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Coupled axial–torsional vibration of thin-walled Z-section beam induced by boundary conditions

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Abstract

The coupled axial–torsional vibration of thin-walled Z-section beam induced by the boundary conditions is investigated. The value of the warping function is not zero at centroid for Z-section beam. If the axial displacement of the pin end is restrained at the centroid of the Z-section for thin-walled Z-section beam, the axial vibration and torsional vibration may be coupled. The governing equations for linear axial and torsional vibration of a thin-walled Z-section beam are derived by the d'Alembert principle and the virtual work principle. The bending vibration is uncoupled from axial and torsional vibrations and is not dealt with in this paper. For harmonic vibration, the general solution of these equations with undetermined constant coefficients may be obtained. Substituting the general solution into the displacement and force boundary conditions, a set of homogeneous equations can be obtained. The natural frequencies and the coefficients of the general solution may be obtained by solving the homogeneous equations using the bisection method.

Numerical examples are studied to verify the accuracy of the proposed method and to investigate the effect of boundary conditions and the value of warping function at centroid on the coupled axial and torsional natural frequency of Z-section beam.

Keywords: Natural frequency; Coupled vibration; Thin-walled beam; Warping

1. Introduction

Beams of thin-walled open cross-section are widely used in structure design. The vibration characteristics are of fundamental importance in the design of thin-walled structures. In general, the shear center and the centroid of cross-section for monosymmetric and asymmetric thin-walled beams are not coincident. Thus, the bending and torsional vibrations are coupled. The doubly coupled bending–torsional vibrations of monosymmetric beams and the triply coupled bending–bending–torsional vibration of asymmetric beam have been investigated by several authors [1–13]. In the literature, the axial vibration is considered to be uncoupled from the bending and torsional vibrations and can be analyzed independently. However, this consideration may be incorrect for Z-section beam, which shear center and the centroid are coincident and the value of the warping function is not zero at centroid. It is well known that for a beam with warping DOF considered, a given external axial load, when referred to loading point, can be replaced by an axial force and a bimoment equal to the product of the external axial load and the value of the warping function at the loaded point [14]. Thus, if the axial displacement of the pin end is restrained at the centroid of the Z-section, the bending and torsional vibrations are uncoupled, but the axial vibration and the torsional vibration may be coupled. To the authors' knowledge, the coupled axial vibration and the torsional vibration induced by the boundary conditions has not been reported in the literature. The object of this paper is to investigate the coupled axial and the torsional vibration of thin-walled Z-section beam induced by the boundary conditions. Because the bending vibration is uncoupled from the axial and torsional vibrations and can be analyzed independently, the bending vibration is not dealt with here. However, it should be noted that in the great majority of

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cases, the minor axis bending vibration mode is the fundamental one. The kinematics of the Vlasov thin-walled beam [14] is employed here. The governing differential equations of motion in free axial and torsional vibration are derived using the virtual work principle and the d'Alembert principle. For harmonic vibration, the general solution of these equations with undetermined constant coefficients can be easily obtained. Substituting the general solution into the displacement and force boundary conditions, a set of homogeneous equations can be obtained. The natural frequencies and the coefficients of the general solution can be obtained by solving the homogeneous equations numerically. Here the bisection method is used.

Numerical examples are studied to verify the accuracy of the proposed method and to investigate the effect of boundary conditions and the value of warping function at centroid on the coupled axial and torsional natural frequency of Z-section beam.

2. Formulation

2.1. Kinematics of beam member

A straight uniform beam member of length L with Z-section shown in Fig. 1 is considered. The centroid axis and the shear center axis of the beam are coincident. The x-axis are chosen to coincide with the centroid axis of the undeformed beam, and the y- and z-axis are chosen to be the principal centroidal axes of the cross-section in the undeformed state. Based on the Vlasov thin-walled beam theory [14], it is assumed that the cross-section of the beam does not deform in its own plane, and the out-of-plane warping of the cross-section is the product of the twist rate of the beam element and the Saint Venant warping function for a prismatic thin-walled beam of the same cross-section. In this study, Prandtl's membrane analogy and the Saint Venant torsion theory [15,16] are used to obtain an approximate Saint Venant warping function for a prismatic thin-walled beam.

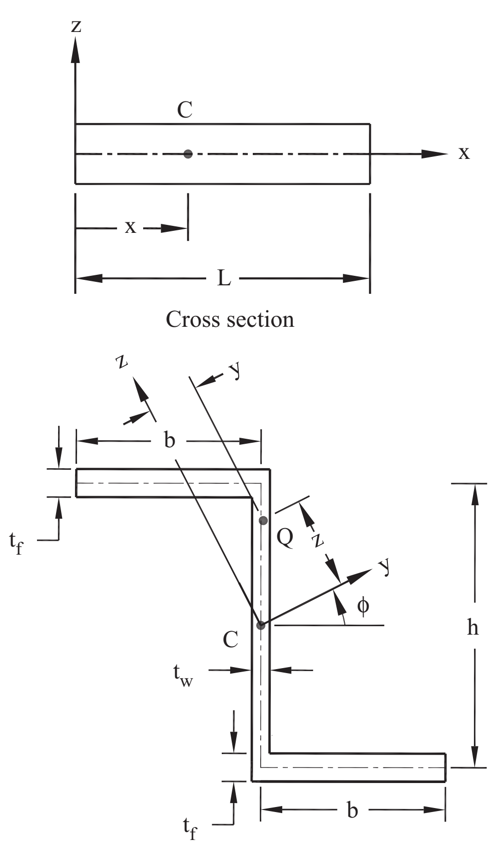

Fig. 1. Z-section beam and coordinate systems.

Because the bending vibrations are uncoupled from the axial and torsional vibrations for Z-section beam, only the axial displacement and axial rotation are considered here. Thus, the deformation of the beam can be described by the displacement along the beam axis and lateral displacements induced by the rotation about the centroid axis of the beam. Let Q (Fig. 1) be an arbitrary point in the beam element, and C be the point corresponding to Q on the centroid axis. The position vector of point Q in the undeformed and deformed configurations may be expressed as

$$\mathbf{r}_0 = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z \tag{1}$$

and

$$\mathbf{r} = (x + u + \psi\theta_{,x})\mathbf{e}_x + (y - z\theta)\mathbf{e}_y + (z + y\theta)\mathbf{e}_z, \tag{2}$$

where x, y and z are coordinates of point Q in the underformed state, $u = u(x,t)$ is the axial displacement of the unwarped cross-section, $\psi = \psi(y,z)$ is the Saint Venant warping function for a prismatic beam of the same cross-section, $\theta = \theta(x,t)$ and $\theta_{,x} = \partial\theta/\partial x$ are the twist angle and the twist rate about the shear center axis, respectively. In this paper, the symbol $(\dot{\ })'$ denotes $(\ )_{,x} = \partial(\ )/\partial x$.

Let u_c denote the axial displacement of the centroid axis of the beam. From Eq. (2), u_c may be expressed by

$$u_c = u_c(x,t) = u + \psi_0\theta_{,x}, \tag{3}$$

where $\psi_0 = \psi(0,0)$ is the value of warping function at the centroid of the cross-section.

If x, y and z in Eq. (1) are regarded as the Lagrangian coordinates, the components of the Green strains ε_{xx},ε_{xy} and ε_{xz} are given by [17]

$$\varepsilon_{xx} = \tfrac{1}{2}(\mathbf{r}^t_{,x}\mathbf{r}_{,x} - 1), \quad \varepsilon_{xy} = \tfrac{1}{2}\mathbf{r}^t_{,x}\mathbf{r}_{,y}, \quad \varepsilon_{xz} = \tfrac{1}{2}\mathbf{r}^t_{,x}\mathbf{r}_{,z}. \tag{4}$$

In this study only infinitesimal free vibration is considered. Thus only the linear part of the Green strains is retained. Substituting Eq. (2) into Eq. (4) and retaining all terms up to the first order, one may obtain

$$\begin{aligned} \varepsilon_{xx} &= u_{,x} + \psi\theta_{,xx}, \\ \varepsilon_{xy} &= \tfrac{1}{2}(\psi_{,y} - z)\theta_{,x}, \\ \varepsilon_{xz} &= \tfrac{1}{2}(\psi_{,z} + y)\theta_{,x}. \end{aligned} \tag{5}$$

2.2. Equations of motion

The equations of motion for axial and torsional vibration of beam member are derived by the d'Alembert principle and the virtual work principle.

Let segment ab denote an arbitrary segment of a beam member with end sections a and b. The virtual work principle for segment ab of a linear elastic beam member may be written as [18]

$$\delta W_E = \delta W_I, \tag{6}$$

$$\begin{aligned}\delta W_E &= (f\delta u_c + m\delta\theta + b\delta\theta_{,x})\big|_{x_a}^{x_b} \\ &= \int_{x_a}^{x_b}\left\{\frac{\mathrm{d}}{\mathrm{d}x}(f\delta u_c) + \frac{\mathrm{d}}{\mathrm{d}x}(m\delta\theta) + \frac{\mathrm{d}}{\mathrm{d}x}(b\delta\theta_{,x})\right\}\mathrm{d}x \\ &= \int_{x_a}^{x_b}\{f_{,x}\delta u_c + f\delta u_{c,x} + b\delta\theta_{1,xx} + m_{,x}\delta\theta \\ &\quad + (m+b_{,x})\delta\theta_{,x}\}\,\mathrm{d}x,\end{aligned} \tag{7}$$

$$\begin{aligned}\delta W_I &= \int_V (E\varepsilon_{xx}\delta\varepsilon_{xx} + 4G\varepsilon_{xy}\delta\varepsilon_{xy} + 4G\varepsilon_{xz}\delta\varepsilon_{xz})\,\mathrm{d}V \\ &\quad + \int_V \rho\delta\mathbf{r}^t\ddot{\mathbf{r}}\,\mathrm{d}V,\end{aligned} \tag{8}$$

where δW_E and δW_1 are the virtual work of the external forces and the internal stresses, respectively, δu_c, and $\delta\theta$ and $\delta\theta_{,x}$ are the variation of u_c in Eq. (3), and the variation of θ and $\theta_{,x}$ in Eq. (2), respectively, f, m, and b are the external axial force, twist moment, and bimoment corresponding to δu_c, $\delta\theta$, and $\delta\theta_{,x}$, respectively, $(\ )\big|_{x_a}^{x_b}$ is the value of ( ) at section b minus the value of ( ) at section a, x_a and x_b are the x_1 coordinates of sections a and b at the undeformed state, $\delta\varepsilon_{xj}$ ($j = x, y, z$) are the variation of ε_{xj} in Eq. (5), $\delta\mathbf{r}$ is the variation of $\mathbf{r}$ given in Eq. (2), $\ddot{\mathbf{r}} = \partial^2 r/\partial t^2$. E is Young's modulus, G is the shear modulus, ρ is the density, V is the volume of the undeformed beam between sections a and b. The differential volume $\mathrm{d}V$ may be expressed as $\mathrm{d}V = \mathrm{d}A\,\mathrm{d}x$, where $\mathrm{d}A$ is the differential cross-section area of the beam. In this paper, the symbol $\dot{(\ )}$ denotes differentiation with respect to time t.

Substituting Eqs. (2), (3) and (5) into Eq. (8), and using $\int y\,\mathrm{d}A = \int z\,\mathrm{d}A = \int yz\,\mathrm{d}A = 0$ and $\int \psi\,\mathrm{d}A = \int y\psi\,\mathrm{d}A = \int z\psi\,\mathrm{d}A = 0$, one may obtain

$$\begin{aligned}\delta W_I &= \int_{x_a}^{x_b}[\delta u_{c,x}(EAu_{,x}) + \delta\theta_{,x}(GJ\theta_{,x}) \\ &\quad + \delta\theta_{,xx}(EI_\psi\theta_{,xx} - EA\psi_0 u_{,x})]\,\mathrm{d}x \\ &\quad + \int_{x_a}^{x_b}[\delta u_c(\rho A\ddot{u}) + \delta\theta(\rho I_p\ddot{\theta}) \\ &\quad + \delta\theta_{,x}(\rho I_\psi\ddot{\theta}_{,x} - \rho\psi_0 A\ddot{u})]\,\mathrm{d}x,\end{aligned} \tag{9}$$

where

$$I_p = I_y + I_z,\quad I_y = \int z^2\,\mathrm{d}A,\quad I_z = \int y^2\,\mathrm{d}A,\quad I_\psi = \int \psi^2\,\mathrm{d}A,$$

$$J = \int_A \{[-(z - z_p) + \psi_{,y}]^2 + [(y - y_p) + \psi_{,z}]^2\}\mathrm{d}A. \tag{10}$$

Substituting Eqs. (7) and (9) into Eq. (6), and equating the terms in both sides of Eq. (6) corresponding to the same generalized virtual displacements, one may obtain

$$f_{,x} = \rho A\ddot{u}, \tag{11}$$

$$m_{,x} = \rho I_p\ddot{\theta}, \tag{12}$$

$$f = EAu_{,x}, \tag{13}$$

$$m + b_{,x} = GJ\theta_{,x} + \rho I_\psi\ddot{\theta}_{,x} - \rho A\psi_0\ddot{u}, \tag{14}$$

$$b = EI_\psi\theta_{,xx} - EAu_{,x}\psi_0. \tag{15}$$

Substituting Eqs. (11), (13) and (15) into Eq. (14), one may obtain

$$m = GJ\theta_{,x} + \rho I_\psi\ddot{\theta}_{,x} - EI_\psi\theta_{,xxx}. \tag{16}$$

Eqs. (11) and (12) may be regarded as equations of motion and Eqs. (13), (15) and (16) may be regarded as generalized constitutive equations. It should be noted that due to the existence of nonnull warping ψ_0 in Eq. (15), axial extension couples with torsion. Otherwise, the bimoment will depend only on the second derivative of twist angle.

Substituting Eqs. (13) and (16) into Eqs. (11) and (12), respectively, one may obtain

$$u_{,xx} = \frac{\rho}{E}\ddot{u}, \tag{17}$$

$$GJ\theta_{,xx} + \rho I_\psi\ddot{\theta}_{,xx} - EI_\psi\theta_{,xxxx} = \rho I_p\ddot{\theta}. \tag{18}$$

At $x = 0$ and at $x = L$, five different boundary conditions called BCI ($I = 1$–5) are considered here and are given by

$$\begin{aligned}\mathrm{BC1}:\ & u_c(0,t) = 0,\quad u_c(L,t) = 0,\quad \theta(0,t) = 0,\quad \theta(L,t) = 0, \\ & b(0,t) = 0,\quad b(L,t) = 0,\end{aligned} \tag{19}$$

$$\begin{aligned}\mathrm{BC2}:\ & u_c(0,t) = 0,\quad u_c(L,t) = 0,\quad \theta(0,t) = 0, \\ & \theta'(0,t) = 0,\quad m(L,t) = 0,\quad b(L,t) = 0,\end{aligned} \tag{20}$$

$$\begin{aligned}\mathrm{BC3}:\ & u_c(0,t) = 0,\quad u_c(L,t) = 0,\quad \theta(0,t) = 0, \\ & \theta(L,t) = 0,\quad \theta'(0,t) = 0,\quad b(L,t) = 0,\end{aligned} \tag{21}$$

$$\begin{aligned}\mathrm{BC4}:\ & u_c(0,t) = 0,\quad f(L,t) = 0,\quad \theta(0,t) = 0, \\ & \theta'(0,t) = 0,\quad m(L,t) = 0,\quad b(L,t) = 0,\end{aligned} \tag{22}$$

$$\begin{aligned}\mathrm{BC5}:\ & u_c(0,t) = 0,\quad u_c(L,t) = 0,\quad \theta(0,t) = 0, \\ & \theta'(0,t) = 0,\quad \theta(L,t) = 0,\quad \theta'(L,t) = 0,\end{aligned} \tag{23}$$

where u_c is defined in Eq. (3), θ and θ' are defined in Eq. (2), f, m and b are given in Eqs. (13), (15) and (16), respectively. BC1 refers to axial extension and torsion restrained at both supports, but warping free; BC2 refers to axial extension restrained at both supports, and torsion and warping restrained at one support but free at the other support; BC3 refers to axial extension and torsion restrained at both supports, and warping restrained at one support but free at the other support; BC4 refers to axial extension, torsion and warping restrained at one support, but free at the other support; BC5 refers to axial extension, torsion and warping restrained at both supports.

It can be seen that $u = u(x,t)$ and $\theta = \theta(x,t)$ are uncoupled in Eqs. (17) and (18). However, from Eqs. (3), (19)–(21), it can be seen that $u = u(x,t)$ and $\theta = \theta(x,t)$ may be coupled.

2.3. Free vibration

The free harmonic axial and torsional vibrations defined by Eqs. (17) and (18) may be expressed in the form

$$u = U(x)\mathrm{e}^{\mathrm{i}\omega t}, \tag{24}$$

$$\theta = \Theta(x)\mathrm{e}^{\mathrm{i}\omega t}, \tag{25}$$

where $\mathrm{i} = \sqrt{-1}$, ω is the natural frequency to be determined, $U(x)$ and $\Theta(x)$ are axial and torsional vibration modes to be determined. Introducing Eqs. (24) and (25) into Eqs. (17) and (18), one may obtain

$$U_{,xx} + \frac{\rho\omega^2}{E} U = 0, \tag{26}$$

$$\Theta_{,xxxx} - \left(\frac{GJ}{EI_\psi} - \frac{\rho\omega^2}{E}\right)\Theta_{,xx} - \frac{\rho I_p \omega^2}{EI_\psi}\Theta = 0. \tag{27}$$

It can be seen that Eqs. (26) and (27) are two uncoupled linear ordinary differential equations with constant coefficients. The general solution of Eqs. (26) and (27) may be expressed as

$$U(x) = C_1 \sin \alpha x + C_2 \cos \alpha x, \tag{28}$$

$$\Theta(x) = C_3 \sinh \beta x + C_4 \cosh \beta x + C_5 \sin \gamma x + C_6 \cos \gamma x, \tag{29}$$

$$\alpha = \sqrt{\frac{\rho\omega^2}{E}}, \tag{30}$$

$$\beta = \left(\frac{1}{2}\left(\frac{GJ}{EI_\psi} - \frac{\rho\omega^2}{E}\right) + \frac{1}{2}\sqrt{\left(\frac{GJ}{EI_\psi} - \frac{\rho\omega^2}{E}\right)^2 + \frac{4\rho I_p \omega^2}{EI_\psi}}\right)^{1/2},$$

$$\gamma = \left(-\frac{1}{2}\left(\frac{GJ}{EI_\psi} - \frac{\rho\omega^2}{E}\right) + \frac{1}{2}\sqrt{\left(\frac{GJ}{EI_\psi} - \frac{\rho\omega^2}{E}\right)^2 + \frac{4\rho I_p \omega^2}{EI_\psi}}\right)^{1/2},$$

where C_i ($i = 1$–6) are undetermined coefficients.

From one of the boundary conditions BCI ($I = 1$–5) given in Eqs. (19)–(23), and Eqs. (24), (25), (28), and (29), one may obtain

$$\mathbf{K}(\omega)\mathbf{C} = \mathbf{0}, \tag{31}$$

$$\mathbf{C} = \{C_1, C_2, C_3, C_4, C_5, C_6\}, \tag{32}$$

where $\mathbf{K}(\omega)$ is a 6×6 matrix. $\mathbf{K}(\omega)$ denotes matrix $\mathbf{K}$ is function of natural frequency ω given in Eq. (24). The explicit form of $\mathbf{K}$ for boundary conditions BCI ($I = 1$–5) is given in Appendix A.

For a nontrivial $\mathbf{C}$, the determinant of the matrix $\mathbf{K}$ in Eq. (31) must be equal to zero. The values of ω which make these determinants vanish are called eigenvalues of matrix $\mathbf{K}$. The bisection method is used here to find the eigenvalues. Let ω_i and $\mathbf{X}$ denote an eigenvalue and the corresponding eigenvector of Eq. (31). Substituting $\mathbf{C} = \mathbf{X}$ into Eqs. (28) and (29), the mode shape of axial vibration and torsional vibration corresponding to ω_i can be calculated.

3. Numerical examples

To investigate the natural frequency of the axial and torsional vibration for Z-section beam, several numerical examples are studied. The geometry and material properties for the beam of Z-section given in Fig. 1 are as follows: $L = 3\,\mathrm{m}$, $b = 0.2\,\mathrm{m}$, $h = 0.3\,\mathrm{m}$, $t_w = t_f = 0.01\,\mathrm{m}$, $\phi = 0.555956\,\mathrm{rad}$, $E = 206\,\mathrm{GPa}$, $v = 0.3$, $\rho = 7800\,\mathrm{kg/m^3}$. The section constants are as follows: $A = 70 \times 10^{-4}\,\mathrm{m}^2$, $I_y = 1.49844 \times 10^{-4}\,\mathrm{m}^4$, $I_z = 1.60473 \times 10^{-5}\,\mathrm{m}^4$, $J = 2.33333 \times 10^{-7}\,\mathrm{m}^4$, $I_\psi = 6.86346 \times 10^{-7}\,\mathrm{m}^6$, $\psi_0 = 85.7143 \times 10^{-4}\,\mathrm{m}^2$.

Table 1
Natural frequencies (rad/s) for the unsymmetric Z-section beam

Mode	BC1A	BC1A (FEM)	BC1B	BC2A	BC2B	BC3A	BC3B	BC4A (B)	BC5A (B)
1	464.31 (AT)	464.31 (AT)	382.63 (T)	170.04 (AT)	154.25 (T)	604.96 (AT)	580.65 (T)	154.25 (T)	831.18 (T)
2	1437.61 (AT)	1437.62 (AT)	1458.26 (T)	876.00 (AT)	834.24 (T)	1850.62 (AT)	1835.59 (T)	834.24 (T)	2257.46 (T)
3	3287.00 (AT)	3287.14 (AT)	3218.94 (T)	2270.50 (AT)	2253.28 (T)	3725.09 (AT)	3767.73 (T)	2253.28 (T)	4360.21 (T)
4	4631.38 (AT)	4633.92 (AT)	5381.64 (A)	4144.93 (AT)	4326.18 (T)	5179.43 (AT)	5381.64 (A)	2690.82 (A)	5381.64 (A)
5	6161.02 (AT)	6163.99 (AT)	5621.06 (T)	5227.63 (AT)	5381.64 (A)	6451.37 (AT)	6331.90 (T)	4326.18 (T)	7084.90 (T)
6	8280.64 (AT)	8286.42 (AT)	8608.86 (T)	7128.37 (AT)	7006.97 (T)	9327.69 (AT)	9470.74 (T)	7006.97 (T)	10373.1 (T)
7	10660.2 (AT)	10694.0 (AT)	10763.3 (A)	9675.28 (AT)	10234.3 (T)	10667.1 (AT)	10763.3 (A)	8072.47 (A)	10763.3 (A)
8	11979.6 (AT)	11992.6 (AT)	12119.1 (T)	10989.0 (AT)	10763.3 (A)	13153.6 (AT)	13119.9 (T)	10234.3 (T)	14159.4 (T)
9	15696.1 (AT)	15812.7 (AT)	16085.5 (T)	13838.8 (AT)	13943.2 (T)	15732.0 (AT)	16144.9 (A)	13454.1 (A)	16144.9 (A)
10	16092.6 (AT)	16123.6 (AT)	16144.9 (A)	15768.6 (AT)	16144.9 (A)	17438.1 (AT)	17212.7 (T)	13943.2 (T)	18376.6 (T)

Here, boundary conditions BCI ($I = 1$–5), defined in Eqs. (19)–(23) are considered. To investigate the effects of the value of ψ_0 (Eq. (2)) on the natural frequency of Z-section beam, cases with and without considering the value of ψ_0 in Eqs. (3) and (15), referred to as case A and case B, respectively, are considered. For convenience, in

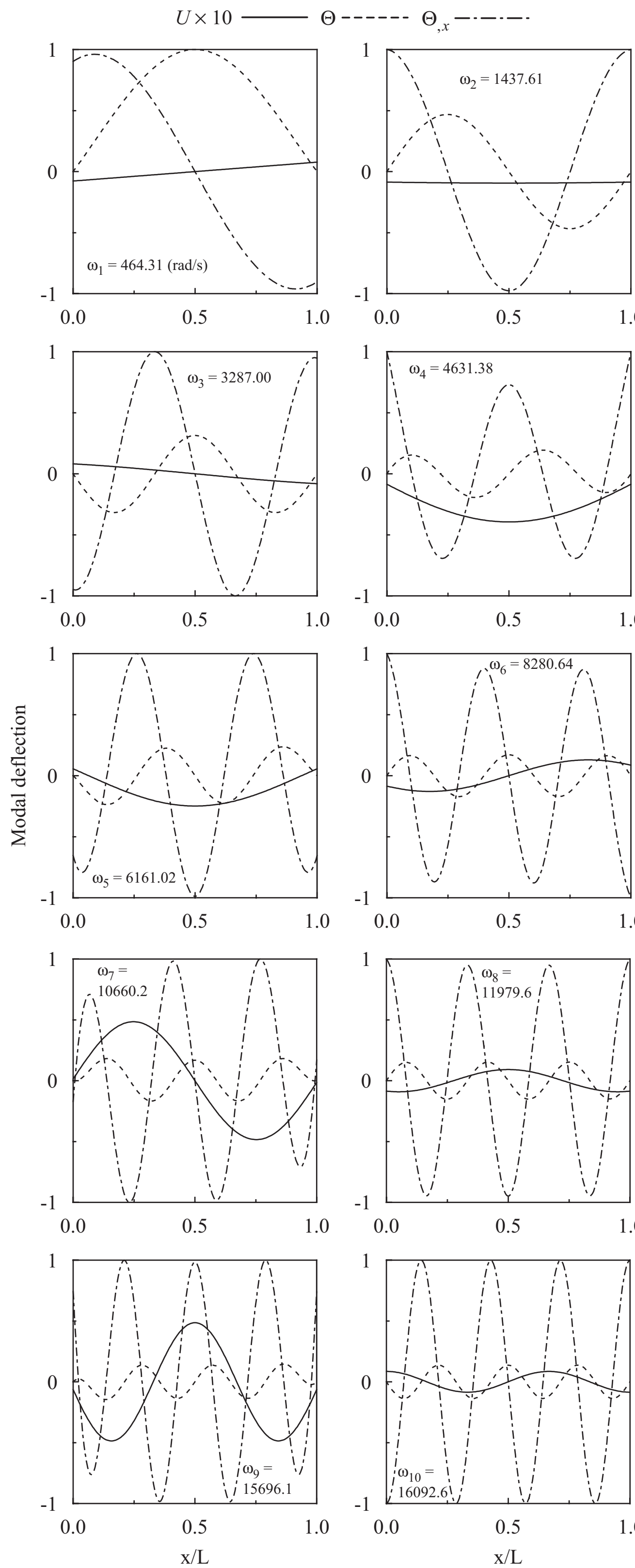

Fig. 2. The first 10 vibration mode shapes for BC1A.

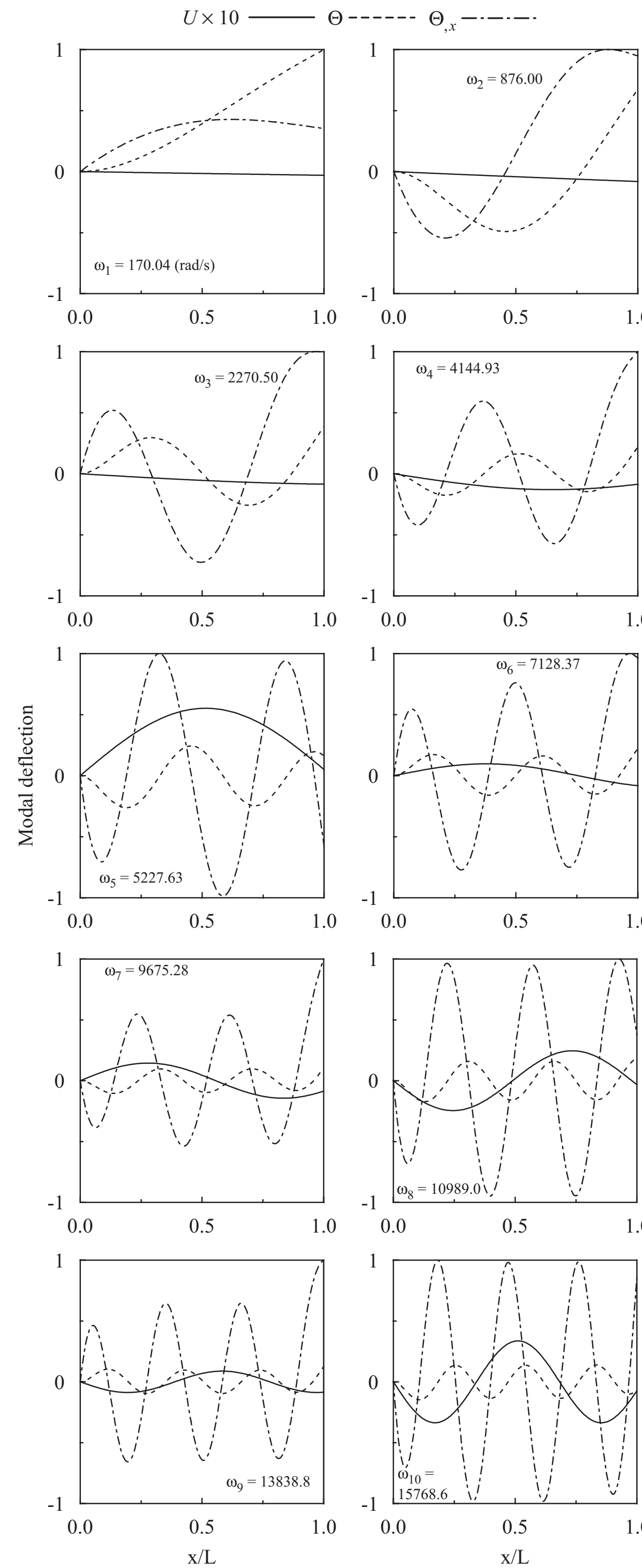

Fig. 3. The first 10 vibration mode shapes for BC2A.

this study, BCIX ($I = 1$–5, X = A, B) is used to denote case X with boundary condition BCI. It can be seen from Eqs. (3), (15) and (19)–(23) that the axial vibration and torsional vibration are uncoupled for BCIB ($I = 1$–5). For BC4, the twist rate $\Theta_{,x} = 0$ at the fixed end and $U_{,x} = 0$ at the free end. Thus, it can be seen from Eqs. (3) and (13)

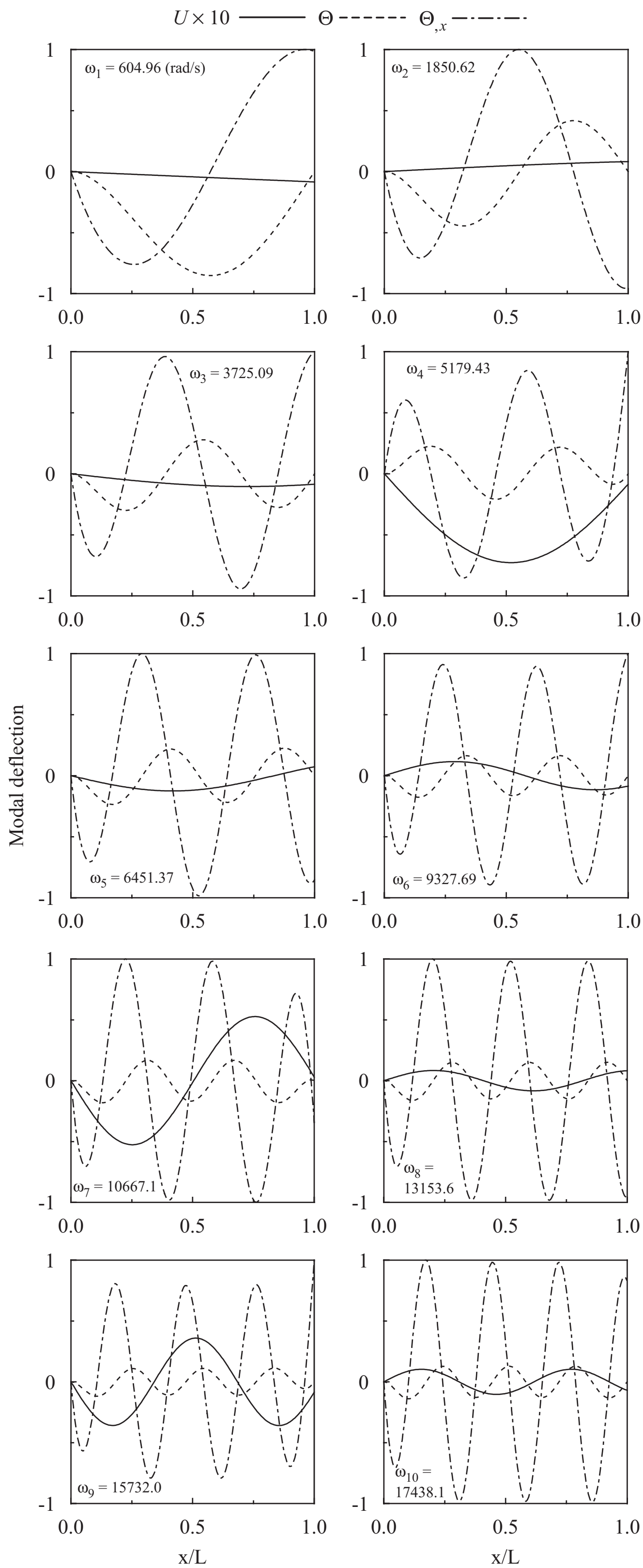

Fig. 4. The first 10 vibration mode shapes for BC3A.

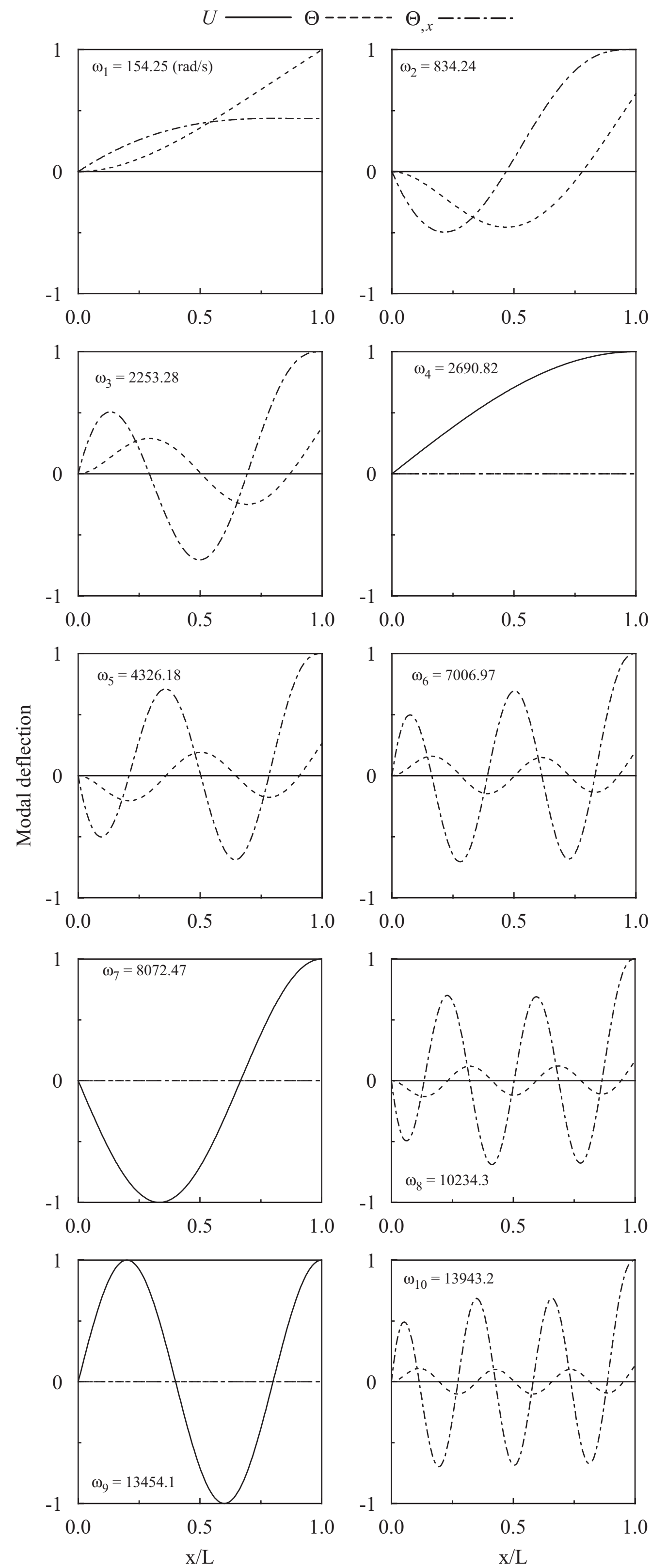

Fig. 5. The first 10 vibration mode shapes for BC4A.

that cases A and case B are equivalent for BC4. For BC5, the twist rate $\Theta_{,x} = 0$ at both fixed ends. Thus, it can be seen from Eq. (3) that cases A and B are equivalent for BC5.

The lowest 10 natural frequencies of the present study are shown in Table 1. In Table 1, (A), (T), and (AT) denotes that the natural frequency corresponds to uncoupled axial vibration, uncoupled torsional vibration, and coupled axial–torsional vibration,. As expected, for BCIA ($I = 1$–3), the axial vibration and torsional vibration are coupled; for BCIB ($I = 1$–5), BC4A and BC5A, the axial vibration and torsional vibration are uncoupled. The natural frequencies are identical for BCIA and BCIB ($I = 4$, 5) as expected. For BC1B, the analytical solution for the natural frequency of torsional vibration may be express by [13]

$$\omega_n = \lambda_n \sqrt{\frac{E}{\rho}} \left(\frac{1 + GJ/\lambda_n^2 EI_\psi}{1 + I_p/\lambda_n^2 I_\psi} \right)^{1/2}, \quad \lambda_n = \frac{n\pi}{L}, \quad n = 1, 2, 3, \ldots. \tag{33}$$

The natural frequencies of torsional vibration for BC1B calculated by Eq. (33) are identical to those given in Table 1. This example is also studied using the finite element method for BC1A. The bar element (corresponding to $\delta\mathbf{q}_c$) developed in Appendix B is employed here. The finite element results given in Table 1 are obtained using 20 elements. It can be seen from Table 1 that the agreement between the results obtained using Eq. (31) and those obtained using the finite element method is very good. The discrepancy between the natural frequencies corresponding to BCIA and BCIB ($I = 1$–3) is not negligible. It indicates that the effects of the value of ψ_0 on the natural frequencies of the Z-section beam may be not negligible.

Figs. 2–6 present the mode shapes corresponding to the lowest 10 natural frequencies for BCIA ($I = 1$–5). The vibration modes plotted are U, the axial displacement of the unwarped cross-section, Θ twist angle and $\Theta_{,x}$, twist rate of the centroid axis. In order to increase the clarity of the axial extension curves, the values of U are amplified 10 times in Figs. 2–4. As expected, for BCIA ($I = 1$–3), the axial and torsional vibrations are coupled, and for BCIA ($I = 4$, 5), the axial and torsional vibrations are uncoupled.

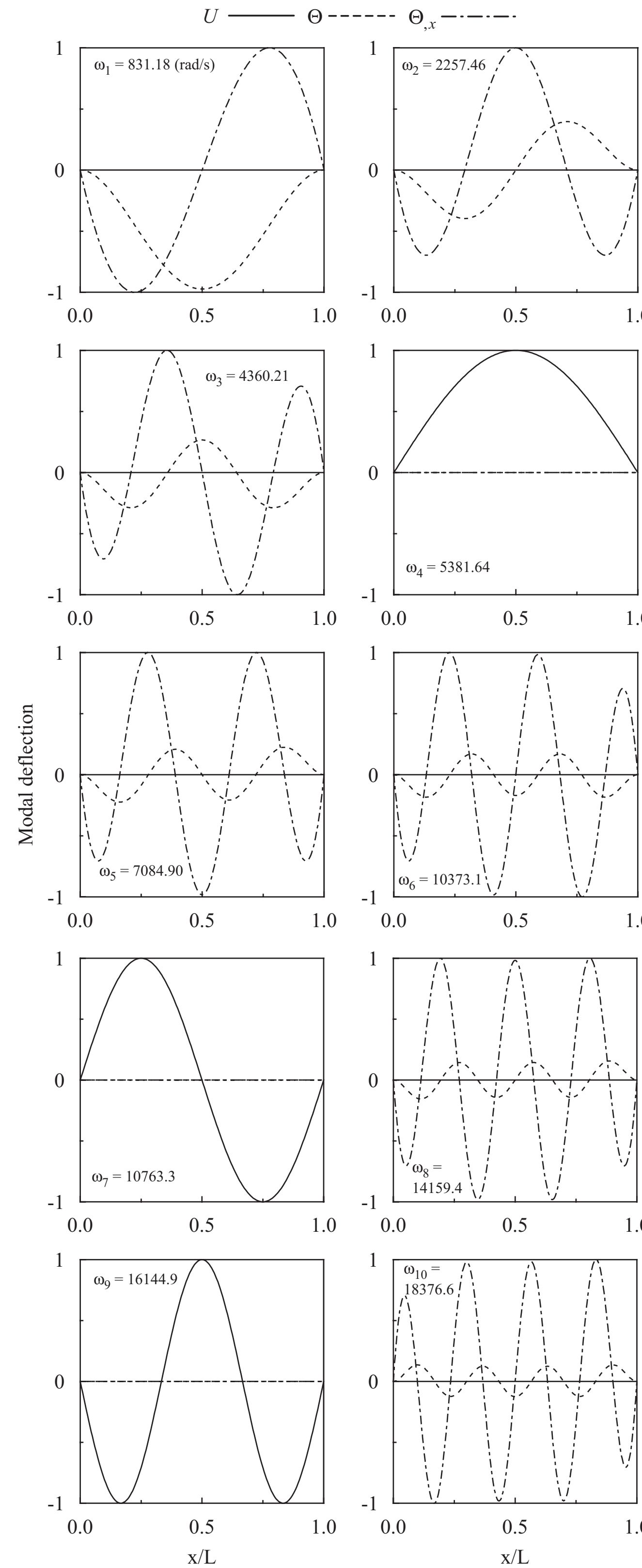

Fig. 6. The first 10 vibration mode shapes for BC5A.

4. Conclusions

In this paper, the correct governing differential equations for the linear axial and torsional vibration of a uniform Z-section beam are derived using the d'Alembert principle and the virtual work principle based on the Vlasov thin-walled beam theory. The value of warping function at centroid of Z-section is considered. The bisection method is used to solve the natural frequency of axial and torsional vibration.

The results of numerical examples show that if the axial displacement of the pin end is restrained at the centroid of Z-section, the effect of the value of warping function at centroid of Z-section on the natural frequencies of the axial and torsional vibration for the Z-section beam may be not negligible. Although only the Z-section beam is studied here, it is suggested that the value of warping function should be considered for the vibration analysis of thin-walled beam if the axial displacement of the pin end is restrained at a point with nonzero value of warping function.

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Appendix A

The matrix **K** in Eq. (31) for boundary conditions BCI ($I = 1$–5) may be given by

BC1: $u_c(0, t) = 0$, $u_c(L, t) = 0$, $\theta(0, t) = 0$, $\theta(L, t) = 0$, $b(0, t) = 0$, $b(L, t) = 0$,

$$\mathbf{K} = \begin{bmatrix} 0 & 1 & \psi_0\beta & 0 & \psi_0\gamma & 0 \\ \sin\alpha L & \cos\alpha L & \psi_0\beta\cosh\beta L & \psi_0\beta\sinh\beta L & \psi_0\gamma\cos\gamma L & -\psi_0\gamma\sin\gamma L \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & \sinh\beta L & \cosh\beta L & \sin\gamma L & \cos\gamma L \\ \alpha & 0 & 0 & -D\beta^2 & 0 & D\gamma^2 \\ \psi_0\alpha\cos\alpha L & -\psi_0\alpha\sin\alpha L & -D\beta^2\sinh\beta L & -D\beta^2\cosh\beta L & D\gamma^2\sin\gamma L & D\gamma^2\cos\gamma L \end{bmatrix},$$

BC2: $u_c(0, t) = 0$, $u_c(L, t) = 0$, $\theta(0, t) = 0$, $\theta'(0, t) = 0$, $m(L, t) = 0$, $b(L, t) = 0$,

$$\mathbf{K} = \begin{bmatrix} 0 & 1 & \psi_0\beta & 0 & \psi_0\gamma & 0 \\ \sin\alpha L & \cos\alpha L & \psi_0\beta\cosh\beta L & \psi_0\beta\sinh\beta L & \psi_0\gamma\cos\gamma L & -\psi_0\gamma\sin\gamma L \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & \beta & 0 & \gamma & 0 \\ 0 & 0 & R_1\cosh\beta L & R_1\sinh\beta L & R_2\cos\gamma L & -R_2\sin\gamma L \\ \psi_0\alpha\cos\alpha L & -\psi_0\alpha\sin\alpha L & -D\beta^2\sinh\beta L & -D\beta^2\cosh\beta L & D\gamma^2\sin\gamma L & D\gamma^2\cos\gamma L \end{bmatrix},$$

BC3: $u_c(0, t) = 0$, $u_c(L, t) = 0$, $\theta(0, t) = 0$, $\theta(L, t) = 0$, $\theta'(0, t) = 0$, $b(0, t) = 0$,

$$\mathbf{K} = \begin{bmatrix} 0 & 1 & \psi_0\beta & 0 & \psi_0\gamma & 0 \\ \sin\alpha L & \cos\alpha L & \psi_0\beta\cosh\beta L & \psi_0\beta\sinh\beta L & \psi_0\gamma\cos\gamma L & -\psi_0\gamma\sin\gamma L \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & \sinh\beta L & \cosh\beta L & \sin\gamma L & \cos\gamma L \\ 0 & 0 & \beta & 0 & \gamma & 0 \\ \psi_0\alpha\cos\alpha L & -\psi_0\alpha\sin\alpha L & -D\beta^2\sinh\beta L & -D\beta^2\cosh\beta L & D\gamma^2\sin\gamma L & D\gamma^2\cos\gamma L \end{bmatrix},$$

BC4: $u_c(0, t) = 0$, $f(L, t) = 0$, $\theta(0, t) = 0$, $\theta'(0, t) = 0$, $m(L, t) = 0$, $b(0, t) = 0$,

$$\mathbf{K} = \begin{bmatrix} 0 & 1 & \psi_0\beta & 0 & \psi_0\gamma & 0 \\ \alpha\cos\alpha L & -\alpha\sin\alpha L & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & \beta & 0 & \gamma & 0 \\ 0 & 0 & R_1\cosh\beta L & R_1\sinh\beta L & R_2\cos\gamma L & -R_2\sin\gamma L \\ \psi_0\alpha\cos\alpha L & -\psi_0\alpha\sin\alpha L & -D\beta^2\sinh\beta L & -D\beta^2\cosh\beta L & D\gamma^2\sin\gamma L & D\gamma^2\cos\gamma L \end{bmatrix},$$

BC5: $u_c(0, t) = 0$, $u_c(L, t) = 0$, $\theta(0, t) = 0$, $\theta'(0, t) = 0$, $\theta(L, t) = 0$, $\theta'(L, t) = 0$,

$$\mathbf{K} = \begin{bmatrix} 0 & 1 & \psi_0\beta & 0 & \psi_0\gamma & 0 \\ \sin\alpha L & \cos\alpha L & \psi_0\beta\cosh\beta L & \psi_0\beta\sinh\beta L & \psi_0\gamma\cos\gamma L & -\psi_0\gamma\sin\gamma L \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & \beta & 0 & \gamma & 0 \\ 0 & 0 & \sinh\beta L & \cosh\beta L & \sin\gamma L & \cos\gamma L \\ 0 & 0 & \beta\cosh\beta L & \beta\sinh\beta L & \gamma\cos\gamma L & -\gamma\sin\gamma L \end{bmatrix},$$

where $D = I_\psi/A,\ R_1 = H\beta - F\beta^3,\ R_2 = H\gamma + F\gamma^3,\ F = EI_\psi/GJ,\ H = 1 - (\rho I_\psi/GJ)\omega^2$.

Appendix B

The bar element developed here has two nodes with six degrees of freedom per node. Two sets of nodal degrees of freedom corresponding to warped and unwarped cross-section are considered here. The relation between two sets of element nodal degrees of freedom is derived. The element stiffness matrix and mass matrix corresponding to unwarped cross-section may be easily derived using standard procedure and not given here. The relation between element matrices referred to different sets of element nodal degrees of freedom is derived.

The axial displacement $u(x, t)$ of the unwarped cross-section and axial rotation $\theta(x, t)$ about the centroid axis are assumed to be the linear and Hermitian polynomials of x, respectively. $u(x, t)$ and $\theta(x, t)$ may be expressed by

$$u(x,t) = \mathbf{N}_a^t \mathbf{u}_a, \quad \theta(x,t) = \mathbf{N}_d^t \mathbf{u}_d, \tag{34}$$

$$\mathbf{N}_a = \left\{ \frac{1-\xi}{2}, \frac{1+\xi}{2} \right\}, \quad \mathbf{u}_a = \{u_1, u_2\},$$
$$\mathbf{N}_d = \{N_1, N_2, N_3, N_4\}, \quad \mathbf{u}_d = \{\theta_1, \theta_{,x1}, \theta_2, \theta_{,x2}\},$$

$$N_1 = \tfrac{1}{4}(1-\xi)^2(2+\xi), \quad N_2 = \tfrac{\ell}{8}(1-\xi^2)(1-\xi),$$
$$N_3 = \tfrac{1}{4}(1+\xi)^2(2-\xi), \quad N_4 = \tfrac{\ell}{8}(-1+\xi^2)(1+\xi),$$

$$\xi = -1 + \frac{2x}{\ell},$$

where u_j ($j = 1, 2$) denote the axial displacement of the unwarped cross-section at element node j, θ_j and $\theta_{,xj}$ ($j = 1, 2$) are nodal values of θ, $\theta_{,x}$ at nodes j, respectively, ℓ is the length of the bar element. In this paper, { } denotes column matrix.

Let δu_j^c ($j = 1, 2$) denote the variation of the axial displacement of the centroid axis of the bar element at node j, and δu_j ($j = 1, 2$) denote the variation of u_j.

Making use of Eq. (3), the relation between δu_j^c and δu_j may be expressed by

$$\delta u_j = \delta u_j^c - \psi_0 \delta\theta_{,xj}, \tag{35}$$

where $\delta\theta_{,xj}$ ($j = 1, 2$) are the variation of $\theta_{,xj}$.

Let $\delta\mathbf{q}$ and $\delta\mathbf{q}_c$ denote the variation of nodal degrees of freedom corresponding to unwarped and warped cross-section. $\delta\mathbf{q}$ and $\delta\mathbf{q}_c$ may be given by

$$\delta\mathbf{q} = \{\delta\mathbf{u}_a, \delta\mathbf{u}_d\}, \quad \delta\mathbf{q}_c = \{\delta\mathbf{u}_a^c, \delta\mathbf{u}_d\},$$
$$\delta\mathbf{u}_a = \{\delta\mathbf{u}_1, \delta\mathbf{u}_2\}, \quad \delta\mathbf{u}_a^c = \{\delta\mathbf{u}_1^c, \delta\mathbf{u}_2^c\},$$
$$\delta\mathbf{u}_d = \left\{\delta\theta_1, \delta\theta_{,x1}, \delta\theta_2, \delta\theta_{,x2}\right\}, \tag{36}$$

where $\delta\theta_j$ ($j = 1, 2$) are the variation of θ_j.

From Eqs. (35) and (36), one may obtain

$$\delta\mathbf{q} = \mathbf{T}_c \delta\mathbf{q}_c, \tag{37}$$

$$\mathbf{T}_c = \begin{bmatrix} 1 & 0 & 0 & -\psi_0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -\psi_0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{38}$$

Let $\mathbf{k}_c$ and $\mathbf{k}$ denote the element stiffness matrices corresponding to $\delta\mathbf{q}_c$ and $\delta\mathbf{q}$, respectively. The relation between $\mathbf{k}_c$ and $\mathbf{k}$ may be expressed by

$$\mathbf{k}_c = \mathbf{T}_c^t \mathbf{k} \mathbf{T}_c, \tag{39}$$

$$\mathbf{k} = \begin{bmatrix} \mathbf{k}_{aa} & \mathbf{k}_{ad} \\ \mathbf{k}_{ad}^t & \mathbf{k}_{dd} \end{bmatrix}, \tag{40}$$

$$\mathbf{k}_{aa} = \frac{AE}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \mathbf{k}_{ad} = \mathbf{0}_{2\times 4},$$

$$\mathbf{k}_{dd} = GJ \int \mathbf{N}_d' \mathbf{N}_d'^t \,\mathrm{d}x + EI_\omega \int \mathbf{N}_d'' \mathbf{N}_d''^t \,\mathrm{d}x,$$

where $\mathbf{N}_d' = \mathrm{d}N_d/\mathrm{d}x$, $\mathbf{N}_d'' = \mathrm{d}^2 N_d/\mathrm{d}x^2$, $\mathbf{N}_d$ is defined in Eq. (34), the range of integration for the integral $\int(\,)\,\mathrm{d}x$ in Eq. (40) is from 0 to ℓ, A is the cross-section area.

Let $\mathbf{m}_c$ and $\mathbf{m}$ denote the element stiffness matrices corresponding to $\delta\mathbf{q}_c$ and $\delta\mathbf{q}$, respectively. The relation between $\mathbf{m}_c$ and $\mathbf{m}$ may be expressed by

$$\mathbf{m}_\mathrm{c} = \mathbf{T}_c^t \mathbf{m} \mathbf{T}_c, \tag{41}$$

$$\mathbf{m} = \begin{bmatrix} \mathbf{m}_{aa} & \mathbf{m}_{ad} \\ \mathbf{m}_{ad}^t & \mathbf{m}_{dd} \end{bmatrix}, \tag{42}$$

$$\mathbf{m}_{aa} = \rho A \int \mathbf{N}_a \mathbf{N}_a^t \,\mathrm{d}x, \quad \mathbf{m}_{ad} = \mathbf{0}_{2\times 4},$$

$$\mathbf{m}_{dd} = \rho(I_y + I_z) \int \mathbf{N}_d \mathbf{N}_d^t \,\mathrm{d}x + \rho I_\omega \int \mathbf{N}_d' \mathbf{N}_d'^t \,\mathrm{d}x,$$

where $\mathbf{N}_a$ and $\mathbf{N}_d$ are defined in Eq. (34), the range of integration for the integral $\int(\,)\,\mathrm{d}x$ in Eq. (42) is from 0 to ℓ, A is the cross-section area.

References

[1] Mei C. Coupled vibrations of thin-walled beams of open section using the finite element method. Int J Mech Sci 1970;12:883–91.

[2] Hallauer WL, Liu RYL. Beam bending–torsion dynamic stiffness method for calculation of exact vibration modes. J Sound Vib 1982;85(1):105–13.

[3] Friberg PO. Beam element matrices derived from Vlasov's theory of open thin-walled elastic beams. Int J Numer Methods Eng 1985;21:1205–28.

[4] Bishop RED, Cannon SM, Miao S. On coupled bending and torsional vibration of uniform beams. J Sound Vib 1989;131:457–64.

[5] Banerjee JR. Coupled bending–torsional dynamic stiffness matrix for beam elements. Int J Numer Methods Eng 1989;28:1283–98.

[6] Leung AYT. Natural shape functions of a compressed Vlasov element. Thin-wall Struct 1991;11:431–8.

[7] Banerjee JR, Williams FW. Clamped–clamped natural frequencies of a bending–torsion coupled beam. J Sound Vib 1994;176(3):301–6.

[8] Banerjee JR, Guo S, Howson WP. Exact dynamic stiffness matrix of a bending–torsion coupled beam including waring. Comput Struct 1996;59(4):613–21.

[9] Yaman Y. Vibrations of open-section channels: a coupled flexural and torsional. J Sound Vib 1997;204(1):131–58.

[10] Arpaci A, Bozdag E. On free vibration analysis of thin-walled beams with nonsymmetrical open cross-sections. Comput Struct 2002;80:691–5.

[11] Kim MY, Yun HT, Kim NI. Exact dynamic and static element stiffness matrices of nonsymmetric thin-walled beam-columns. Comput Struct 2003;81:1425–48.

[12] Arpaci A, Bozdag SE, Sunbuloglu E. Triply coupled vibrations of thin-walled open-section beams including rotary inertia effects. J Sound Vib 2003;260:889–900.

[13] Prokic A. On triply coupled vibrations of thin-walled beams with arbitrary cross-section. J Sound Vib 2005;279:723–37.

[14] Vlasov VZ. Thin walled elastic beams. English translation published for US Science Foundation by Israel Program for Scientific Translations, 1961.
[15] Chen HH, Lin WY, Hsiao KM. Co-rotational finite element formulation for thin-walled beams with generic open section. Comput Methods Appl Mech Eng 2006;195:2334–70.
[16] Lin WY, Hsiao KM. More general expression for the torsional warping of a thin-walled open-section beam. Int J Mech Sc 2000; 45:831–49.
[17] Chung TJ. Continuum mechanics. NJ: Prentice-Hall; 1988.
[18] Malvern LJ. Introduction to the Mechanics of the Continuous Medium. Englewood Cliffs, NJ: Prentice-Hall; 1969.