Large-time solutions for groundwater flow problems using the relationship of small *p* versus large *t*

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[1] An approximate solution is useful if the corresponding analytical solution is complicated and difficult to accurately evaluate. In the past, the relationship of small *p* versus large *t* was commonly applied to the Laplace domain solution and could successfully obtain a large-time solution in the groundwater area. The large-time solution usually has a simpler form than the analytical solution and is much easier for estimating the transient behavior of the groundwater flow system. However, Chen and Stone (1993) pointed out that the use of this relationship might fail to yield the correct solution in calculating the wellbore flux for the constant head test problem. Later, Mathias and Zimmerman (2003) indicated that a poor result was obtained by Gerke and van Genuchten (1993) when using the relationship of small *p* versus large *t* to derive the water transfer coefficient for the dual-porosity media problem. This note is to show the detailed mathematical derivations involved in the issues that Chen and Stone (1993) and Gerke and van Genuchten (1993) addressed and to ensure that the relationship of small *p* versus large *t* is correct to obtain a large-time solution for transient groundwater flow problems.

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1. Introduction

[2] The constant head aquifer test is suitable for use in estimating the hydraulic parameters of low-permeability aquifer in groundwater areas. This test keeps a constant head at a pumping/injection well throughout the test period, and the change of flow rate across the wellbore versus time is recorded [*Batu*, 1998]. The head distribution for constant head test in a confined aquifer is analogous to the heat distribution for constant temperature maintained at a bounded circular cylinder; thus the head solutions for both Laplace domain and time domain can be obtained from *Carslaw and Jaeger* [1959]. The wellbore flux can then be derived on the basis of the solution of head distribution and Darcy's law [e.g., *Yang and Yeh*, 2002; *Peng et al.*, 2002]. However, the time domain solution of the wellbore flux is complex and difficult to accurately evaluate. Therefore it was common to derive the approximate solutions for small or large value of the time for the wellbore flux [e.g., *Carslaw and Jaeger*, 1959, p.336].

[3] One way for obtaining the small- or large-time solution is to apply the relationship of large *p* versus small *t* (hereinafter referred to as LPST) or small *p* versus large *t* (hereinafter referred to as SPLT), respectively, to the Laplace domain solution. This concept is based on a symbolic relation between the derivative operator of time, i.e., *d*/*dt*, in the time domain and the dummy variable, *p*, in the Laplace domain [*van Everdingen and Hurst*, 1949]. Then, one may obtain a small- or large-time solution by taking inverse Laplace transform on the reduced Laplace domain solution while the dummy variable is large or small, respectively. Some of the successful illustrations of applying this concept can be found in the groundwater literature. *van Everdingen and Hurst* [1949] used the relationships of LPST and SPLT to derive the pressure head of groundwater flow in a reservoir for time which were small and large, respectively. *Hantush* [1960] considered the aquitard storage for flow in a leaky aquifer system and obtained small- and large-time drawdown solutions by applying those two relationships. *Neuman and Witherspoon* [1969] studied the problem for flow in a confined two-aquifer system by considering the aquitard storage and drawdown in the unpumped aquifer. They used the LPST relationship to obtain the small-time solution for their problem. *Singh and Sagar* [1980] proposed approximate solutions of head to the linearized flow equation of slightly compressible fluids by using the relationships of LPST and SPLT. *Javandel and Witherspoon* [1983] and *Butler and Liu* [1993] provided a large-time solution for pumping-induced drawdown in a vertical and horizontal nonuniform aquifer, respectively, based on the SPLT relationship. *Chakrabarty et al.* [1993] provided a nonlinear pressure distribution of compressible liquid in a homogeneous formation and the corresponding small- and large-time solutions obtained using the relationships of LPST and SPLT, respectively. In addition, a number of approximate solutions for small and/or large value of time were derived on the basis of these relationships in areas such as the solute transport problem [*van Genuchten et al.*, 1984; *Chen*, 1985, 1986; *Yates*, 1990], dual-porosity media problem [*Barker*, 1985], and unsteady infiltration problem [*Philip*, 1986].

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[4] However, one may also find two articles indicating that erroneous results were obtained when applying the SPLT relationship. On the constant head test problem, *Chen and Stone* [1993] presented a calculation for the flow rate across the wellbore from the constant head test problem and concluded that the SPLT relationship failed to yield a correct large-time solution in this case. For the dual-porosity media problem, *Gerke and van Genuchten* [1993] solved a large-time water transfer coefficient by comparing two Laplace domain solutions, one was the "first-order" flux equation between fracture and matrix and the other was Richards's equation within the matrix block, while p became small. Moreover, *Mathias and Zimmerman* [2003] also obtained a large-time water transfer coefficient for the dual-porosity media problem based on the time domain approach. Their solution differs from the one obtained by *Gerke and van Genuchten* [1993] using the Laplace domain approach based on the SPLT relationship. When comparing both Laplace and time domain solutions with the exact solution, they implicitly indicated that the SPLT relationship did not hold.

[5] The relationships of LPST and SPLT had been widely used in the groundwater literature for more than 40 years. Previous researches indicated that the SPLT relationship could successfully yield correct large-time solutions if applied to the related Laplace domain solutions. The contrary results of *Chen and Stone* [1993] and *Mathias and Zimmerman* [2003] stimulate our curiosity and further study on this issue. The objectives of this note are to go through detailed mathematical derivations involved in the issues that *Chen and Stone* [1993] and *Gerke and van Genuchten* [1993] addressed and to resolve the dispute on the validity of the SPLT relationship. In addition, on the basis of the Laplace domain solution and the SPLT relationship, we will derive a correct large-time solution of the wellbore flux rate for the constant head test and the correct water transfer coefficient for the dual-porosity media problem.

2. Problem of Wellbore Flux for the Constant Head Test

2.1. Analytical Solution of Wellbore Flux

[6] A constant head test conducted in a confined aquifer is considered in this section. Assume that the confined aquifer is homogeneous, isotropic, laterally infinite, and with a constant thickness. The flow rate across the wellbore at the test well can be obtained by applying Darcy's law to head distribution for the constant head test problem [*Carslaw and Jaeger*, 1959] and the results in Laplace domain and time domain are, respectively,

$$\overline{Q}(r_w, p) = 2\pi r_w T \frac{h_w}{p} \frac{qK_1(qr_w)}{K_0(qr_w)} \tag{1}$$

$$Q(r_w, t) = \frac{8Th_w}{\pi} \int_0^\infty \exp\left(-\frac{T}{S}u^2 t\right) \frac{du}{u\left[J_0^2(r_w u) + Y_0^2(r_w u)\right]} \tag{2}$$

where r_w is the well radius; p is the Laplace variable; h_w is the constant head maintained at test well; $q = \sqrt{pS/T}$ in which S is the storage coefficient and T is the transmissivity of the confined aquifer; $K_0(\cdot)$ and $K_1(\cdot)$ are modified Bessel functions of the second kinds of order zero and order one, respectively; t is time variable; $J_0(\cdot)$ and $Y_0(\cdot)$ are Bessel functions of the first and second kinds of order zero, respectively. Note that the negative sign of Q corresponds to withdrawal and the positive sign corresponds to injection.

2.2. Large-Time Solution of Wellbore Flux

[7] With the limiting forms of $K_0(x) \cong -[\ln(x/2) + \gamma]$ and $K_1(x) \cong 1/x$, where $\gamma = 0.57722 \ldots$ is Euler's constant, for small value of x [*Abramowitz and Stegun*, 1970, p.375], the wellbore flux in Laplace domain, (1), for small p can be reduced to

$$\overline{Q}(r_w, p) \cong -4\pi T h_w \frac{1}{p \ln(p/\lambda)} \tag{3}$$

where $\lambda = 4T/(c^2 r_w^2 S)$ and $c = \exp(\gamma)$. The large-time wellbore flux is subsequently obtained by taking the inverse Laplace transform on (3). *Chen and Stone* [1993] used a inverse Laplace transform formula given by the *Oberhettinger and Badii* [1973, p. 276, equation (6.75)] for the inverse Laplace transform of term $1/[p \ln(p/\lambda)]$ in (3) as

$$L^{-1}\left[\frac{1}{p\ln(p/\lambda)}\right] = \int_0^\infty \frac{(\lambda t)^x}{\Gamma(x+1)} dx \tag{4}$$

where $\Gamma(x)$ is the gamma function. They derived the integration of (4) and obtained the large-time wellbore flux as $Q(r_w, t) = -\infty$. In addition, they showed this large-time wellbore flux was contradictory to the result obtained by applying the Tauberian theorem (also called final value theorem [*Spiegel*, 1965]) to (3), i.e., $Q(r_w, t \to \infty) = 0$. On the basis of this study, they concluded that the application of the SPLT relationship should be used with care because of possible failure to yield a correct large-time solution.

[8] In fact, we found that the use of inverse Laplace transform formula of (4) should be under a constraint of $p > \lambda$ which is proven in the Appendix A. For most confined aquifers, the value of the storage coefficient falls in the range $10^{-5} \sim 10^{-3}$ and for sand and silt formations the values of hydraulic conductivity falls in the range of $10^{-2} \sim 10^{1}$ m/day [*Todd and Mays*, 2005]. Assuming that the thickness of confined aquifer is 10 m and the radius of test well is 5 cm; then, the value of λ ranges from 10^4/day to 10^9/day. Notice that the Laplace variable p is required to be small in (3) and λ should be small too. Yet, the value of λ is larger than 10^4/day for the real-world problem as demonstrated above; accordingly, (4) does not hold at all.

[9] Hereafter, we propose an alternative formula for the inverse Laplace transform of term $1/[p \ln(p/\lambda)]$ in (3). *Ritchie and Sakakura* [1956] presented an article on the approximate expansions of solutions of the heat conduction equation in an internally bounded cylindrical solid. They gave the inverse Laplace transform for the term $1/[p \ln(p/\lambda)]$ when p is small as

$$L^{-1}\left[\frac{1}{p\ln(p/\lambda)}\right] = \sum_{s=0}^{N}\left[\left(\frac{-1}{\ln\eta}\right)^{s+1}\binom{-1}{s}\frac{d^s}{d\nu^s}\left(\frac{1}{\Gamma(1-\nu)}\right)\bigg|_{\nu=0}\right] \tag{5}$$

where the dimensionless variable $\eta = \lambda t$, column vector $(-1, s)^T$ is the binomial coefficient, and N is the number of truncated terms depending on the values of the remainder.

[10] The right hand side (RHS) of (5) is a summation of products of the dimensionless variable $\ln\eta$ and the constant

value of the Gamma function. For the large value of time, the terms of $1/(\ln\eta)^5$ and higher-order terms may be truncated since η is proportional to the dimensionless time. Therefore the large-time wellbore flux can be obtained on the basis of (3) and (5). Appendix B shows the detailed expansion of (5) and the large-time solution for the wellbore flux is obtained approximately as

$$Q(r_w,t) = 4\pi T h_w \left[\frac{1}{\ln\eta} - \frac{\gamma}{(\ln\eta)^2} + \frac{\gamma^2 - \frac{\pi^2}{6}}{(\ln\eta)^3} - \frac{\gamma^3 - \frac{\pi^2}{2}\gamma + 2\xi(3)}{(\ln\eta)^4}\right] \quad (6)$$

[11] The numerators of the RHS terms of (6) are all constants and the denominators are the function of dimensionless time, $\eta = 4Tt/(c^2 r_w^2 S)$. The value of $\ln\eta$ reaches infinity if t approaches infinity; therefore (6) becomes zero; that is, the steady state solution for the wellbore flux is zero. Notice that *Jaeger* [1943] and *Carslaw and Jaeger* [1959, p. 336] gave a large-time wellbore flux which has the first three terms and first two terms of (6), respectively.

3. Problem of the Water Transfer Coefficient for the Dual-Porosity System

3.1. Large-Time Water Transfer Coefficient From Time Domain Approach

[12] In this section we consider the dual-porosity system with parallel rectangular slabs which are matrix blocks separated by a fracture pore studied by *Gerke and van Genuchten* [1993]. The water flow of dual-porosity system is described by two Richards's equations, one for the fracture pore and the other for the matrix block. Those two flow equations are coupled by means of a sink/source term to account for water transfer between the dual-porosity systems. *Gerke and van Genuchten* [1993] proposed a first-order model to calculate the water transfer which is proportional to the difference in pressure head between the fracture and matrix pore systems. The water transfer coefficient, denoted as α_w, was defined as the ratio of water transfer and head difference.

[13] If the pressure head of the fracture pore is considered to be a constant in time, the water transfer coefficient can be obtained by comparing two rearranged flow equations in Laplace domain as [*Gerke and van Genuchten*, 1993]

$$\overline{h_m}(p) = \left(\frac{h_f - h_{m,i}}{p}\right)\frac{\tanh(\xi)}{\xi} + \frac{h_{m,i}}{p} \quad (7)$$

$$\overline{h_m}(p) = \frac{h_f}{p}\left(\frac{1}{1+\zeta}\right) + \frac{h_{m,i}}{p}\left(\frac{1}{1+\zeta^{-1}}\right) \quad (8)$$

where $\overline{h_m}$ and $h_{m,i}$ are the Laplace domain head and the initial head, respectively, in the matrix block; h_f is the imposed head at the fracture boundary; and variables $\xi = a[(1-w_f)C_m p/K_a]^{0.5}$ and $\zeta = (1-w_f)C_m p/\alpha_w$ while a is the characteristic half width of the matrix block, w_f is the fracture porosity, K_a is the hydraulic conductivity of the matrix block near the fracture/matrix interface, and C_m is the specific water capacity at the matrix.

[14] *Mathias and Zimmerman* [2003] applied the Laurent-type expansion to (7) and (8) and transferred those series to the time domain by the asymptotic formula of *Doetsch* [1961] when time is large. On the basis of the time domain approach, they obtained an exact large-time water transfer coefficient of $\alpha_w = \pi^2 K_a/4a^2 (\cong 2.47\ K_a/a^2)$ which differs from the Laplace domain approach of $\alpha_w = 3K_a/a^2$ obtained by *Gerke and van Genuchten* [1993] derived from (7) and (8) on the basis of the SPLT relationship.

[15] Once the water transfer coefficient is obtained, the normalized head difference for large value of time is [*Mathias and Zimmerman*, 2003]

$$\frac{h_m(t) - h_f}{h_{m,i} - h_f} = \exp\left[\frac{-\alpha_w t}{(1-w_f)C_m}\right] \quad (9)$$

where h_m is the time domain head. Equation (9) can be verified by comparing it with the exact solution given by *Crank* [1956, p. 48] which was shown by *Mathias and Zimmerman* [2003] as

$$\frac{h_m(t) - h_f}{h_{m,i} - h_f} = \frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\exp\left[\frac{-(2n+1)^2\pi^2 K_a t}{(2a)^2(1-w_f)C_m}\right] \quad (10)$$

It is clear that the normalized head difference approaches zero while dimensionless time $K_a t/[(2a)^2(1-w_f)C_m]$ on the RHS of (10) goes to very large.

[16] After comparing with the exact solution given by *Crank* [1956], *Mathias and Zimmerman* [2003] concluded that the discrepancy of the water transfer coefficient from the time domain approach and Laplace domain approach arise from the use of the SPLT relationship. However, following careful investigations, we find that this relationship is indeed correct and the defect of *Gerke and van Genuchten* [1993] is mainly caused by neglecting the convergent requirements of series expansion for (7) and (8).

3.2. Large-Time Water Transfer Coefficient From the Laplace Domain Approach

[17] *Gerke and van Genuchten* [1993] used series expansion for $\tanh(\xi)$ and $1/(1+\zeta)$ in (7) and (8), respectively. Those two equations were then respectively expressed as

$$\begin{aligned}\overline{h_m}(p) = {} & \frac{h_f}{p}\left[1 - \frac{a^2(1-w_f)C_m p}{3K_a} + \frac{2a^4(1-w_f)^2C_m^2p^2}{15K_a^2} - \frac{17a^6(1-w_f)^3C_m^3p^3}{315K_a^3} + \ldots\right] \\ & + \frac{h_{m,i}}{p}\left[\frac{a^2(1-w_f)C_m p}{3K_a} - \frac{2a^4(1-w_f)^2C_m^2p^2}{15K_a^2} + \frac{17a^6(1-w_f)^3C_m^3p^3}{315K_a^3} - \ldots\right]\end{aligned} \quad (11)$$

$$\begin{aligned}\overline{h_m}(p) = {} & \frac{h_f}{p}\left[1 - \frac{(1-w_f)C_m p}{\alpha_w} + \frac{(1-w_f)^2C_m^2p^2}{\alpha_w^2} - \frac{(1-w_f)^3C_m^3p^3}{\alpha_w^3} + \ldots\right] \\ & + \frac{h_{m,i}}{p}\left(\frac{(1-w_f)C_m p}{\alpha_w} - \frac{(1-w_f)^2C_m^2p^2}{\alpha_w^2} + \frac{(1-w_f)^3C_m^3p^3}{\alpha_w^3} - \ldots\right)\end{aligned} \quad (12)$$

The water transfer coefficient can then be obtained as $\alpha_w = 3K_a/a^2$ if one compares (11) with (12) and sets the second term on RHS of these two equations are equal. Similarly, the water transfer coefficients obtained from the third and fourth terms are $\alpha_w = \sqrt{15/2}K_a/a^2$ ($\cong 2.74K_a/a^2$) and $\alpha_w = \sqrt[3]{315/17}K_a/a^2$ ($\cong 2.65K_a/a^2$), respectively. It seems that the estimated water transfer coefficient appears monotonously decreasing from 3 and asymptotically approaches 2.47. Notice that *Gerke and van Genuchten* [1993] truncated the third and remaining higher-order terms of p in (11) and (12) because of small p.

[18] The series expansion for $\tanh(\xi)$ in (7) and $1/(1 + \zeta)$ in (8) should be restricted to the convergent criteria $|\xi| < \pi/2$ and $|\zeta| < 1$ [*Abramowitz and Stegun*, 1970, pp. 15, 85], respectively. Accordingly, the Laplace variable, p, has to satisfy the constraints of $p < \pi^2 K_a/[4a^2(1 - w_f)C_m]$ and $p < \alpha_w/[(1 - w_f)C_m]$ respectively for (7) and (8). In order to satisfy these two constraints simultaneously, the water transfer coefficient has to be $\alpha_w = \pi^2 K_a/4a^2$ which is indeed equal to that of *Mathias and Zimmerman* [2003]. Obviously, the coefficient of $\alpha_w = 3K_a/a^2$ obtained by *Gerke and van Genuchten* [1993] is larger than the one of $\alpha_w = \pi^2 K_a/4a^2$ and consequently violates the former constraint. A detailed derivation for large-time water transfer coefficient is shown in Appendix C and the result also yields $\alpha_w = \pi^2 K_a/4a^2$. This result obtained from the Laplace domain approach based on the SPLT relationship is exactly the same as that of *Mathias and Zimmerman* [2003] derived from the time domain approach. Therefore the dispute in the discrepancy that was calculated using Laplace domain approach by making use of the SPLT relationship and those found by working in the time domain is clearly resolved.

4. Conclusions

[19] In this note we demonstrate that the inverse Laplace transform formula of *Oberhettinger and Badii* [1973, pp. 276, 424] adopted to invert Laplace domain solution by *Chen and Stone* [1993] should be under the constraint $p > \lambda$ where λ is a finite value. One will obtain an erroneous solution if applying that inverse Laplace transform formula without satisfying this necessary constraint. Therefore the inconsistent results obtained by *Chen and Stone* [1993] from the SPLT relationship and the Tauberian theorem that arose from a violation the constraint occurred when applying the inverse Laplace transform formula rather than using the SPLT relationship. In addition, we also derive a large-time solution for the dimensionless wellbore flux of the constant head test based on the SPLT relationship and the work of *Ritchie and Sakakura* [1956]. Moreover, our large-time solution obtains the steady state result, when the time approaches infinity, which is exactly the same as that obtained by *Chen and Stone* [1993] by applying the Tauberian theorem to the Laplace domain solution.

[20] In regard to the estimation of water transfer coefficient in a dual-porosity media problem, the series expansion for $\tanh(\xi)$ and $1/(1 + \zeta)$ used by *Gerke and van Genuchten* [1993] should satisfy the convergent criteria of $|\xi| < \pi/2$ and $|\zeta| < 1$, respectively. By neglecting these two convergent criteria, *Gerke and van Genuchten* [1993] obtained a poor estimated result for the water transfer coefficient. We also derived a large-time water transfer coefficient from the Laplace domain solution based on series expansion approaches and the SPLT relationship. Our water transfer coefficient obtained from the Laplace domain approach is exactly the same as that of *Mathias and Zimmerman* [2003] derived from the time domain approach. Therefore the long-standing discrepancy in the estimated water transfer coefficients in dual-porosity systems obtained using the time domain approach and the Laplace domain approach is clearly and thoroughly resolved. We therefore conclude that the SPLT relationship is correct and applicable to the Laplace domain solution in obtaining a large-time solution for transient groundwater flow problems.

Appendix A: Derivation of Equation (4)

[21] By applying the Laplace transform, the integral function of (4) can be expressed as

$$L\left[\int_0^\infty \frac{(\lambda t)^x}{\Gamma(x+1)}dx\right] = \int_0^\infty e^{-pt}\int_0^\infty \frac{(\lambda t)^x}{\Gamma(x+1)}dxdt \tag{A1}$$

The RHS of (A1) is a double integral and can be rearranged as

$$\int_0^\infty e^{-pt}\int_0^\infty \frac{(\lambda t)^x}{\Gamma(x+1)}dxdt = \frac{1}{p}\int_0^\infty \frac{(\lambda/p)^x}{\Gamma(x+1)}\left[\int_0^\infty e^{-pt}(pt)^x d(pt)\right]dx \tag{A2}$$

[22] The Gamma function is defined as [*Abramowitz and Stegun*, 1970, p. 255]

$$\Gamma(x+1) = \int_0^\infty e^{-u}u^x du \tag{A3}$$

Replacing the second integral in (A2) by Gamma function, the RHS of (A2) after the integration gives

$$\frac{1}{p}\int_0^\infty \left(\frac{\lambda}{p}\right)^x dx = -\lim_{z\to\infty}\frac{1}{p}\frac{1}{\ln(p/\lambda)}\left(\frac{p}{\lambda}\right)^{-x}\Bigg|_{x=0}^{x=z} \tag{A4}$$

[23] If $p > \lambda$, the term $(p/\lambda)^{-z}$ approaches zero as $z \to \infty$, then (A4) reduces to

$$L\left[\int_0^\infty \frac{(\lambda t)^x}{\Gamma(x+1)}dx\right] = \frac{1}{p\ln(p/\lambda)} \tag{A5}$$

This derivation shows that (4) is hold only under the condition that $p > \lambda$.

Appendix B: Derivation of Equation (6)

[24] The first four terms of (5) can be rewritten using the notation of *Ritchie and Sakakura* [1956] as

$$L^{-1}\left[\frac{1}{p\ln(p/\lambda)}\right] = -\left[\frac{B_0^{0,-1}}{\ln\eta} + \frac{B_1^{0,-1}}{(\ln\eta)^2} + \frac{B_2^{0,-1}}{(\ln\eta)^3} + \frac{B_3^{0,-1}}{(\ln\eta)^4}\right] \tag{B1}$$

$$\mathrm{B}_s^{0,-1} = (-1)^s \binom{-1}{s} \frac{d^s}{d\nu^s}\left[\frac{1}{\Gamma(1-\nu)}\right]\Bigg|_{\nu=0}, \quad s = 0, 1, 2, 3 \quad \text{(B2)}$$

where the coefficient $\mathrm{B}_s^{0,-1}$ relates to the Gamma function. The selected properties of binomial coefficient, Gamma function, and Polygamma function for following derivation are, respectively, [*Abramowitz and Stegun*, 1970]

$$\binom{-1}{s} = (-1)^s \quad \text{(B3)}$$

$$\Gamma(1-\nu) = -\nu\Gamma(-\nu) \quad \text{(B4)}$$

$$\frac{d^s}{d\nu^s}\psi(\nu) = \frac{d^{s+1}}{d\nu^{s+1}}[\ln\Gamma(\nu)] \quad \text{(B5)}$$

[25] The values of $\mathrm{B}_s^{0,-1}$ for s equaling 0, 1, 2, and 3 can be derived as, respectively,

$$\mathrm{B}_0^{0,-1} = \binom{-1}{0}(-1)^0\frac{1}{\Gamma(1)} = \frac{1}{\Gamma(1)} = 1 \quad \text{(B6)}$$

$$\mathrm{B}_1^{0,-1} = \binom{-1}{1}(-1)^1\frac{d}{d\nu}\left[\frac{1}{\Gamma(1-\nu)}\right]\Bigg|_{\nu=0} = \frac{\Gamma'(1)}{\Gamma^2(1)} = -\gamma \quad \text{(B7)}$$

$$\begin{aligned}\mathrm{B}_2^{0,-1} &= \binom{-1}{2}(-1)^2\frac{d^2}{d\nu^2}\left[\frac{1}{\Gamma(1-\nu)}\right]\Bigg|_{\nu=0} \\ &= -\frac{\Gamma''(1)}{\Gamma^2(1)} + 2\frac{(\Gamma'(1))^2}{\Gamma^3(1)} = \gamma^2 - \frac{\pi^2}{6}\end{aligned} \quad \text{(B8)}$$

$$\begin{aligned}\mathrm{B}_3^{0,-1} &= \binom{-1}{3}(-1)^3\frac{d^3}{d\nu^3}\left[\frac{1}{\Gamma(1-\nu)}\right]\Bigg|_{\nu=0} \\ &= \frac{\Gamma'''(1)\Gamma^2(1) - 2\Gamma''(1)\Gamma'(1)\Gamma(1)}{\Gamma^4(1)} \\ &\quad - 2\frac{2\Gamma''(1)\Gamma'(1)\Gamma^3(1) - 3(\Gamma'(1))^2\Gamma^2(1)\Gamma(1)}{\Gamma^6(1)} \\ &= -\gamma^3 + \frac{\pi^2}{2}\gamma - 2\xi(3)\end{aligned} \quad \text{(B9)}$$

where the Riemann Zeta function $\xi(3) = 1.2020569032$.

[26] Substituting (B6)–(B9) into (B1) gives

$$L^{-1}\left[\frac{1}{p\ln(p/a)}\right] = -\left[\frac{1}{\ln\eta} - \frac{\gamma}{(\ln\eta)^2} + \frac{\gamma^2 - \frac{\pi^2}{6}}{(\ln\eta)^3} - \frac{\gamma^3 - \frac{\pi^2}{2}\gamma + 2\xi(3)}{(\ln\eta)^4}\right] \quad \text{(B10)}$$

Thus the inverse Laplace transform of (3) results in (6) when truncating high-order terms of (5).

Appendix C: Derivations of the Water Transfer Coefficient

[27] The series expansion of tanh x can be expressed as [*Abramowitz and Stegun*, 1970, p.85]

$$\begin{aligned}\tanh x &= x - \frac{x^3}{3} + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \ldots \\ &\quad + \frac{2^{2n}(2^{2n}-1)B_{2n}}{(2n)!}x^{2n-1} + \ldots, \quad n = 1, 2, \ldots\end{aligned} \quad \text{(C1)}$$

where B_{2n} is the nth Bernoulli number and the convergent criterion of tanh x is $|x| < \pi/2$.

[28] Subsequently, one uses the Fourier expansion of the Bernoulli number [*Abramowitz and Stegun*, 1970, p. 805] and obtains

$$B_{2n} = \frac{(-1)^{n-1}2(2n)!}{(2\pi)^{2n}}\sum_{k=1}^{\infty}\frac{1}{k^{2n}}, \quad n = 1, 2, \ldots \quad \text{(C2)}$$

Therefore the nth term of water transfer coefficient is obtained by letting (11) equal (12) as

$$\begin{aligned}\alpha_{w,n} &= \left[\frac{(-1)^{n-1}(2n)!}{2^{2n}(2^{2n}-1)B_{2n}}\right]^{\frac{1}{n-1}}\frac{\mathrm{K}_a}{a^2} = \left[\frac{\pi^{2n}}{(2^{2n+1}-2)\sum_{k=1}^{\infty}\frac{1}{k^{2n}}}\right]^{\frac{1}{n-1}} \\ &\quad \cdot\frac{\mathrm{K}_a}{a^2}, \quad n = 2, 3, \ldots\end{aligned} \quad \text{(C3)}$$

[29] The limit of α_w for $n \to \infty$ is

$$\alpha_w = \lim_{n\to\infty}\left[\frac{\pi^{2n}}{(2^{2n+1}-2)\sum_{k=1}^{\infty}\frac{1}{k^{2n}}}\right]^{\frac{1}{n-1}}\frac{\mathrm{K}_a}{a^2} = \frac{\pi^2}{4}\frac{\mathrm{K}_a}{a^2} \quad \text{(C4)}$$

where the limit of Riemann Zeta function, $\lim_{n\to\infty}\sum_{k=1}^{\infty}k^{-2n}$, equals 1.

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