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ON WEAK RESIDUAL ERROR ESTIMATION*

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Abstract. A general framework for weak residual error estimators applying to various types of boundary value problems in connection with finite element and finite volume approximations is developed. Basic ideas commonly shared by various applications in error estimation and adaptive computation are presented and illustrated. Some numerical results are given to show the effectiveness and efficiency of the estimators.

Key words. adaptivity, a posteriori error estimates, finite element, finite volume, variational problems

AMS subject classifications. 65N30, 65N50, 35J20, 35J60

1. Introduction.

Adaptivity is a trend in contemporary computational science. The remarkable advances in adaptive methodology for finite element applications since the pioneer work by Babuška and Rheinboldt [7] have had a profound impact on practical, large-scale computations.

There are four types of adaptivity which can be identified by the letters r (relocating nodes), h (mesh size h refinement), p (spectral order p increment), and hp (both h refinement and p increment). All adaptive methods require more or less a posteriori information about the computed solution for optimizing overall computational efforts in the sense that the methods deliver a given level of accuracy with a minimum of degrees of freedom. In essence, the *a posteriori error estimation* can be regarded as the heart of the adaptive mechanism. In the development of a posteriori error estimators, three main approaches may be distinguished [22], [29], namely, those based on residual, postprocessing, or interpolation techniques. Our estimators here follow the first approach.

In the first approach it has become practically standard to specify the interior residuals in terms of the governing differential equation and to measure the boundary residuals by the jumps in the normal derivatives on the interfaces between elements; see [3], [4], [6], [7], [8], [11], [19]. The various error estimators then differ essentially in the way the jumps are handled. In contrast, we consider here error equations (or inequalities) in which the right side is a residual of the computed solution in weak form. It appears to be natural to call the resulting error estimates *weak residual* estimators. A special case of the estimators seems to be first proposed by Adjerid and Flaherty in [1]. The weak residual error estimators tend to be more widely applicable since, in many applications, the governing equation may not be available in differential form. In recent years, there has been growing evidence, in theory as well as in application, showing the promise of the use of weak residual estimators; see [1], [2], [9], [21], [25].

This paper attempts to give an overall view of the weak residual error estimation in connection with the adaptive process, different numerical methods, and various types of boundary value problems. The numerical schemes of particular interest belong to two families of widely used methods—the finite element method (FEM) and the finite volume method (FVM). All estimators presented here can be extracted to two basic components which are *weak residuals* and *complementary finite element (FE) spaces*. Two types of complementary spaces can be classified. One is the conforming type which together with the original FE space preserves the continuity across adjacent FEs. This in turn corresponds to an elementwise error estimation using error residuals only interior to elements. The estimators used in [1], [2], [9],

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[25] are of this type. The other is the nonconforming type which instead uses both interior and boundary residuals for each element. It is shown in [21] that the nonconforming type is independent of the FE order used for the computed solution, whereas the conforming type considers more closely the effect of the order of the FEs [1], [8]. While this study is not comprehensive, we do illustrate the two components in error estimation and the adaptive process by four model problems; namely, we consider elliptic boundary value problems, parametrized nonlinear equations, symmetric hyperbolic equations, and variational inequalities.

In §2, the error estimators are derived in a general framework by means of abstract variational formulations. Definitions, notation, and basic ideas are then introduced along the course of the derivation. Although the estimators are not restricted to any particular type of adaptivity, we also discuss briefly a standard h-refinement strategy in two space dimensions. In §3, the four model problems and the two numerical methods are specifically used to demonstrate how they can be cast into the general framework. Finally, in §4, numerical results are given with respect to each subsection in §3 to show the effectiveness and efficiency of the estimators.

2. Background and basic ideas. The aim of this paper is to offer, to the extent possible, a global view of the use of weak residual error estimators. It is instructive to summarize some fundamental features which constitute various adaptive methods for various model problems considered herein.

2.1. Abstract variational problems. Let Ω be a bounded region in the plane with a Lipschitz boundary $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$ and $H^k(\Omega)$ and $H^r(\Gamma)$, $\Gamma \subset \partial\Omega$, be the usual Sobolev spaces equipped with the norms $\|\cdot\|_k$ and $|\cdot|_{r,\Gamma}$, respectively. Let $H(\Omega) \subset L^2(\Omega)$ be a Sobolev space equipped with the norm $\|\cdot\|$. For simplicity, we assume that all functions in $H(\Omega)$ satisfy a homogeneous Dirichlet boundary condition on $\partial\Omega_D$, if any. Let K be a closed convex subset of $H(\Omega)$. Let $F : H(\Omega) \to R$ be a continuous linear form. Let $B(\cdot,\cdot) : H(\Omega) \times H(\Omega) \to R$ be a bilinear form such that there exist two positive constants β, δ and a nonnegative constant α for which

$$B(u,v) \le \beta\|u\|\|v\|, \quad u, v \in H(\Omega), \tag{2.1}$$

$$B(u,u) \ge \delta\|u\|^2 - \alpha\|u\|_0^2, \quad u \in H(\Omega). \tag{2.2}$$

We shall consider a class of very general problems in an abstract variational setting: Find $u \in K$ such that

$$B(u,v) - F(v) \ge B(u,u) - F(u) \quad \forall v \in K. \tag{2.3}$$

Depending on the definition of the closed convex set K and the properties of the bilinear form B, the abstract variational problem (2.3) can give various equivalent formulations of problems which will be exemplified in §3. Discussions of well-posedness of the problem (2.3), in some selective settings, can be found, e.g., in [13], [18].

2.2. FE spaces. To discretize (2.3), we introduce S a finite-dimensional subspace of $H(\Omega)$ characterized by a mesh size h and associated with a regular, but not necessarily quasi-uniform, triangulation $\mathcal{T}$ on Ω. To approximate K, we construct a closed convex subset K_s of S. The approximate problem of (2.3) is then to find $u_s \in K_s$ such that

$$B(u_s, v_s) - F(v_s) \ge B(u_s, u_s) - F(u_s) \quad \forall v_s \in K_s. \tag{2.4}$$

Our objective is to present various error formulations on which various a posteriori error estimators assessing the exact error between the solutions u and u_s of (2.3) and (2.4), respectively, are based. All estimators can be extracted to two basic components—weak residuals and complementary FE spaces. We first discuss the complementary spaces.

For the sake of efficiency, the estimators will be based on local computations. We introduce some local function spaces that we will require. Associated with $\mathcal{T}$, let $\bar{S}$ be another larger FE subspace of $H(\Omega)$, i.e., $S \subset \bar{S}$. Let $H(\tau)$ denote the restriction of $H(\Omega)$ to $\tau \in \mathcal{T}$ and $H_{\mathcal{T}}(\Omega) = \prod_{\tau\in\mathcal{T}} H(\tau)$ denote the space of piecewise $H(\Omega)$ functions. For instance, if $H(\Omega) \equiv H^1(\Omega)$, then $H_{\mathcal{T}}(\Omega)$ will be the space of piecewise H^1 functions. For v, w in $H_{\mathcal{T}}(\Omega)$, we define the broken L^2 inner product and norm by $(v, w) = \sum_{\tau\in\mathcal{T}}(v, w)_\tau$, $\|v\|_0^2 = (v, v)$, analogously, the broken $H(\Omega)$ norm, $\|v\|^2 = \sum_{\tau\in\mathcal{T}} \|v\|^2_{H(\tau)}$. Note that $H(\Omega) \subset H_{\mathcal{T}}(\Omega) \subset L^2(\Omega)$ and the above definitions reduce to the usual ones whenever v, w are in $H(\Omega)$. Let $\bar{S}$ be split into two subspaces $\bar{S} = S \oplus S^c$, $S \cap S^c = \{0\}$. Let S^c_τ denote the restriction of S^c to $\tau \in \mathcal{T}$ and let $S^c_{\mathcal{T}} = \prod_{\tau\in\mathcal{T}} S^c_\tau$. We shall consider in particular the splitting

$$\bar{S}_{\mathcal{T}} = S \oplus S^c_{\mathcal{T}}, \quad S \cap S^c_{\mathcal{T}} = \{0\}, \quad S^c_{\mathcal{T}} \neq \emptyset. \tag{2.5}$$

The spaces $S^c_{\mathcal{T}}$ and $\bar{S}_{\mathcal{T}}$ are spaces of piecewise polynomials locally defined in each element τ in $\mathcal{T}$. Note the inclusions $\bar{S} \subset \bar{S}_{\mathcal{T}} \subset H_{\mathcal{T}}(\Omega)$ and $S^c \subset S^c_{\mathcal{T}}$. The error estimators given in §3 will be calculated in the complementary space $S^c_{\mathcal{T}}$. We assume that the bilinear form B can define an inner product $B(\cdot, \cdot)$ on $\bar{S}_{\mathcal{T}}$ and with it the energy norm $|||w|||^2 = B(w, w)$. All error estimators below are measured in this norm, although, in theory, they are not strictly restricted to this norm.

2.3. Abstract variational error formulation. Let $e = u - u_s$ denote the exact error of the approximate solution u_s. We now derive a general formulation for the exact error in terms of the approximate solution. Substituting $u = e + u_s$ into problem (2.3), we have

$$\begin{aligned} &B(e, v) - F(v) + B(u_s, v) \\ &\geq B(e, e) + B(e, u_s) + B(u_s, e) + B(u_s, u_s) - F(e) - F(u_s) \quad \forall v \in K. \end{aligned}$$

Note that the bilinear form B can be nonsymmetric. By rearranging terms in the above inequality, we can rewrite this as

$$\begin{aligned} &B(e, v - u_s) - [F(v - u_s) - B(u_s, v - u_s)] \\ &\geq B(e, e) - [F(e) - B(u_s, e)] \quad \forall v \in K. \end{aligned}$$

The left-hand side of the inequality clearly suggests that we can define the following new closed convex subset by translating the original convex set K with respect to the computed solution u_s:

$$K' = K - u_s \subset H(\Omega).$$

Moreover, by virtue of the boundedness of the bilinear form B and the linear functional F on $H(\Omega)$, the Riesz representation theorem shows that there exists a unique linear functional G on $H(\Omega)$ defined by

$$G(w) = F(w) - B(u_s, w) \ \forall w \in H(\Omega). \tag{2.6}$$

We call G a *weak residual* in contrast to the usual formulation in which the residual is in terms of the governing differential equation used by many authors [8], [11], [19].

The error estimation is then based on the following new variational (error) problem: Determine $e \in K'$ such that

$$B(e, w) - G(w) \geq B(e, e) - G(e) \quad \forall w \in K'. \tag{2.7}$$

The reason we use the weak residual form G instead of the governing equation is twofold. First, since u_s is itself an approximate solution, its second derivative, for second-order partial

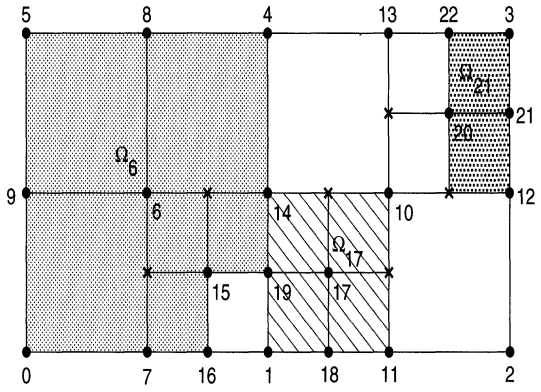

FIG. 2.1. *1-irregular mesh refinement.*

differential equations (PDEs), in the sense of distribution will further incur error. Second, for many applications, the governing differential equation may not be available, i.e., only the integral form is available. For such cases, the formulation of error problems like (2.7) is certainly more general.

2.4. A mesh refinement strategy. There are many refinement strategies proposed in the literature. The error estimators presented here are not restricted to any particular adaptive refinement. In fact, it has been shown in [21] that there is no order restriction for the FE spaces used in the approximate solution or in the error estimation; that is, the error estimators can be used in connection with any one of the h-, p-, or hp-versions of the FEM. In the numerical experiments, to test our error estimators (see §4), we use in particular the so-called 1-irregular mesh refinement strategy first proposed in [7] and later detailed in [14]. Since the refinement scheme has been extended to include adaptive finite volume computations, we briefly discuss its basic features.

Recall that a node is called *regular* if it constitutes a vertex for each of the neighboring elements; otherwise it is *irregular*. Figure 2.1 shows a particular 1-irregular mesh where irregular nodes marked by $\times$ are all of index-1 irregularity, that is, the maximum number of irregular nodes on an element side is one. In implementation, no degrees of freedom will be associated with these irregular nodes. Hence, supports of the shape functions defining a basis for an FE space change adaptively with mesh refinements; for example, the shaded subdomains Ω_6, Ω_{17}, and Ω_{21} are the supports of the shape functions corresponding, respectively, to the regular nodes 6, 17, and 21 in Fig. 2.1. The FE spaces so constructed preserve the conformality required by the standard FE approximation provided that some special element constraint methods are used to invoke continuity across interelement boundaries of elements of different size and with shape functions of differing polynomial degree [14].

For finite volume approximation, control volumes have to adapt accordingly to their dual elements. Let $\mathcal{B}$ denote the dual mesh for $\mathcal{T}$. Note that the FVM requires a control volume for each regular node where degrees of freedom are defined. Thus, in particular, 23 control volumes in the dual mesh of Fig. 2.1 are constructed and shown, by dotted lines, in Fig. 2.2. Notice that the pattern of the boundaries of control volumes appearing in elements may differ element by element. This plays an essential role in the implementation since most computations, approximations, as well as error estimations, are to be performed elementwise before the assembling process.

3. Model problems and numerical methods. We now apply the general formulations and the basic ideas discussed above to four model problems in connection with FEM and FVM.

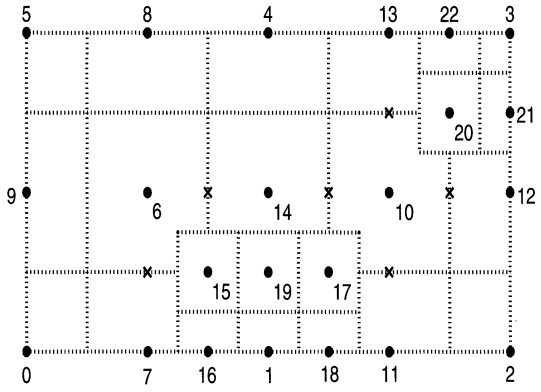

FIG. 2.2. *Adaptive control volumes.*

Since the a posteriori error estimation is the heart of a complete selfadaptive mechanism, we stress particularly various error equations or inequalities on which the error estimators are based. Some rigorous theories of the estimators can be found in [2], [21].

We first note that if the closed convex set K in (2.3) is itself the Sobolev space $H(\Omega)$, then the inequality (2.3) reduces to a *variational equation*. In §§3.1–3.3, 3.5 we will address this equality form.

3.1. Elliptic boundary value problems. Consider the boundary value problem

$$\begin{aligned} Lu &:= -\nabla \cdot (a(x)\nabla u(x)) + b(x)u(x) = f(x) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega_D, \qquad a(x)\frac{\partial u}{\partial n} = g(x) \quad \text{on } \partial\Omega_N. \end{aligned} \tag{3.1}$$

The associated variational problem is to find $u \in H(\Omega)$ such that

$$B(u, v) = (f, v) + \langle g, v\rangle_{\partial\Omega_N} \quad \forall v \in H(\Omega), \tag{3.2}$$

where

$$H(\Omega) := \{u \in H^1(\Omega) : u = 0 \text{ on } \partial\Omega_D\}, \quad \|\cdot\| := \|\cdot\|_1,$$

$$B(u, v) := \int_\Omega (a\nabla u\nabla v + buv)\, dx,$$

$$(f, v) := \int_\Omega fv\, dx, \quad \langle g, v\rangle_{\partial\Omega_N} := \int_{\partial\Omega_N} gv\, ds.$$

To begin with and to avoid technical details, we always assume the unique solvability of the variational problems considered here and below. Note that we do not assume the coefficient function $b(x)$ in (3.1) to be strictly positive in Ω, hence the error estimators to be given apply to indefinite problems as well [21].

Corresponding to (2.4) and (2.7), the approximation and error problems for (3.2) are to determine $u_s \in S$ and $e \in H(\Omega)$ such that

$$B(u_s, v) = (f, v) + \langle g, v\rangle_{\partial\Omega_N} \quad \forall v \in S \tag{3.3}$$

and

$$B(e, v) = -B(u_s, v) + (f, v) + \langle g, v\rangle_{\partial\Omega_N} \quad \forall v \in H(\Omega), \tag{3.4}$$

respectively, where the computed FE solution u_s can only be assessed a posteriori, i.e., after its availability.

Since the space $H(\Omega)$ is still infinite dimensional and the discretization of (3.4) in the original FE space S only produces trivial estimated errors, (3.4) has to be solved in a larger space $\bar{S}$. However, this would cause the error calculation to be impractically expensive since (3.4) will result in a larger system of equations than that of (3.3). Hence, the use of the complementary subspaces appears to be natural. Nevertheless, if the subspace is chosen to be S^c, the error calculation may involve a global solution to a system of equations. We are therefore led to consider the complementary subspaces $S_{\mathcal{T}}^c \subset H_{\mathcal{T}}(\Omega)$ and, consequently, to solve the following reduced problem: Determine $\tilde{e} \in S_{\mathcal{T}}^c(\tau_i)$, in each element $\tau_i \in \mathcal{T}$, such that

$$B(\tilde{e}, v) = -B(u_s, v) + (f, v) + \langle g, v\rangle_{\partial\Omega_N} \quad \forall v \in S_{\mathcal{T}}^c(\tau_i). \tag{3.5}$$

Note that if $S_{\mathcal{T}}^c \subset H(\Omega)$, a conforming subspace, the unique solvability of (3.5) can be ensured as that of (3.3) and only the interior residual in each element will be used. On the other hand, if $S_{\mathcal{T}}^c \subset H_{\mathcal{T}}(\Omega)$ but not in $H(\Omega)$, a nonconforming subspace, then (3.5) results in a nonconforming solution scheme [13] and the estimation will include both interior and boundary residuals for each element. With some moderate assumptions on the bilinear form and the complementary spaces (sufficiently large), it is shown in [21] that the unique solvability for such problems still holds.

As noted already, the error residuals can be expressed in either differential or weak form. We briefly describe the differences between these two approaches. For more detailed theoretical investigation we refer the reader to [11], [21].

Let E be the collection of curves which forms an edge of an element τ in T. The set of edges may be decomposed as the union of two disjoint sets $E = E_B \cup E_I$, where E_B is the set of edges on $\partial\Omega$ and E_I is the set of edges in the interior of Ω. For each edge ε in E, we define a normal direction $n = n_\varepsilon$. More specifically, n_ε is the usual outward normal when $\varepsilon \in E_B$ while, for $\varepsilon \in E_I$, its choice is arbitrary. Let τ_{in}, τ_{out} be two elements sharing an edge ε in E_I and suppose that the normal n is outward from τ_{in}. Then, for x on ε, the jump and the average of v on ε are defined, respectively, by

$$[v]_J(x) = v(x)|_{\tau_{\text{out}}} - v(x)|_{\tau_{\text{in}}} \quad \text{and} \quad [v]_A(x) = \frac{1}{2}\{v(x)|_{\tau_{\text{out}}} + v(x)|_{\tau_{\text{in}}}\}.$$

Substitute $u = e + u_s$ into (3.1) and multiply by a test function v; then we obtain, in each element τ,

$$(Le, v)_\tau = -(Lu_s, v)_\tau + (f, v)_\tau, \tag{3.6}$$

where Lu_s is defined in the sense of distributions. Now integration by parts of the left term in (3.6) yields

$$(Le, v)_\tau = B(e, v)_\tau - \left\langle a\frac{\partial u}{\partial n}, v\right\rangle_{E_\tau} + \left\langle a\frac{\partial u_s}{\partial n}, v\right\rangle_{E_\tau}, \tag{3.7}$$

and after summing (3.6) over all elements, we find that

$$B(e, v) = (f - Lu_s, v) + \left\langle g - a\frac{\partial u_s}{\partial n}, v\right\rangle_{\partial\Omega_N} + \left\langle \left[a\frac{\partial u_s}{\partial n}\right]_J, v\right\rangle_{E_I}. \tag{3.8}$$

This is the standard formulation for estimating errors used by many authors [7], [8], [11], [19]. On the other hand, if the term $(Lu_s, v)_\tau$ in (3.6) is rewritten as in (3.7), then in each element

τ, we obtain

$$B(e, v)_\tau - \left\langle a\frac{\partial u}{\partial n}, v\right\rangle_{E_\tau} = -B(u_s, v)_\tau + (f, v)_\tau.$$

Hence a summation over all elements leads to

$$B(e, v) - \left\langle a\frac{\partial u}{\partial n}, v\right\rangle_{E_B} - \left\langle \left[a\frac{\partial u}{\partial n}\right]_J, [v]_A\right\rangle_{E_I} = -B(u_s, v) + (f, v),$$

which gives (3.4) with more general test functions $v \in H_{\mathcal{T}}(\Omega)$ if the exact solution u is in $H^2(\Omega)$.

It is necessary to reuse the differential operator L when the estimators are calculated based on (3.8). In contrast, in terms of (3.4), the same bilinear formulation (3.2) can be used in both approximation and error estimation. This is certainly preferable, from the user's viewpoint, in adaptive implementation. Moreover, the formal approach can only utilize the complementary spaces $S_{\mathcal{T}}^c$ in a nonconforming setting while the weak residual approach may be used in either a conforming or a nonconforming setting. Finally, in [11], the following saturation condition is introduced for the spaces $\bar{S}$ used in the error analysis:

$$\|u - u_{\bar{s}}\|_1^2 + \left| h_\varepsilon^{1/2}\left[a\frac{\partial(u - u_{\bar{s}})}{\partial n}\right]_J\right|_{0,E_I}^2 \le \rho(h)^2\|u - u_s\|_1^2, \tag{3.9}$$

where $u_{\bar{s}}$ is the approximated solution (for analytical purposes only) sought in the larger spaces $\bar{S}$. It is assumed that $\lim_{h\to 0}\rho = 0$; that is, $\bar{S}$ has higher order than S. On the other hand, in weak form, the saturation condition is as follows:

$$\|u - u_{\bar{s}}\|_1 \le \rho\|u - u_s\|_1, \quad 0 \le \rho < 1. \tag{3.10}$$

Thus the condition (3.10) is somewhat weaker than (3.9) since it allows the FE orders of S and $\bar{S}$ to be equal (see, e.g., [21] and Example 4.2 below). The introduction of the jumps in the normal derivative of the computed solutions at interelement boundaries actually forces ρ to become larger and hence deteriorates the quality of the error estimator [21]. Almost all differential-type residual error estimators [3], [4], [7], [8], [11], [19] require some compatibility conditions or auxiliary local problems to overcome this intrinsic difficulty simply because the distribution and jump terms are included in the residual, i.e., the right-hand side of (3.8). These conditions or problems are somewhat ad hoc depending strongly on the model problem under consideration; see the above references. The complementary spaces are essential for both differential and weak residual approaches. With the weak residual form, one can concentrate primarily on the construction of the shape functions of $S_{\mathcal{T}}^c$, which can still handle jumps across element boundaries (with a nonconforming setting), if they are dominant errors.

The norm $\||\tilde{e}|\|_{\tau_i} =: \eta_i$, for each element $\tau_i \in \mathcal{T}$, is called the *error indicator* of the element which assesses the quality of the approximate solution u_s in this element and indicates whether the element needs to be refined, derefined, or unchanged. Summing over all elements, the *error estimator* for u_s can be defined by

$$\||\tilde{e}|\| = \left[\sum_i \eta_i^2\right]^{1/2}.$$

The error estimator can serve as one of the major stopping criteria for an entire adaptive process. The quality of a proposed error estimator is usually tested by various model problems to which

the exact solutions are explicitly known. A computable *effectivity index*

$$\theta = \frac{|||\tilde{e}|||}{|||e|||}$$

is usually introduced to quantify the quality of the estimator and, consequently, the quality of the approximate solution.

3.2. Parametrized nonlinear equations. Stationary problems for many scientific and engineering problems are modeled by a parameter-dependent equation

$$F(u, \lambda) = 0, \tag{3.11}$$

where u is a state variable, λ is a d-dimensional parameter vector, and F denotes some differential operator on a suitable state space. Typically, the solution set $M = F^{-1}(0)$ turns out to be a differentiable submanifold M of dimension d of the product X of the state space and the parameter space. This can be ensured, for instance, when F is a Fredholm map of index d on X [23].

All standard discretizations of a parametrized boundary value problem (3.11) leave the parameter vector untouched and hence approximate the equations by some finite-dimensional system $F_s(u_s, \lambda) = 0$. Hence, under suitable conditions, we may expect the solution set $M_s = F_s^{-1}(0)$ to be a d-dimensional submanifold of some discretization space X_s. Frequently, X_s can be embedded in X and then the discretization error represents some measure of the distance between M and M_s in X.

Our goal now is to estimate the discretization error. There are two major issues for the error estimation of M_s. The first issue is efficiency. It is intrinsically more expensive for nonlinear parametrized problems than it is for linear problems. The second issue is consistency. It is well known [23] that the parameter dependence causes the discretization error to become a local concept which depends on the choice of the local coordinate system on the manifold. For instance, in the continuation method, we must often fix a local coordinate system for calculating several points on the manifold and then change the coordinate system as needed, and so on. However, for error estimation, the coordinate system is usually fixed, [23], throughout the entire manifold and hence is not applicable near any foldpoint with respect to the parameter space. The main way to resolve these difficulties is to use linearization and a local coordinate system.

At any $x_0 = (u_0, \lambda_0) \in M$, we define a local coordinate system that satisfies the following conditions:

$$X = W \oplus T, \quad \dim T = d, \quad W \cap \ker DF(x_0) = \{0\}. \tag{3.12}$$

It is shown in [25] that the constrained linearized (infinite-dimensional) problem

$$F(x_s) + DF(x_s)\omega = 0, \quad \pi(\omega) = 0, \tag{3.13}$$

for (3.11) has a unique solution $\omega = x_0 - x_s \in W$ which is the exact error of the approximate solution $x_s = (u_s, \lambda_s)$. Here $\pi \in L(X)$ is a natural projection of X onto T along W. The FE approximation of (3.11) is of interest here. We consider, in particular, the following mildly nonlinear problem: Find $(u, \lambda) \in H_0^1(\Omega) \times \Lambda$ such that

$$\langle F(u, \lambda), v\rangle := \int_\Omega [a(\lambda, \xi)\nabla u \cdot \nabla v + g(u, \lambda, \xi)v]\, d\xi = 0 \quad \forall v \in H_0^1(\Omega), \tag{3.14}$$

where $F : X = H_0^1(\Omega) \times \Lambda \to Y = H^{-1}(\Omega)$ and the coefficient functions a and g are given so that the problem is well posed.

Apparently, the nonlinear problem (3.14) does not fit into the general variational formulation given in the previous section. However, its weak residual form can be cast in the framework of the variational error setting.

In weak form (3.13) requires the determination of $(w, \mu) \in H_0^1 \times \Lambda$ such that

$$B(w, v) + C(\mu, v) = -\langle F(u_s, \lambda_s), v\rangle \quad \forall v \in H_0^1(\Omega), \tag{3.15}$$

$$\pi(w, \mu) = 0, \tag{3.16}$$

where

$$B(w, v) := \int_\Omega (a(\lambda_s, \xi)\nabla w \cdot \nabla v + g_u(u_s, \lambda_s, \xi)wv)\, d\xi,$$

$$C(\mu, v) := \int_\Omega ((a_\lambda(\lambda_s, \xi) \cdot \mu)\nabla u_s \cdot \nabla v + (g_\lambda(u_s, \lambda_s, \xi) \cdot \mu)v)\, d\xi.$$

At any computed point $(u_s, \lambda_s) \in M_s$, our a posteriori error estimates thus require the determination of $(\tilde{w}, \tilde{\mu}) \in S_{\mathcal{T}}^c(\tau_i) \times \Lambda$ such that

$$B(\tilde{w}, v) + C(\tilde{\mu}, v) = -\langle F(u_s, \lambda_s), v\rangle \quad \forall v \in S_{\mathcal{T}}^c(\tau_i), \tag{3.17}$$

$$\pi(\tilde{w}, \tilde{\mu}) = 0. \tag{3.18}$$

The choice of the local coordinate system at any point x_0 on M is arbitrary as long as it satisfies the conditions in (3.12). By definition, the local coordinate system (3.12) and hence the constraint $\pi(\omega, \mu) = 0$ can change from point to point on the solution manifold. We choose $T = \ker DF(x_0)$ in particular. The constraint (3.18) is then equivalent to $(\tilde{w}, \tilde{\mu}) \in W_s = X_s \cap W$ which is orthogonal to the d-dimensional subspace $T = \ker DF_s(x_s)$ of X_s corresponding to the tangent space of M_s at the computed point x_s. If Λ is one dimensional and a standard continuation process is used, then a normalized tangent vector is usually available at each computed point. Analogously, in the multiparameter case, if a triangulation of M_s is computed by the method of [24], then again orthonormal bases of the tangent space are available at the computed points on the manifold. This uniform treatment of the a posteriori error estimation along the computed manifold M_s avoids aforementioned difficulties. Moreover, the introduction of the linear, local, solution scheme ensures that the method is computationally relatively inexpensive.

An a posteriori error estimation has been developed for strongly nonlinear equations with a scalar parameter in [27]. There the asymptotic exactness of a residual estimator was proved under suitable hypotheses. Tsuchiya's approach requires one to fix the local coordinate system in two stages. In the first stage before the turning point, the system is defined by the natural parameter. In the second stage when the continuation process is near and after the turning point, the system is then rotated by 90 degrees thus allowing a more elaborate error estimate near the turning point. In the sense of the definition of the local coordinate system, this is a special case of our approach. Moreover, Tsuchiya's approach requires a global solution to the linearized residual equations for error estimation.

3.3. Symmetric hyperbolic equations. In the theory of PDEs there is a fundamental distinction between those of elliptic, hyperbolic, and parabolic types. The theory of symmetric positive differential equations developed by Friedrichs [17] is known for its unified treatment, analytically as well as numerically, for PDEs that change type within the domain of interest such as the Tricomi problem and forward–backward heat equations. In the development of the error estimator for the Friedrichs system, we use in particular the FEM proposed by Lesaint [20].

An unknown p-dimensional vector-valued function defined on Ω is given by $\mathbf{u} = (u_1, u_2, \ldots, u_p)^T$. Let $\mathbf{f} = (f_1, f_2, \ldots, f_p)^T$ be a given p-dimensional vector-valued function defined on Ω. Let the operators $\mathbf{L}$ and $\mathbf{M}$ be defined by and consider the following systems:

$$\mathbf{Lu}(x) := \sum_{i=1}^{2} A_i(x)\frac{\partial \mathbf{u}}{\partial x_i} + A_0(x)\mathbf{u} = \mathbf{f}(x) \quad \text{for } x \in \Omega, \tag{3.19}$$

$$\mathbf{Mu}(x) := (\mu(x) - \beta(x))\mathbf{u}(\mathbf{x}) = 0 \quad \text{for } x \in \partial\Omega, \tag{3.20}$$

where $\beta = \sum_{i=1}^{2} n_i A_i$, the n_i's being the components of the outer normal on $\partial\Omega$. The matrices A_i, $i = 1, 2$, are symmetric, Lipschitz continuous in $x = (x_1, x_2)$ for $x \in \bar{\Omega}$. The coefficients of the matrix $A_0(x)$ of $L(R^p)$ are bounded in Ω. The matrix $\mu(x)$ of $L(R^p)$ is defined for $x \in \partial\Omega$ so that the boundary condition (3.20) is *admissible* and the operator $\mathbf{L}$ is *positive* in the sense of Friedrichs [17]; i.e.,

$$\begin{cases} \text{(i)} & \mu(x) + \mu^*(x) \text{ is positive semidefinite on } \partial\Omega, \\ \text{(ii)} & \operatorname{Ker}(\mu - \beta) \oplus \operatorname{Ker}(\mu + \beta) = R^p \text{ on } \partial\Omega, \text{ and} \\ \text{(iii)} & A_0 + A_0^* - \sum_{i=1}^{2} \frac{\partial A_i}{\partial x_i} \geq c_0 I \quad \forall x \in \Omega, \end{cases}$$

where μ^* and A_0^* are adjoint matrices of μ and A_0, respectively, c_0 is a positive constant, and I is the identity matrix.

The adjoint operators $\mathbf{L}^*$ and $\mathbf{M}^*$ of $\mathbf{L}$ and $\mathbf{M}$ are defined, respectively, by

$$\mathbf{L}^*\mathbf{v}(x) := -\sum_{i=1}^{2} \frac{\partial}{\partial x_i}(A_i(x)\mathbf{v}(x)) + A_0^*(x)\mathbf{v}(x) \quad \forall x \in \Omega, \tag{3.21}$$

$$\mathbf{M}^*\mathbf{v}(x) := (\mu^*(x) + \beta(x))\mathbf{v}(x) \quad \forall x \in \partial\Omega. \tag{3.22}$$

Let $(\mathbf{q}, \mathbf{g}) := \int_\Omega \mathbf{q}(x) \cdot \mathbf{g}(x)\, dx$ and $\langle \mathbf{q}, \mathbf{g} \rangle := \int_{\partial\Omega} \mathbf{q} \cdot \mathbf{g}\, ds$, where $\mathbf{q} \cdot \mathbf{g} = \sum_{i=1}^{p} q_i g_i$. One variational formulation of (3.19) and (3.20) is to find $\mathbf{u} \in (H^1(\Omega))^p$ such that

$$B(\mathbf{u}, \mathbf{v}) := \frac{1}{2}(\mathbf{Lu}, \mathbf{v}) + \frac{1}{2}(\mathbf{u}, \mathbf{L}^*\mathbf{v}) + \frac{1}{2}\langle \mu\mathbf{u}, \mathbf{v} \rangle = (\mathbf{f}, \mathbf{v}) \; \forall \mathbf{v} \in (H^1(\Omega))^p. \tag{3.23}$$

For any given approximate solution $\mathbf{u}_s \in (S)^p \subset (H^1(\Omega))^p$ of (3.23), again analogous to (2.7), the error estimator (3.23) can then be calculated by solving the reduced error problem: Determine $\tilde{\mathbf{e}} \in (S_T^c)^p$ such that

$$B(\tilde{\mathbf{e}}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) - B(\mathbf{u}_s, \mathbf{v}) \; \forall \mathbf{v} \in (S_T^c)^p. \tag{3.24}$$

Since the boundary conditions (3.20) and (3.22) for test and trial functions $\mathbf{u}$ and $\mathbf{v}$, respectively, are different, this would make it impossible to show the coercivity in a simple manner if the adjoint operator $\mathbf{L}^*$ were not included in (3.23). With the formulation of (3.23) and Friedrichs's identities [17], the coercivity is guaranteed in [20] but only in the lower-order norm, i.e., $\|\cdot\|_0$, instead of the general Gårding-type inequality (2.2).

3.4. Variational inequalities. In the previous subsections, in terms of abstract settings, the closed convex set K is the Sobolev $H(\Omega)$ itself. This in turn leads to variational equations for the preceding model problems. In this subsection, we deal with variational inequalities formulated in the general form (2.3) where now the K is indeed a closed convex subset of $H(\Omega)$. As we would in a typical problem, we shall now also consider the abstract minimization problem: Find $u \in K$ such that

$$J(u) = \inf_{v \in K} J(v), \tag{3.25}$$

where the functional $J : H(\Omega) \to R$ is defined by

$$J(v) = \frac{1}{2}B(v, v) - F(v),$$

provided that the bilinear form $B(\cdot, \cdot)$ is symmetric and $H(\Omega)$-elliptic, i.e., $\alpha = 0$ in (2.2).

Corresponding to (2.4), (3.25) reduces to the finite-dimensional approximate problem

$$J(u_s) = \inf_{v_s \in K_s} J(v_s), \tag{3.26}$$

which, under suitable conditions on the approximate convex set K_s [16], [18] can be solved by mathematical programming subject to a finite number of constraints induced by K_s.

Clearly, the variational error problem (2.7) suggests the needed formulation for error estimation; that is, determine $\tilde{e} \in K_c'$ such that

$$B(\tilde{e}, w) - G(w) \geq B(\tilde{e}, \tilde{e}) - G(\tilde{e}) \quad \forall w \in K_c', \tag{3.27}$$

where K_c' is a complementary closed convex set of $S_{\mathcal{T}}^c$ under similar conditions as those of K_s. Inequality (2.7) also suggests that a new functional can be defined in terms of the weak residual G, namely,

$$E(w) = \frac{1}{2}B(w, w) - G(w).$$

Consequently, we have the following reduced minimization problem: Determine $\tilde{e} \in K_c'$ such that

$$E(\tilde{e}) = \inf_{w \in K_c'} E(w). \tag{3.28}$$

3.5. The finite volume element method. There are many variants of FVMs. We consider specifically the finite volume element method (FVEM). As noted in [12], the FVEM was developed as an attempt to use FE ideas to create a more systematic finite volume (FV) methodology. The basic idea is to approximate the discrete fluxes needed in FV by replacing the unknown PDE solution by an FE approximation. It turns out that the approximate solution by FVEM is in fact sought in a standard FE trial function space whereas the corresponding test function space consists of volumewise constant step functions (zero-order polynomials). In [10], Bank and Rose termed the FVEM the *box method* and showed that, under reasonable hypotheses, the solution u_b generated by the FVEM is of comparable accuracy to the solution u_s generated by the standard Galerkin procedure using piecewise linear FEs. More precisely, the a priori errors of u_b and u_s are of the same order in the energy norm.

As far as a posteriori error estimation is concerned, there are surprisingly fewer results available for the FVMs than for their FE counterparts. From the above observations and by the actual implementation features of the FVEM, we can see that FEMs and FVMs do have many important similarities in error estimation and the adaptive process.

First of all, the FVEM solution u_b is itself an element of the standard FE space S associated with the regular mesh $\mathcal{T}$. Therefore, it is perfectly all right to replace u_s by u_b in (2.6) so that the weak residual G is now in terms of the FV solution u_b. Second, since the solution u_b was computed with degrees of freedom defined at the nodal points of elements instead of volumes, it is quite reasonable to do the error estimation on an element-by-element basis. Third, most computations in practice are customarily carried out elementwise, including control volumes which are constructed according to their dual elements; see, e.g., [10], [12]. Finally, it is interesting to explore how the well-established adaptive features of FE technology can be utilized in FV computations. In short, it is plausible to develop error estimators for the FVEM elementwise instead of volumewise.

We consider again the selfadjoint elliptic boundary value problem (3.1). For the FV element approximation of (3.1), we follow the formulation proposed in [10] which is in a more general setting than that in [12]. The FVEM (or the box method) for (3.1) is defined as follows: Find $u_b \in S$ such that

$$\bar{B}(u_b, \bar{v}) + (b\bar{u}_b, \bar{v}) = (f, \bar{v}) \quad \forall \bar{v} \in P_0^0(\mathcal{B}), \tag{3.29}$$

where

$$\bar{B}(u_b, \bar{v}) = -\sum_{b_i \in \mathcal{B}} \int_{\partial b_i} a \frac{\partial u_b}{\partial n} \bar{v}\, ds,$$

and $P_0^0(\mathcal{B})$ denotes the space of discontinuous piecewise constants with respect to the control volumes (or boxes), whose elements are zero on $\partial\Omega_D$. Note that to avoid a nondiagonal (and generally nonsymmetric) matrix, $\bar{u}_b$ is used instead of u_b in the second term on the left-hand side of (3.29); see [10]. Here, the $\bar{u}_b$ is defined as a volumewise constant and has the same values as u_b at vertices of $\mathcal{T}$.

From the above observation, we see that the error estimator of the FV solution u_b is proposed by determining $\tilde{e} \in S_{\mathcal{T}}^c(\tau_i)$ such that

$$B(\tilde{e}, v) = -B(u_b, v) + (f, v) + \langle g, v\rangle_{\partial\Omega_N} \quad \forall v \in S_{\mathcal{T}}^c(\tau_i), \tag{3.30}$$

where the bilinear form B is defined exactly as in (3.5).

The FV solution u_b is obtained quite differently in the sense of the abstract setting (2.4). Nevertheless, the error estimation based on (3.30) is still in the unifying theme presented so far. Note also that there is no direct link between (3.29) and (3.30). It is precisely our intention to use the weak residual error estimation. Otherwise, if the boundary integrals $\bar{B}$ were used in (3.30), we would then be forced to perform the estimation volumewise. However, the theoretical investigation of the estimators would certainly involve the a priori results concerning u_b and yet remains open. Equation (3.30) is in exactly the same form as (3.5). As a result, the error indicators and estimators can be calculated in exactly the same way as those of the FEM. Further, the remarks made in §3.1 apply here.

4. Numerical examples. The numerical examples presented in this section correspond with those of the previous subsections. Some examples were also considered in the places cited. Although we intend to stress the performance (the effectivity index) of the respective error estimators, some adaptive computational results are also presented.

There are two different types of the complementary spaces. The conforming shape functions of $S_{\mathcal{T}}^c$ vanish on boundaries of elements; consequently, the error indicators ignore interelement jumps in fluxes. The shape functions for the nonconforming case are discontinuous across element boundaries and hence the error indicators will include both the interior errors and the jumps in, of course, weak form.

Although most applications require only the use of conforming shape functions [1], [9], [25], there are some cases in which this approach would fail (see Example 4.1) if the FE order of S were not properly considered [1], [8]. We shall illustrate both conforming (Examples 4.2, 4.3) and nonconforming (Examples 4.1, 4.4, 4.5) error calculations. General algorithms in implementing these error estimators, for equality-type problems, are detailed in [21], [25]. We shall present an algorithm for the variational inequalities in Example 4.4.

Example 4.1. Following [15] we consider Laplace's equation on an L-shaped domain:

$$\begin{aligned} &\Delta u = 0 \quad \text{in } \Omega = (-1, 1) \times (-1, 1)\backslash(0, 1) \times (-1, 0), \\ &u = 0 \quad \text{on } \partial\Omega_D, \qquad \frac{\partial u}{\partial n} = g \quad \text{on } \partial\Omega_N, \end{aligned} \tag{4.1}$$

where $\partial\Omega_D = \{(x, y) : x \in [0, 1],\ y = 0\} \cup \{(x, y) : x = 0,\ y \in [-1, 0]\}$, $\partial\Omega_N = \partial\Omega \backslash \partial\Omega_D$ and g is defined so that, in polar coordinates, the exact solution u becomes $u = r^{2/3} \sin \frac{2\theta}{3}$ and hence has a point singularity at the origin.

For our computations, bilinear elements are used to define S. Hence, by the nature of the problem, the computed solution u_s is a harmonic function in each element and therefore satisfies (4.1) on each element. This means that the errors occur purely on the edges of elements; that is, the complementary space S_T^c has to be nonconforming. For this, we construct four edge midpoint basis functions defined, on the reference element $\hat{\tau} = \{(\xi, \eta) : |\xi| \leq 1,\ |\eta| \leq 1\}$, by

$$\text{(4.2)} \qquad \begin{cases} \psi_1(\xi, \eta) = (1 - \xi^2)(1 - \eta)/2, & \psi_2(\xi, \eta) = (1 + \xi)(1 - \eta^2)/2, \\ \psi_3(\xi, \eta) = (1 - \xi^2)(1 + \eta)/2, & \psi_4(\xi, \eta) = (1 - \xi)(1 - \eta^2)/2. \end{cases}$$

For the p-version FEM, this type of construction for the complementary spaces can be readily extended to high-order shape functions. For instance, given basis functions defining S on each element, one can simply add one or more shape functions to the element by higher-order side mode or internal mode or both (see [26]) for S_T^c. In essence, this corresponds to a hierarchical basis method [2].

Of course, the conforming-type shape functions can still be used for this kind of problem provided that the original space S is chosen so that the effects of the boundary residuals are negligible. In [8] it was shown that for linear elliptic problems the discretization error of odd-order FE solutions is mainly due to the jumps, while that of even-order approximations occurs principally in the interior of the elements, allowing the jumps across element boundaries to be neglected. In line with this, eight-node biquadratic FEs were used in [1] for the approximation while two fifth-degree polynomials were introduced for the error estimation in weak residual form.

Figure 4.1 is a given initial mesh on the domain. The adaptive computation shown in Fig. 4.2 is very similar to those using the code FEARS (finite element adaptive research solver); see, e.g., [6]. Our approach however is much simpler in terms of implementing the error indicators. Depending on the model problem, the location of singularity, the quantity of interest (displacement, stress, etc.), and the spectral order of the FE, different extraction expressions for an auxiliary problem associated with the model problem should be chosen accordingly for Babuška–Miller indicators; see [6] for more details. Our error indicators are not confined by the above effects; namely, ours do not depend on singularity and spectral order and there are no auxiliary problems. In fact, based on the ideas in the present paper, we are able to develop a very general and robust code which we call **AdaptC++**. So far, we have successfully tested our code for linear elasticity problems, mixed-type problems, flows in porous media, obstacle problems, Navier–Stokes equations, and semiconductor device simulation with FE, FV, and least-squares FE methods. Among other things, one of its advantageous features is the simplicity in implementing the error indicators which disregard input model problems and see only a very general setting of linear and bilinear forms and boundary conditions. From the user's viewpoint, use of the code is even simpler partly due to the refinement scheme and the object-oriented programming language. All numerical data presented in this section except that in Example 4.2 were produced by **AdaptC++**.

Tables 4.1 and 4.2 show that adaptive computations are clearly superior to uniform mesh reductions. For instance, if a tolerance is set to 3% of the relative error (r.e. $= \frac{|||e|||}{|||u|||}$), the uniform approach requires over 10 times the degrees of freedom (DOF) of the adaptive approach. Also, the effectivity indices θ show reliable error estimators for both uniform and adaptive approximations.

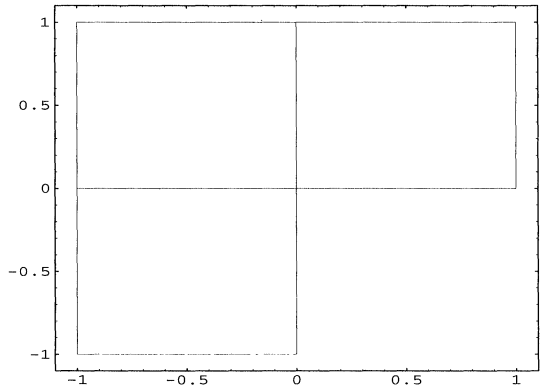

FIG. 4.1. *Initial mesh for Example* 4.1.

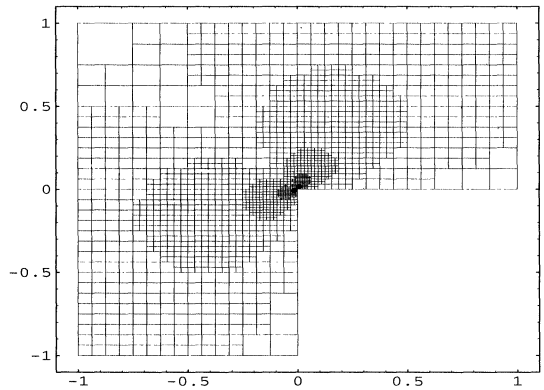

FIG. 4.2. *Adaptive final mesh (FEM) for Example* 4.1.

TABLE 4.1
Example 4.1 *using uniform meshes (FEM).*

DOF	$\|\|\|e\|\|\|$	r.e.	θ
8	0.284368	0.210	0.732
21	0.196695	0.145	0.801
65	0.128699	0.095	0.821
225	0.082777	0.061	0.830
833	0.052781	0.039	0.835
3201	0.033493	0.025	0.837

Example 4.2. The discussions in §3.2 are illustrated by the nonlinear boundary value problem

$$\begin{aligned} -\Delta u &= \lambda e^u, \quad u = u(x, y) \quad \forall (x, y) \in \Omega = (0, 1) \times (0, 1), \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{4.3}$$

The weak formulation of (4.3) is given as

$$\langle F(u, \lambda), v\rangle := \int_\Omega (u_x v_x + u_y v_y - \lambda e^u v)\, dx\, dy = 0 \ \ \forall v \in H_0^1(\Omega) \tag{4.4}$$

and we assume that $\lambda \in R^1$, which means that the solutions of (4.4) form a one-dimensional manifold M.

TABLE 4.2
Example 4.1 *using adaptive meshes (FEM).*

DOF	$\|\|\|e\|\|\|$	r.e.	θ
8	0.284368	0.210	0.732
21	0.196695	0.145	0.801
34	0.135229	0.100	0.849
53	0.096849	0.071	0.869
78	0.072604	0.053	0.877
119	0.054446	0.040	0.905
301	0.031254	0.023	0.945
469	0.024136	0.017	0.957
765	0.018578	0.013	0.966
1024	0.010675	0.011	0.973
1847	0.011501	0.009	0.980

We use a uniform mesh, $\mathcal{T}$, of 16 biquadratic elements on Ω. For the computation of the one-dimensional solution manifold M_s of the discretized problem, a continuation process (PITCON [23]) is applied starting from $(u_s^0, \lambda_s^0) = (0, 0)$, and our aim is to determine a posteriori estimates of the error between M and M_s at all computed solutions $(u_s, \lambda_s) \in M_s$.

In line with (3.15) and (3.16), the linearized problem at $(u_s, \lambda_s) \in M_s$ is to determine $(w, \mu) \in H_0^1(\Omega) \times \Lambda$ such that

$$\int_\Omega \left[w_x v_x + w_y v_y - \lambda_s e^{u_s} w v + e^{u_s} \mu v\right] dx\, dy \tag{4.5}$$

$$= -\int_\Omega (u_{h_x} v_x + u_{h_y} v_y - \lambda_s e^{u_s} v)\, dx\, dy \quad \forall v \in H_0^1(\Omega),$$

$$\langle (w, \mu), t_s \rangle = 0. \tag{4.6}$$

As noted, the t_s are chosen as normalized tangent vectors on M_s at (u_s, λ_s). Such tangent vectors are available at each step of the continuation process and hence the equation (4.6) involves little additional computational cost.

For local solutions each one, $\tau_i \in \mathcal{T}$, of the 16 elements of Ω is divided into $m = (k+1)^2$ biquadratic subelements with $k = 1, 4$. That means that on each subelement, τ_{ij}, $i = 1, \ldots, 16$, $j = 1, \ldots, m$, a bubble-shaped function $\psi_{ij}(x, y)$ is constructed via the mapping of the shape function

$$\psi(\xi, \eta) = (1 - \xi^2)(1 - \eta^2)$$

defined on the reference element. We thus have $S_{\mathcal{T}}^c(\tau_i) = span\{\psi_{ij}\}_{j=1}^m \subset H_0^1(\tau_i)$ for each element $\tau_i \in \mathcal{T}$ and hence $S_{\mathcal{T}}^c \subset H_0^1(\Omega)$ is a conforming FE subspace. The more subelements are used, the more accurate auxiliary condition (4.6) becomes and a better quality estimator can be obtained.

The resulting error estimates are shown in Table 4.3, where $\|\|w_{\bar{s}_k}\|\|$ denotes the computed error norms for the two cases of k. The computations are very cost-effective, since each local problem involves only a fixed number of degrees of freedom depending on the value of k. The table also shows that, as expected, the estimated errors vary smoothly along the solution path M_s and show no sudden increases near the limit point $\lambda = 6.804524$. As mentioned earlier, if the natural coordinate system induced by the parameter space Λ is chosen, then we expect the resulting error estimates $\|\|\hat{w}\|\|$ to become unduly large near the limit point. This is indeed the case as the last column of Table 4.3 shows. At the same time, it should be noted that the computational cost of the two approaches is practically identical.

TABLE 4.3
Example 4.2 *using uniform meshes.*

λ	$\|\|\|w_{\tilde{s}_1}\|\|\|$	$\|\|\|w_{\tilde{s}_4}\|\|\|$	$\|\|\|\hat{w}\|\|\|$
5.907655	0.017914	0.019054	0.022905
6.193552	0.018368	0.019562	0.024439
6.434373	0.019144	0.020404	0.027792
6.620862	0.020640	0.021986	0.041992
6.745397	0.023317	0.024787	0.159595
6.804524	0.027539	0.029185	0.176171
6.800451	0.033461	0.035345	0.086353
6.740221	0.041066	0.043255	0.069906
6.633252	0.050283	0.052838	0.065171
6.489009	0.061065	0.064041	0.065099
6.328924	0.072480	0.075894	0.067672

Example 4.3. For mixed-type PDEs, we consider the forward–backward heat equation

$$x\phi_t(x,t) - \phi_{xx}(x,t) = f(x,t) \quad \forall (x,t) \in \Omega = (-1,1) \times (0,1), \tag{4.7}$$

$$\begin{cases} \phi(\pm 1, t) = 0 & \forall t \in [0,1], \\ \phi(x,0) = 0 & \forall x \in [0,1], \\ \phi(x,1) = 0 & \forall x \in [-1,0]. \end{cases} \tag{4.8}$$

Note that the equation changes type as x changes sign in Ω. There have been a number of papers addressing this kind of mixed-type heat equation; for further references see [5], [28].

For our computations, the exact solution ϕ is chosen as

$$\phi(x,t) = (x^2-1)t^2[(t-1)^2 - 4x^2] \ \forall x \geq 0, \ t \in [0,1],$$
$$\phi(x,t) = (x^2-1)(t^2-4x^2)(t-1)^2 \ \forall x \leq 0, \ t \in [0,1].$$

Denote the boundary $\partial\Omega$ by $\Gamma_1 \cup \cdots \cup \Gamma_6$,

$$\Gamma_1 = \{(x,t) : x \in [-1,0], \ t = 0\},$$
$$\Gamma_2 = \{(x,t) : x = -1, \ t \in [0,1]\},$$
$$\Gamma_3 = \{(x,t) : x \in [-1,0], \ t = 1\},$$
$$\Gamma_4 = \{(x,t) : x \in [0,1], \ t = 1\},$$
$$\Gamma_5 = \{(x,t) : x = 1, \ t \in [0,1]\},$$
$$\Gamma_6 = \{(x,t) : x \in [0,1], \ t = 0\}.$$

By a change of dependent variables,

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} e^{-0.1t}\phi \\ e^{-0.1t}\phi_x \end{pmatrix},$$

(4.7) can then be expressed in symmetric positive form

$$L\mathbf{u} := A_1\mathbf{u}_x + A_2\mathbf{u}_t + A_0\mathbf{u} = \mathbf{f}, \tag{4.9}$$

with boundary condition

$$M\mathbf{u} := (\mu - \beta)\mathbf{u} = 0 \quad \forall (x,t) \in \partial\Omega, \tag{4.10}$$

TABLE 4.4
Example 4.3: *Boundary matrices.*

	2μ	2β	M
Γ_1	$\begin{pmatrix} -x & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -x & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
Γ_2	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$
Γ_3	$\begin{pmatrix} -x & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -x & 0 \\ 0 & 0 \end{pmatrix}$
Γ_4	$\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
Γ_5	$\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & -1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$
Γ_6	$\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -x & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$

TABLE 4.5
Example 4.3 *using adaptive meshes.*

DOF	$\|\|\|\mathbf{e}\|\|\|$	r.e.	θ
20	1.310	0.804	1.035
72	0.344	0.211	0.985
156	0.155	0.096	0.972
272	0.087	0.054	0.966
420	0.056	0.034	0.963
600	0.039	0.024	0.961
812	0.029	0.017	0.961

where

$$A_1 = \begin{pmatrix} -x & -1 \\ -1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}, \quad A_0 = \begin{pmatrix} 0.1x & x \\ 0 & 1 \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} e^{-0.1t} f \\ 0 \end{pmatrix},$$

μ, β, and M are given in Table 4.4. It can be readily shown that the system (4.9), (4.10) is symmetric positive.

The adjoint operator $\mathbf{L}^*$ of $\mathbf{L}$ is defined by

$$\mathbf{L}^*\mathbf{v} := \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix} \mathbf{v}_x - \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \mathbf{v}_t + \begin{pmatrix} 0.1x + 1 & 0 \\ x & 1 \end{pmatrix} \mathbf{v}.$$

Now, the weak formulation corresponding to (3.23) for (4.7), (4.8) is complete.

For FE approximation of (3.23), a uniform mesh is introduced on Ω and bilinear elements are used. On the other hand, the bubble-shaped functions

$$\psi(\xi, \eta) = \xi(1 - \xi^2)(1/4 - \xi^2)\eta(1 - \eta^2)(1/4 - \eta^2) \tag{4.11}$$

are used to define $(S_T^c)^2$. The effectivity of error estimates using (3.24) is shown in Table 4.5.

In view of error equations (3.5) and (3.24), the calculation of error indicators for the present example proceeds in a similar way as that of Example 4.1 except that we now have, in each element, a 2×2 system induced by (3.24) and by the complementary functions (4.11) for the vector-valued functions. However, by the nature of symmetric positive linear differential equations, the bilinear form associated with the system is coercive only in the L^2 norm instead of in the usual H^1 norm for elliptic problems. Hence, there is no equivalence of the energy

TABLE 4.6
Example 4.4 *using adaptive meshes.*

DOF	$\|\|\|e\|\|\|$	r.e.	θ
32	0.0223	0.1930	0.890
92	0.0117	0.1015	0.956
345	0.0053	0.0462	0.958
1489	0.0024	0.0211	0.980
5156	0.0012	0.0108	0.986
6833	0.0011	0.0096	0.987

norm found in elliptic problems. Our numerical experience has shown that the quadratic order of the complementary basis functions used as in Example 4.1 gives rather pessimistic error estimates. The only remedy to this difficulty seems to be to use higher order $(S_{\mathcal{T}}^2)^2$, e.g., (4.11). Note that this will not cause more computational cost, in fact, the cost is almost the same as when lower orders are used. Apparently, the a posteriori error analysis for mixed-type PDEs remains largely to be further investigated.

Example 4.4. Our error estimators for variational inequalities (2.3) are tested by the model problem given in [4] with the convex set $K = \{v(x, y) \in H^1(\Omega) : v(x, y) \geq 0 \text{ and } v(x, y) = g(x, y) \text{ on } \partial\Omega\}$, where $\Omega = (0, \frac{1}{2}) \times (0, 1)$ and g is chosen so that we have the exact solution

$$u = \begin{cases} 0, & \text{if } (x+1)^2 + y^2 \geq 2, \\ \dfrac{1}{4}[(x+1)^2 + y^2] - \dfrac{1}{2} - \dfrac{1}{2}\ln\{[(x+1)^2 + y^2]\}, & \text{otherwise.} \end{cases}$$

The bilinear and linear forms are defined by

$$B(u, v) := \int_\Omega \nabla u \cdot \nabla v \, dxdy, \quad F(v) := \int_\Omega v \, dxdy.$$

ALGORITHM.

1. Given an initial mesh Ω_h on Ω.
2. Construct a convex set K_s with linear shape functions on Ω_h.
3. Use the Gauss–Seidel–SOR method [18] with the relaxation parameter and the relative error chosen to be 1.2 and 10^{-6}, respectively, to obtain an approximate solution $u_s \in K_s$ of the minimization problem (3.26).
4. Construct the complementary convex set $K_c' := \{w \in S_{\mathcal{T}}^c \subset H_{\mathcal{T}}(\Omega) \text{ and } w \geq -u_s\}$, where $S_{\mathcal{T}}^c$ is defined via (4.2).
5. In each element τ_i, use the method in Step 3 to solve the reduced problem (3.28) for $\tilde{e} \in K_c'$, which involves only four equations with four unknowns, and then calculate the error indicator η_i for that element. Calculate the error estimator for u_s. If r.e. > 0.01 then refine all elements with $\eta_i \geq 0.1\eta_{\max}$, $\eta_{\max} = \max_i \eta_i$, and go back to Step 2, otherwise stop.

The numerical results are shown in Table 4.6.

Example 4.5. To compare FV and FE computations, we consider again the L-shaped problem (4.1).

For any particular (1-irregular) mesh, e.g., Fig. 4.3, on Ω, the FV element approximation of (4.1) is to find $u_b \in S$ such that

$$-\sum_{b_i \in \mathcal{B}} \int_{\partial b_i} \frac{\partial u_b}{\partial n} \, ds = 0,$$

where the FE space S is defined as in Example 4.1. Note that the test functions defined on the volumes $b_i \in \mathcal{B}$ are equal to one.

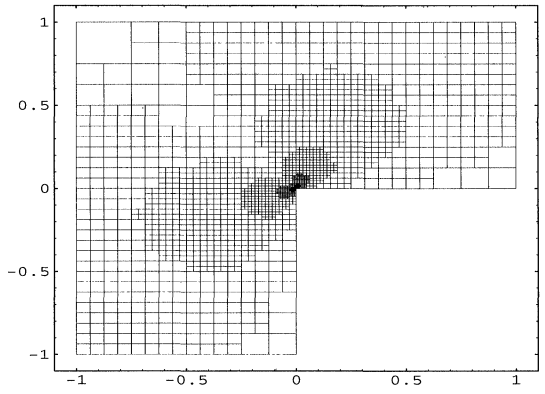

FIG. 4.3. *Adaptive final mesh (FVEM) for Example* 4.5.

TABLE 4.7
Example 4.5 *using uniform meshes (FVEM).*

DOF	$\|\|\|e\|\|\|$	r.e.	θ
8	0.293127	0.217	0.748
21	0.199400	0.147	0.795
65	0.130269	0.096	0.817
225	0.083735	0.062	0.827
833	0.053375	0.039	0.832

TABLE 4.8
Example 4.5 *using adaptive meshes (FVEM).*

DOF	$\|\|\|e\|\|\|$	r.e.	θ
8	0.293127	0.217	0.748
21	0.199400	0.147	0.795
34	0.136928	0.101	0.852
53	0.098177	0.072	0.874
78	0.073441	0.054	0.884
119	0.055017	0.041	0.910
188	0.041540	0.031	0.930
301	0.031497	0.023	0.947
459	0.024512	0.018	0.960
749	0.018927	0.014	0.965
1188	0.014646	0.011	0.973
1801	0.011699	0.009	0.979

Corresponding to Tables 4.1 and 4.2, the data of the FV computations are shown in Tables 4.7 and 4.8, respectively. The final adaptive mesh is shown in Fig. 4.3.

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