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Chaos in a modified van der Pol system and in its fractional order systems

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Abstract

Chaos in a modified van der Pol system and in its fractional order systems is studied in this paper. It is found that chaos exists both in the system and in the fractional order systems with order from 1.8 down to 0.8 much less than the number of states of the system, two. By phase portraits, Poincaré maps and bifurcation diagrams, the chaotic behaviors of fractional order modified van der Pol systems are presented.

1. Introduction

The topic of fractional calculus is enjoying growing interest not only among mathematicians, but also among physicists and engineers. In recent years, many scholars have devoted themselves to study the applications of the fractional order system to physics and engineering such as viscoelastic systems [1], dielectric polarization, and electromagnetic waves. More recently, there is a new trend to investigate the control [2] and dynamics [3–10] of the fractional order dynamical systems [11–13,5]. In [1] it has been shown that nonlinear chaotic systems can still behave chaotically when their models become fractional. In [11], chaos control was investigated for fractional chaotic systems by the ''backstepping'' method of nonlinear control design. In [12,13], it was found that chaos exists in a fractional order Chen system with order less than 3. Linear feedback control of chaos in this system was also studied. In [5], chaos synchronization of fractional order chaotic systems were studied. The existence and uniqueness of solutions of initial value problems for fractional order differential equations have been studied in the literature [14–17]. In this paper, chaotic behaviors of a fractional order modified van der Pol system are studied by phase portraits [18–23], Poincaré maps [24–27] and bifurcation diagrams [28–37]. It is found that chaos exists in this system with order from 1.8 down to 0.8 much less than the number of states of the system. Linear transfer function approximations of the fractional integrator block are calculated for a set of fractional orders in [0.1, 0.9] based on frequency domain arguments [38].

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This paper is organized as follows. In Section 2, the fractional derivative and its approximation are introduced. In Section 3, a modified van der Pol system and the corresponding fractional order system are presented. In Section 4, numerical simulations are given. In Section 5, conclusions are drawn.

2. A fractional derivative and its approximation

There are several definitions of fractional derivatives. The commonly used definition for a general fractional derivative is the Riemann–Liouville definition [39], which is given by

$$\frac{\mathrm{d}^q f(t)}{\mathrm{d}t^q} = \frac{1}{\Gamma(n-q)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t \frac{f(\tau)}{(t-\tau)^{q-n+1}} \,\mathrm{d}\tau \tag{1}$$

where $\Gamma(\cdot)$ is the gamma function and n is an integer such that $n-1<q<n$. This definition is different from the usual intuitive definition of derivative. Fortunately, the basic engineering tool for analyzing linear systems, the Laplace transform, is still applicable and works as one would expect

$$L\left\{\frac{\mathrm{d}^q f(t)}{\mathrm{d}t^q}\right\} = s^q L\{f(t)\} - \sum_{k=0}^{n-1} s^k \left[\frac{\mathrm{d}^{q-1-k} f(t)}{\mathrm{d}t^{q-1-k}}\right]_{t=0}, \quad \text{for all } q, \tag{2}$$

where n is an integer such that $n-1<q<n$. Upon considering the initial conditions to be zero, this formula reduces to the more expected form

$$L\left\{\frac{\mathrm{d}^q f(t)}{\mathrm{d}t^q}\right\} = s^q L\{f(t)\} \tag{3}$$

An efficient method is to approximate fractional operators by using standard integer order operators. In [40–44], an effective algorithm is developed to approximate fractional order transfer functions. Basically the idea is to approximate the system behavior based on frequency domain arguments. By utilizing frequency domain techniques based on Bode diagrams, one can obtain a linear approximation of the fractional order integrator, the order of which depends on the desired bandwidth and discrepancy between the actual and the approximate magnitude Bode diagrams. In Table 1 of [38], approximations for $\frac{1}{s^q}$ with $q = 0.1$–0.9 in steps 0.1 are given, with errors of approximately 2 dB. These approximations are used in the following simulations.

3. A modified van der Pol system and the corresponding fractional order system

Firstly, a van der Pol [45–47] oscillator driven by a periodic force is considered. The equation of motion can be written as

$$\ddot{x} + \varphi x + a\dot{x}(x^2-1) - b\sin\omega t = 0 \tag{4}$$

In Eq. (4), the linear term stands for a conservative harmonic force which determines the intrinsic oscillation frequency. The self-sustaining mechanism which is responsible for the perpetual oscillation rests on the nonlinear term. Energy exchange with the external agent depends on the magnitude of displacement $|x|$ and on the sign of velocity $\dot{x}$. During a complete cycle of oscillation, the energy is dissipated if displacement $x(t)$ is large than one, and that energy is fed-in if $|x|<1$. The time-dependent term stands for the external driving force with amplitude b and frequency ω. Eq. (4) can be rewritten as two first order equations

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\varphi x + a(1-x^2)y + b\sin\omega t \end{cases} \tag{5}$$

The modified van der Pol system and its fractional order system studied in this paper are

$$\begin{cases} \dfrac{\mathrm{d}^\alpha x}{\mathrm{d}t^\alpha} = y \\ \dfrac{\mathrm{d}^\beta y}{\mathrm{d}t^\beta} = -x + a(1-x^2)y + bz \\ \dot{z} = w \\ \dot{w} = -cz - dz^3 \end{cases} \tag{6}$$

where α, β are integer numbers and fractional numbers, respectively. System (6) can be separated into two parts

$$\begin{cases} \frac{d^\alpha x}{dt^\alpha} = y \\ \frac{d^\beta y}{dt^\beta} = -x + a(1-x^2)y + bz \end{cases} \tag{7}$$

and

$$\begin{cases} \dot{z} = w \\ \dot{w} = -cz - dz^3 \end{cases} \tag{8}$$

In Eq. (5) changing the integral order derivatives to the fractional order derivations and replacing $\sin\omega t$ by z which is the periodic time function solution of the nonlinear oscillator (8), we obtain system (7). In Eq. (8) if $d = 0$, z is a sinusoidal function of time. Now $d \neq 0$, z is a periodic motion of time but not a sinusoidal function of time. As a result, system (7) can be considered as a nonautonomous system with two states, while system (6) consisting of Eq. (7) and Eq. (8) can be considered as an autonomous system with four states. When $\alpha = \beta = 1$, Eq. (6) is the modified van der Pol system.

4. Numerical simulations

In this section, phase portraits and bifurcation diagrams are studied for system (6) for $\alpha + \beta \leqslant 2$. Three parameters a, c, d are chosen as $a = 5$, $c = 0.01$, $d = 0.001$. A time step of 0.001 is used.

Nine cases are studied:

Case 1: Let $\alpha = \beta = 1$. Fig. 1 shows the bifurcation diagram of the 2 order system. It is shown that chaos exists when $b \in [0, 1.0]$. Fig. 2 is the phase portrait of chaotic motion with $b = 1.0$. Figs. 3–5 are phase portraits of periodic motions with $b = 1.1, 1.5, 3$, respectively.

Case 2: Let $\alpha = 0.9$, $\beta = 0.9$. Fig. 6 shows the bifurcation diagram of the 1.8 order system. It is shown that chaos exists when $b \in [0, 9.7]$. Fig. 7 is the phase portrait of chaotic motion with $b = 9.7$. Figs. 8–11 are phase portraits of periodic motions with $b = 9.8, 14, 23, 40$, respectively.

Case 3: Let $\alpha = 0.9$, $\beta = 0.8$. Fig. 12 shows the bifurcation diagram of the 1.7 order system. It is shown that chaos exists when $b \in [0, 9.8]$. Fig. 13 is the phase portrait of chaotic motion with $b = 9.8$. Figs. 14–17 are phase portraits of periodic motions with $b = 9.9, 12, 25, 30$, respectively.

Case 4: Let $\alpha = 0.8$, $\beta = 0.9$. Fig. 18 shows the bifurcation diagram of the 1.7 order system. It is shown that chaos exists when $b \in [0, 10.1]$. Fig. 19 is the phase portrait of chaotic motion with $b = 10.1$. Figs. 20–24 are phase portraits of periodic motions with $b = 10.2, 15, 23, 30, 45$, respectively.

Case 5: Let $\alpha = 0.8$, $\beta = 0.8$. Fig. 25 shows the bifurcation diagram of the 1.6 order system. It is shown that chaos exists when $b \in [0, 10.1]$. Fig. 26 is the phase portrait of chaotic motion with $b = 10.1$. Figs. 27–31 are phase portraits of periodic motions with $b = 10.2, 14.5, 20, 35, 40$, respectively.

Case 6: Let $\alpha = 0.7$, $\beta = 0.7$. Fig. 32 shows the bifurcation diagram of the 1.4 order system. It is shown that chaos exists when $b \in [0, 7.9]$. Fig. 33 is the phase portrait of chaotic motion with $b = 7.9$. Figs. 34–37 are phase portraits of periodic motions with $b = 8.0, 15, 35, 40$, respectively.

Case 7: Let $\alpha = 0.6$, $\beta = 0.6$. Fig. 38 shows the bifurcation diagram of the 1.2 order system. It is shown that chaos exists when $b \in [0, 6.0]$. Fig. 39 is the phase portrait of chaotic motion with $b = 6.0$. Figs. 40–44 are phase portraits of periodic motions with $b = 6.1, 6.5, 9.5, 20, 45$, respectively.

Case 8: Let $\alpha = 0.5$, $\beta = 0.5$. Fig. 45 shows the bifurcation diagram of the 1.0 order system. It is shown that chaos exists when $b \in [0, 4.2]$. Fig. 46 is the phase portrait of chaotic motion with $b = 2$. Figs. 47–49 are phase portraits of periodic motions with $b = 6.0, 10, 15$, respectively.

Case 9: Let $\alpha = 0.4$, $\beta = 0.4$. Fig. 50 shows the bifurcation diagram of the 0.8 order system. It is shown that chaos exists when $b \in [0, 1.8]$. Fig. 51 is the phase portrait of chaotic motion with $b = 0.7$. Figs. 51–53 are phase portraits of periodic motions with $b = 3.5, 6.0$ respectively. When we tried to reduce the total order to 0.6, the phase portraits become periodic motions, as shown in Fig. 54 and Fig. 55, for any b value.

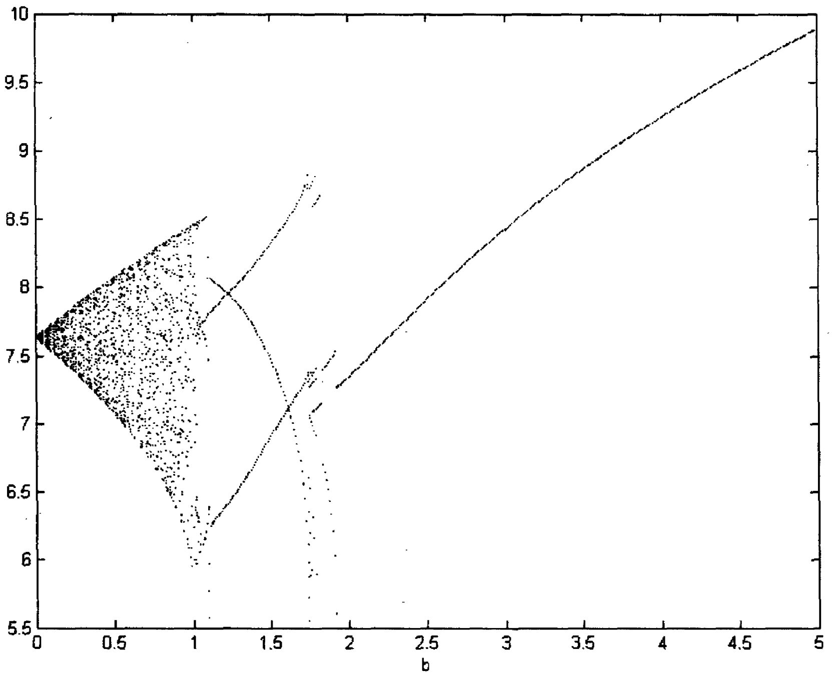

Fig. 1. The bifurcation diagram for $\alpha = \beta = 1$.

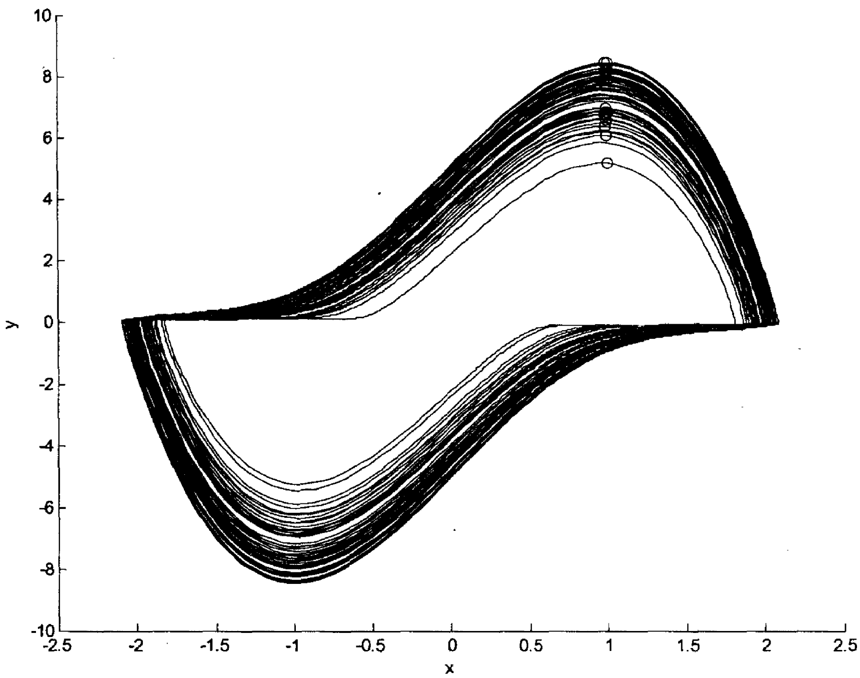

Fig. 2. The phase portrait for $\alpha = \beta = 1$, $b = 1.0$.

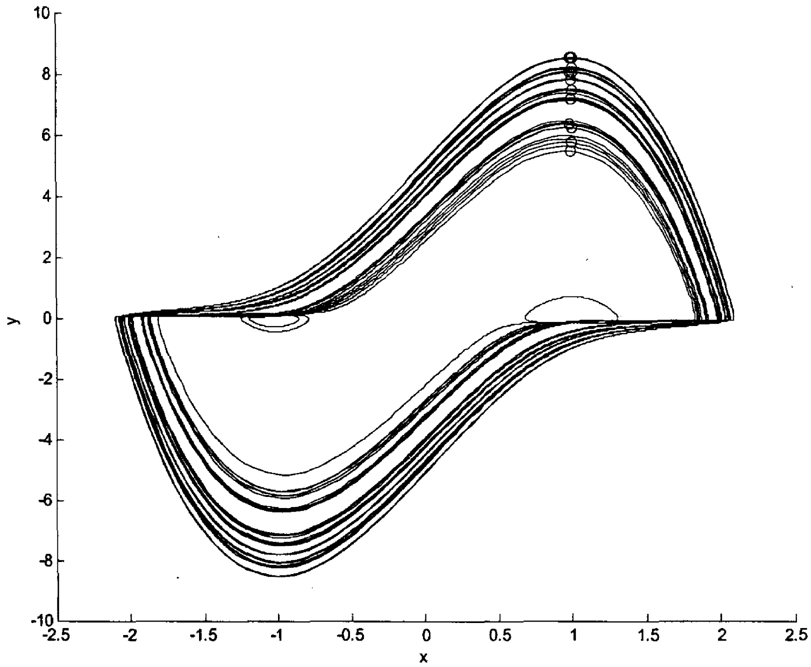

Fig. 3. The phase portrait for $\alpha = \beta = 1$, $b = 1.1$.

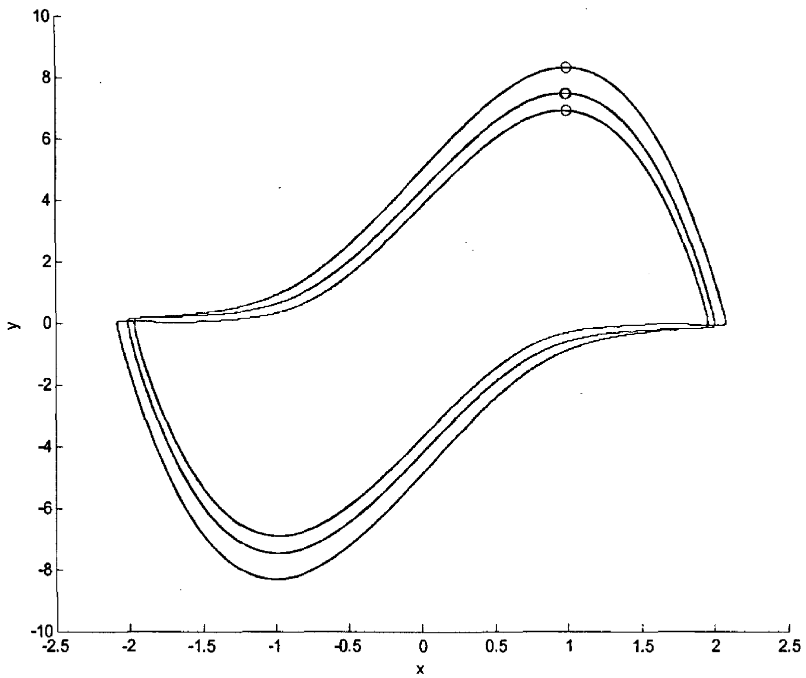

Fig. 4. The phase portrait for $\alpha = \beta = 1$, $b = 1.5$.

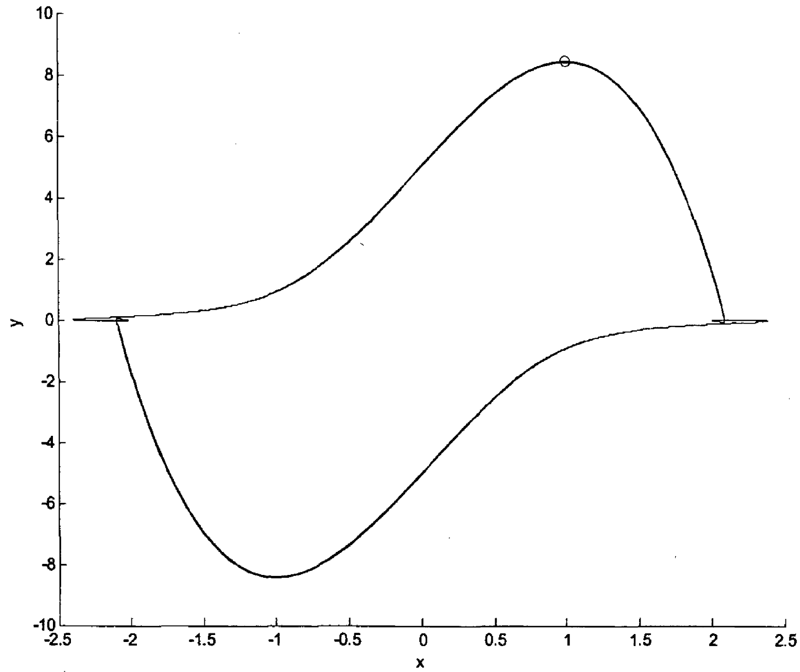

Fig. 5. The phase portrait for $\alpha = \beta = 1$, $b = 3$.

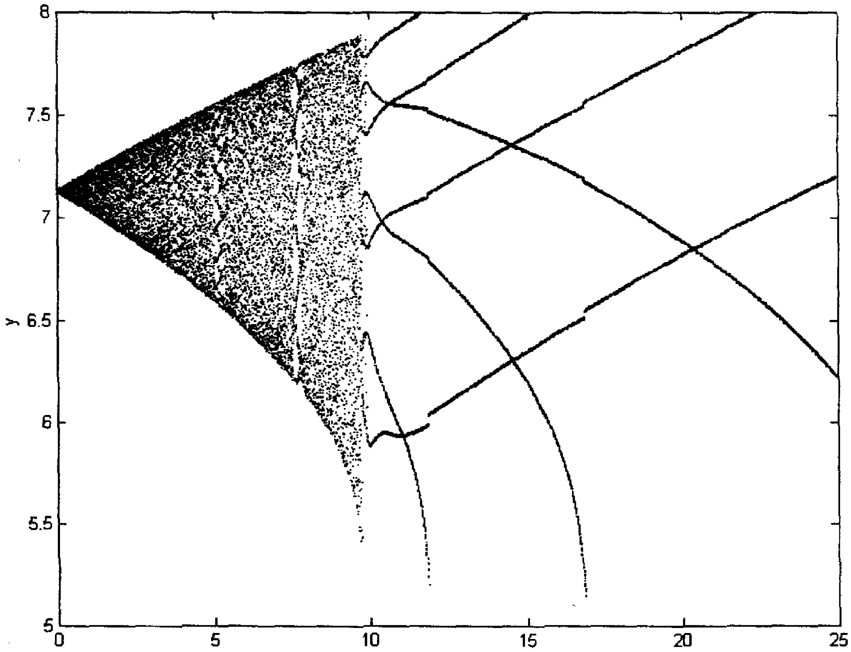

Fig. 6. The bifurcation diagram for $\alpha = 0.9$, $\beta = 0.9$.

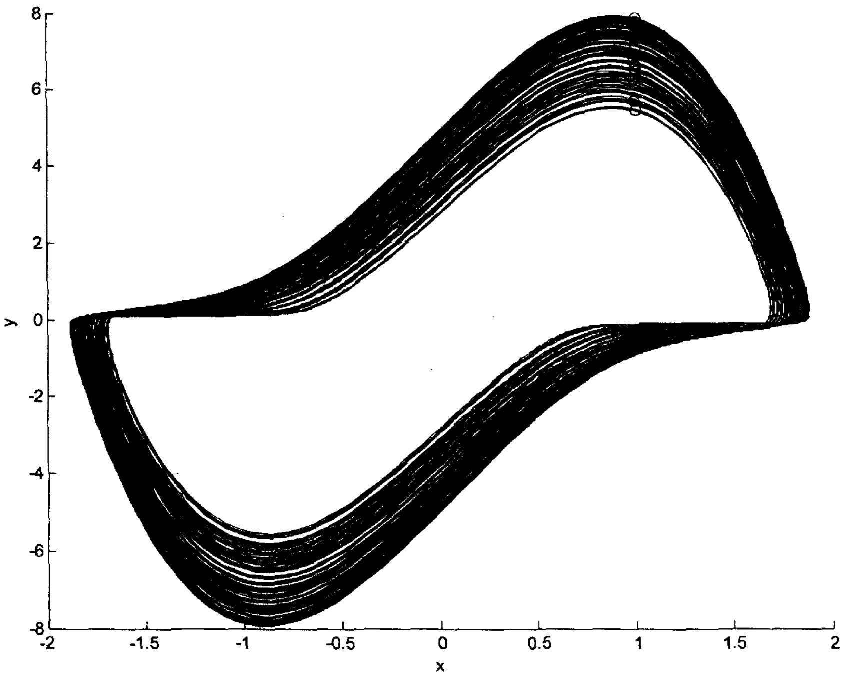

Fig. 7. The phase portrait for $\alpha = 0.9$, $\beta = 0.9$, $b = 9.7$.

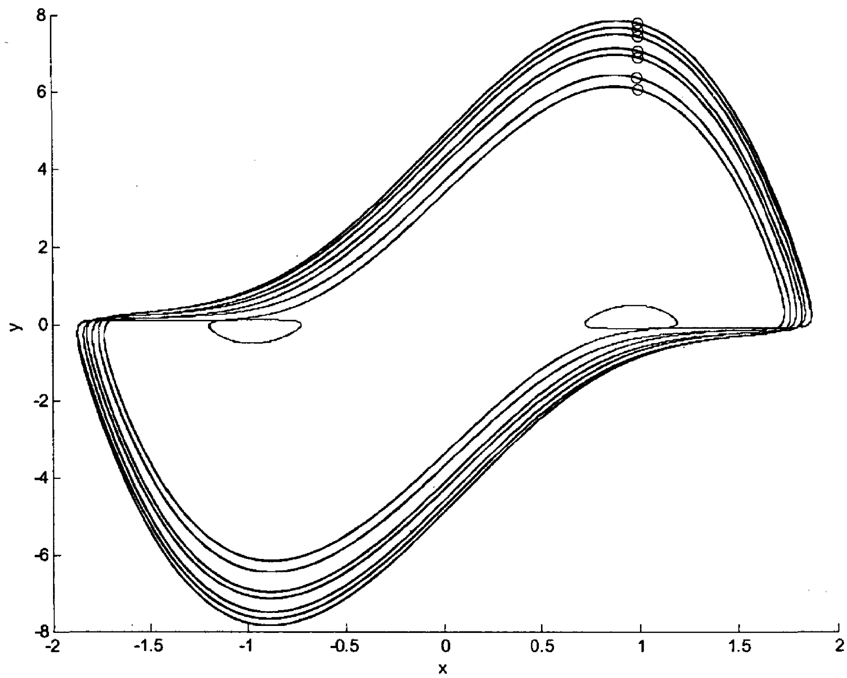

Fig. 8. The phase portrait for $\alpha = 0.9$, $\beta = 0.9$, $b = 9.8$.

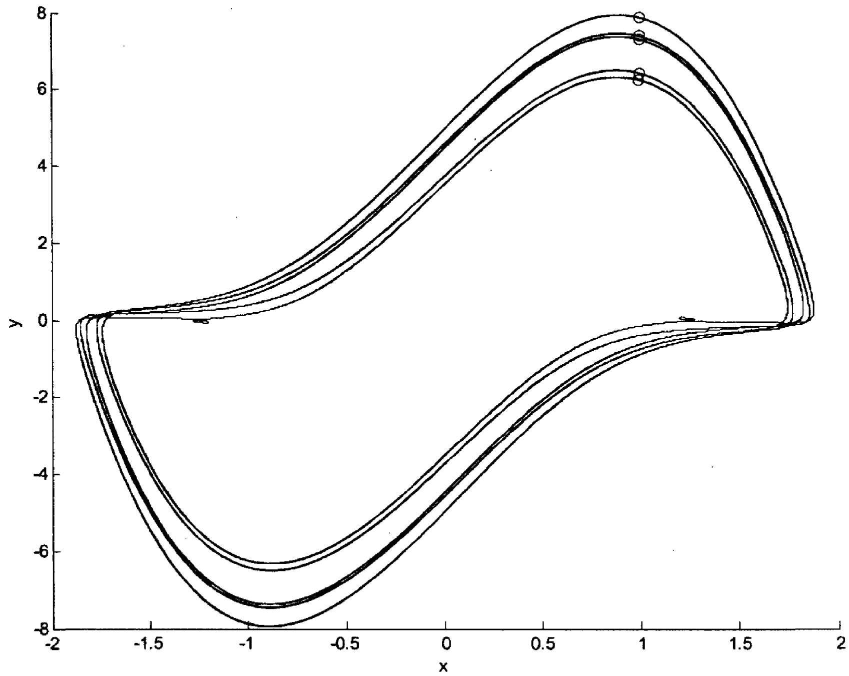

Fig. 9. The phase portrait for $\alpha = 0.9$, $\beta = 0.9$, $b = 14$.

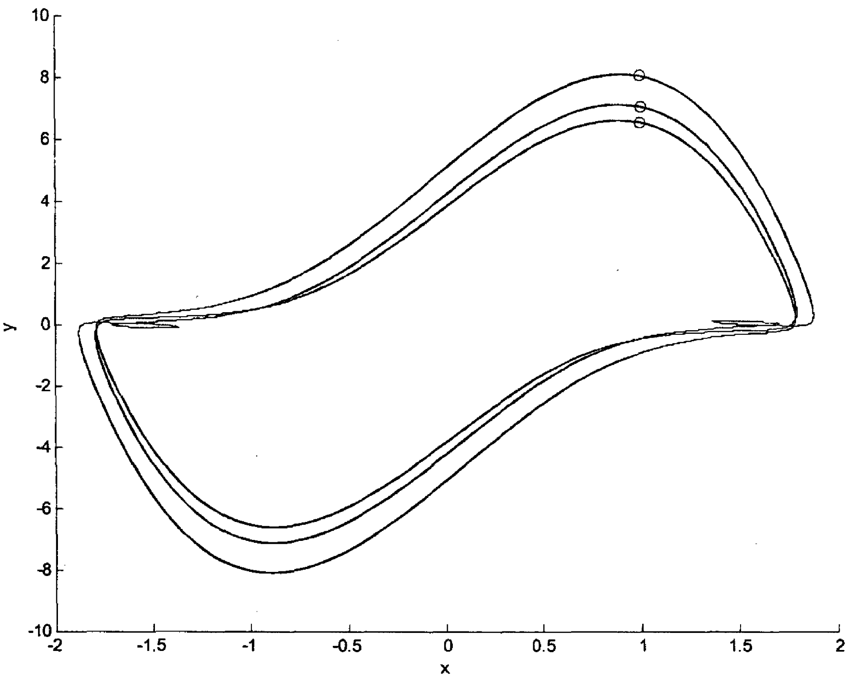

Fig. 10. The phase portrait for $\alpha = 0.9$, $\beta = 0.9$, $b = 23$.

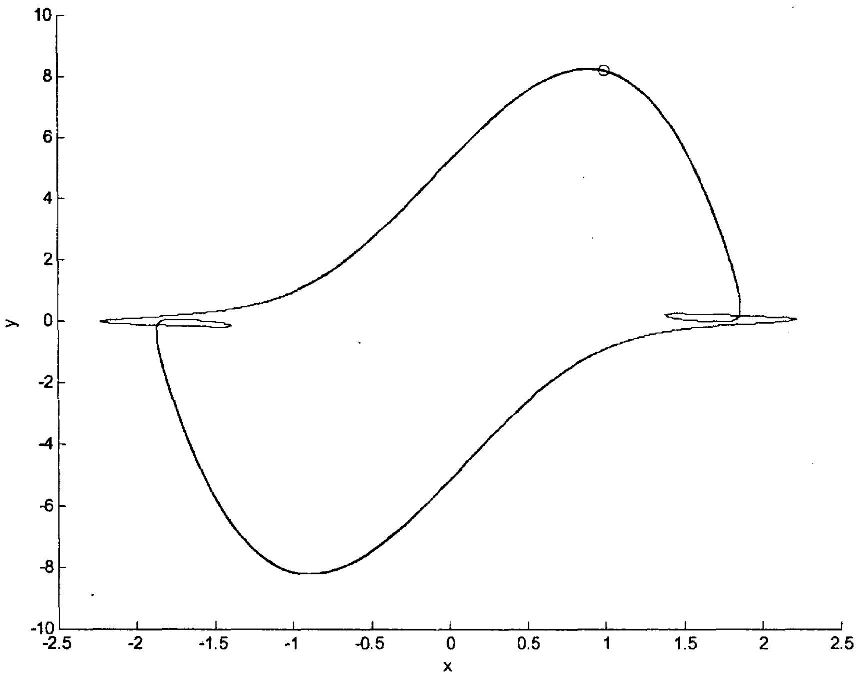

Fig. 11. The phase portrait for $\alpha = 0.9$, $\beta = 0.9$, $b = 40$.

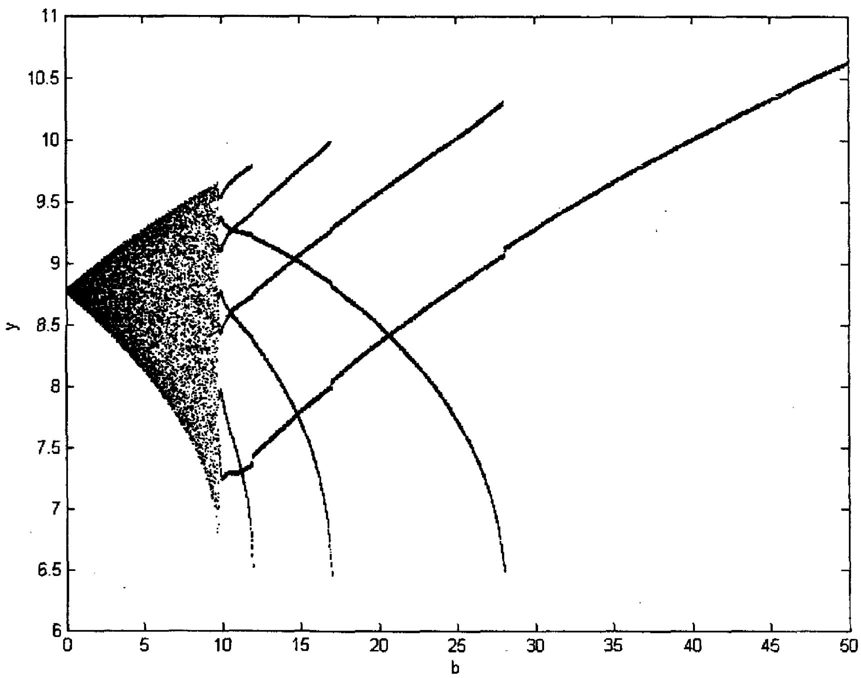

Fig. 12. The bifurcation diagram for $\alpha = 0.9$, $\beta = 0.8$.

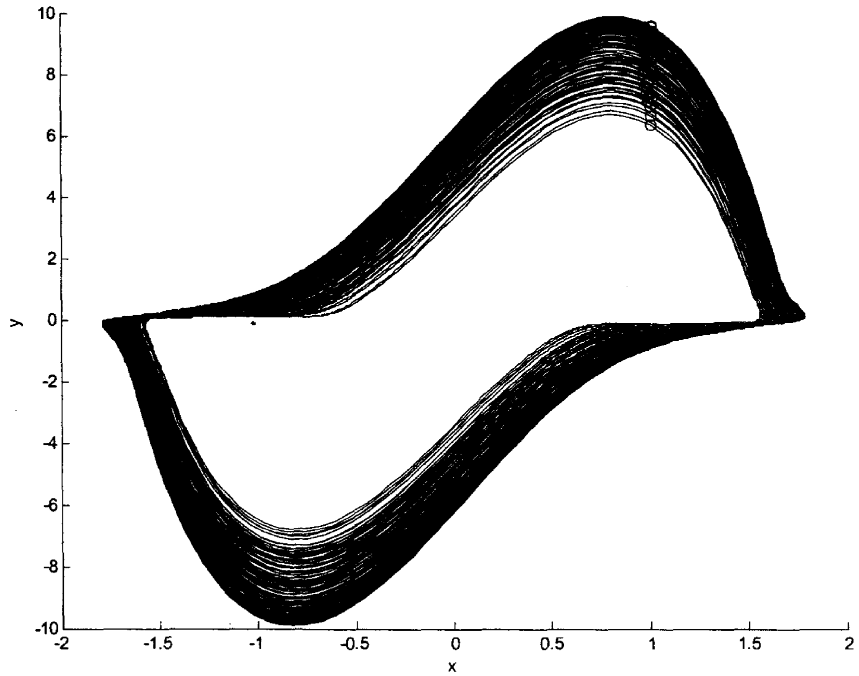

Fig. 13. The phase portrait for $\alpha = 0.9$, $\beta = 0.8$, $b = 9.8$.

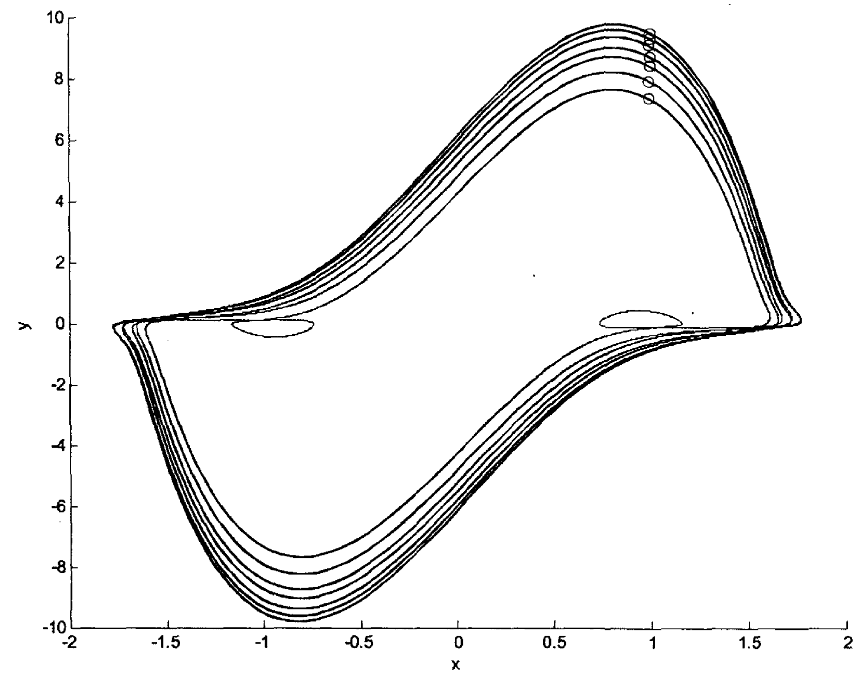

Fig. 14. The phase portrait for $\alpha = 0.9$, $\beta = 0.8$, $b = 9.9$.

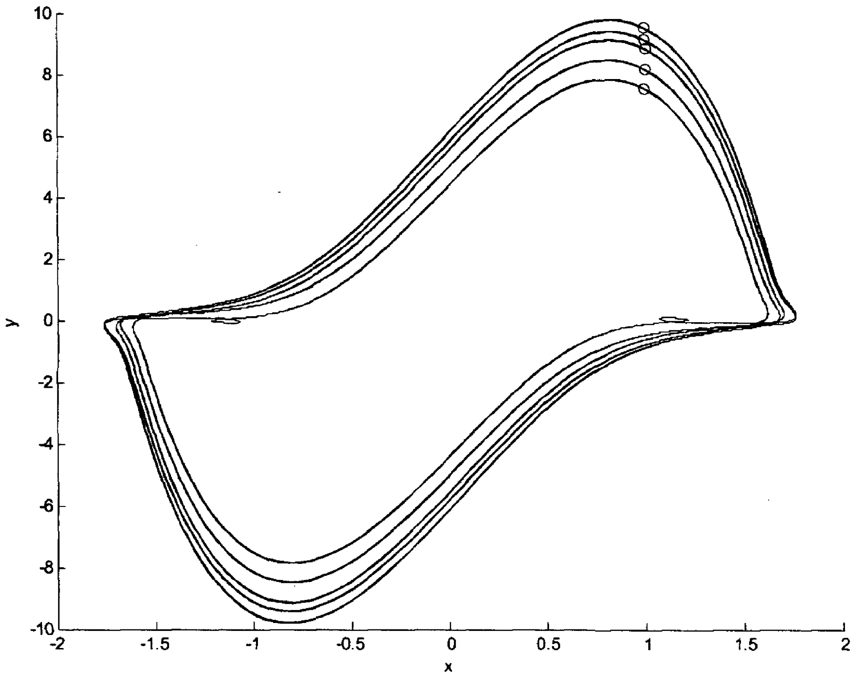

Fig. 15. The phase portrait for $\alpha = 0.9$, $\beta = 0.8$, $b = 12$.

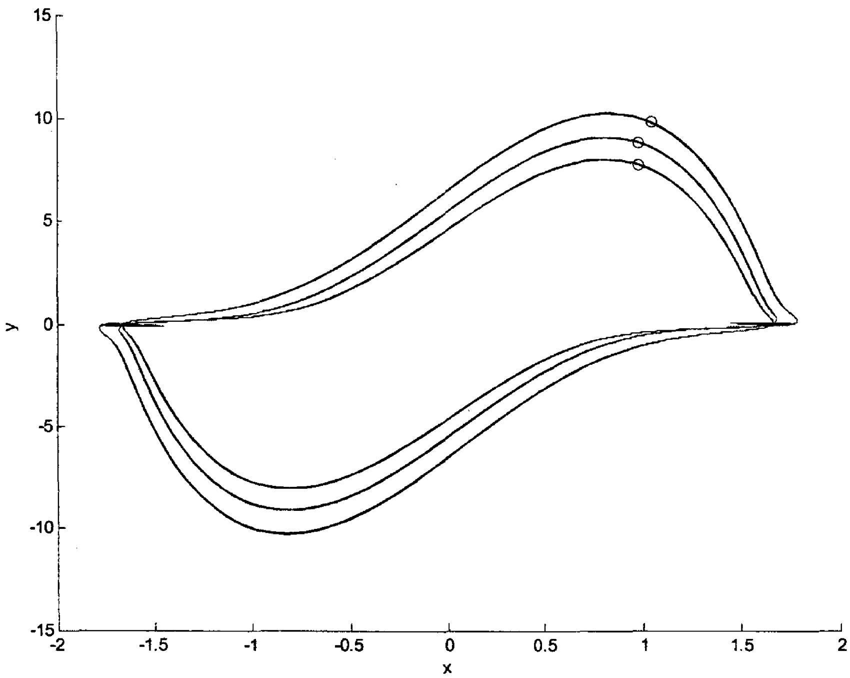

Fig. 16. The phase portrait for $\alpha = 0.9$, $\beta = 0.8$, $b = 25$.

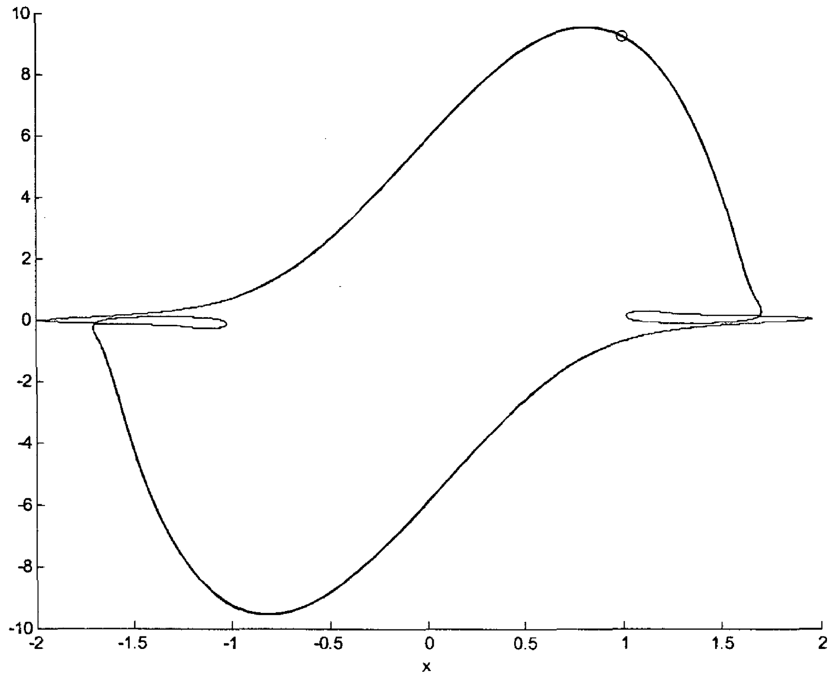

Fig. 17. The phase portrait for $\alpha = 0.9$, $\beta = 0.8$, $b = 30$.

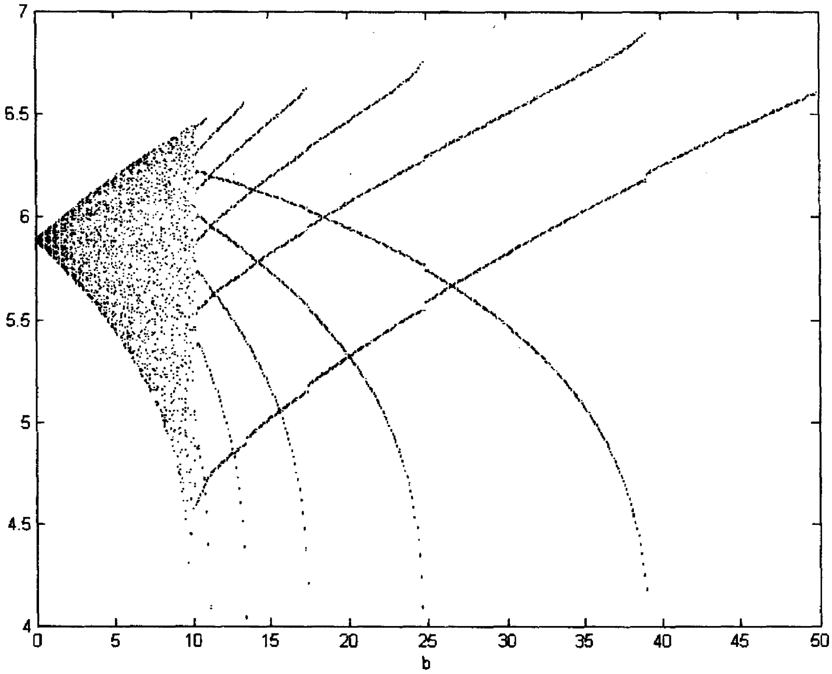

Fig. 18. The bifurcation diagram for $\alpha = 0.8$, $\beta = 0.9$.

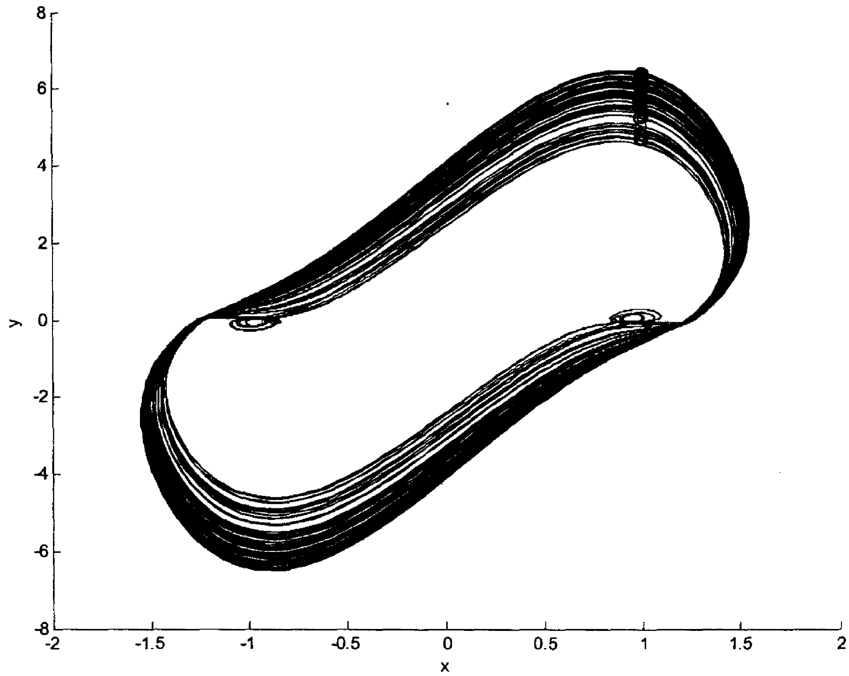

Fig. 19. The phase portrait for $\alpha = 0.8$, $\beta = 0.9$, $b = 10.1$.

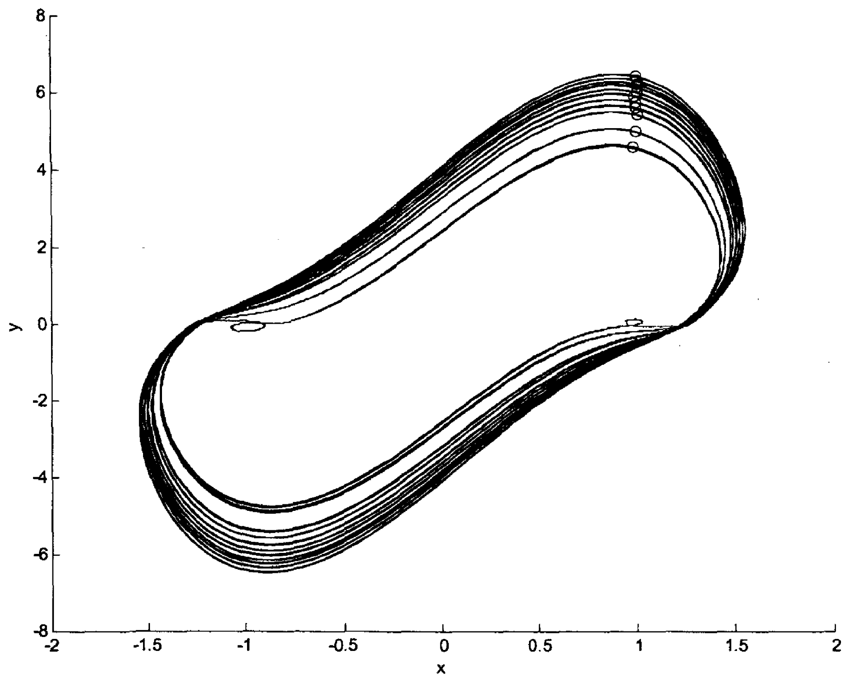

Fig. 20. The phase portrait for $\alpha = 0.8$, $\beta = 0.9$, $b = 10.2$.

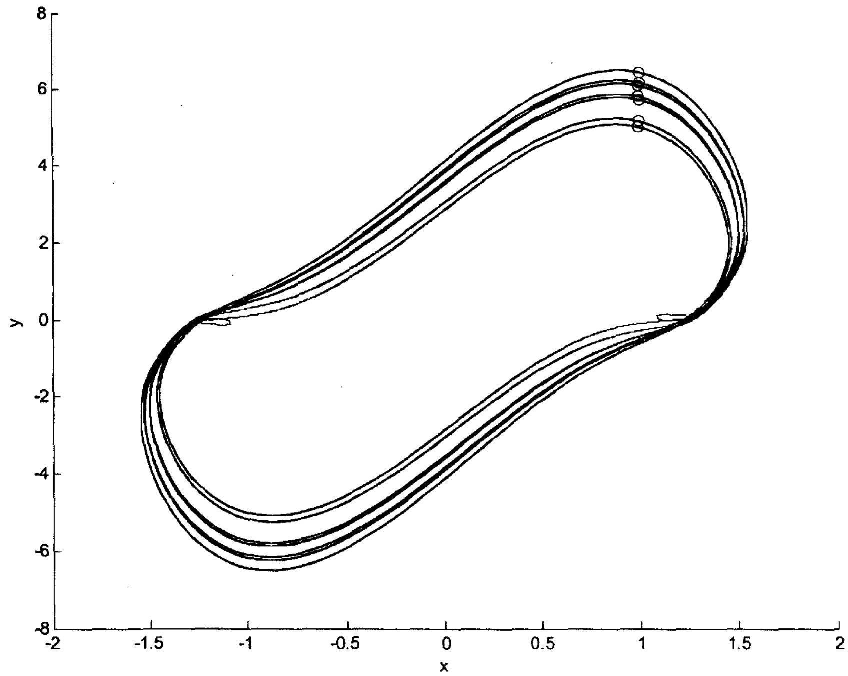

Fig. 21. The phase portrait for $\alpha = 0.8$, $\beta = 0.9$, $b = 15$.

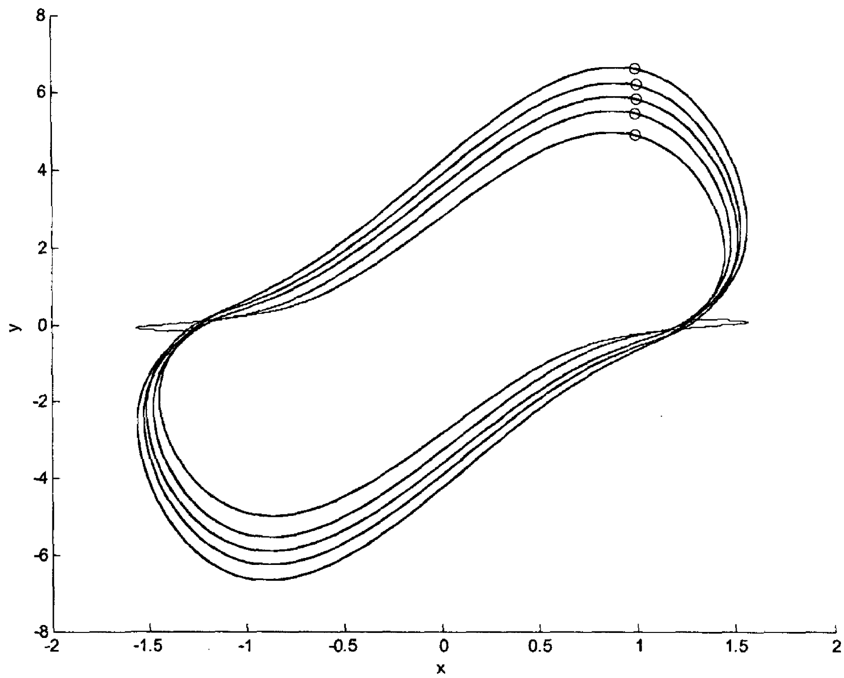

Fig. 22. The phase portrait for $\alpha = 0.8$, $\beta = 0.9$, $b = 23$.

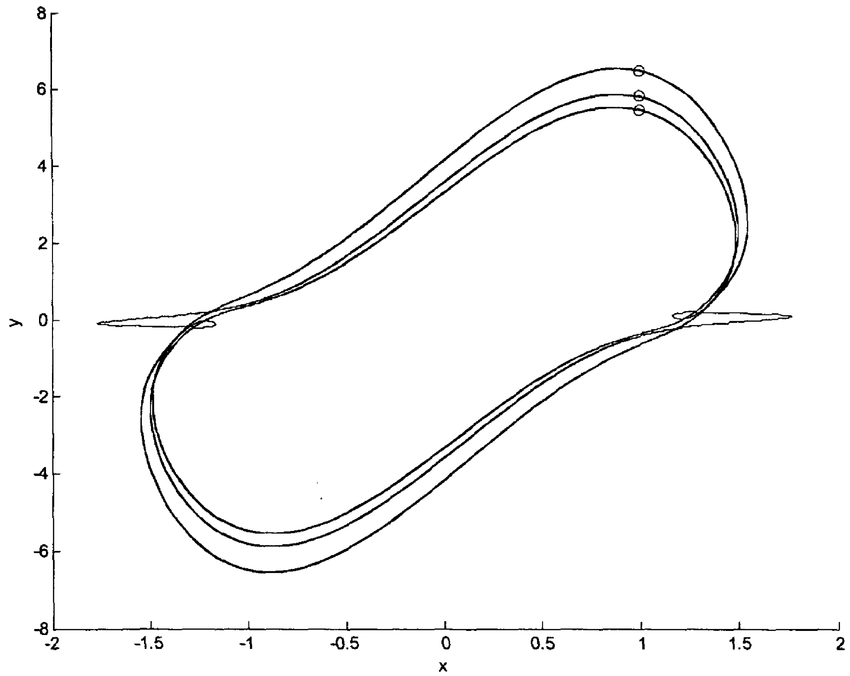

Fig. 23. The phase portrait for $\alpha = 0.8$, $\beta = 0.9$, $b = 30$.

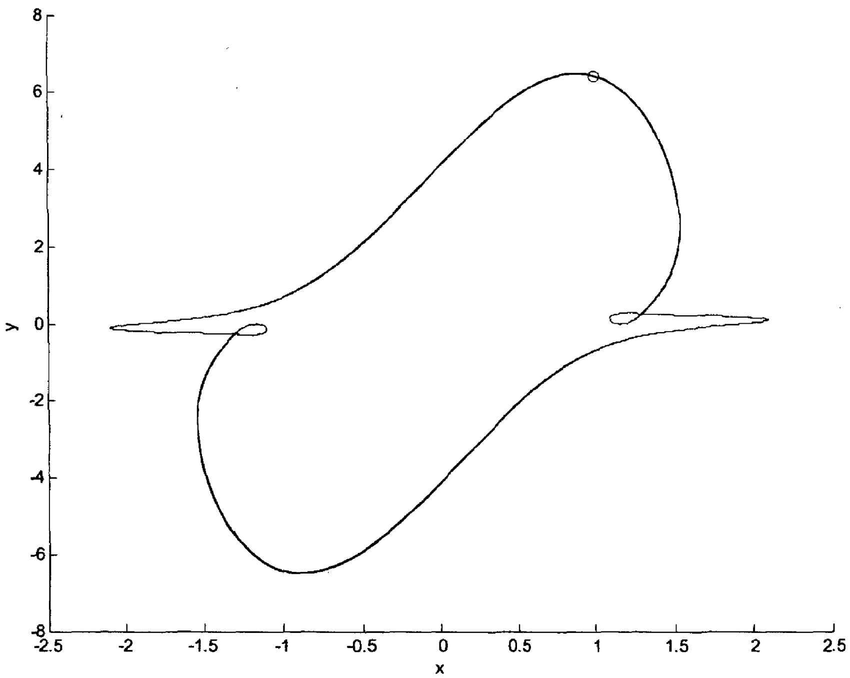

Fig. 24. The phase portrait for $\alpha = 0.8$, $\beta = 0.9$, $b = 45$.

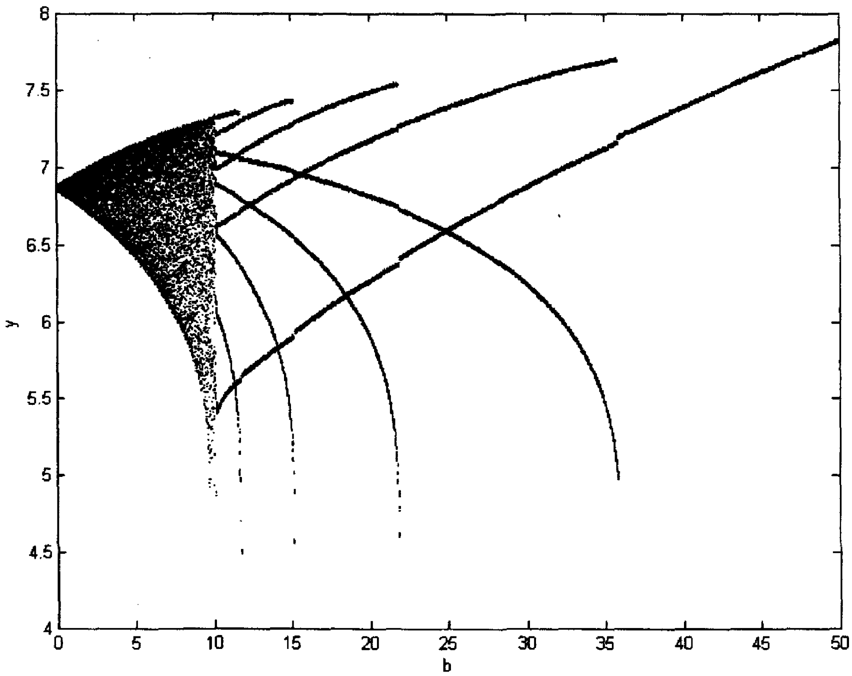

Fig. 25. The bifurcation diagram for $\alpha = 0.8$, $\beta = 0.8$.

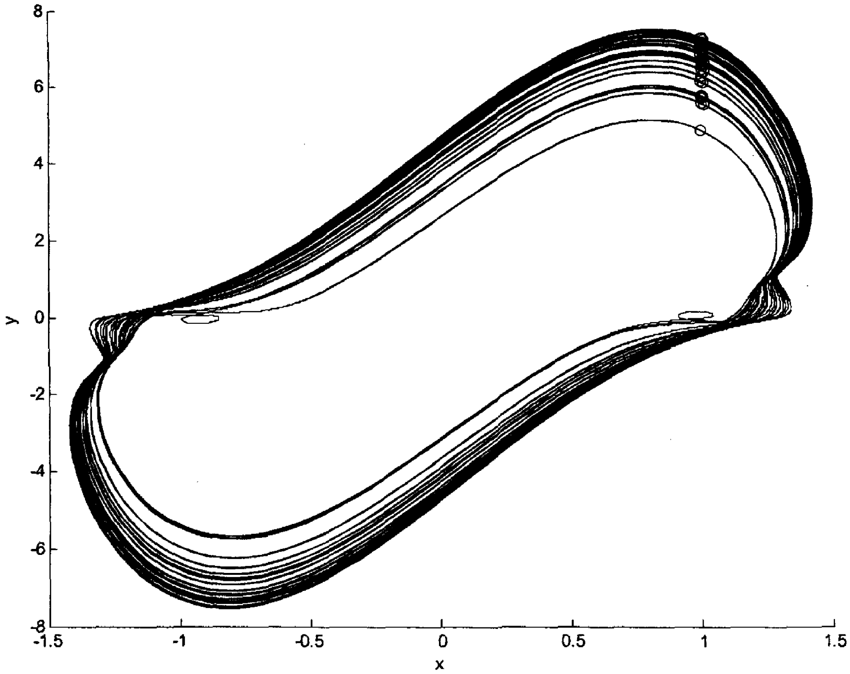

Fig. 26. The phase portrait for $\alpha = 0.8$, $\beta = 0.8$, $b = 10.1$.

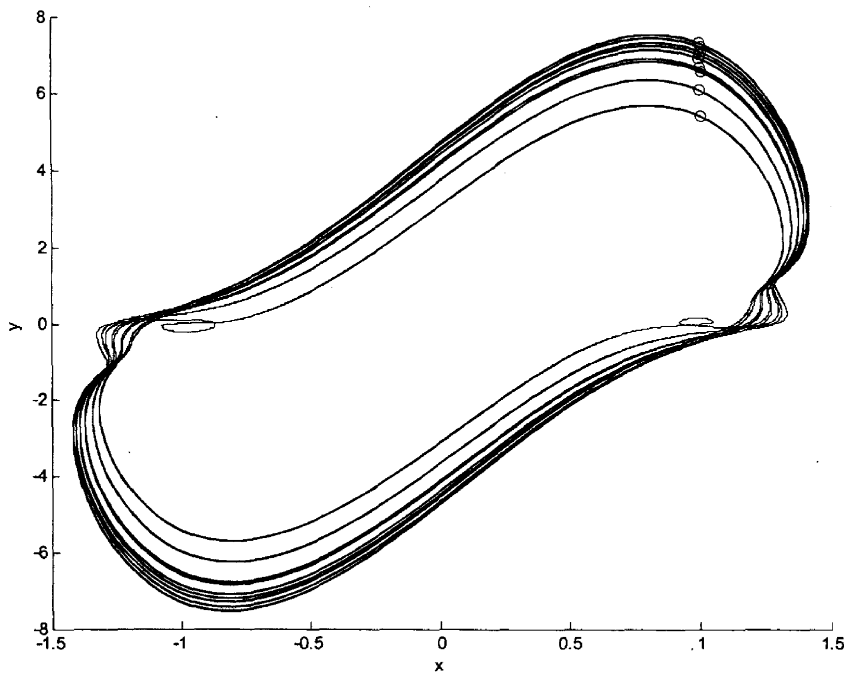

Fig. 27. The phase portrait for $\alpha = 0.8$, $\beta = 0.8$, $b = 10.2$.

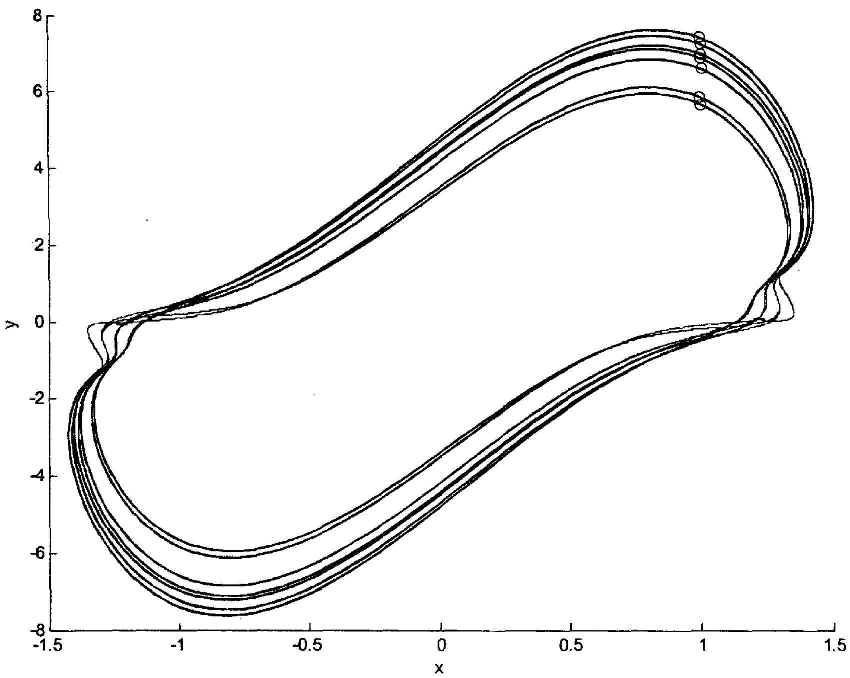

Fig. 28. The phase portrait for $\alpha = 0.8$, $\beta = 0.8$, $b = 14.5$.

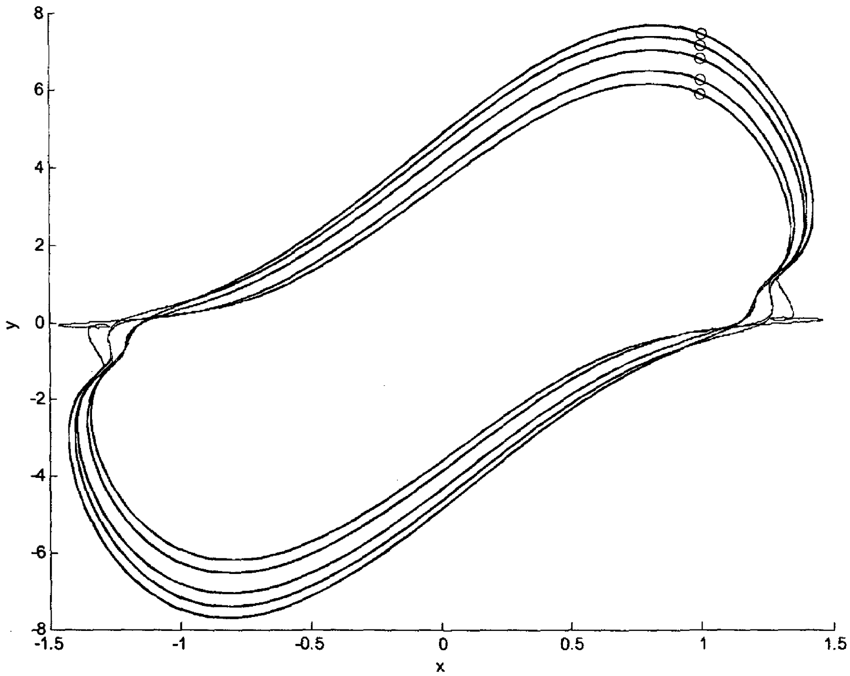

Fig. 29. The phase portrait for $\alpha = 0.8$, $\beta = 0.8$, $b = 20$.

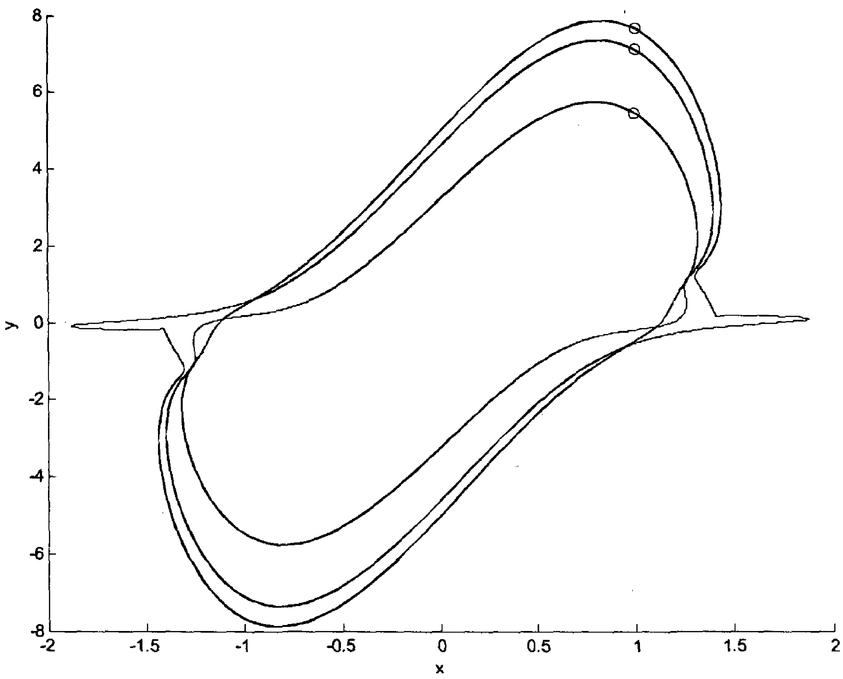

Fig. 30. The phase portrait for $\alpha = 0.8$, $\beta = 0.8$, $b = 35$.

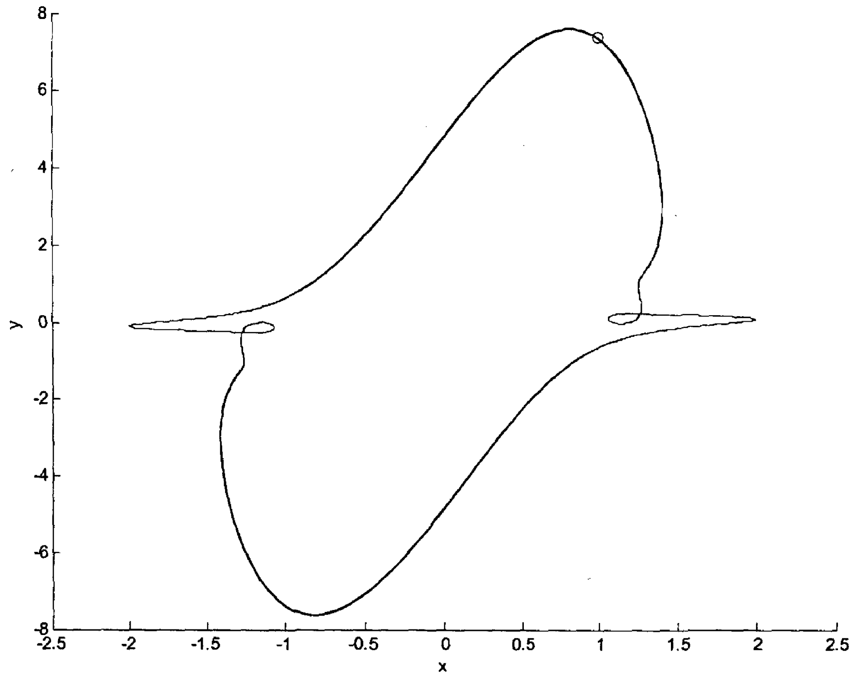

Fig. 31. The phase portrait for $\alpha = 0.8$, $\beta = 0.8$, $b = 40$.

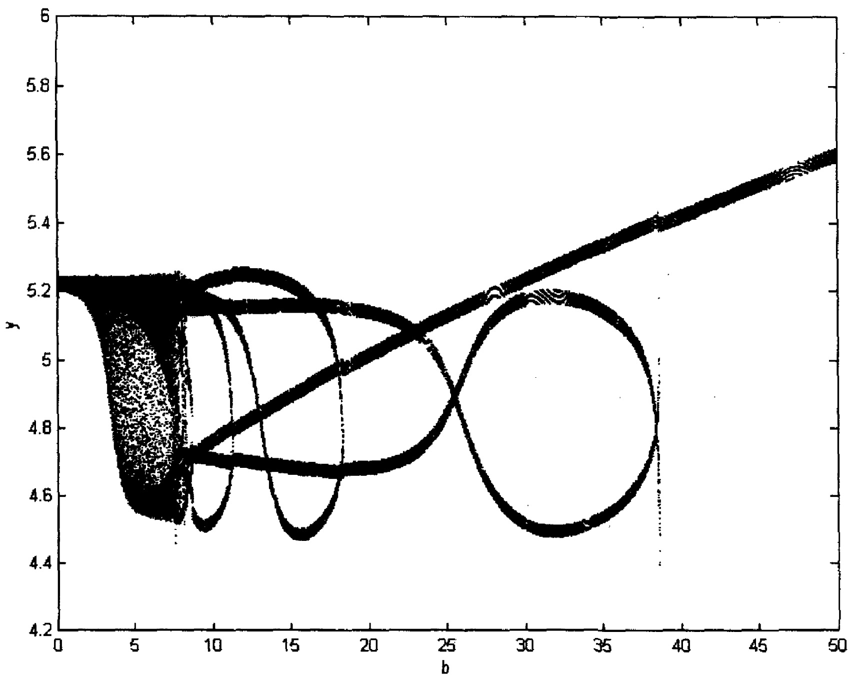

Fig. 32. The bifurcation diagram for $\alpha = 0.7$, $\beta = 0.7$.

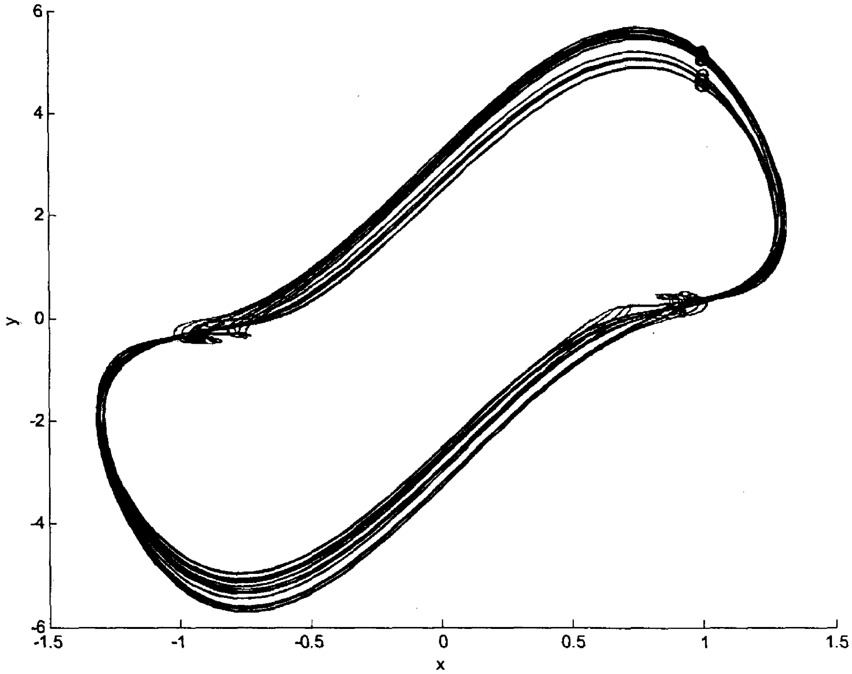

Fig. 33. The phase portrait for $\alpha = 0.7$, $\beta = 0.7$, $b = 7.9$.

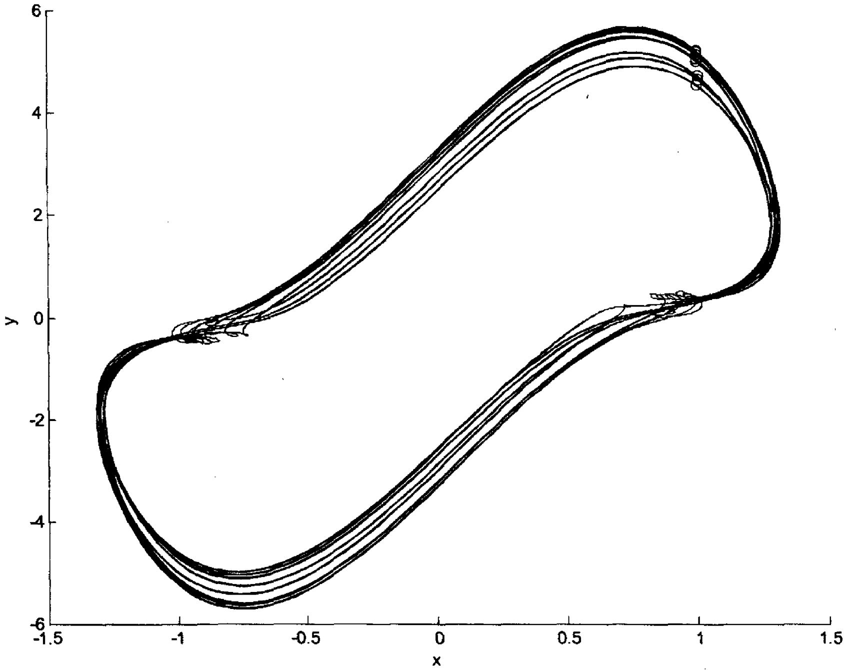

Fig. 34. The phase portrait for $\alpha = 0.7$, $\beta = 0.7$, $b = 8.0$.

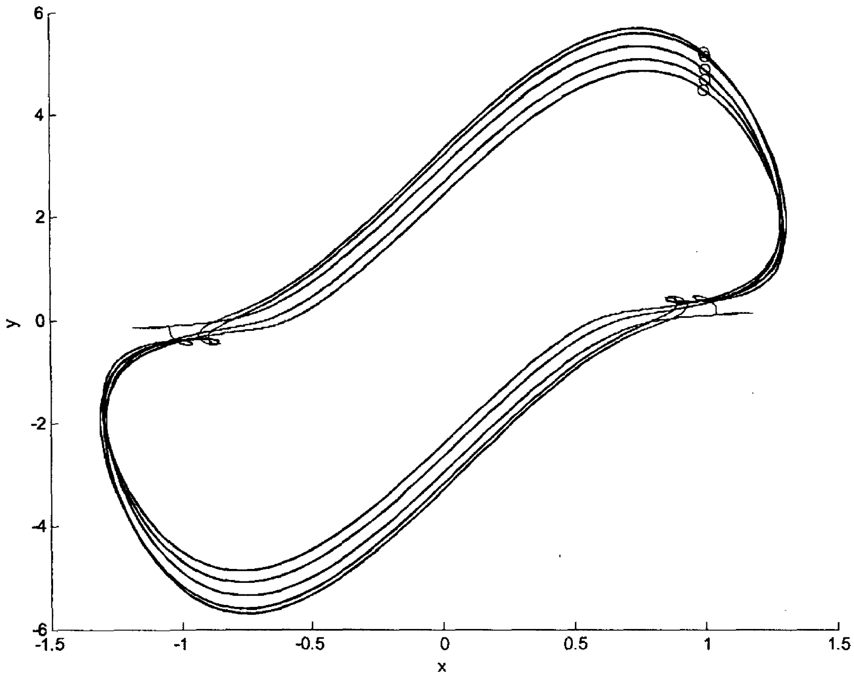

Fig. 35. The phase portrait for $\alpha = 0.7$, $\beta = 0.7$, $b = 15$.

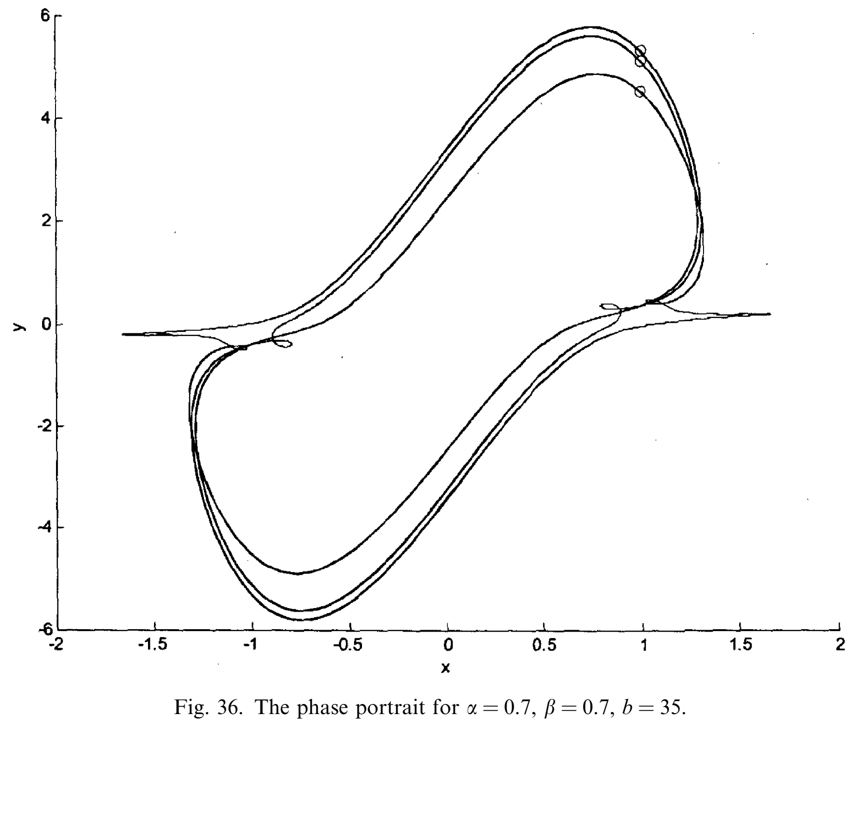

Fig. 36. The phase portrait for $\alpha = 0.7$, $\beta = 0.7$, $b = 35$.

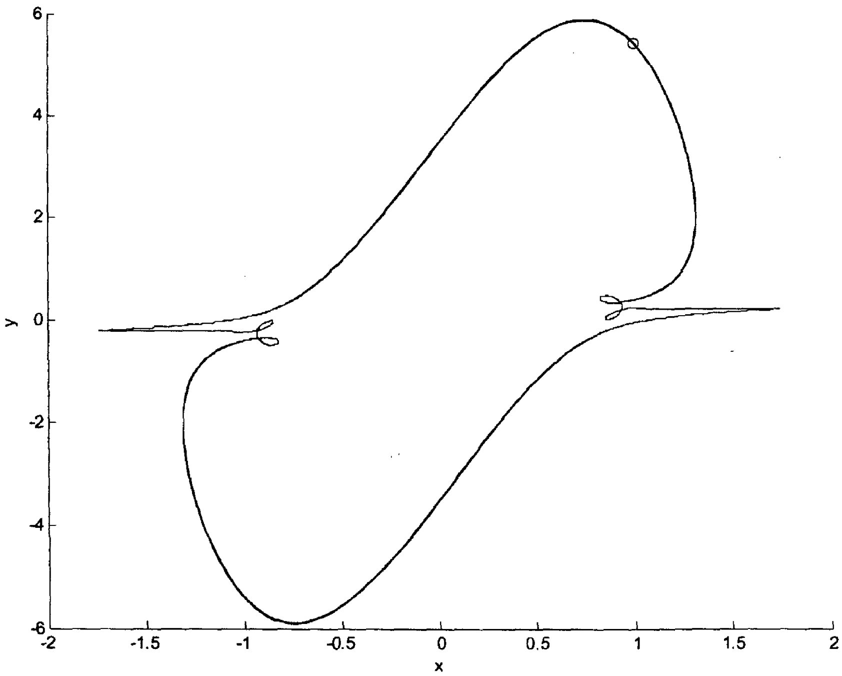

Fig. 37. The phase portrait for $\alpha = 0.7$, $\beta = 0.7$, $b = 40$.

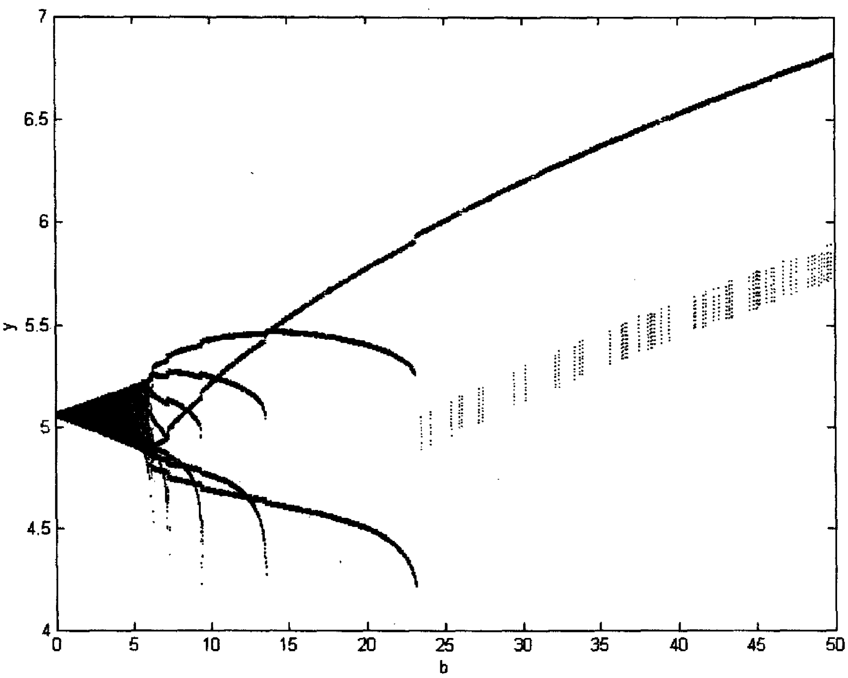

Fig. 38. The bifurcation diagram for $\alpha = 0.6$, $\beta = 0.6$.

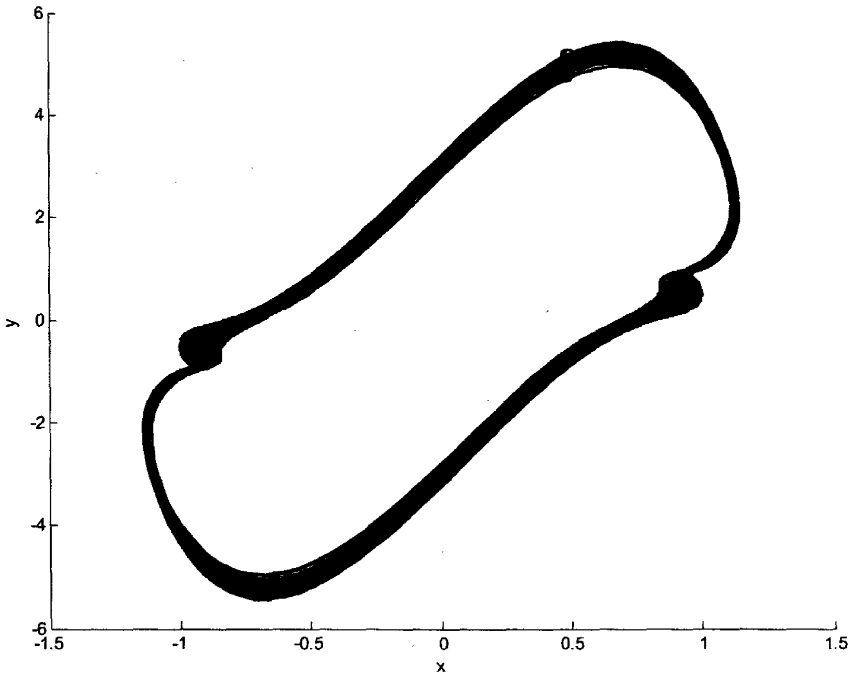

Fig. 39. The phase portrait for $\alpha = 0.6$, $\beta = 0.6$, $b = 6.0$.

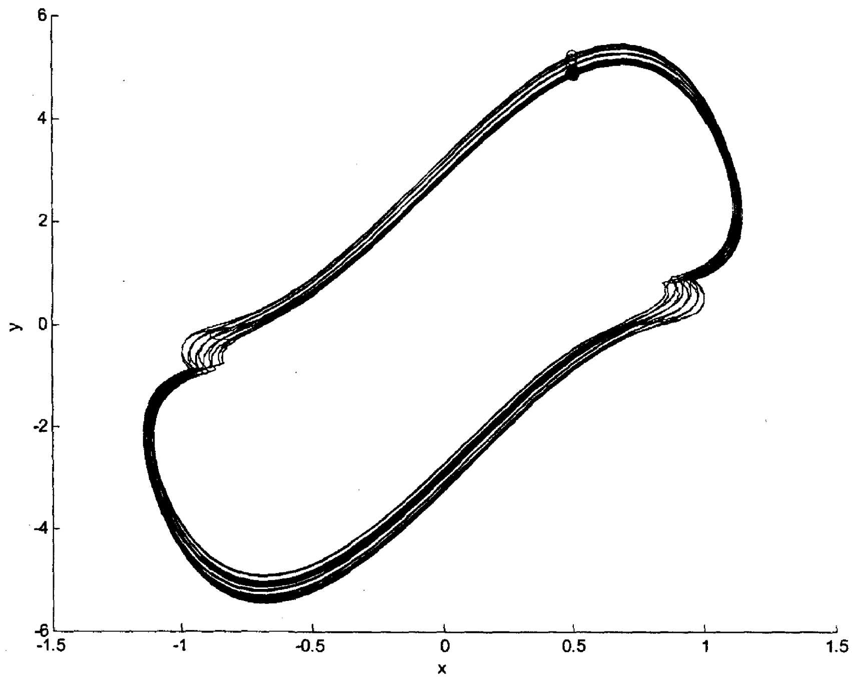

Fig. 40. The phase portrait for $\alpha = 0.6$, $\beta = 0.6$, $b = 6.1$.

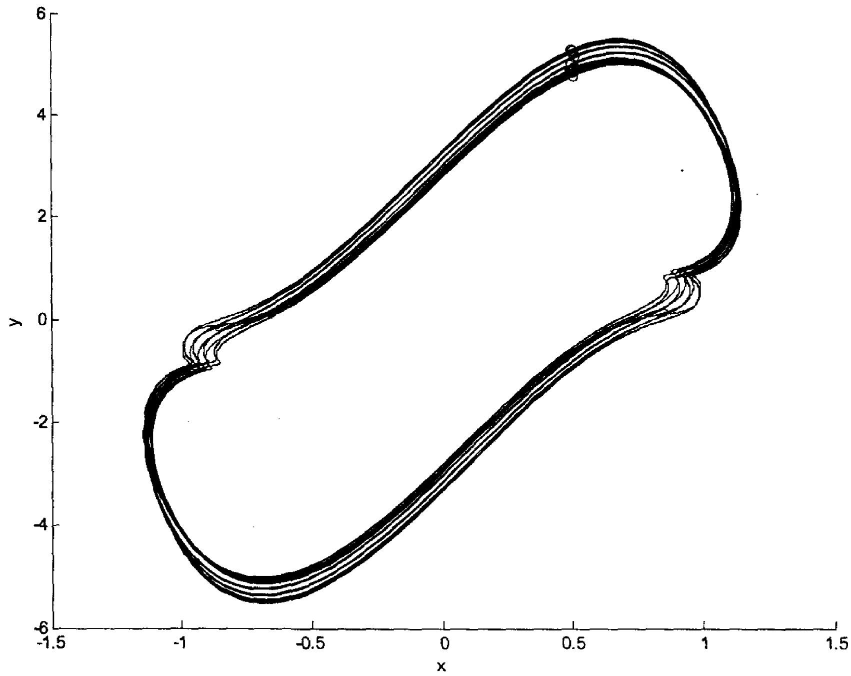

Fig. 41. The phase portrait for $\alpha = 0.6$, $\beta = 0.6$, $b = 6.5$.

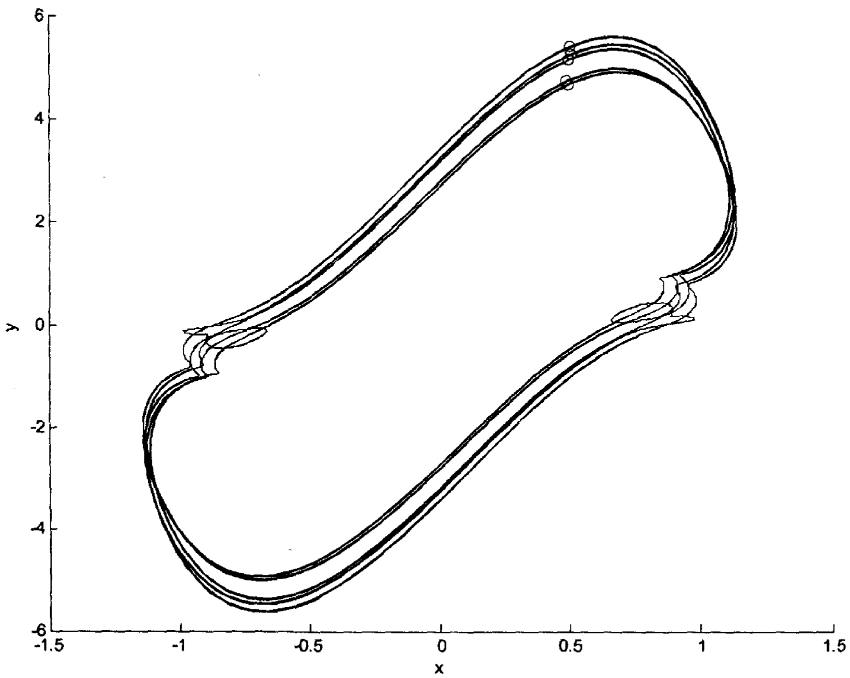

Fig. 42. The phase portrait for $\alpha = 0.6$, $\beta = 0.6$, $b = 9.5$.

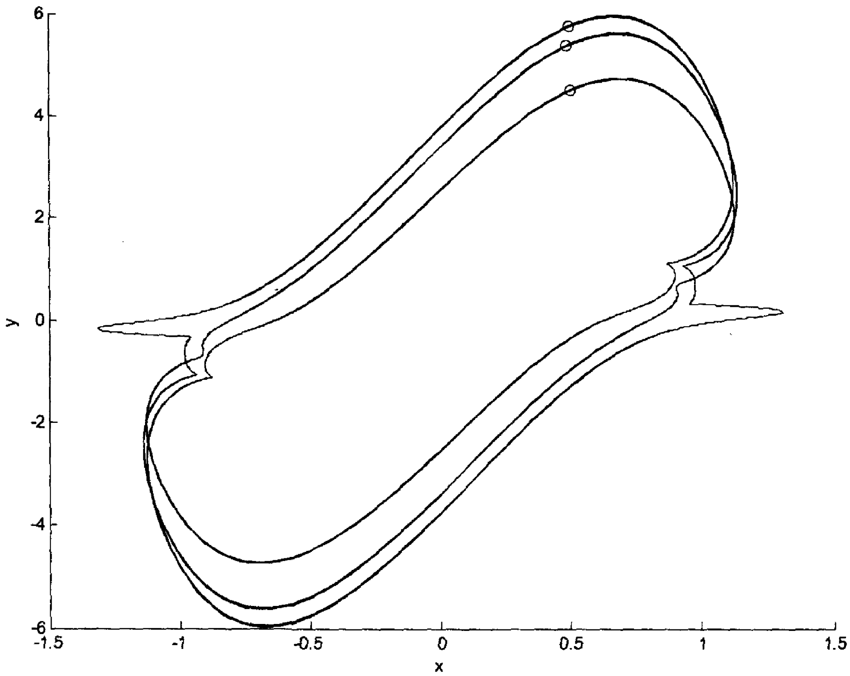

Fig. 43. The phase portrait for $\alpha = 0.6$, $\beta = 0.6$, $b = 20$.

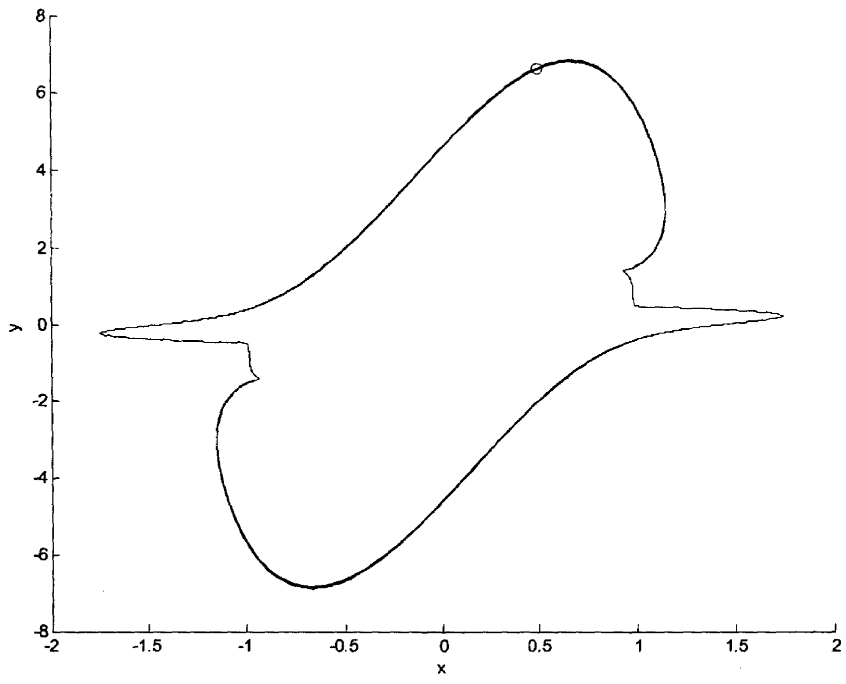

Fig. 44. The phase portrait for $\alpha = 0.6$, $\beta = 0.6$, $b = 45$.

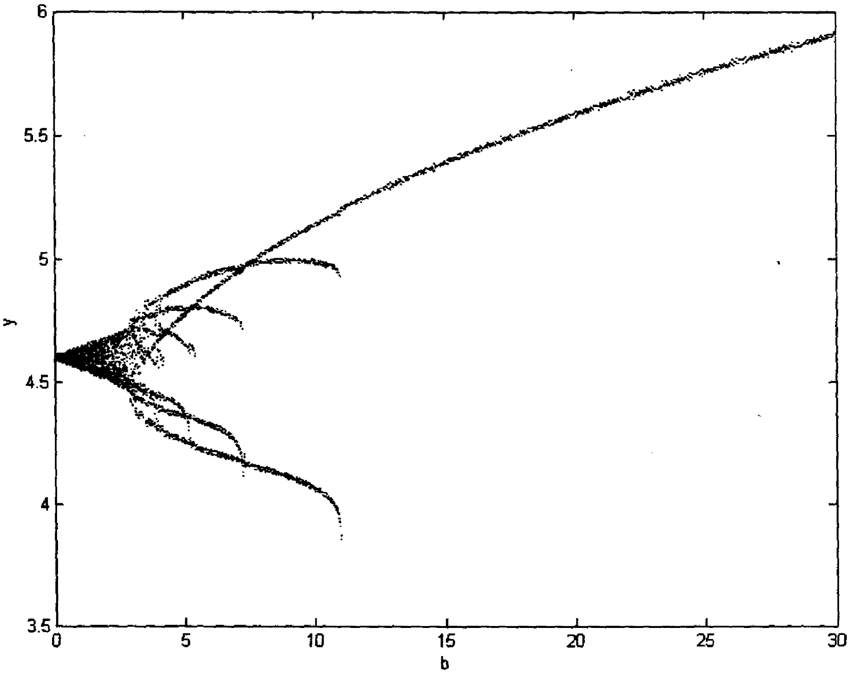

Fig. 45. The bifurcation diagram for $\alpha = 0.5$, $\beta = 0.5$.

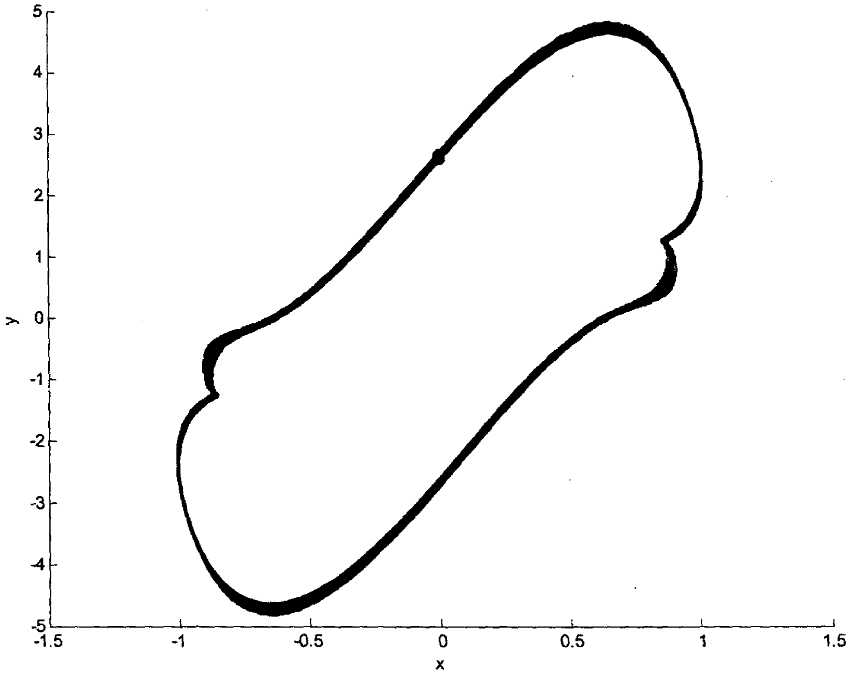

Fig. 46. The phase portrait for $\alpha = 0.5$, $\beta = 0.5$, $b = 2$.

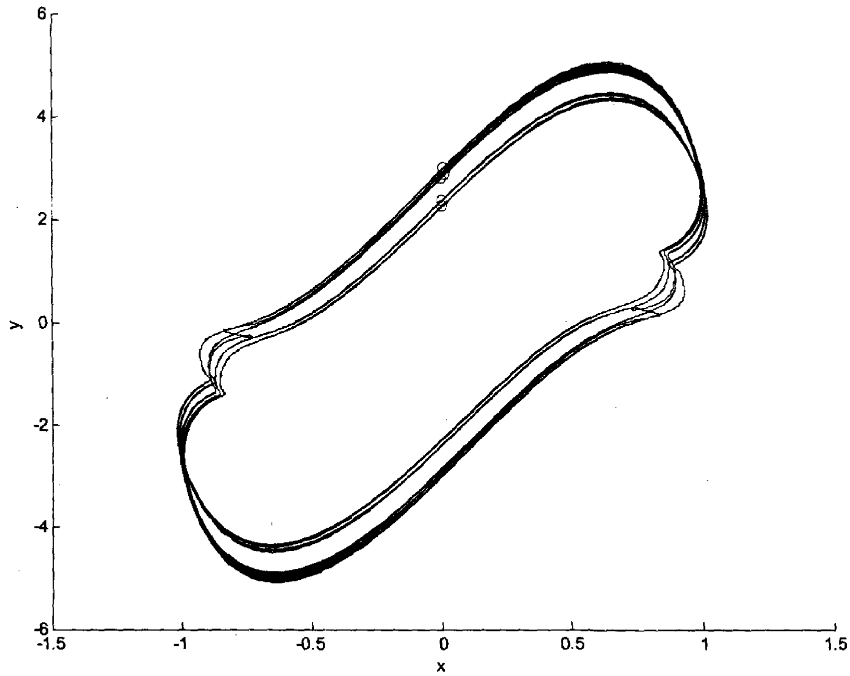

Fig. 47. The phase portrait for $\alpha = 0.5$, $\beta = 0.5$, $b = 6$.

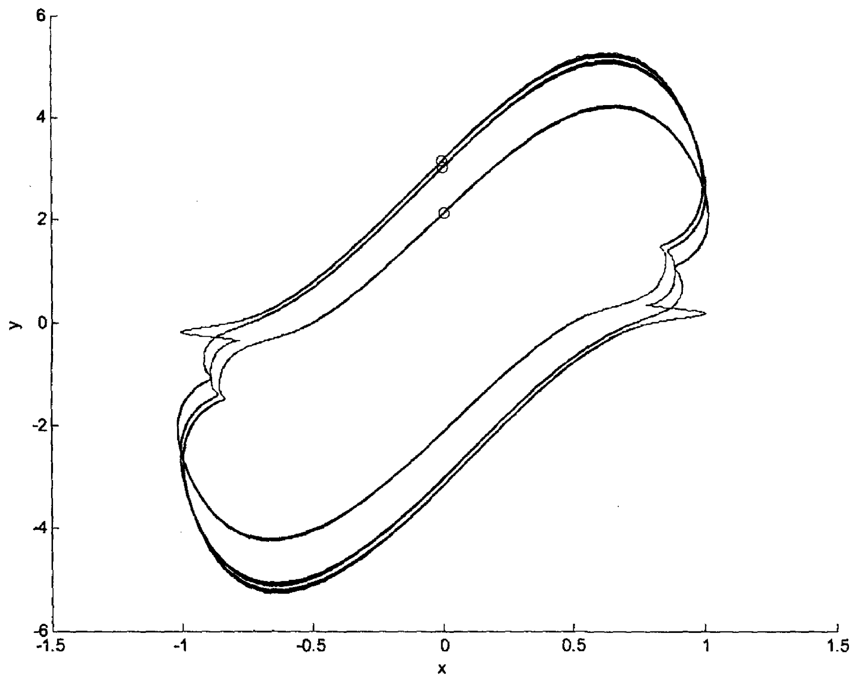

Fig. 48. The phase portrait for $\alpha = 0.5$, $\beta = 0.5$, $b = 10$.

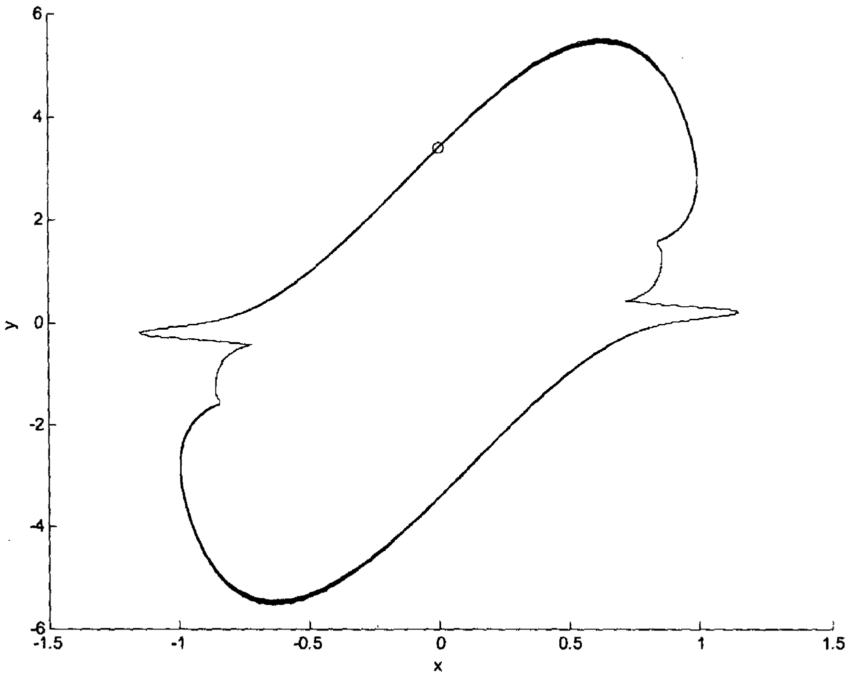

Fig. 49. The phase portrait for $\alpha = 0.5$, $\beta = 0.5$, $b = 15$.

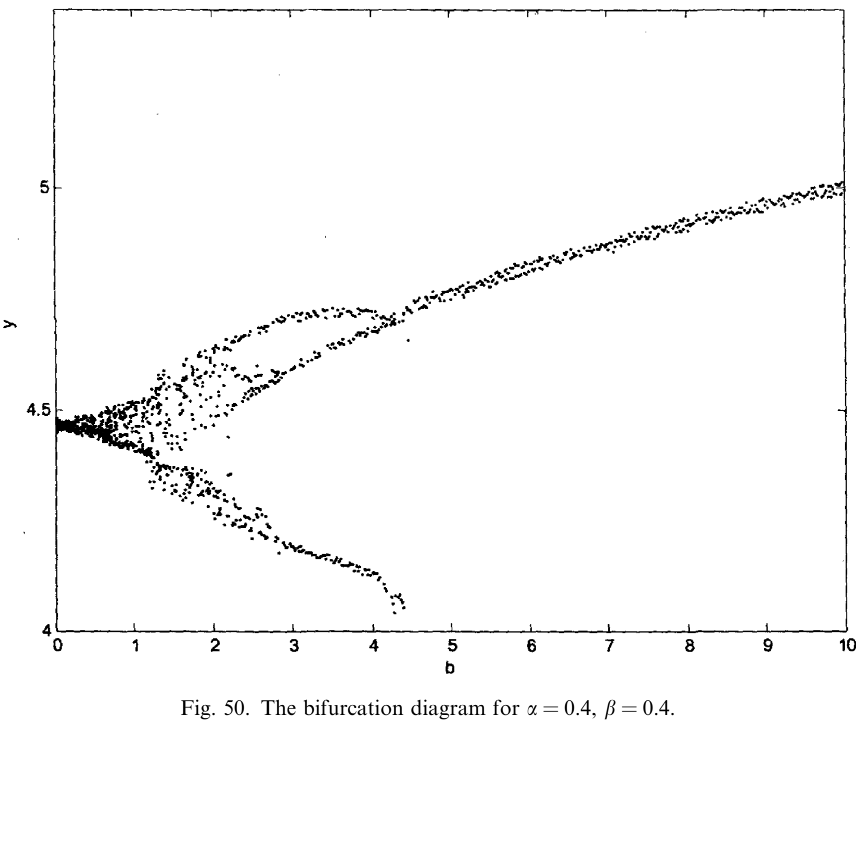

Fig. 50. The bifurcation diagram for $\alpha = 0.4$, $\beta = 0.4$.

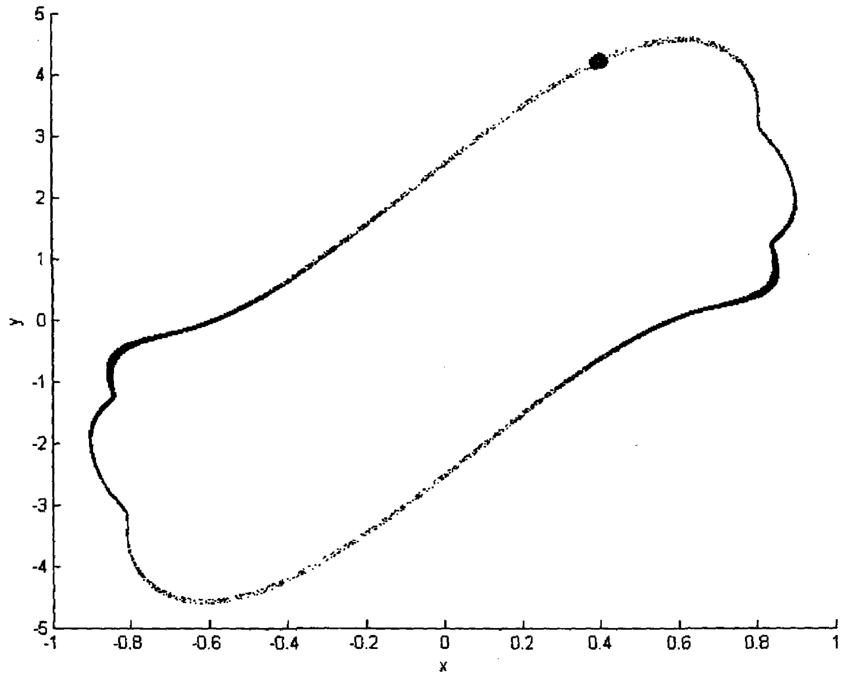

Fig. 51. The phase portrait for $\alpha = 0.4$, $\beta = 0.4$, $b = 0.7$.

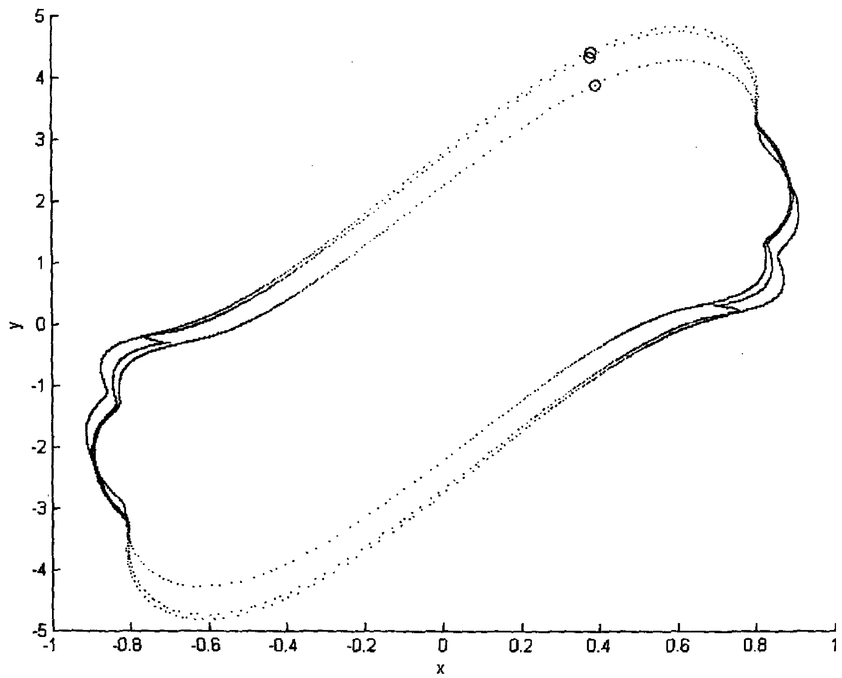

Fig. 52. The phase portrait for $\alpha = 0.4$, $\beta = 0.4$, $b = 3.5$.

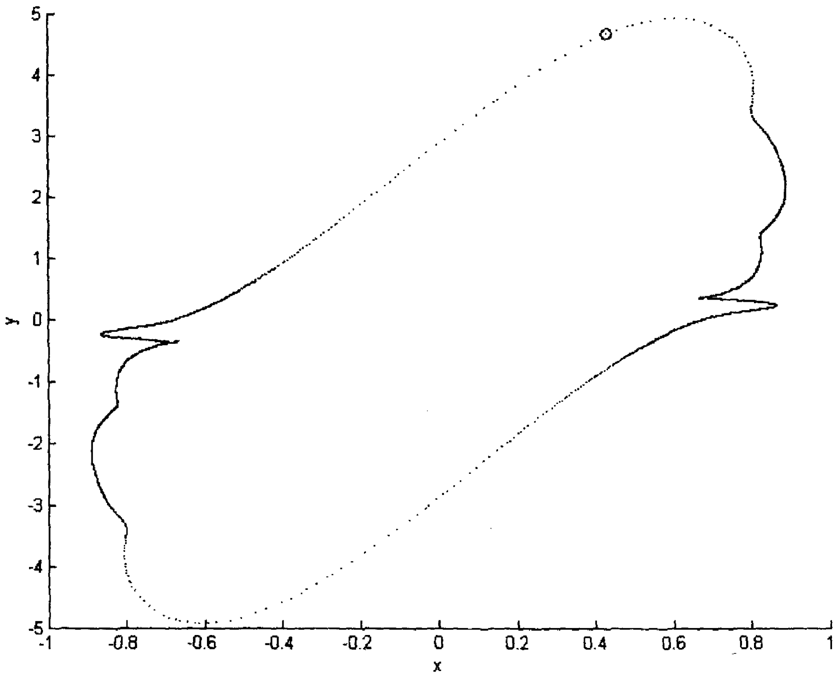

Fig. 53. The phase portrait for $\alpha = 0.4$, $\beta = 0.4$, $b = 6.0$.

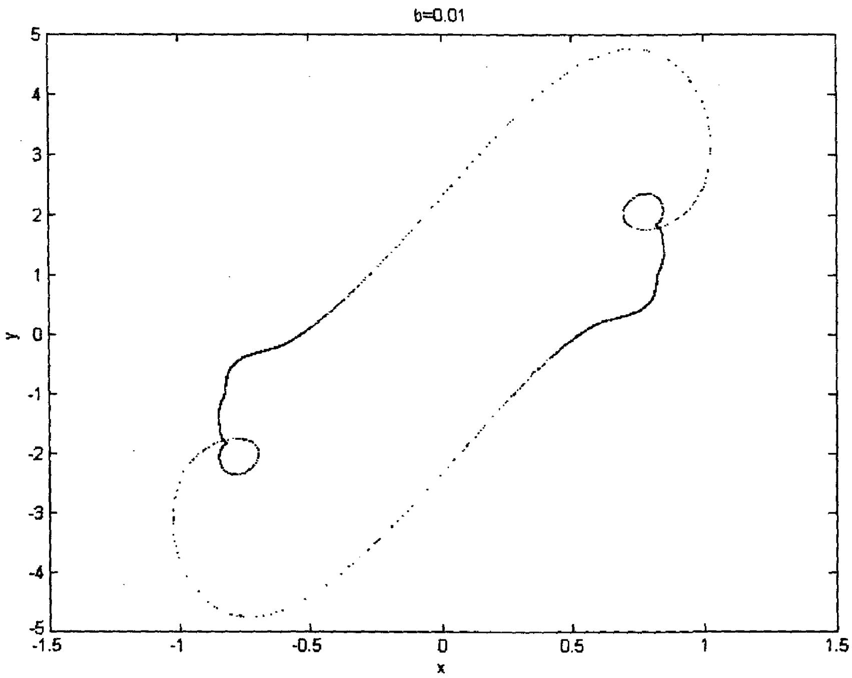

Fig. 54. Phase portrait ($b = 0.01$).

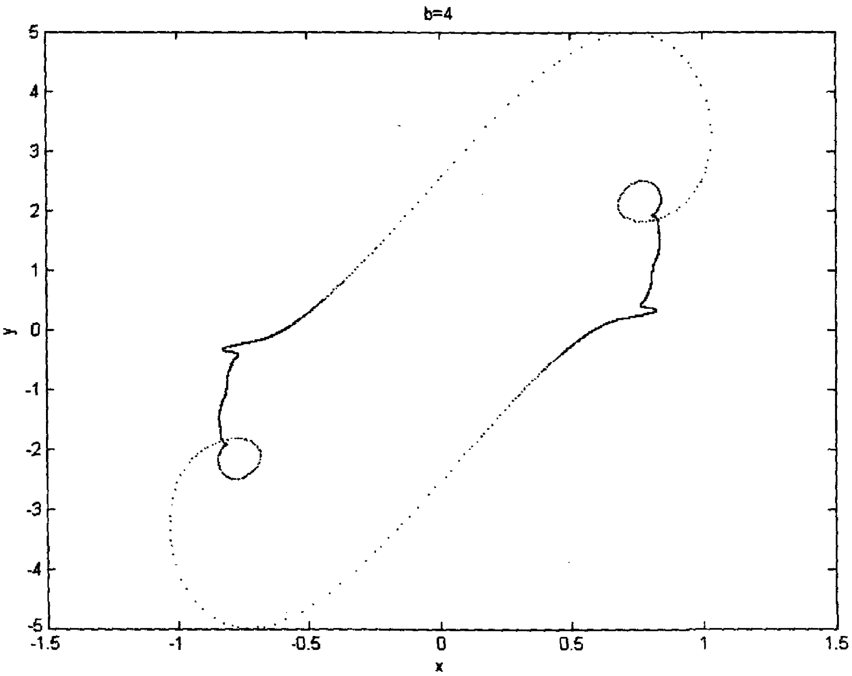

Fig. 55. Phase portrait ($b = 4$).

5. Conclusions

Chaos in modified van der Pol system and in its fractional order systems is studied in this paper. It is found that the range of the chaos in the system gradually decreases as the total order number $\alpha + \beta$ decreases. Nine cases for $0.8 \leqslant (\alpha + \beta) \leqslant 2.0$ are studied. The lowest total order for chaos existence in the system is found to be 0.8.

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