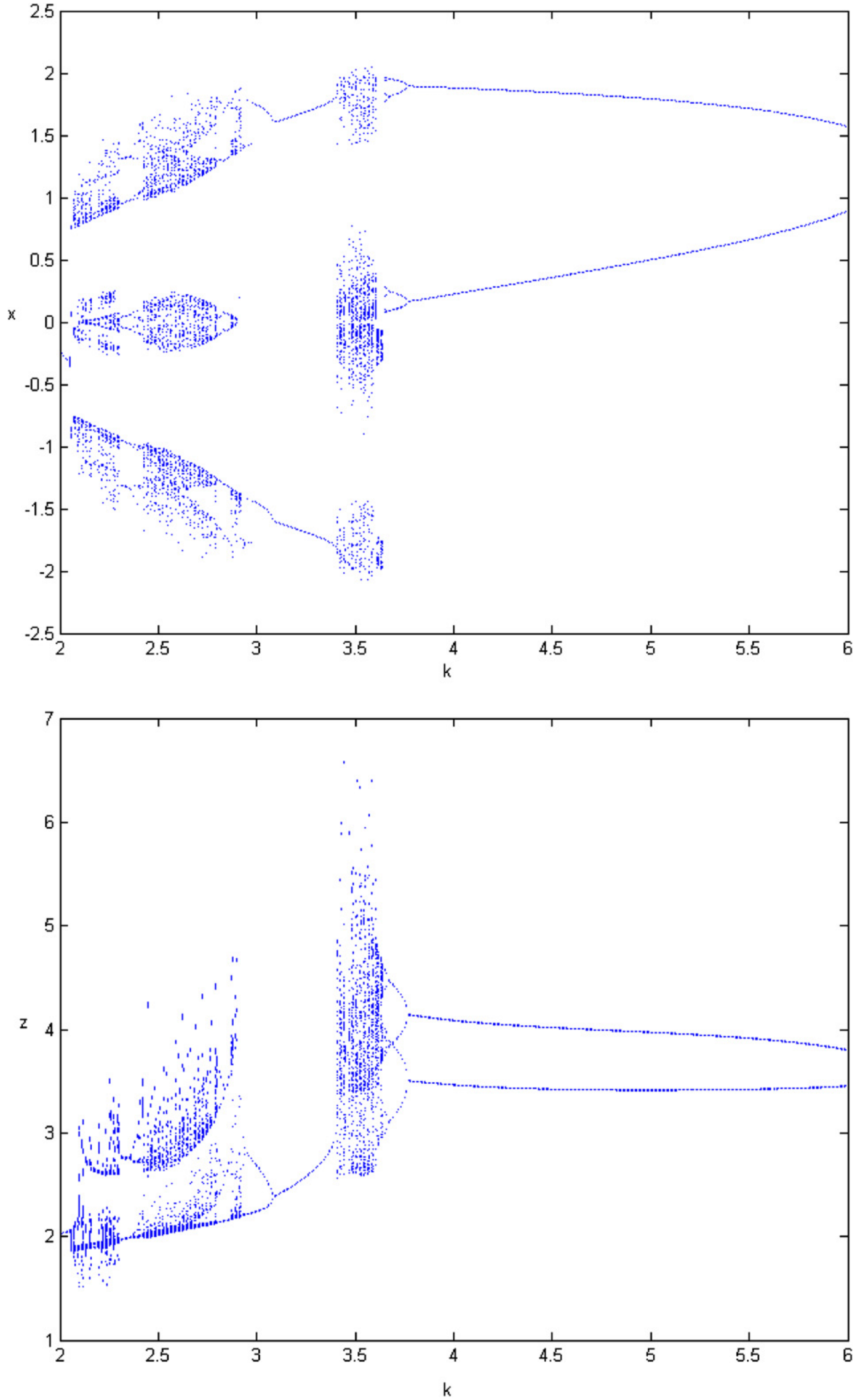

Fig. 3. The bifurcation diagrams of the autonomous fractional order system with order $q = 0.9$.

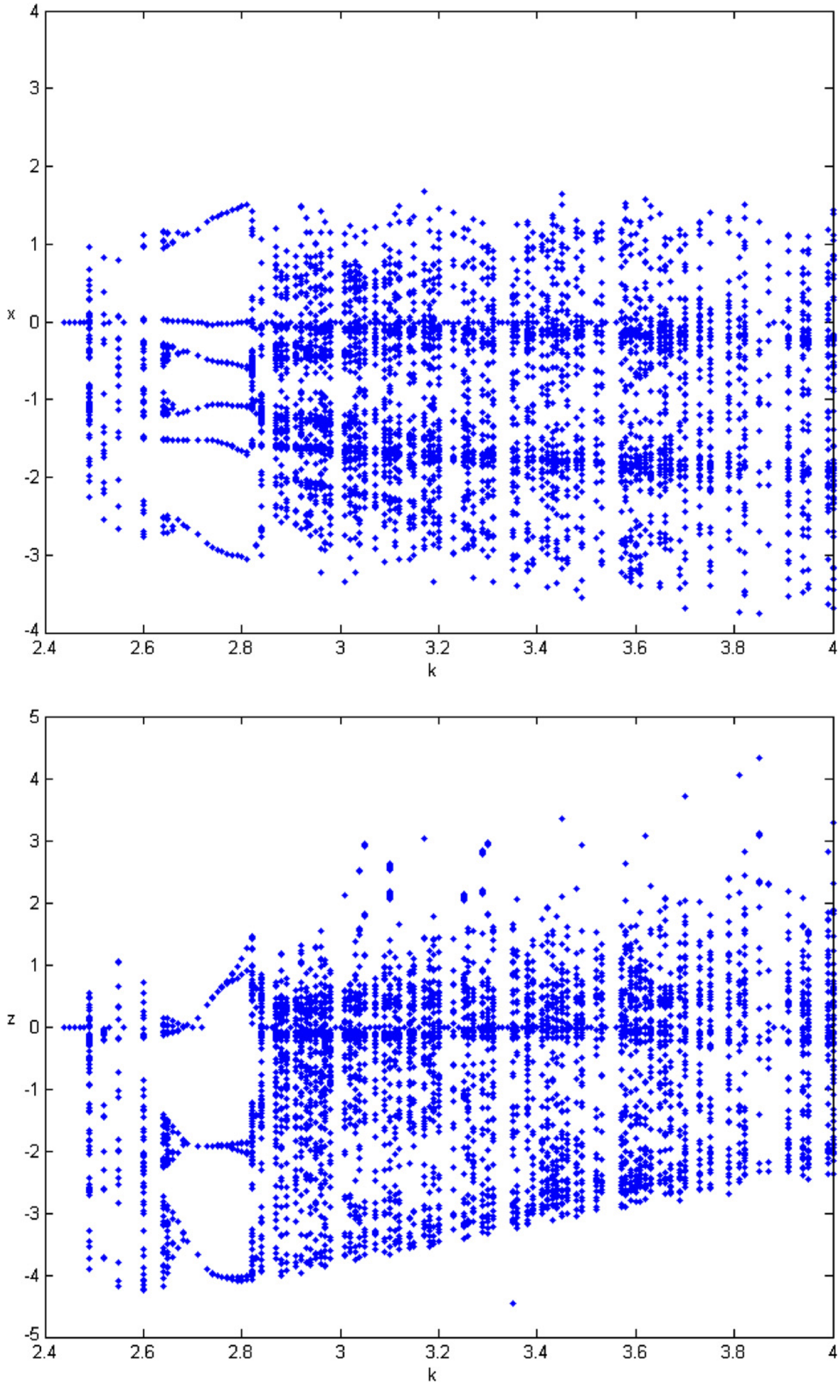

Fig. 4. The bifurcation diagrams of the autonomous fractional order system with order $q = 1.1$.

$$\frac{d^q X}{dt^q} = f(X) \tag{2}$$

with $X = [x, y, z]^T$. With a linear state feedback controller, Eq. (2) can be written as

$$\frac{d^q X}{dt^q} = f(X) + u, \tag{3}$$

where u is the linear state feedback controller and has the following form:

$$u = K(X - \bar{X}),$$

where $K = \text{diag}(k_1, k_2, k_3)$, $\overline{X}$ is the control target and k_1, k_2, k_3 are constant parameters. Clearly, $(0, 0, 0)$ is an equilibrium point of system (1). In the following simulation, we stabilize system (2) to this equilibrium point. Standard stability analysis easily shows that with $(k_1, k_2, k_3) = (-1, 0, -1)$, the equilibrium $(0, 0, 0)$ of the controlled integral order chaotic system is locally stable and is shown in Fig. 5. Simulation results show that this controller can also stabilize the fractional order chaotic system to this equilibrium. The trajectories of the controlled fractional order chaotic system with $q = 0.9$ and $q = 1.1$ are shown in Figs. 6 and 7, respectively. The control signals are added at $t = 500$ s and $t = 200$ s, respectively. The designed chaos controller controls the chaotic motion of the fractional order chaotic system to its equilibrium point $\overline{X} = (0, 0, 0)$ effectively.

4.2. Chaos synchronization of the autonomous systems with same fractional order

In this subsection, chaos synchronization of system (1) is studied. Consider the drive-response synchronization scheme of autonomous chaotic systems

$$\frac{d^q X}{dt^q} = f(X), \tag{4}$$

$$\frac{d^q X'}{dt^q} = f(X') + u, \tag{5}$$

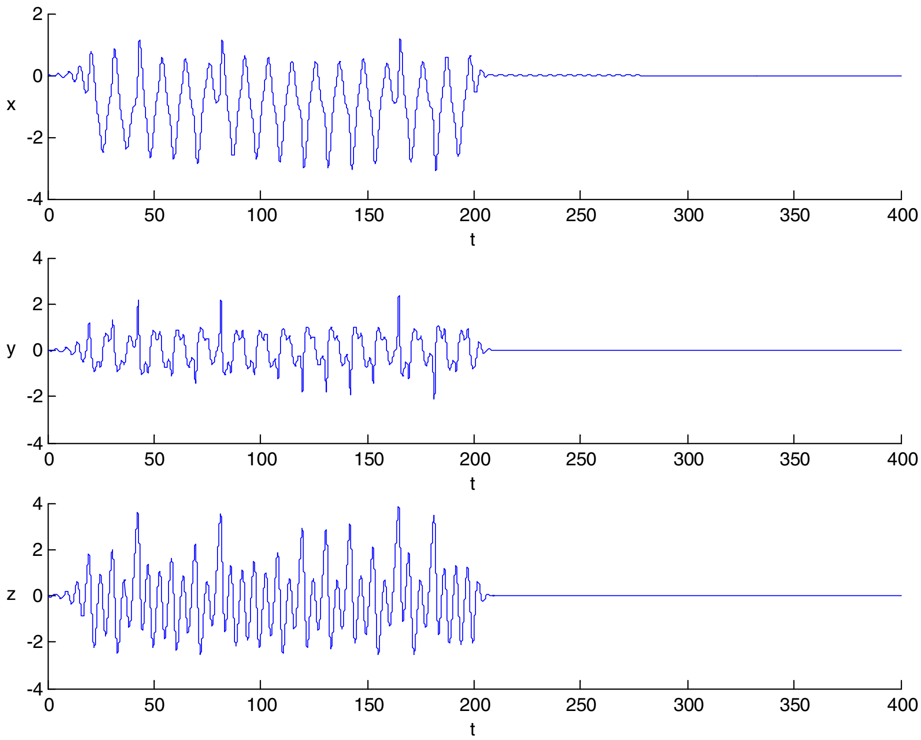

Fig. 5. The time histories of the state variables of the controlled autonomous integral order system with order $q = 1$ and $k = 3.5$.

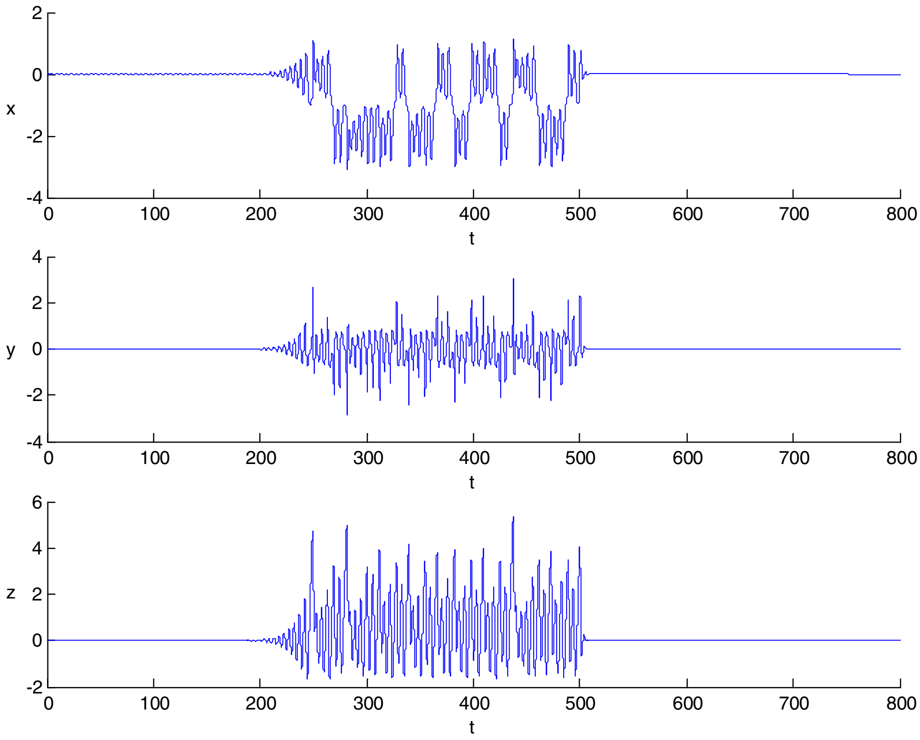

Fig. 6. The time histories of the state variables of the autonomous controlled fractional order system with order $q = 0.9$ and $k = 3.5$.

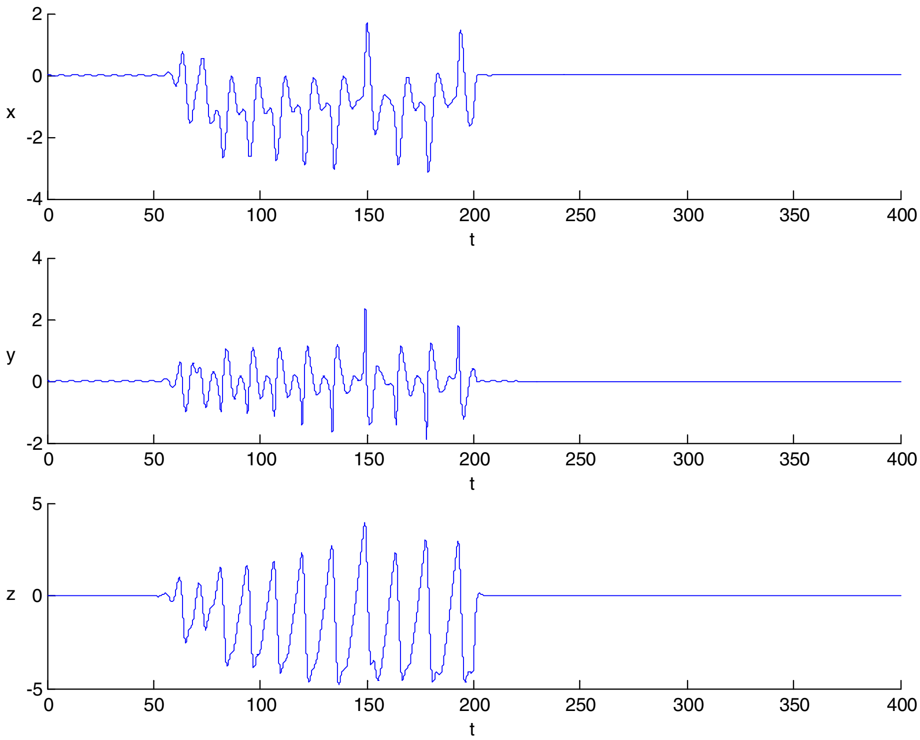

Fig. 7. The time histories of the state variables of the controlled autonomous fractional order system with order $q = 1.1$ and $k = 3.5$.

where q is the fractional order, and u is a linear state feedback controller which is in the following form:

$$u = K(X' - X),$$

where $K = \mathrm{diag}(k_1, k_2, k_3)$, k_1, k_2, k_3 are constant parameters. Define the error state as $e = X' - X$, synchronization is achieved when $\|e(t)\| \to 0$ as $t \to \infty$.

Next, we numerically study the synchronizations in two cases.

Case 1. $q = 0.9$, $K = \mathrm{diag}(-1, 0, -1)$.
The controller is added at $t = 300$ s, and the response system is synchronized at $t = 341$ s as shown in Fig. 8.

Case 2. $q = 1.1$, $K = \mathrm{diag}(-1, 0, -1)$.
The controller is added at $t = 300$ s, and the response system is synchronized at $t = 315$ s as shown in Fig. 9.

4.3. Chaos synchronization of the autonomous systems with different fractional orders

In this subsection, chaos synchronization of different order systems is discussed.

Consider the drive-response synchronization scheme of autonomous chaotic systems

$$\frac{\mathrm{d}^q X}{\mathrm{d}t^q} = f(X), \tag{6}$$

$$\frac{\mathrm{d}^p X'}{\mathrm{d}t^p} = f(X') + u, \tag{7}$$

where q and p are different fractions, and $u = K(X - X')$ is a linear state feedback controller. The synchronization can be practically achieved.

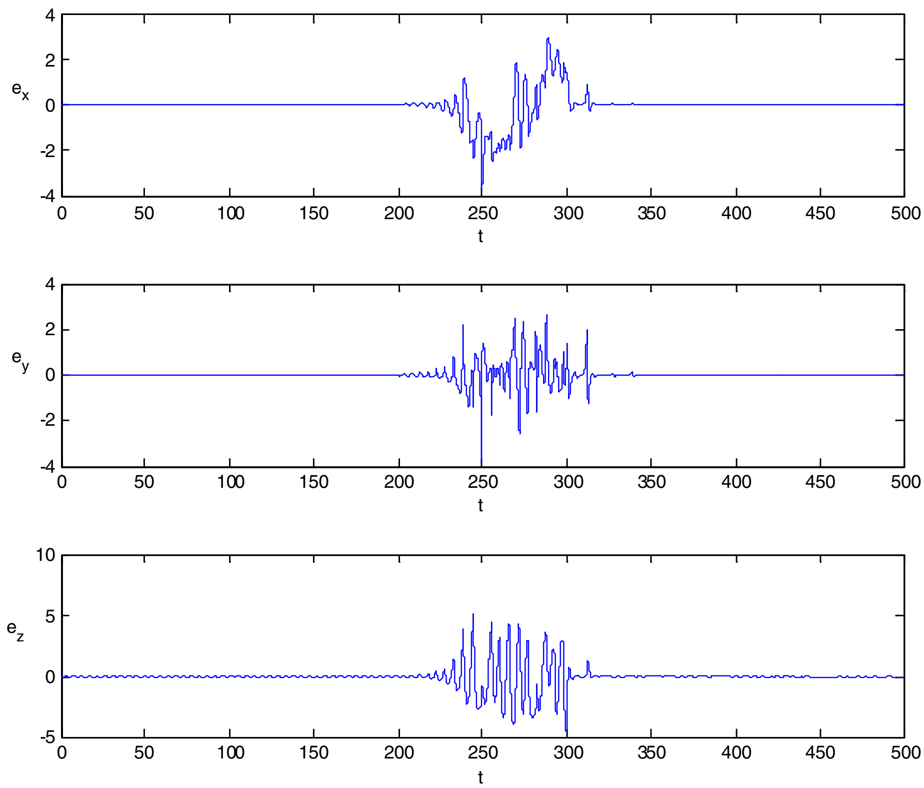

Fig. 8. The time histories of the errors of the autonomous fractional order system with order $q = 0.9$ and $k = 3.5$.

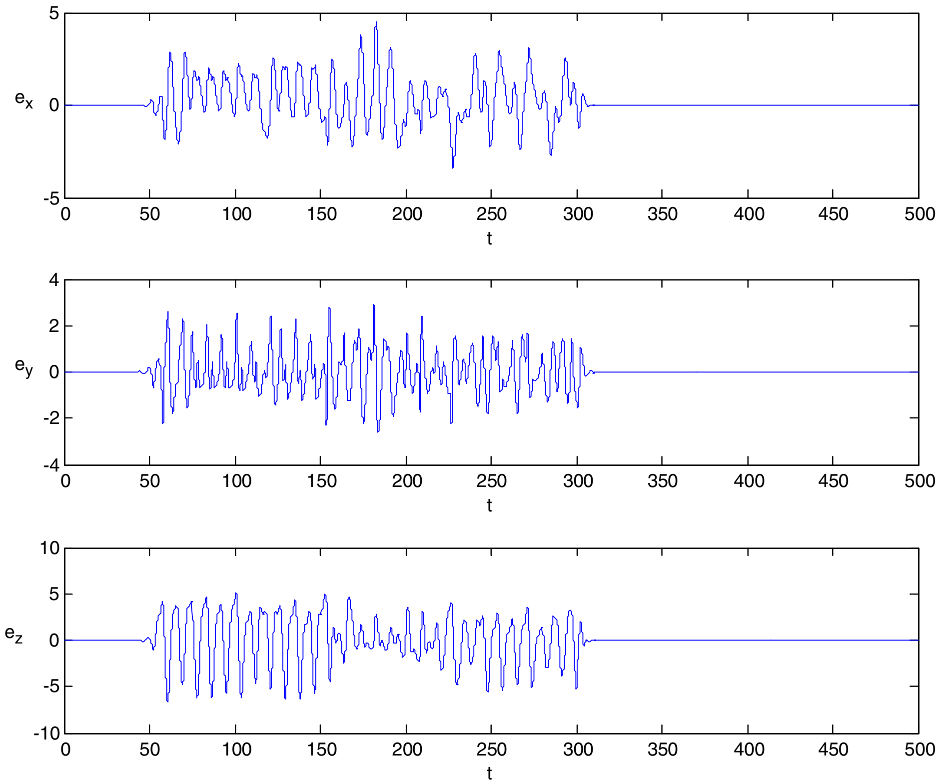

Fig. 9. The time histories of the errors of the autonomous fractional order system with order $q = 1.1$ and $k = 3.5$.

The synchronizations are studied in two cases.

Case 1. $q = 0.9$, $p = 1.1$, $K = \mathrm{diag}(-1000, -1000, -1000)$.
The controller is added at $t = 200$ s, and response system is practically synchronized as shown in Fig. 10.

Case 2. $q = 1.1$, $p = 0.9$, $K = \mathrm{diag}(-1000, -1000, -1000)$.
The controller is added at $t = 200$ s, and response system is practically synchronized as shown in Fig. 11.

5. Chaos in a nonautonomous fractional order system

A nonautonomous system [9] is studied in this section. The standard derivatives are replaced by the fractional derivatives as follows::

$$\begin{cases} \dfrac{d^q x}{dt^q} = y, \\ \dfrac{d^q y}{dt^q} = r(z + \omega_0)^2 \sin(x - \varphi_0)\cos(x - \varphi_0) - \sin(x - \varphi_0) - Cy, \\ \dfrac{d^q z}{dt^q} = k\cos(x - \varphi_0) - F - v\sin\bar{\omega}t, \end{cases} \tag{8}$$

where q is the fractional order, $C = 0.7$, $r = 0.25$, $k = 2.8$, $F = 1.942$, $v = 0.5$, $\bar{\omega} = 3$, $\cos\varphi_0 = \frac{F}{k}$ and $\omega_0^2 = \frac{k}{rF}$. Simulations are given for $q = 0.8$, 0.9, 1.1 and 1.2, respectively. The simulation results show that chaos indeed exist in the nonautonomous fractional order system with order less than 3 and with order larger than 3, respectively. When $q = 0.9$ and 1.1, chaotic attractors are found and the phase portraits are shown in Figs. 12 and 13, respectively. Bifurcation diagrams which assure the existence of chaos are shown in Figs. 14 and 15. When $q = 0.8$ and 1.2, no chaotic behavior is found. This indicates that the lowest and the highest limits of the fractional order for the existence of chaos in this system may be $q = 0.8$–0.9 and $q = 1.1$–1.2, respectively. Thus, the lowest total order and the highest total order found for the existence of chaos in this system are 2.7 and 3.3, respectively.

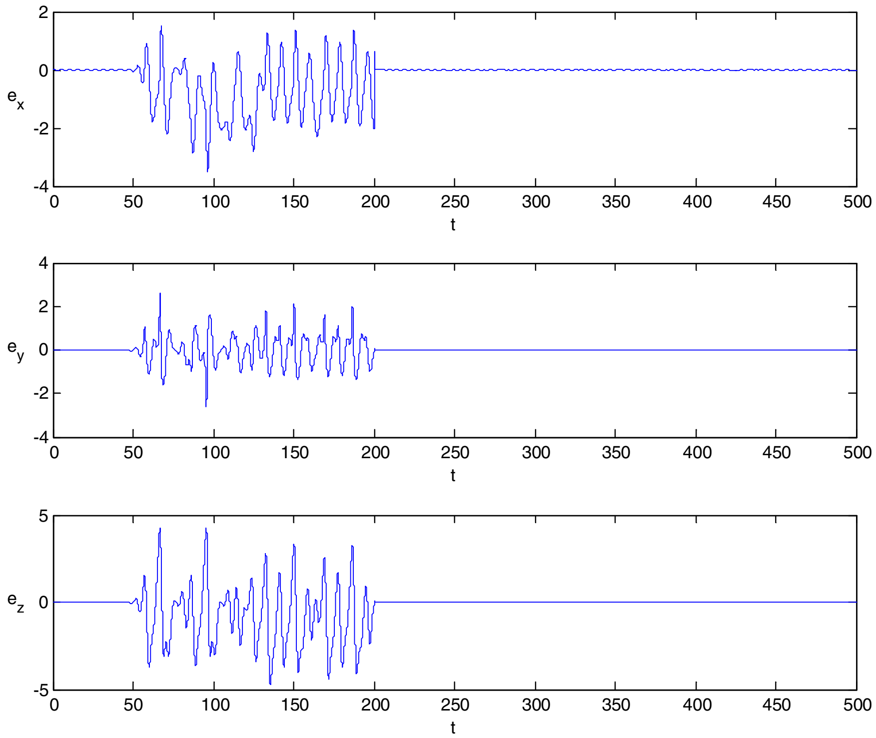

Fig. 10. The time histories of the errors of the autonomous systems with different fractional orders $q = 0.9$, $p = 1.1$ and $k = 3.5$.

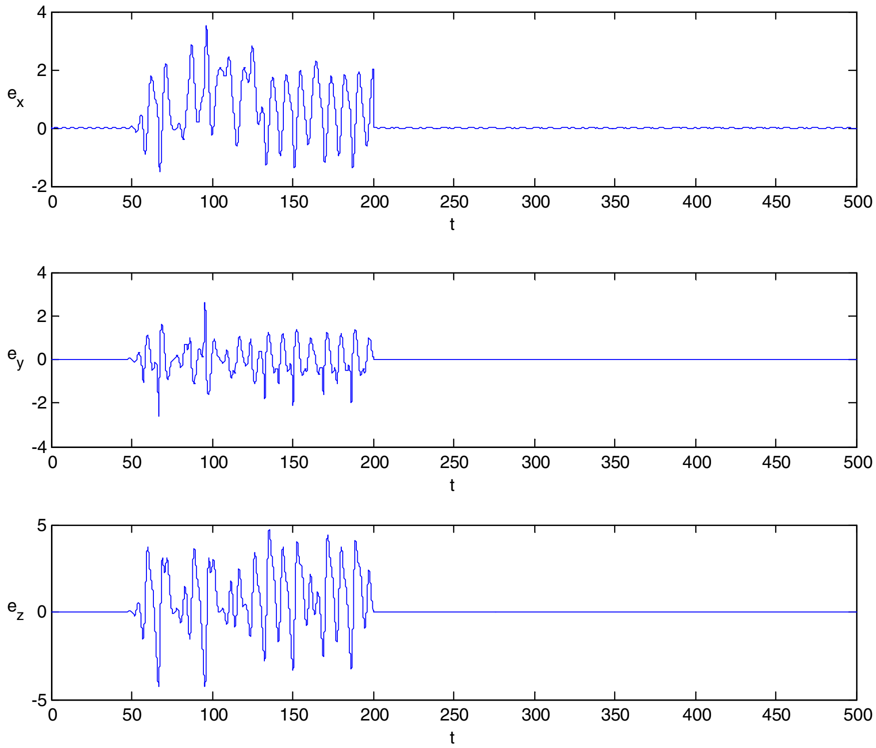

Fig. 11. The time histories of the errors of the autonomous systems with different fractional orders $q = 1.1$, $p = 0.9$ and $k = 3.5$.

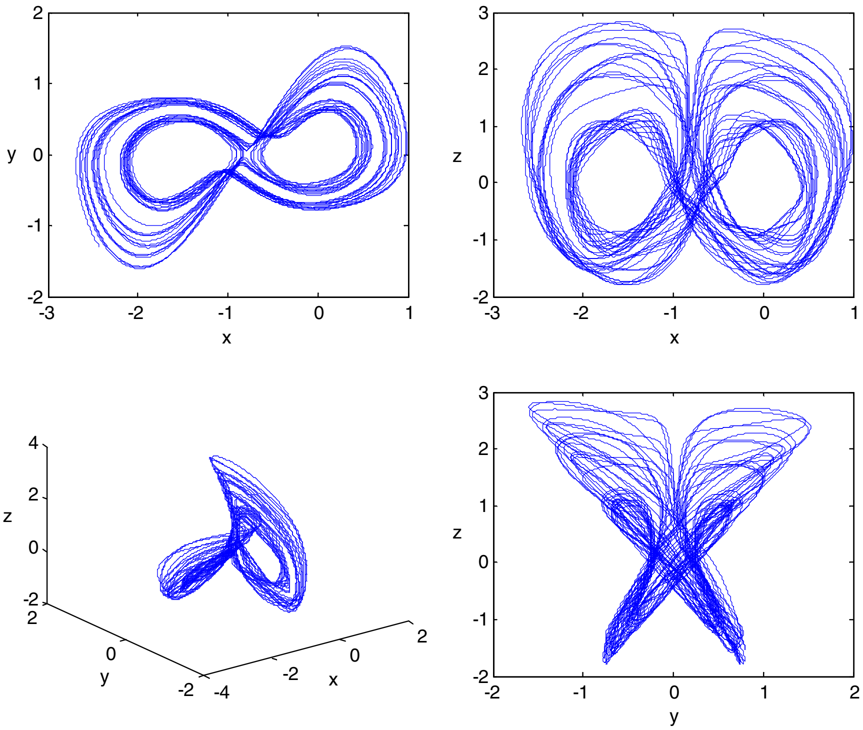

Fig. 12. The phase portraits of the nonautonomous fractional order system with order $q = 0.9$ and $k = 2.8$.

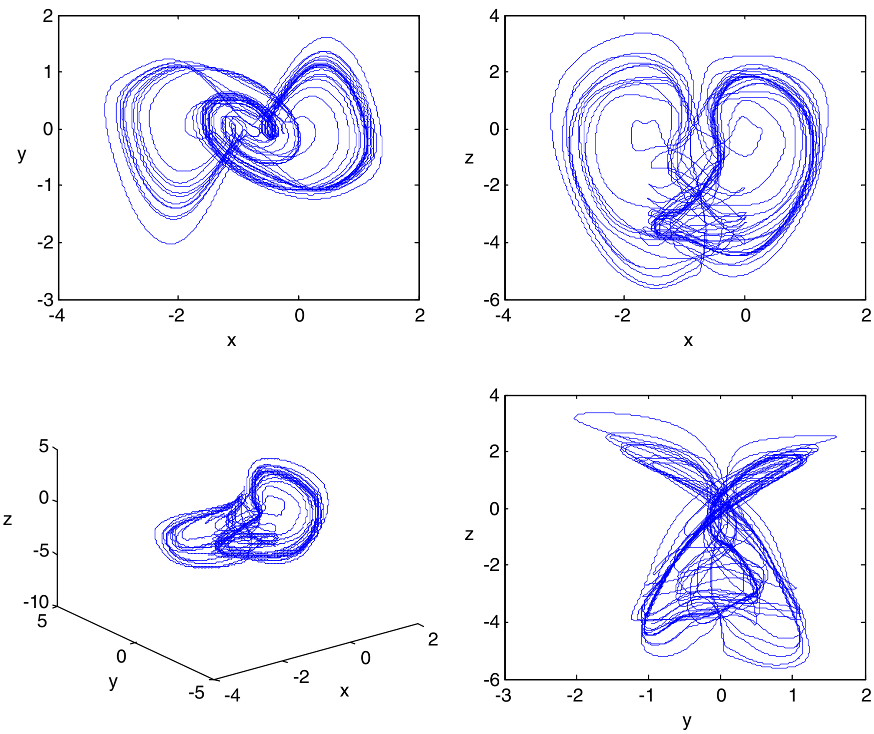

Fig. 13. The phase portraits of the nonautonomous fractional order system with order $q = 1.1$ and $k = 2.8$.

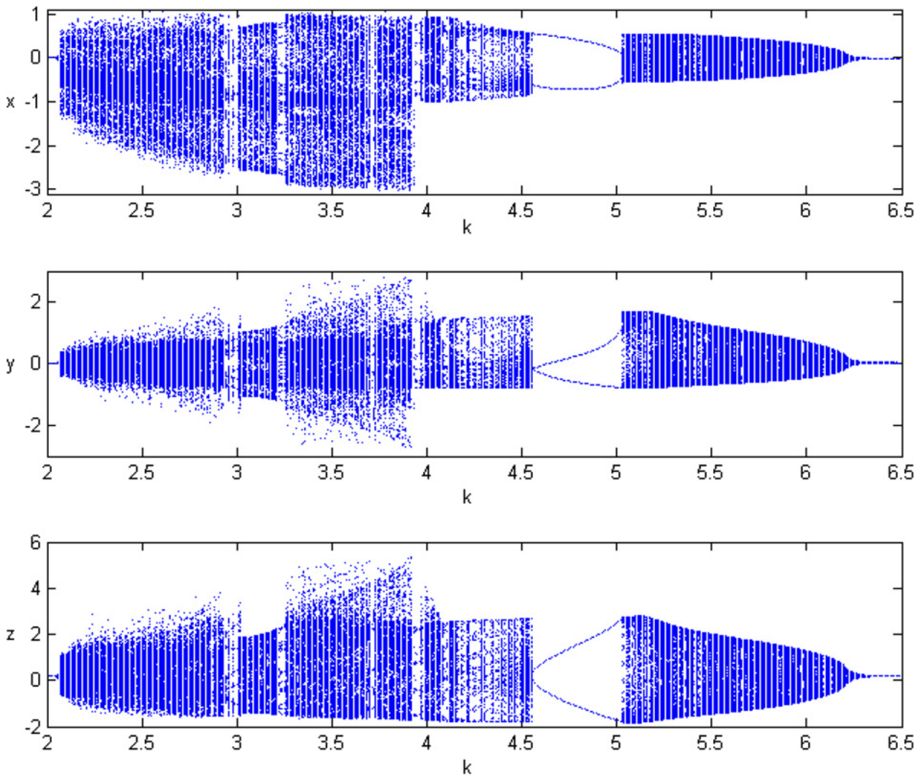

Fig. 14. The bifurcation diagram of the nonautonomous fractional order system with order $q = 0.9$.

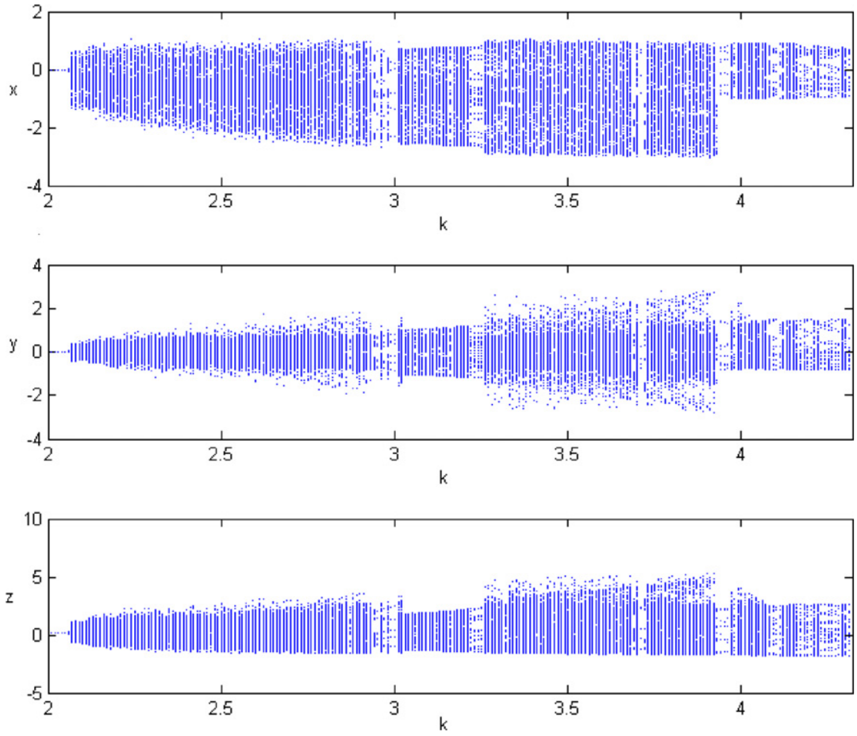

Fig. 15. The bifurcation diagram of the nonautonomous fractional order system with order $q = 1.1$.

6. Chaos control and chaos synchronization of the nonautonomous fractional order system

6.1. Chaos control of the nonautonomous fractional order system

In this subsection, controlling the chaotic motion of the nonautonomous fractional order chaotic system to a periodic motion is studied. Simplifying the nonautonomous fractional order chaotic system (8) to a compact vector form, we have

$$\frac{d^q X}{dt^q} = f(X) + g(t), \tag{9}$$

where $X = [x, y, z]^T$, $g(t) = [0, 0, -v \sin \overline{\omega} t]$, and $f(X)$ corresponds to the right hand side terms in the three equations of Eq. (8), except $-v \sin \overline{\omega} t$ term. With a linear state feedback controller, Eq. (9) can be written as

$$\frac{d^q X}{dt^q} = f(X) + g(t) + u, \tag{10}$$

where u is the linear state feedback controller which is in the following form:

$$u = KX,$$

where $K = \text{diag}(k_1, k_2, k_3)$, k_1, k_2, k_3 are constant parameters. In the following simulation, we control system (10) with $q = 1$ to a period motion. With $(k_1, k_2, k_3) = (-1, 0, -1)$, the chaotic motion of the integral order chaotic system can be controlled to periodic motion and is shown in Fig. 16. Simulation results show that this controller can also control the fractional order chaotic system to a periodic motion. The time histories and the phase portrait of the states of the controlled fractional order chaotic system with $q = 0.9$ and $q = 1.1$ are shown in Figs. 17 and 18, respectively. The control signals are added at $t = 40$ s. The designed chaos controller controls the fractional order chaotic system to a periodic motion effectively.

6.2. Chaos synchronization of the nonautonomous systems with the same fractional order

In this subsection, chaos synchronization of system (8) is studied. Synchronization is obtained by linear feedback. Consider the drive-response synchronization scheme of two nonautonomous chaotic systems

$$\frac{d^q X}{dt^q} = f(X) + g(t), \tag{11}$$

$$\frac{d^q X'}{dt^q} = f(X') + g(t) + u, \tag{12}$$

where q is the fractional order, and u is a linear state feedback controller which is in the following form:

$$u = K(X' - X),$$

where $K = \text{diag}(k_1, k_2, k_3)$, k_1, k_2, k_3 are constant parameters. Define the error state as $e = X' - X$, synchronization is obtained when $\|e(t)\| \to 0$ as $t \to \infty$.

The synchronizations are studied in two cases.

Case 1. $q = 0.9$, $K = \text{diag}(-1, 0, -1)$.
The controller is added at $t = 300$ s and the response system is synchronized at $t = 310$ s as shown in Fig. 19.

Case 2. $q = 1.1$, $K = \text{diag}(-1, 0, -1)$.
The controller is added at $t = 300$ s and the response system is synchronized at $t = 320$ s as shown in Fig. 20.

6.3. Chaos synchronization of the nonautonomous systems with different fractional orders

In this subsection, chaos synchronization of the nonautonomous systems with the different fractional orders is discussed.

Consider the drive-response synchronization scheme of nonautonomous chaotic systems

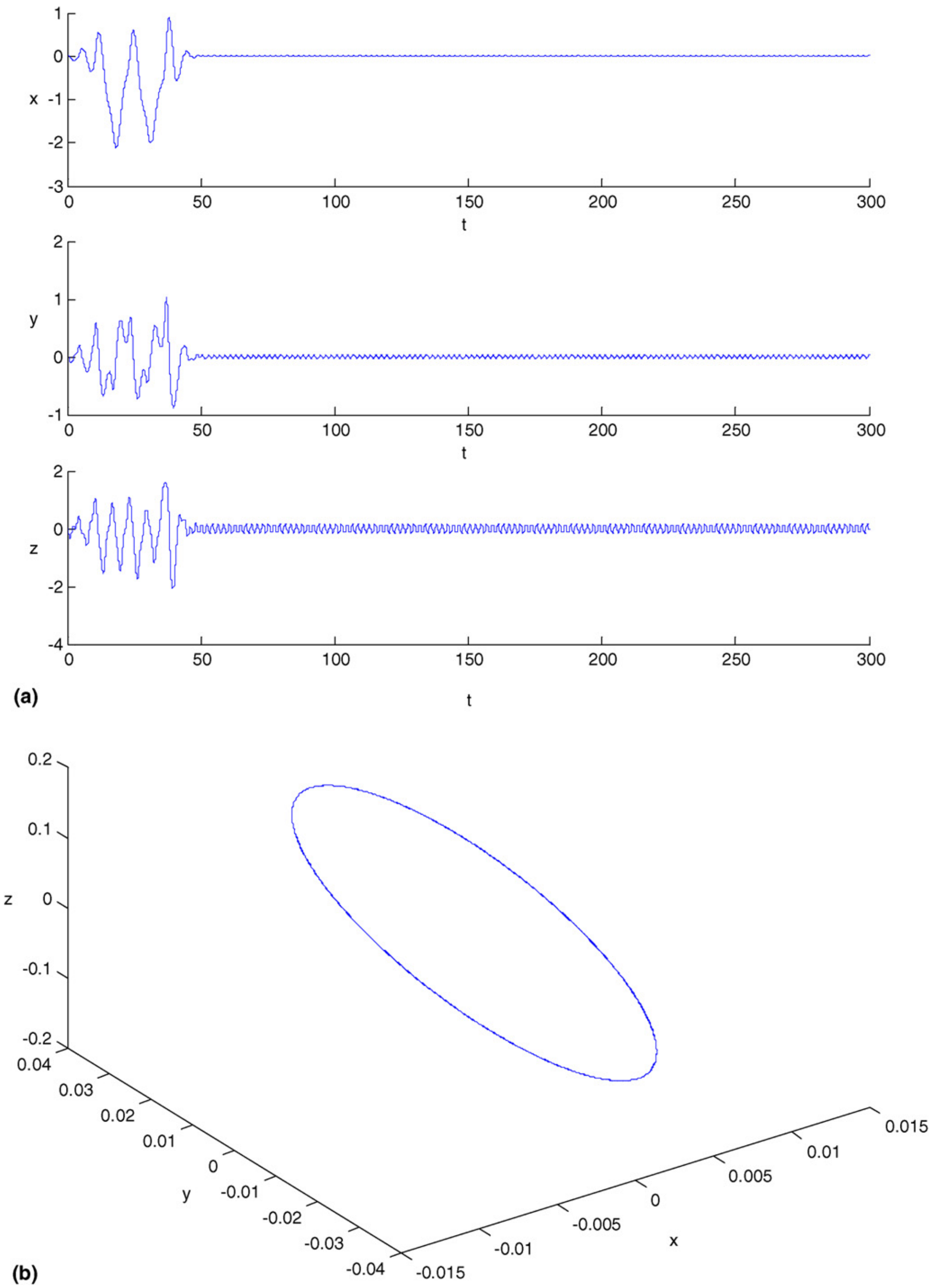

Fig. 16. (a) The time histories and (b) the phase portrait of the states of the controlled nonautonomous integral order system with order $q = 1$.

$$\frac{d^q X}{dt^q} = f(X) + g(t), \tag{13}$$

$$\frac{d^p X'}{dt^p} = f(X') + g(t) + u, \tag{14}$$

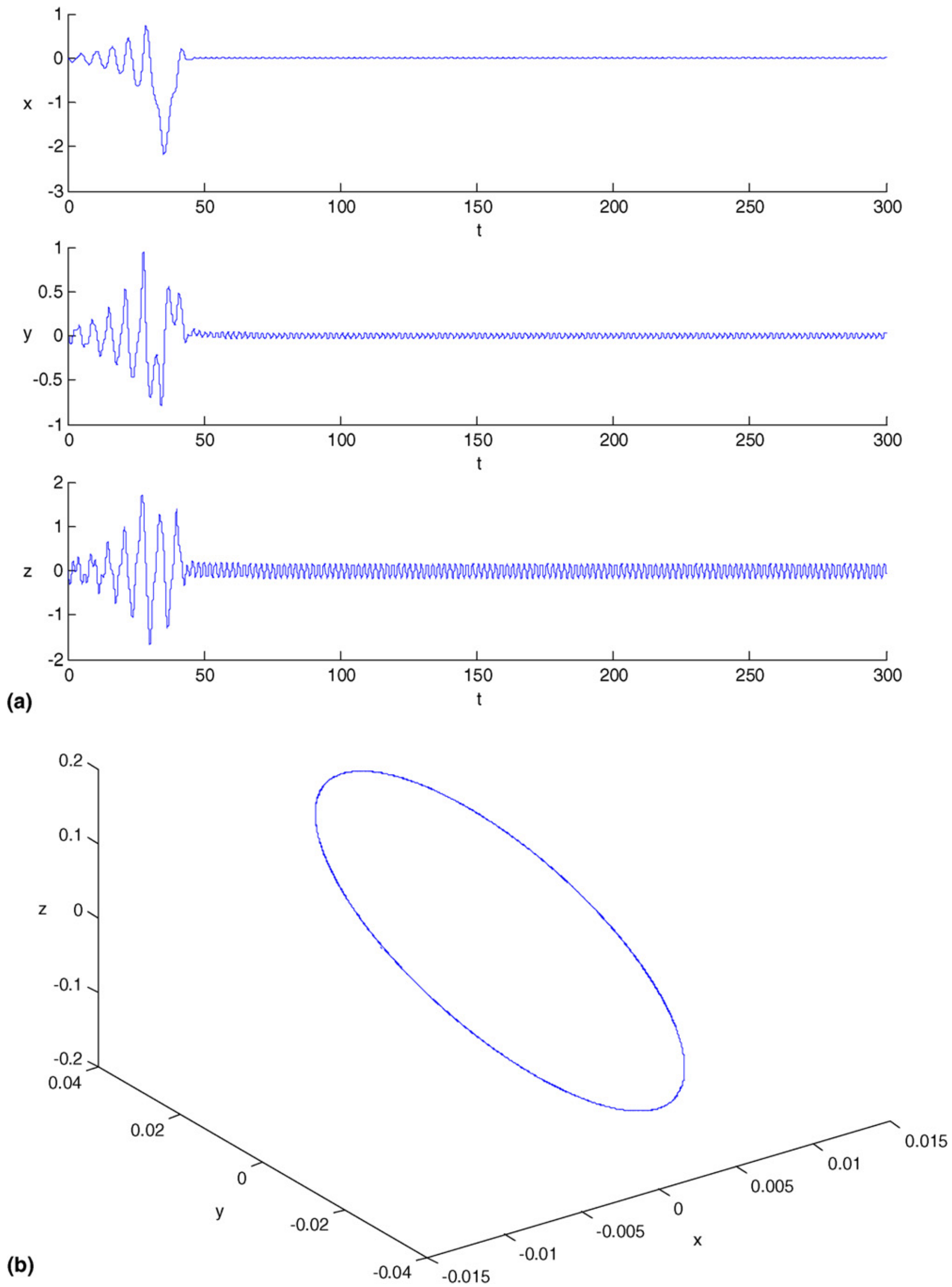

Fig. 17. (a) The time histories and (b) the phase portrait of the states of the controlled nonautonomous integral order system with order $q = 0.9$.

where q and p are different fractions and u is a linear state feedback controller as that in Section 6.2. Synchronization can be practically achieved.

The synchronizations are studied in two cases.

Case 1. $q = 0.9$, $p = 1.1$, $K = \text{diag}(-1000, -1000, -1000)$.

The controller is added at $t = 300$ s, and the response system is practically synchronized as shown in Fig. 21.

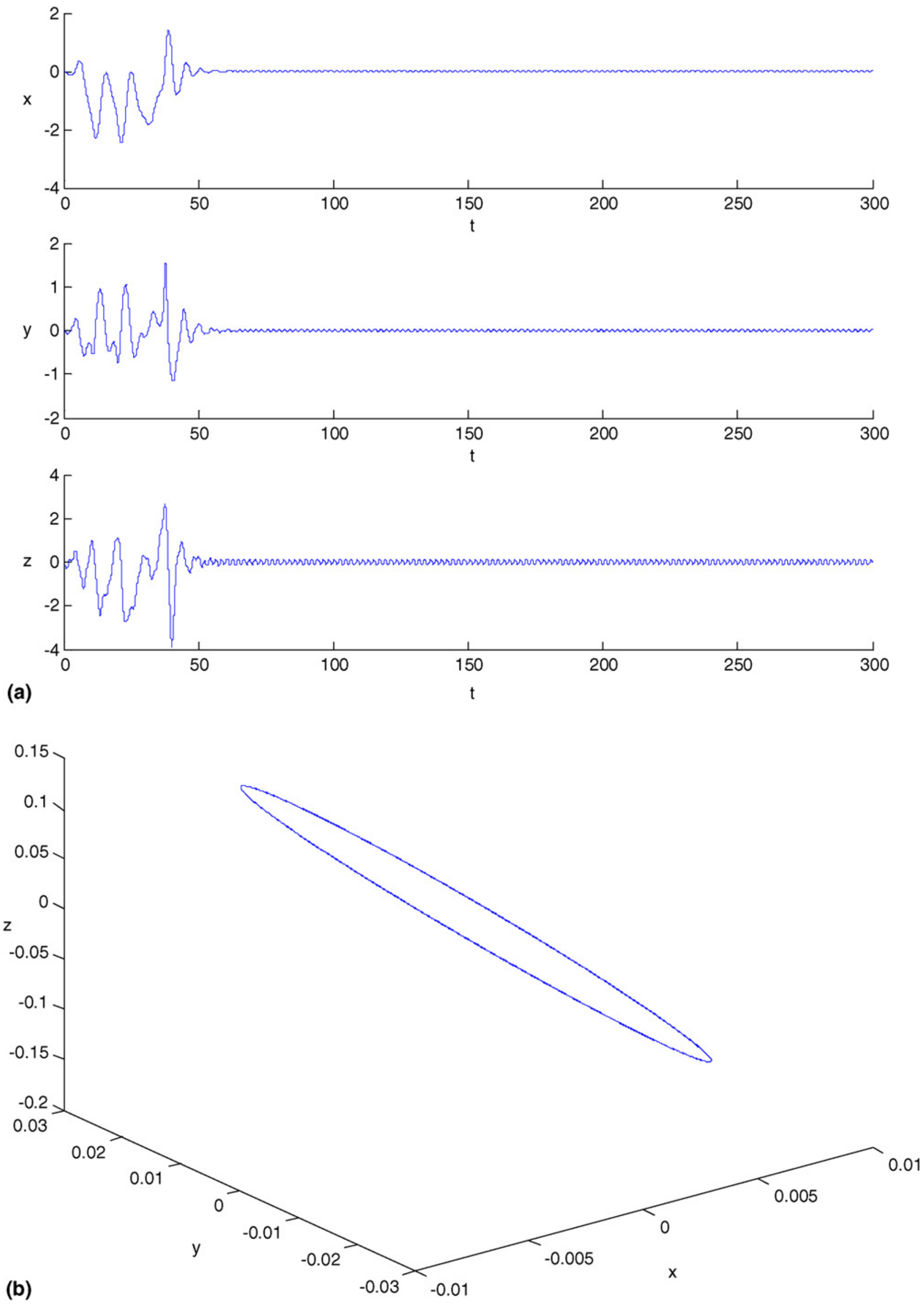

Fig. 18. (a) The time histories and (b) the phase portrait of the states of the controlled nonautonomous fractional order system with order $q = 1.1$.

Case 2. $q = 1.1$, $p = 0.9$, $K = \text{diag}(-1000, -1000, -1000)$.

The controller is added at $t = 300$ s, and the response system is practically synchronized as shown in Fig. 22.

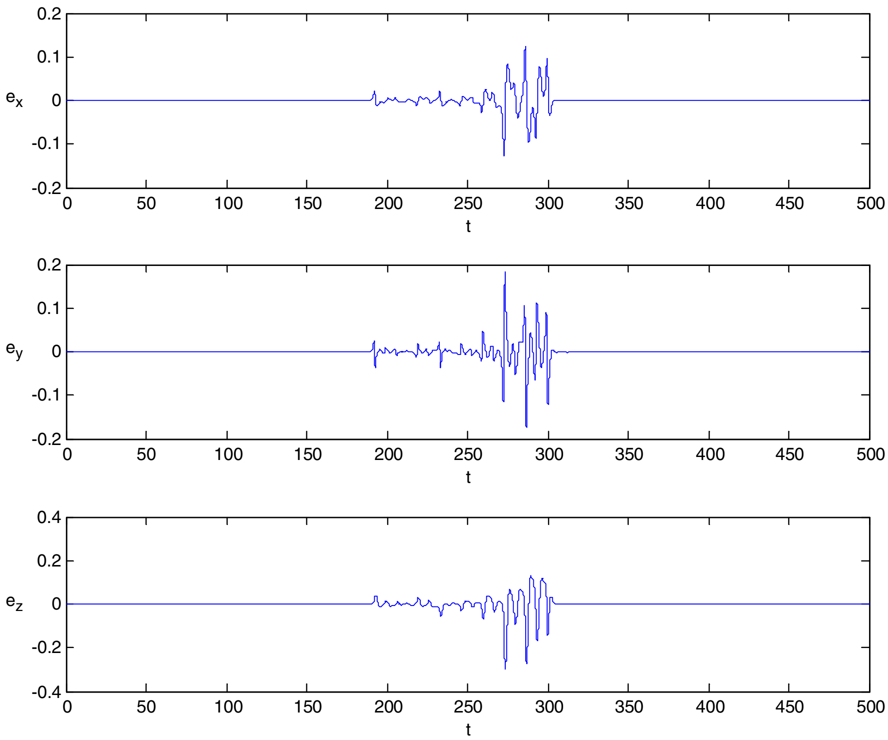

Fig. 19. The time histories of the errors of the states of the nonautonomous fractional order system with order $q = 0.9$.

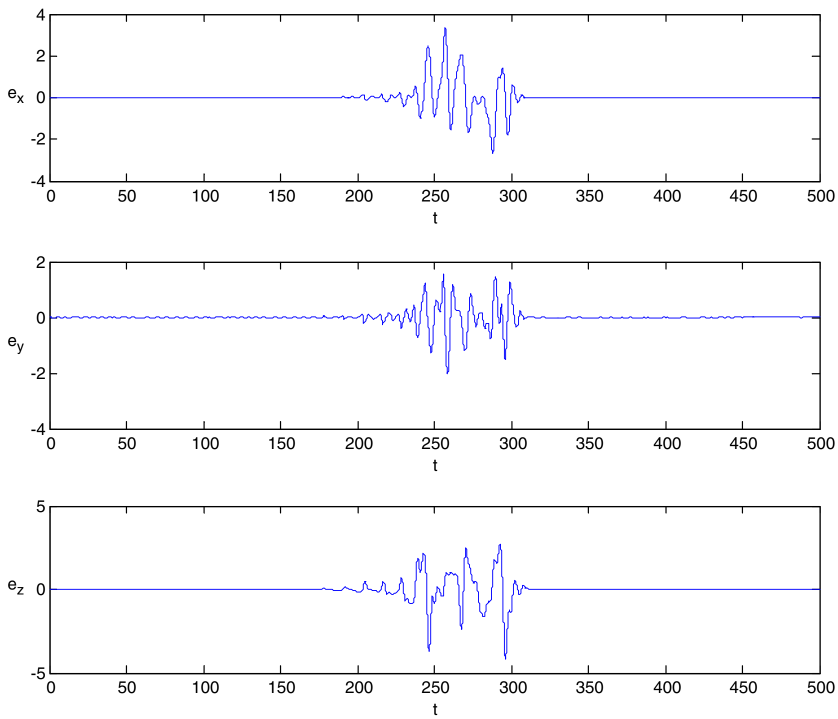

Fig. 20. The time histories of the errors of the states of the nonautonomous fractional order system with order $q = 1.1$.

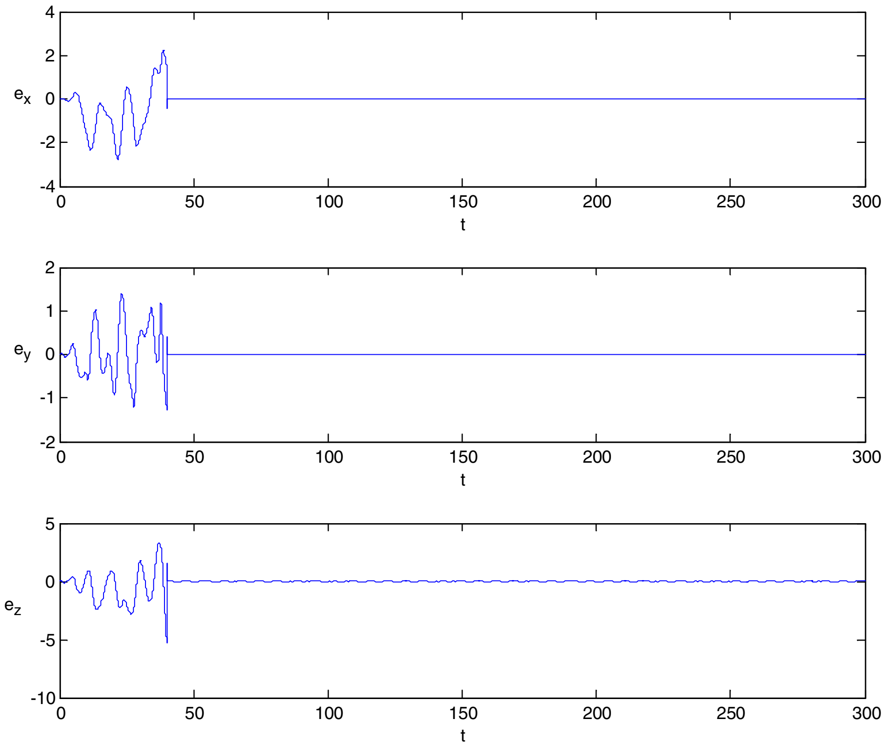

Fig. 21. The time histories of the errors of the states of nonautonomous systems with different fractional orders $q = 0.9$ and $p = 1.1$.

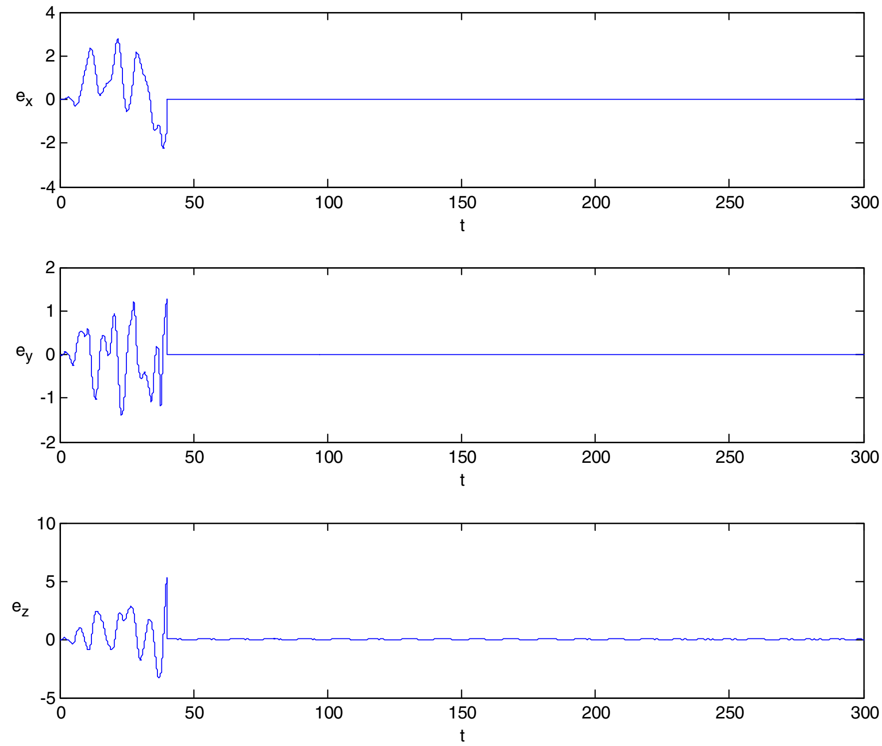

Fig. 22. The time histories of the errors of the states of nonautonomous systems with different fractional orders $q = 1.1$ and $p = 0.9$.

7. Conclusions

The chaotic behaviors in the fractional order autonomous and nonautonomous nonlinear systems of rotational mechanical system with a centrifugal governor are studied. It is shown that systems with total order less than and more than the number of state variables exhibit chaos as well as their corresponding integral order system. Phase portraits and bifurcation diagrams assure the existence of chaotic phenomena. By using the linear feedback controller scheme, chaos controls and chaos synchronizations are obtained. However, practical chaos synchronizations of different fractional order systems need large coupling strength.

Acknowledgement

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