International Journal of Bifurcation and Chaos, Vol. 17, No. 6 (2007) 1969–1983

SNAP-BACK REPELLERS AND CHAOTIC TRAVELING WAVES IN ONE-DIMENSIONAL CELLULAR NEURAL NETWORKS

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Received June 12, 2006; Revised September 7, 2006

In 1998, Chen *et al.* [1998] found an error in Marotto's paper [1978]. It was pointed out by them that the existence of an expanding fixed point $\mathbf{z}$ of a map $\mathbf{F}$ in $B_r(\mathbf{z})$, the ball of radius r with center at $\mathbf{z}$ does not necessarily imply that $\mathbf{F}$ is expanding in $B_r(\mathbf{z})$. Subsequent efforts (see e.g. [Chen *et al.*, 1998; Lin *et al.*, 2002; Li & Chen, 2003]) in fixing the problems have some discrepancies since they only give conditions for which $\mathbf{F}$ is expanding "locally". In this paper, we give sufficient conditions so that $\mathbf{F}$ is "globally" expanding. This, in turn, gives more satisfying definitions of a snap-back repeller. We then use those results to show the existence of chaotic backward traveling waves in a discrete time analogy of one-dimensional Cellular Neural Networks (CNNs). Some computer evidence of chaotic traveling waves is also given.

Keywords: Snap-back repellers; traveling waves; cellular neural networks.

1. Introduction

The study of traveling wave and standing wave solutions for partial differential equations and lattice dynamical systems has drawn considerable attention in the past decades. For instance, the existence and stability of such solutions for lattice dynamical systems has been much studied by many authors. (See, e.g. [Afraimovich *et al.*, 1994; Afraimovich & Nekorkin, 1994; Chow *et al.*, 1998; Erneux & Nicolis, 1993; Hsu & Lin, 2000; Hsu *et al.*, 1999; Hsu & Yang, 2004; Hudson & Zinner, 1994; Keener, 1987; Mallet-Paret, 1999a, 1999b; Wu & Zou, 1997; Zinner, 1992; Zinner *et al.*, 1993; Zou & Wu, 1998] and many references cited therein.) On the other hand, the study of the discrete (in time) analog of such systems has only focused on diffusion Huxley–Nagumo (see, e.g. [Afraimovich *et al.*, 1994; Afraimovich & Nekorkin 1994]) equation. Specifically, they proved the existence of traveling waves of a chaotic profile by establishing the existence of the Smale-Horseshoe of a corresponding two-dimensional map. In this paper, we study a chaotic profile of stationary traveling wave solutions of a discrete analog of one-dimensional Cellular Neural Networks (CNNs) by showing the presence of snap-back repellers. The dynamics of one-dimensional CNNs (see, e.g. [Ban *et al.*, 2002; Ban *et al.*, 2001; Chua, 1998; Chua & Yang, 1998a, 1998b; Hänggi & Chua, 2000; Itoh *et al.*, 2001; Juang & Lin, 2000; Thiran, 1997] and the references cited therein) is of the form

$$\frac{d\overline{x}_i}{dt} = -\overline{k}\,\overline{x}_i + \overline{z} + \overline{\alpha} f(\overline{x}_{i-1}) + \overline{a} f(\overline{x}_i) + \overline{\beta} f(\overline{x}_{i+1}), \quad i \in \mathbb{Z}. \tag{1a}$$

Here f is a piecewise linear output function defined by

$$f(x) = \begin{cases} rx + 1 - r, & \text{if } x \geq 1, \\ x, & \text{if } |x| \leq 1, \\ rx - 1 + r, & \text{if } x \leq -1, \end{cases} \tag{1b}$$

where r is a non-negative constant, $\overline{k}$ is positive. The quantity $\overline{z}$ is called a threshold or biased term. The constants $\overline{\alpha}$, $\overline{a}$ and $\overline{\beta}$ are the interaction weights between neighboring cells.

Discretizing equation (1a) by Euler method, we have the discrete-time CNNs of the form

$$\overline{x}_i(t+1) = k\overline{x}_i(t) + z + \alpha f(\overline{x}_{i-1}(t)) + af(\overline{x}_i(t)) + \beta f(\overline{x}_{i+1}(t)). \tag{2}$$

Here $k = 1 - \triangle t\overline{k}$, $z = \triangle t\overline{z}$, $\alpha = \triangle t\overline{\alpha}$, $a = \triangle t\overline{a}$, $\beta = \triangle t\overline{\beta}$, and $\triangle t$ is a step size. We will refer to the solutions of system (2) of the form $\overline{x}_i(n) = \varphi(i+cn), c \in \mathbb{Z}$ being a wave speed, as stationary waves by analogy with the continuous case. Apparently, the function $\varphi(i+cn)$ must satisfy the equation

$$\begin{aligned}\varphi(i+cn+c) &= k\varphi(i+cn) + z + \alpha f(\varphi(i-1+cn)) \\ &\quad + af(\varphi(i+cn)) + \beta f(\varphi(i+1+cn)).\end{aligned} \tag{3}$$

Setting the "iteration index" $j = i + cn$, we see that (3) becomes

$$y_{j+c} = ky_j + \alpha f(y_{j-1}) + af(y_j) + \beta f(y_{j+1}) + z. \tag{4}$$

where $y_j = \varphi(i+cn) = \varphi(j)$. For $c > 1$, Eq. (4) induces a $(c+1)$-dimensional map T of the form

$$\begin{aligned}T(x_1, x_2, \ldots, x_{c+1}) &= (x_2, \ldots, x_{c+1}, kx_2 + \alpha f(x_1) + af(x_2) \\ &\quad + \beta f(x_3) + z).\end{aligned} \tag{5}$$

For $c = 1$, Eq. (4) becomes

$$\begin{aligned}x_{j+1} := g(y_{j+1}) &:= y_{j+1} - \beta f(y_{j+1}) \\ &= ky_j + af(y_j) + \alpha f(y_{j-1}) + z.\end{aligned} \tag{6a}$$

or

$$x_{j-1} := f(y_{j-1}) = \frac{-k}{\alpha}y_j - \frac{a}{\alpha}x_j + \frac{1}{\alpha}y_{j+1} - \frac{\beta}{\alpha}x_{j+1} - \frac{z}{\alpha}. \tag{6b}$$

If we assume momentarily that g (resp. f) is invertible, then Eq. (6a) (resp. (6b)) can be represented by a two-dimensional map $\mathbf{F}$ (resp. $\mathbf{B}$) of the form

$$\mathbf{F}(x,y) = (y, f_1(y) + f_2(x) + z). \tag{7a}$$

$$\left(\text{resp. } \mathbf{B}(x,y) = \left(y, g_1(y) + g_2(x) - \frac{z}{\alpha}\right)\right). \tag{7b}$$

Here,

$$f_1(x) = kg^{-1}(x) + af(g^{-1}(x)), \tag{8a}$$

$$\left(\text{resp. } g_1(x) = -\frac{k}{\alpha}f^{-1}(x) - \frac{a}{\alpha}x\right), \tag{8b}$$

and,

$$f_2(x) = \alpha f(g^{-1}(x)). \tag{9a}$$

$$\left(\text{resp. } g_2(x) = \frac{1}{\alpha}f^{-1}(x) - \frac{\beta}{\alpha}x\right). \tag{9b}$$

The map $\mathbf{F}$ (resp. $\mathbf{B}$) generates the forward (resp. backward) wave solutions of (2). Assuming $(1-\beta r)(1-\beta) > 0$, we see that g is invertible. After some calculations, we obtain that, for $1 - \beta > 0$,

$$f_1(x) = \begin{cases} \dfrac{k+ar}{1-\beta r}x + \dfrac{(1-r)(k\beta+a)}{1-\beta r}, & \text{if } x \geq 1-\beta, \\[2ex] \dfrac{k+a}{1-\beta}x, & \text{if } |x| \leq 1-\beta, \\[2ex] \dfrac{k+ar}{1-\beta r}x - \dfrac{(1-r)(k\beta+a)}{1-\beta r}, & \text{if } x \leq -1+\beta, \end{cases} \tag{10a}$$

and,

$$f_2(x) = \begin{cases} \dfrac{\alpha r}{1-\beta r}x + \dfrac{\alpha(1-r)}{1-\beta r}, & \text{if } x \geq 1-\beta, \\[2ex] \dfrac{\alpha}{1-\beta}x, & \text{if } |x| \leq 1-\beta, \\[2ex] \dfrac{\alpha r}{1-\beta r}x - \dfrac{\alpha(1-r)}{1-\beta r}, & \text{if } x \leq -1+\beta, \end{cases} \tag{10b}$$

for $1-\beta<0$,

$$f_1(x)=\begin{cases}\dfrac{k+ar}{1-\beta r}x-\dfrac{(1-r)(k\beta+a)}{1-\beta r}, & \text{if } x\geq -1+\beta,\\[2mm] \dfrac{k+a}{1-\beta}x, & \text{if } |x|\leq -1+\beta,\\[2mm] \dfrac{k+ar}{1-\beta r}x+\dfrac{(1-r)(k\beta+a)}{1-\beta r}, & \text{if } x\leq 1-\beta,\end{cases} \tag{10c}$$

and,

$$f_2(x)=\begin{cases}\dfrac{\alpha r}{1-\beta r}x+\dfrac{\alpha(r-1)}{1-\beta r}, & \text{if } x\geq -1+\beta,\\[2mm] \dfrac{\alpha}{1-\beta}x, & \text{if } |x|\leq -1+\beta,\\[2mm] \dfrac{\alpha r}{1-\beta r}x-\dfrac{\alpha(r-1)}{1-\beta r}, & \text{if } x\leq 1-\beta.\end{cases} \tag{10d}$$

Replacing x by $(1-\beta)x$ or $(\beta-1)x$ depending upon the sign of $1-\beta$, we have that

$$f_1(x)=\begin{cases}a_1x+a_{10}-a_1, & \text{if } x\geq 1,\\ a_{10}x, & \text{if } |x|\leq 1,\\ a_1x-a_{10}+a_1, & \text{if } x\leq -1,\end{cases} \tag{11a}$$

and,

$$f_2(x)=\begin{cases}a_2x+a_{20}-a_2, & \text{if } x\geq 1,\\ a_{20}x, & \text{if } |x|\leq 1,\\ a_2x-a_{20}+a_2, & \text{if } x\leq -1.\end{cases} \tag{11b}$$

Here,

$$a_1=\begin{cases}\dfrac{(1-\beta)(k+ar)}{1-\beta r}, & \text{if } 1-\beta>0,\\[2mm] \dfrac{(\beta-1)(k+ar)}{1-\beta r}, & \text{if } 1-\beta<0.\end{cases}$$

$$a_{10}=\begin{cases}k+a, & \text{if } 1-\beta>0,\\ -(k+a), & \text{if } 1-\beta<0.\end{cases}$$

$$a_2=\begin{cases}\dfrac{(1-\beta)\alpha r}{1-\beta r}, & \text{if } 1-\beta>0,\\[2mm] \dfrac{(\beta-1)\alpha r}{1-\beta r}, & \text{if } 1-\beta<0.\end{cases}$$

and,

$$a_{20}=\begin{cases}\alpha, & \text{if } 1-\beta>0,\\ -\alpha, & \text{if } 1-\beta<0.\end{cases}$$

Assuming $r>0$, which in turn guarantees the invertibility of f, we have that

$$g_1(x)=\begin{cases}b_1x+b_{10}-b_1, & \text{if } x\geq 1,\\ b_{10}x, & \text{if } |x|\leq 1,\\ b_1x-b_{10}+b_1, & \text{if } x\leq -1,\end{cases} \tag{12a}$$

and,

$$g_2(x)=\begin{cases}b_2x+b_{20}-b_2, & \text{if } x\geq 1,\\ b_{20}x, & \text{if } |x|\leq 1,\\ b_2x-b_{20}+b_2, & \text{if } x\leq -1,\end{cases} \tag{12b}$$

Here $b_1=-(k+ra)/r\alpha$, $b_{10}=-(k+a)/\alpha$, $b_2=(1-\beta r)/r\alpha$, $b_{20}=(1-\beta)/\alpha$.

To consider various possibilities of the graphs of $f_i(x)$ and $g_i(x)$, $i=1,\ 2$, we need the following notions.

Given a piecewise function $f(x)$ in the form of $f_1(x)$, as given in (11), then $f(x)$ is said to be of Types (I)–(IV), respectively, if $a_1<0$ and $a_{10}>0$; $a_1>0$ and $a_{10}<0$; $a_1,a_{10}>0$; and $a_1,a_{10}<0$. See Fig. 1.

To find out the type of functions of $f_i(x)$ and $g_i(x)$, $i=1,\ 2$, we group the parameters as follows.

(I) Dividing β into two parts: we have (i) $1<\beta$ (ii) $1>\beta$.

(II) Dividing k into four parts: we have

- (i) $k>-ar, a>0$ (or $k>-a, a<0$),
- (ii) $k<-a, a>0$ (or $k<-ar, a<0$),
- (iii) $-a<k<-ar<0$,
- (iv) $0<-ar<k<-a$.

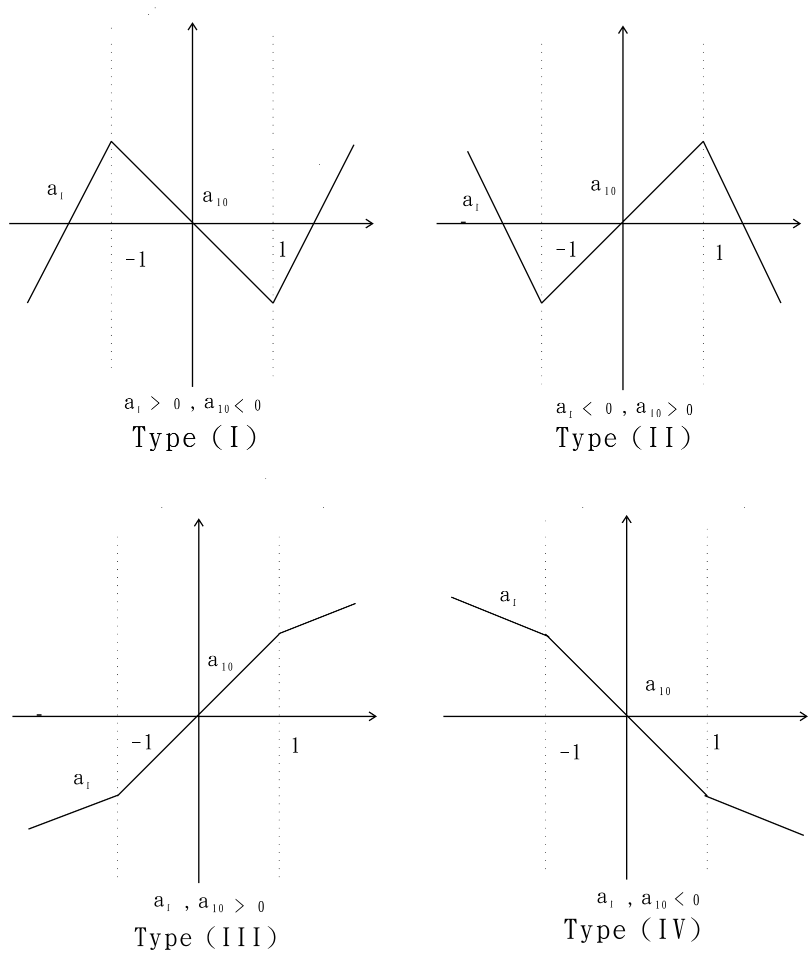


Fig. 1. The graph of each type of f_1.

(III) Dividing α into two parts: we get (i) $\alpha > 0$ (ii) $\alpha < 0$.
(IV) Dividing β into three parts: we get (i) $\beta > 1/r$ (ii) $1 < \beta < 1/r$ (iii) $\beta < 1$.

For the applications purpose in CNNs, r is either zero or a small positive constant. Thus, we assume $-ar > -a$ when $a > 0$. Likewise, $-ar < -a$ when $a < 0$. We also note that to insure the existence of the forward map $\mathbf{F}$, we must have $(1-\beta)(1-\beta r) > 0$.

We next provide conditions of parameters for which the type of functions $f_i(x)$ and $g_i(x)$, $i = 1, 2$, is characterized accordingly. Via the selection of parameters' ranges, the forward map $\mathbf{F}$ can be grouped into eight different types. Likewise, there are 16 different types for the backward map $\mathbf{B}$. See Tables 1 and 2 for those function types and their corresponding parameters' ranges.

If $f_i(x)$ and $g_i(x)$, $i = 1, 2$ are monotonic, such as in the cases **F**1–**F**4 and **B**1–**B**4, then the forward map **F** and the backward map **B** generate no chaotic dynamics. All other cases may produce chaotic dynamics. To have chaotic dynamics via the presence of a snap-back repeller, we need to consider the noninvertible maps, such as **B**5–**B**7 and **B**13–**B**16. In the paper, we will only consider **B**15.

In 1998, Chen *et al.* [1998], found an error in Marotto's paper [1978]. It was pointed out by them that the existence of an expanding fixed point $\mathbf{z}$ (see Definition 2.1) of a map $\mathbf{F}$ in $B_r(\mathbf{z})$, the ball of radius r with center at $\mathbf{z}$, does not necessarily imply that $\mathbf{F}$ is expanding in $B_r(\mathbf{z})$ (see Definition 2.1). Subsequent efforts (see e.g. [Chen *et al.*, 1998; Lin *et al.*, 2002; Li & Chen, 2003]) in fixing the problems have some discrepancies since they only give conditions for which $\mathbf{F}$ is expanding "locally". In this paper, we give sufficient conditions so that $\mathbf{F}$ is "globally" expanding. This, in turn, gives more satisfying definitions of a snap-back repeller. We then use those results to show the existence of chaotic backward traveling waves in a discrete time analogy of one-dimensional Cellular Neural Networks

Table 1.

Cases of Forward Map $\mathbf{F}$	Function Type $f_1(x)$	$f_2(x)$	Conditions Satisfied
$\mathbf{F}1$	III	III	(I-i)+(II-i)+(III-i) or (I-ii)+(II-ii)+(III-ii)
$\mathbf{F}2$	III	IV	(I-i)+(II-i)+(III-ii) or (I-ii)+(II-ii)+(III-i)
$\mathbf{F}3$	IV	III	(I-i)+(II-ii)+(III-i) or (I-ii)+(II-i)+(III-ii)
$\mathbf{F}4$	IV	IV	(I-i)+(II-ii)+(III-ii) or (I-ii)+(II-i)+(III-i)
$\mathbf{F}5$	II	III	(I-i)+(II-iii)+(III-i) or (I-ii)+(II-iv)+(III-ii)
$\mathbf{F}6$	II	IV	(I-i)+(II-iii)+(III-ii) or (I-ii)+(II-iv)+(III-i)
$\mathbf{F}7$	I	III	(I-i)+(II-iv)+(III-i) or (I-ii)+(II-iii)+(III-ii)
$\mathbf{F}8$	I	IV	(I-i)+(II-iv)+(III-ii) or (I-ii)+(II-iii)+(III-i)

Table 2.

Case of Backward Map $\mathbf{B}$	Function Type $g_1(x)$	$g_2(x)$	Conditions Satisfied
$\mathbf{B}1$	III	III	(II-iv)+(III-i)+(IV-iii) or (II-i)+(III-ii)+(IV-i)
$\mathbf{B}2$	III	IV	(II-iv)+(III-i)+(IV-i) or (II-i)+(III-ii)+(IV-iii)
$\mathbf{B}3$	IV	III	(II-i)+(III-i)+(IV-iii) or (II-iv)+(III-ii)+(IV-i)
$\mathbf{B}4$	IV	IV	(II-i)+(IV-i)+(III-i) or (II-iv)+(III-ii)+(IV-iii)
$\mathbf{B}5$	III	I	(II-iv)+(IV-ii)
$\mathbf{B}6$	III	II	(II-i)+(III-ii)+(IV-ii)
$\mathbf{B}7$	IV	I	(II-i)+(IV-ii)
$\mathbf{B}8$	IV	II	(II-iv)+(III-ii)+(IV-ii)
$\mathbf{B}9$	I	III	(II-ii)+(III-i)+(IV-iii) or (II-iii)+(III-ii)+(IV-i)
$\mathbf{B}10$	I	IV	(II-ii)+(III-i)+(IV-i) or (II-iii)+(III-ii)+(IV-iii)
$\mathbf{B}11$	II	III	(II-iii)+(III-i)+(IV-iii) or (II-ii)+(III-ii)+(IV-i)
$\mathbf{B}12$	II	IV	(II-iii)+(III-i)+(IV-i) or (II-ii)+(III-ii)+(IV-iii)
$\mathbf{B}13$	I	I	(II-ii)+(IV-ii)
$\mathbf{B}14$	I	II	(II-iii)+(III-ii)+(IV-ii)
$\mathbf{B}15$	II	I	(II-iii)+(IV-ii)
$\mathbf{B}16$	II	II	(II-ii)+(III-ii)+(IV-ii) or

(CNNs). We conclude this introductory section by mentioning that in Sec. 2, we will review the problems in the early definitions of a snap-back repeller. Moreover, we will point out what would be the more satisfying definitions of snap-back repeller. The sufficient conditions under which $\mathbf{F}$ is "globally" expanding are recorded in Sec. 2 as well. Section 3 contains the applications of those results to a discrete time analogy of CNNs. Moreover, some computer evidence of chaotic traveling waves is also given.

2. Snap-Back Repellers

In 1975, Li and Yorke [1975] proved a celebrated result "period three implies chaos". This result plays an important role in predicting and analyzing one-dimensional chaotic systems. Motivated by Li–Yorke's work, Marotto [1978] generalized such notion of chaos to higher-dimensional discrete dynamical systems. Specifically, he proved that Snap-Back repellers imply chaos in $\mathbb{R}^n$. Because the existence of a snap-back repeller is much easier (as compared to a homoclinic point) to verify, this theorem was widely applied ever since. However, in 1998, Chen *et al.* [1998] found that there was an error in Marotto's paper [1978]. Specifically, let (**A**) and (**B**) be as follows.

(**A**): All eigenvalues of the Jacobian $D\mathbf{F}(\mathbf{x})$, where $\mathbf{x} \in B_r(\mathbf{z}) = \{\mathbf{y} \in \mathbb{R}^n : \|\mathbf{y}-\mathbf{z}\| < r\}$, $r > 0$ and $\mathbf{z}$ is a fixed point of $\mathbf{F}$, are greater than 1 in norm.

(**B**): There exists some $s > 1$ such that any $\mathbf{x} \in B_r(\mathbf{z})$, $\mathbf{x} \neq \mathbf{z}$, $\|\mathbf{F}(\mathbf{x}) - \mathbf{z}\| > s\|\mathbf{x}-\mathbf{z}\|$.

Definition 2.1. A fixed point $\mathbf{z}$ of $\mathbf{F}$ satisfying (**A**) is called an expanding fixed point of $\mathbf{F}$ in $B_r(\mathbf{z})$. A map $\mathbf{F}$ satisfying (**B**) is said to be expanding in $B_r(\mathbf{z})$.

Note that both statements of (**A**) and (**B**) depend on how one chooses the norm $\|\cdot\|$. If necessary, we shall call such $\mathbf{F}$ is expanding (resp. has a fixed point $\mathbf{z}$) in $B_r(\mathbf{z})$ with respect to a certain norm $\|\cdot\|$.

It was pointed out in [Chen *et al.*, 1998] that the existence of an expanding fixed point $\mathbf{z}$ of a high-dimensional map $\mathbf{F}$ in $B_r(\mathbf{z})$ does not necessarily guarantee that $\mathbf{F}$ is expanding in $B_r(\mathbf{z})$. Even in the case that $\mathbf{F}$ is linear, (**A**) does not necessarily imply (**B**). See Fig. 2 (Fig. 1 of [Li & Chen, 2003]). Here in Fig. 2, the norm chosen is the Euclidean norm. Consequently, for $\mathbf{x} \in B_r(\mathbf{z})$, $\mathbf{x}$ does not necessarily lie on the local unstable manifold $W^u_{\mathrm{loc}}(\mathbf{z})$. To fix such problem, Chen *et al.* imported a new norm different from the Euclidean norm to guarantee the map $\mathbf{F}$'s expansibility in the neighborhood of its fixed point.

However, as observed by Li and Chen [2003], and Lin *et al.* [2002] that the incorporation of a new norm by Chen *et al.*, into Marotto Theorem does not close the gap of the proof. Because the new norm depends on the points $\mathbf{x} \in B_r(\mathbf{z})$ and the map $\mathbf{F}$. Thus, it is unclear how such new norm can be used to prove the assertion in (**B**) when $\mathbf{F}$ is nonlinear. Nevertheless, Chen *et al.* also gave a modified definition of a snap-back repeller, which refined the Marotto's Theorem in the spirit of Devaney's Theorem [Devaney, 1989]. Their proof seems to be correct (see also [Li & Chen, 2003; Lin *et al.*, 2002]). We next describe their definition and results.

Chen's Definition. $\mathbf{F} : \mathbb{R}^N \to \mathbb{R}^N$. Let $\mathbf{z}$ be a fixed point of $\mathbf{F}$ such that all eigenvalues of $D\mathbf{F}(\mathbf{z})$ have absolute values larger than 1. We say that $\mathbf{z}$ is a snap-back repeller if there exists a point $\mathbf{x}_0$ in $W^u_{\mathrm{loc}}(\mathbf{z})$, the local unstable set of $\mathbf{z}$, and some integer m, such that $\mathbf{F}^m(\mathbf{x}_0) = \mathbf{z}$ and $\det D\mathbf{F}^m(\mathbf{x}_0) \neq 0$.

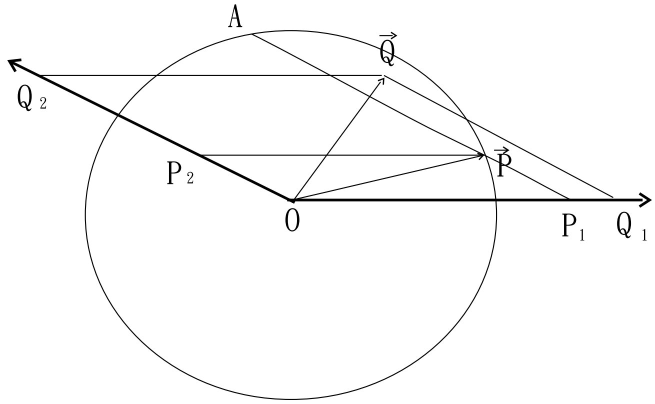


Fig. 2.

Chen–Hsu–Zhou Theorem. Let $\mathbf{F} : \mathbb{R}^N \to \mathbb{R}^N$ be C^1, and $\mathbf{z}$ be a snap-back repeller of $\mathbf{F}$. Then for each neighborhood U of $\mathbf{z}$, there is an integer $m > 0$ such that $\mathbf{F}^m$ has a hyperbolic invariant subset in U on which $\mathbf{F}^m$ is topologically conjugate to the shift map on the binary symbol space $\sum_2$.

We remark that Chen's definition of a snap-back repeller is a special case of a transverse homoclinic point. Moreover, $\mathbf{F}$ is a diffeomorphism on a sufficiently small neighborhood of $\mathbf{z}$. Consequently, their theorem above resembles the results induced by the presence of a transverse homoclinic point (see, e.g. Theorem 4.5 of [Robinson, 1995]). As an effort to prove the existence of chaos in the sense of Marotto, Lin *et al.* [2002] proposed another modified definition of snap-back repeller to ensure chaos in the sense of Marotto. However, as mentioned by Li and Chen [2003], the proof of their corresponding theorem is incorrect. What is at fault is that they used a differential mean value theorem, which generally does not exist for high dimensional vector-valued functions. Li and Chen [2003] then gave their definition of a snap-back repeller as follows.

Li and Chen's Definition. A fixed point $\mathbf{z}$ of system $\mathbf{x}_{k+1} = \mathbf{F}(\mathbf{x}_k), k = 0, 1, 2, \ldots$ is called a snap-back repeller if

(i) $\mathbf{F}(\mathbf{x})$ is continuously differentiable in $B_r(\mathbf{z})$;
(ii) all eigenvalues of $(D\mathbf{F}(\mathbf{z}))^T D\mathbf{F}(\mathbf{z})$ are greater than 1 in norm;
(iii) there exists a point $\mathbf{x}_0 \in B_r(\mathbf{z})$ with $\mathbf{x}_0 \neq \mathbf{z}$ such that $\mathbf{F}^m(\mathbf{x}_0) = \mathbf{z}$, and $\det D\mathbf{F}^m(\mathbf{x}_0) \neq 0$ for some positive integer m.

We next record a Lemma of Li and Chen [2003], which showed the "local" expansibility of $\mathbf{F}$.

Lemma 2.1 (Lemma 5 of [Li & Chen, 2003]). *Suppose that* $\mathbf{z}$ *is a fixed point of system* $\mathbf{x}_{k+1} = \mathbf{F}(\mathbf{x}_k), k = 0, 1, 2, \ldots$ *and* $\mathbf{F}$ *is continuously differentiable in some closed ball* $B_r(\mathbf{z})$. *Also, assume, that, all eigenvalues of* $(D\mathbf{F}(\mathbf{z}))^T D\mathbf{F}(\mathbf{z})$ *are larger than* 1 *in norm. Then, there exist some* $s > 1$ *and* $r' \in (0, r]$ *such that*

(i) $\|\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{y})\| > s\|\mathbf{x} - \mathbf{y}\|$ *for* $\mathbf{x} \neq \mathbf{y} \in B_{r'}(\mathbf{z})$;
(ii) *all eigenvalues of* $(D\mathbf{F}(\mathbf{x}))^T D\mathbf{F}(\mathbf{x})$ *exceed 1 in norm for all* $\mathbf{x} \in B_{r'}(\mathbf{z})$.

Unfortunately, Li and Chen's Definition of a snap-back repeller still contains some discrepancies. Specifically, they proved, though correctly, in the lemma above that $\mathbf{F}$ is expanding in a small neighborhood, $B_{r'}(\mathbf{z})$. However, for $\mathbf{x}_0 \in B_r(\mathbf{z})$ with $r \geq r'$, there is no guarantee that $\mathbf{F}^{-k}(\mathbf{x}_0)$ is to be in $B_{r'}(\mathbf{z})$, for some $k \in \mathbb{N}$. Therefore, the gap appeared in the paper of Marotto is still there. In the paper of Marotto, the following alternative definition of a snap-back repeller was also given.

Definition 2.2. Let $\mathbf{F} : \mathbb{R}^N \to \mathbb{R}^N$ be continuous and $\mathbf{z}$ be a fixed point of $\mathbf{F}$. We say that $\mathbf{z}$ is a snap-back repeller if there exists a sequence of compact sets $\{B_k\}_{k=-\infty}^{m}$ (homeomorphic to the unit ball in $\mathbb{R}^N$) which satisfy: (a) $B_k \to \{\mathbf{z}\}$ as $k \to -\infty$; (b) $\mathbf{F}(B_k) = B_{k+1}$; (c) $\mathbf{F}$ is 1-1 in B_k; (d) $B_k \cap B_m = \emptyset$ for $1 \leq k < m$; and (e) $\mathbf{z} \in B_m^0$, the interior of B_m.

In the original proof of Marotto's chaos, the property that $\mathbf{F}$ is expanding in $B_r(\mathbf{z})$ was used to show the existence of such sequence of compact sets. Thus, if one assumes such existence of a sequence of compact sets as the definition of a snap-back repeller, then the existence of Marotto's chaos holds.

In light of the comment above, we will also define a snap-back repeller as follows

Definition 2.3. Let $\mathbf{z} \in \mathbb{R}^N$ be a fixed point of $\mathbf{F}$. We say that $\mathbf{z}$ is a snap-back repeller if

(i) $\mathbf{F}$ is expanding in $B_r(\mathbf{z})$, for some $r > 0$;
(ii) There exists a point $\mathbf{x}_0 \in B_r(\mathbf{z})$ with $\mathbf{x}_0 \neq \mathbf{z}$, $\mathbf{F}^m(\mathbf{x}_0) = \mathbf{z}$ and $\det D\mathbf{F}^m(\mathbf{x}_0) \neq 0$ for some positive integer m.

For such definitions (Definition 2.2 or Definition 2.3) of snap-back repellers, the following notion of Marotto's chaos, indeed, can be achieved. Thus, from here on, when we say a point $\mathbf{z}$ is a snap-back repeller it means that $\mathbf{z}$ satisfies either Definition 2.2 or Definition 2.3. For completeness, we next recall Marotto's chaos [1978] with the presence of such snap-back repeller.

Theorem 2.1 (Marotto's Chaos). *Suppose* $\mathbf{F} : \mathbb{R}^N \to \mathbb{R}^N$, *and* $\mathbf{z}$ *is a snap-back repeller, defined as in Definition 2.2 or Definition 2.3. Then the map* $\mathbf{F}$ *is chaotic in the sense of Li–Yorke*:

(i) *There is a positive integer* N *such that for each integer* $p \geq N$, $\mathbf{F}$ *has a point of period* p.
(ii) *There is a "scrambled set" of* $\mathbf{F}$, *i.e. an uncountable set* S *containing no periodic points of* $\mathbf{F}$ *such that*

 (b_1) $\mathbf{F}(S) \subset S$,

 (b_2) *for every* $\mathbf{x}_S, \mathbf{y}_S \in S$ *with* $\mathbf{x}_S \neq \mathbf{y}_S$,
$$\lim_{k\to\infty} \sup \|\mathbf{F}^k(\mathbf{x}_S) - \mathbf{F}^k(\mathbf{y}_S)\| > 0.$$
 (b_3) *for every* $\mathbf{x}_S \in S$ *and any periodic point* $\mathbf{y}_{\text{per}}$ *of* $\mathbf{F}$,
$$\lim_{k\to\infty} \sup \|\mathbf{F}^k(\mathbf{x}_S) - \mathbf{F}^k(\mathbf{y}_{\text{per}})\| > 0,$$
(iii) *There is an uncountable subset* S_0 *of* S *such that for every* $\mathbf{x}_{S_0}, \mathbf{y}_{S_0} \in S_0$:
$$\lim_{k\to\infty} \sup \|\mathbf{F}^k(\mathbf{x}_{S_0}) - \mathbf{F}^k(\mathbf{y}_{S_0})\| = 0.$$

In the following, we will give sufficient conditions for which the "global" expansibility of a map can be obtained. Thus, the verification of the existence of a snap-back repeller should be made more friendly.

Theorem 2.2. *Let* $\mathbf{F} = (f_1, f_2, \ldots, f_n)$ *be a smooth vector-valued function from* $\mathbb{R}^N \to \mathbb{R}^N$, *and* $\mathbf{z}$ *be a fixed point of* $\mathbf{F}$. *Suppose* $D\mathbf{F}(\mathbf{z})$ *is a normal matrix. Let* α *and* β *be defined as*
$$\begin{aligned}\alpha &= \min_{1\leq i\leq n} |\lambda_i|,\\ \beta &= \max_{1\leq i\leq n} \max_{\mathbf{x}\in\mathbf{B_r(z)}} \max_{1\leq j\leq n} |\beta_{i,j}(\mathbf{x})|\end{aligned} \tag{13}$$
where $\lambda_i, i = 1, \ldots, n$, *are eigenvalues of* $D\mathbf{F}(\mathbf{z})$ *and* $\beta_{i,j}(\mathbf{x})$, $j = 1, 2, \ldots, n$, *are eigenvalues of Hessian matrices* $H_{f_i}(\mathbf{x}) = (\partial_k\partial_l f_i(\mathbf{x}))_{k\times l}$ *and* $B_r(\mathbf{z})$ *is a closed ball with center at* $\mathbf{z}$ *and radius* $r > 0$. *If* $\alpha - (r\beta/2) > 1$, *then* $\mathbf{F}$ *is expanding in* $B_r(\mathbf{z})$.

Proof. For $\mathbf{y} \in B_r(\mathbf{z})$, we have, see e.g. 3.3.11 of [Ortega & Rheinbaldt, 1970] on p. 80, that
$$\begin{aligned}\mathbf{F}(\mathbf{y}) - \mathbf{z} &= D\mathbf{F}(\mathbf{z})(\mathbf{y} - \mathbf{y})\\ &\quad + \int_0^1 (1-t)\mathbf{F}''(\mathbf{z} + t(\mathbf{y} - \mathbf{z}))\\ &\quad \times (\mathbf{y} - \mathbf{z})(\mathbf{y} - \mathbf{z})dt.\end{aligned} \tag{14}$$
We next estimate the first term on the right-hand side of (14). Now,
$$\begin{aligned}\|D\mathbf{F}(\mathbf{z})(\mathbf{y} - \mathbf{z})\|_2 &= \|T^{-1}\Lambda T(\mathbf{y} - \mathbf{z})\|_2\\ &= \|\Lambda T(\mathbf{y} - \mathbf{z})\|_2 \geq \alpha\|\mathbf{y} - \mathbf{z}\|_2,\end{aligned} \tag{15}$$
where T is a unitary matrix and
$$\Lambda = \begin{bmatrix}\lambda_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \lambda_n\end{bmatrix}.$$

Since Hessian matrices $H_{f_i}(\mathbf{x})$ is symmetric, for all $\mathbf{x} \in B_r(\mathbf{z})$ and $\|\overline{\mathbf{y}}\|_2 \leq r$, we have

$$|\overline{\mathbf{y}}^T H_{f_i}(\mathbf{x})\overline{\mathbf{y}}| \leq \beta \parallel \overline{\mathbf{y}} \parallel_2^2 \leq \beta r \|\overline{\mathbf{y}}\|_2 \tag{16}$$

Using (13)–(15) and the fact that $[\mathbf{F}''(\mathbf{x})hk]^T = (k^T H_{f_1}(\mathbf{x})h, k^T H_{f_2}(\mathbf{x})h, \ldots, k^T H_{f_n}(\mathbf{x})h)$, we see that

$$\|\mathbf{F}(\mathbf{y}) - \mathbf{z}\|_2 \geq \left(\alpha - \frac{r\beta}{2}\right) \|\mathbf{y} - \mathbf{z}\|_2. \tag{17}$$

Thus, $\mathbf{F}$ is expanding in $B_r(\mathbf{z})$. ■

We next give a global expanding theorem without the restriction that $D\mathbf{F}(\mathbf{z})$ is normal.

Theorem 2.3. *Let $\mathbf{F} = (f_1, f_2, \ldots, f_n)$ be a smooth vector-valued function from $\mathbb{R}^N \to \mathbb{R}^N$, and $\mathbf{z}$ be a fixed point of $\mathbf{F}$. Suppose all eigenvalues of $D\mathbf{F}(\mathbf{z})$ have absolute values greater than α. Here we assume that $\alpha > 1$. Let β be defined as in (13). Assume the radius r in (13) is measured in the Euclidean norm. Let P be a matrix for which $P^{-1}D\mathbf{F}(\mathbf{z})P = J$. Here J is a Jordan canonical form of $D\mathbf{F}(\mathbf{z})$. We assume further that c is a positive constant for which*

$$\|\mathbf{x}\|_2 \leq c\|\mathbf{x}\|_p, \quad \textit{where } \|\mathbf{x}\|_p = \|P^{-1}\mathbf{x}\|_2 \tag{18}$$

If

$$\alpha - \frac{r}{2}\beta c\sqrt{n}\|P^{-1}\|_2 > 1, \tag{19}$$

then $\mathbf{F}$ is expanding in $B_r(\mathbf{z})$ with respect to the p-norm, defined in (18).

Proof. It follows from Theorem IV.2 of [Chen *et al.*, 1998] that

$$\|D\mathbf{F}(\mathbf{z})(\mathbf{y} - \mathbf{z})\|_p \geq \alpha \| \mathbf{y} - \mathbf{z} \|_p. \tag{20}$$

Here, the p-norm is defined as in (18). Now, for all $\mathbf{x} \in B_r(\mathbf{z})$ and $\|\mathbf{y}\|_2 \leq r$, we have

$$|\overline{\mathbf{y}}^T H_{f_i}(\mathbf{x})\overline{\mathbf{y}}| \leq \beta \parallel \overline{\mathbf{y}} \parallel_2^2 \leq \beta r \|\overline{\mathbf{y}}\|_2 \leq c\beta r \|\overline{\mathbf{y}}\|_p. \tag{21}$$

Applying (20), (21) to equality (14), we get

$$\| \mathbf{F}(\mathbf{y}) - \mathbf{z} \|_p \geq \left(\alpha - \frac{r}{2}\beta c\sqrt{n}\|P^{-1}\|_2\right) \| \mathbf{y} - \mathbf{z} \|_p. \tag{22}$$

We have just completed the proof of the theorem. ■

In view of (21) we have the following locally expanding theorem.

Corollary 2.1. *Suppose $\alpha > 1$. Then $\mathbf{F}$ is locally expanding. That is, there exists a $r' > 0$ such that $\mathbf{F}$ is expanding in $B_{r'}(\mathbf{z})$ with respect to the p-norm.*

Remarks

1. If $\mathbf{F}$ is linear, $\beta = 0$. Thus $\mathbf{F}$ is expanding in $\mathbb{R}^n$ with respect to the p-norm provided that $\alpha > 1$. This is essentially due to Chen *et al.* [1998].
2. When $\mathbf{F}$ is nonlinear, it is not a small task to verify the assumptions of Theorem 2.3. However, in applications, the choice of $\mathbf{x}_0$, where $\mathbf{x}_0$ is given as in Definition 2.3, often depends on a certain parameter(s), say d. Thus, if α, as given in Theorem 2.3, is greater than one and $\mathbf{x}_0$ can be made arbitrarily close to $\mathbf{z}$, where $\mathbf{z}$ is as given in Definition 2.3, as one varies the parameter d, then there must exist a $r > 0$ sufficiently small so that the following assertion holds: $\mathbf{F}$ is expanding in $B_r(\mathbf{z})$ with respect to the p-norm. Thus, in applications if there exists a $d_0 \in \mathbb{R} \cup \{\pm\infty\}$ (resp. for certain ranges of d) so that

$$\lim_{d \to d_0} \mathbf{x}_0(d) = \mathbf{z}(\text{resp., } \lim_{n \to \infty} \mathbf{F}^{-n}(\mathbf{x}_0(d)) = \mathbf{z}), \tag{23}$$

   then we may choose d sufficiently close to d_0(resp. $k \in \mathbb{N}$ sufficiently large) so that $\mathbf{x}_0(d) \in B_{r'}(\mathbf{z})$ (resp. $\mathbf{F}^{-k}(\mathbf{x}_0(d)) \in B_{r'}(\mathbf{z})$). Here r' is chosen as in Corollary 2.3. Under such circumstances, the verification of Definition 2.3. is much friendly.

3. Chaotic Backward Map

In this section, we consider the backward map $\mathbf{B}$ with $1 < \beta < (1/r)$, $-(k/r) > a > -k > 0$, and $\alpha > 0$. (see **B**15 in Table 2) Under the circumstances,

$$b_1 < 0, b_{10} > 0, b_2 > 0, \quad \text{and} \quad b_{20} < 0. \tag{24}$$

We denote by $\Omega_1, \Omega_0, \Omega_{-1}, \Omega_{1,-1}$ and $\Omega_{-1,1}$ the regions $\Omega_1 = \{(x,y) : x, y \geq 1\}$, $\Omega_0 = \{(x,y) : -1 \leq x, y \leq 1\}$, $\Omega_{-1} = \{(x,y) : x, y \leq -1\}$, $\Omega_{1,-1} = \{(x,y) : x \geq 1, y \leq -1\}$, and $\Omega_{-1,1} = \{(x,y) : x \leq -1, y \geq 1\}$, respectively.

Lemma 3.1. *Suppose (24) holds. Let $b_1 + b_2 > 1$, (resp. $b_1 + b_2 < 1$) and $-1 + b_{10} + b_{20} < c < 1 - b_{10} - b_{20}$ (resp. $1 - b_{10} - b_{20} < c < -1 + b_{10} + b_{20}$), then the map $\mathbf{B}$ has exactly three fixed points*

$$(\overline{x}_1, \overline{x}_1) =: \overline{\mathbf{x}}_1, \quad (\overline{x}_0, \overline{x}_0) =: \overline{\mathbf{x}}_0 \\ \text{and} \quad (\overline{x}_{-1}, \overline{x}_{-1}) =: \overline{\mathbf{x}}_{-1}, \tag{25}$$

in Ω_1, Ω_0, Ω_{-1}, respectively. Here

$$\overline{x}_1 = \frac{b_{10} + b_{20} - b_1 - b_2 + c}{1 - b_1 - b_2}, \overline{x}_0 = \frac{c}{1 - b_{10} - b_{20}} \\ \overline{x}_{-1} = \frac{-b_{10} - b_{20} + b_1 + b_2 + c}{1 - b_1 - b_2}. \tag{26}$$

Lemma 3.2. *Suppose the first set of assumptions in Lemma 3.1 holds. Then $\overline{\mathbf{x}}_1$ and $\overline{\mathbf{x}}_{-1}$ are repelling fixed points.*

Proof. It is obvious that

$$D\mathbf{B}(\mathbf{x}_{\pm 1}) = \begin{bmatrix} 0 & 1 \\ b_2 & b_1 \end{bmatrix}.$$

The eigenvalue of $D\mathbf{B}(\mathbf{x}_{\pm 1})$ are $(b_1 + \sqrt{b_1^2 + 4b_2})/2$ and $(b_1 - \sqrt{b_1^2 + 4b_2})/2$. Moreover, $(b_1 + \sqrt{b_1^2 + 4b_2})/2 > 1$ provided that $b_1 + b_2 > 1$. We thus complete the proof of lemma. ■

We are next to find a point $\mathbf{p} = (x_0, y_0)$ for which $\mathbf{B}(\mathbf{p}) \in \Omega_{1,-1}$, $\mathbf{B}^2(\mathbf{p}) \in \Omega_{-1,1}$, $\mathbf{B}^3(\mathbf{p}) = \overline{\mathbf{x}}_\mathbf{1}$. To this end, we first compute a pre-image $\mathbf{q} = (q_1, q_2)$ of $\overline{\mathbf{x}}_\mathbf{1}$ for which $\mathbf{q}$ lies in $\Omega_{-1,1}$. Clearly, $q_2 = \overline{x}_1$ and q_1 must satisfy equation $g_1(\overline{x}_1) + g_2(q_1) + c = \overline{x}_1$, or equivalently,

$$\begin{aligned} b_2 q_1 &= (1 - b_1)\overline{x}_1 - b_{10} + b_{20} + b_1 - b_2 - c \\ &= b_2 \overline{x}_1 + 2b_{20} - 2b_2. \end{aligned}$$

Thus,

$$q_1 = \overline{x}_1 + 2\frac{b_{20}}{b_2} - 2, \quad \text{and} \quad q_2 = \overline{x}_1. \tag{27}$$

Now,

$$\mathbf{p} = (\mathbf{x_0}, \mathbf{y_0}), \tag{28}$$

must satisfy the following equations

$$g_1(y_0) + g_2(x_0) + c = q_1, \tag{29a}$$

$$g_1(q_1) + g_2(y_0) + c = q_2. \tag{29b}$$

From (29b), we see that

$$\begin{aligned} b_2 y_0 &= (1 - b_1)\overline{x}_1 - 2\frac{b_1 b_{20}}{b_2} + b_1 + b_{10} - b_{20} + b_2 - c \\ &= b_2 \overline{x}_1 - 2\frac{b_1 b_{20}}{b_2} + 2b_{10}. \end{aligned}$$

So,

$$y_0 = \overline{x}_1 - 2\frac{b_1 b_{20}}{b_2^2} + 2\frac{b_{10}}{b_2} := \overline{x}_1 + d_1. \tag{30a}$$

Substituting (30a) into (29a), we obtain that

$$x_0 = \overline{x}_1 + 2\frac{b_{20}}{b_2^2} - \frac{2}{b_2} - 2\frac{b_1 b_{10}}{b_2^2} + 2\frac{b_1^2 b_{20}}{b_2^3} := \overline{x}_1 + d_2. \tag{30b}$$

We then need to show that there is a nonempty set of parameters for which

$$x_0, \quad y_0 > 1, \tag{31a}$$

and

$$g_1(y_0) + g_2(x_0) + c = x_1 + 2\frac{b_{20}}{b_2} - 2 < -1. \tag{31b}$$

Proposition 3.1. *Let $b_{10} = -b_{20} = q > 0$ and $b_2 = -pb_1 > 0$, where $p > 0$. Suppose $b_1(1 - p) > 1$, $-1 < c < 1$, $b_1 \leq -3$, $q \geq 2p \geq 12$. Then $x_0 \geq \overline{x}_1$ and $y_0 \geq \overline{x}_1$. Consequently, (31) holds.*

Proof. Clearly, y_0, given as in (30a), is greater than $\overline{x}_1$.

Now,

$$\begin{aligned} x_0 &= \frac{c + (p-1)b_1}{1 + (p-1)b_1} - \frac{2q}{p^2 b_1}\left[1 + \frac{1}{b_1} - \frac{1}{p}\right] + \frac{2}{pb_1} \\ &\geq \frac{c + (p-1)b_1}{1 + (p-1)b_1} - \frac{q}{p^2 b_1} + \frac{2}{pb_1} \\ &\geq \frac{c + (p-1)b_1}{1 + (p-1)b_1} \\ &= \overline{x}_1 \\ &\geq 1. \end{aligned}$$

To complete the proof of the proposition, we see, via (31b), that

$$\begin{aligned} \overline{x}_1 + 2\frac{b_{20}}{b_2} - 1 &= \frac{c - 1}{1 + (p-1)b_1} + \frac{2q}{pb_1} \\ &= \frac{1}{pb_1(1 + (p-1)b_1)}[pb_1(c - 1) \\ &\quad + 2q(1 + (p-1)b_1)] \\ &\leq \frac{1}{b_1(1 + (p-1)b_1)} \\ &\quad \times [-2b_1 + 4(1 + 5b_1)] \\ &= \frac{1}{b_1(1 + (p-1)b_1)}[18b_1 + 4] \\ &< 0. \end{aligned}$$

■

We next show that there are parameter values for which $\mathbf{B}$ has a snap-back repeller.

Theorem 3.1. *Let $b_{10} = -b_{20} = q > 0$ and $b_2 = -pb_1 > 0$, where $p > 0$.*

Suppose,

$$-1 < c < 1, q \geq 2p \geq 12 \\ \text{and} - b_1 \text{is sufficiently large.} \tag{32}$$

Then $\mathbf{B}$ has a snap-back repeller.

Proof. Let

$$r := \overline{x}_1 - 1 = \frac{c + (p-1)b_1}{1 + (p-1)b_1} = \frac{c-1}{1+(p-1)b_1}. \tag{33}$$

Note that β, as defined in (13), is zero in $B_r(\overline{\mathbf{x}}_\mathbf{1})$. Here r is given as in (33) and $\overline{\mathbf{x}}_\mathbf{1}$ is defined in (25). In view of remark related to (23), it suffices to show that $\lim_{n\to\infty} \mathbf{B}^{-n}(\mathbf{p}) = \overline{\mathbf{x}}_\mathbf{1}$, where $\mathbf{p}$ is defined in (28). To this end, we make a change of variables $x' = x - \overline{x}_1$ and $y' = y - \overline{x}_1$ on $\mathbf{B}$ in the region Ω_1. The resulting map $\mathbf{B}$ then has the form

$$\mathbf{B}(x', y') = (y', b_2 x' + b_1 y'). \tag{34}$$

In the new coordinate systems, (x_0, y_0) becomes (x'_0, y'_0), where

$$\begin{aligned}(x'_0, y'_0) &:= (x_0 - \overline{x}_1, y_0 - \overline{x}_1) \\ &= (d_2, d_1) \\ &= \left(\frac{-2q}{p^2 b_1^2} + \frac{2}{pb_1} - \frac{2q}{p^2 b_1}\right. \\ &\qquad \left. + \frac{2q}{p^3 b_1}, \frac{2q}{p^2 b_1} - \frac{2q}{pb_1}\right);\end{aligned} \tag{35}$$

where d_1 and d_2 are given as in (30). Note also that $d_1, d_2 > 0$. Let the pre-image (x'_0, y'_0),located in Ω_0, be denoted by (x'_{-1}, y'_{-1}). We then denote, inductively, by the pre-image of (x'_{-i}, y'_{-i}), located in Ω_0, (x'_{-i-1}, y'_{-i-1}), for any $i \in \mathbb{N}$. Using (34) and (35), we see immediately that

$$x'_{-i-1} = -\frac{b_1}{b_2} x'_{-i} + \frac{1}{b_2} y'_{-i} = \frac{x'_{-i}}{p} - \frac{y'_{-i}}{pb_1}, \tag{36a}$$

$$y'_{-i-1} = x'_{-i}. \tag{36b}$$

Let $-1 < c < 1$ and $q \geq 2p \geq 16$. By making $-b_1$ sufficiently large, we see that

$$r > x'_0 = d_2 > 0. \tag{37a}$$

and

$$r > y'_0 = d_1 > 0. \tag{37b}$$

Using (36) and (37), we may prove inductively that

$$0 < x'_{-k}, y'_{-k} < r, \\ \text{and} \lim_{k\to\infty} x'_{-k} = \lim_{k\to\infty} y'_{-k} = 0. \tag{38}$$

We have just proved that $\lim_{n\to\infty} \mathbf{B}^{-n}(\mathbf{p}) = \overline{\mathbf{x}}_\mathbf{1}$. Thus there exists a $r > 0$ sufficiently small so that $\mathbf{z}$ is a snap-back repeller in $B_r(\mathbf{z})$. ■

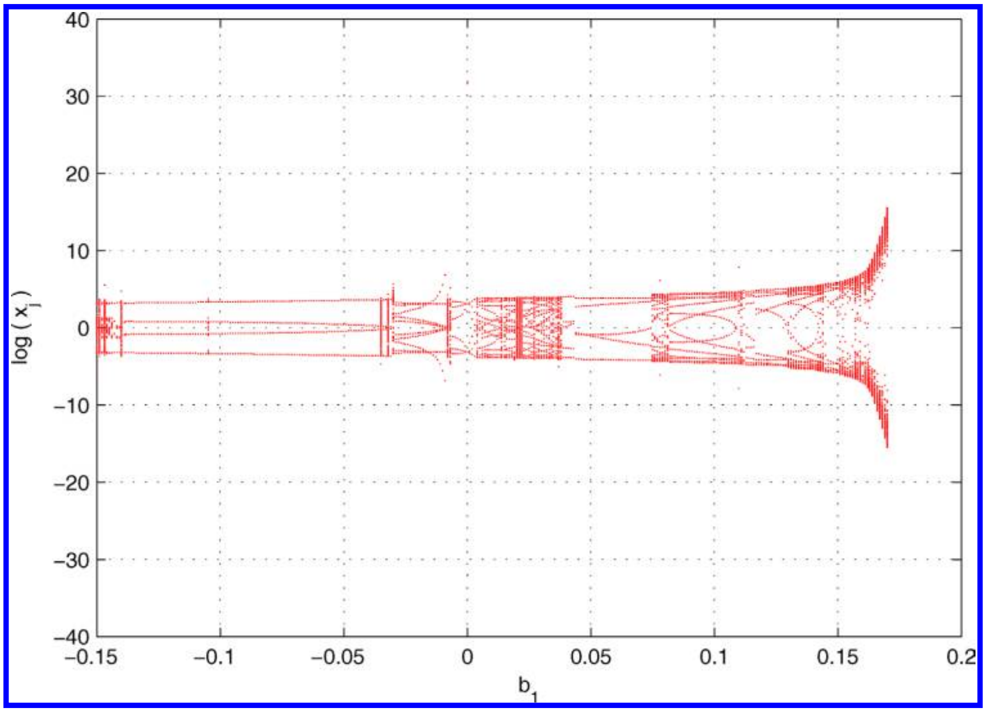
Theorem 3.2. *Let $b_{10} = -b_{20} = q > 0$ and $b_2 = -pb_1 > 0$, where $p > 0$. Suppose (32) holds. Then the system (2) exists as backward traveling waves of a chaotic profile.*

Inspired by the assumption in (32), we first treat b_1 as a bifurcation parameter and fix all other parameters. Specifically, we fix parameters $z = 0$, $\alpha = 1$, $p = 6.1$, $b_{10} = 23$, and $b_{20} = -23$. The bifurcation parameter b_1 is shown on the horizontal axis of the following plots and the vertical axis shows the logarithmical values of the x axis of the backward map $\mathbf{B}$. The periodic solutions, for example, is apparent in Fig. 5 for $|b_1|$ small and is shown in magnified form in Fig. 3 for $-0.15 \leq b_1 \leq 0.2$. When $|b_1|$ is large, the orbit of $\{y_i\}$ seems to be dense on two separate intervals and is shown in magnified form in Fig. 4 As predicted in the assumption (32), if $-b_1$ is sufficiently large, then a snap-back repeller exists. Figure 3, suggests that when $-b_1$ is small, no chaotic traveling wave would occur.

Letting $z = 0$, $\alpha = 1$, $p = 6.1$, $b_{10} = 23$, $b_{20} = -23$, we then illustrate the effectiveness of the main theorems by producing some (chaotic) profiles of the traveling waves graphically by choosing $b_1 = -0.1$ and -5, respectively. The shapes of the wave solutions y_j with $b_1 = -0.1$, -5 are shown in Figs. 6 and 7, respectively. And the amplitudes in Fig. 6 show that the oscillation is close to period four, but in Fig. 7, the amplitudes oscillate acutely.

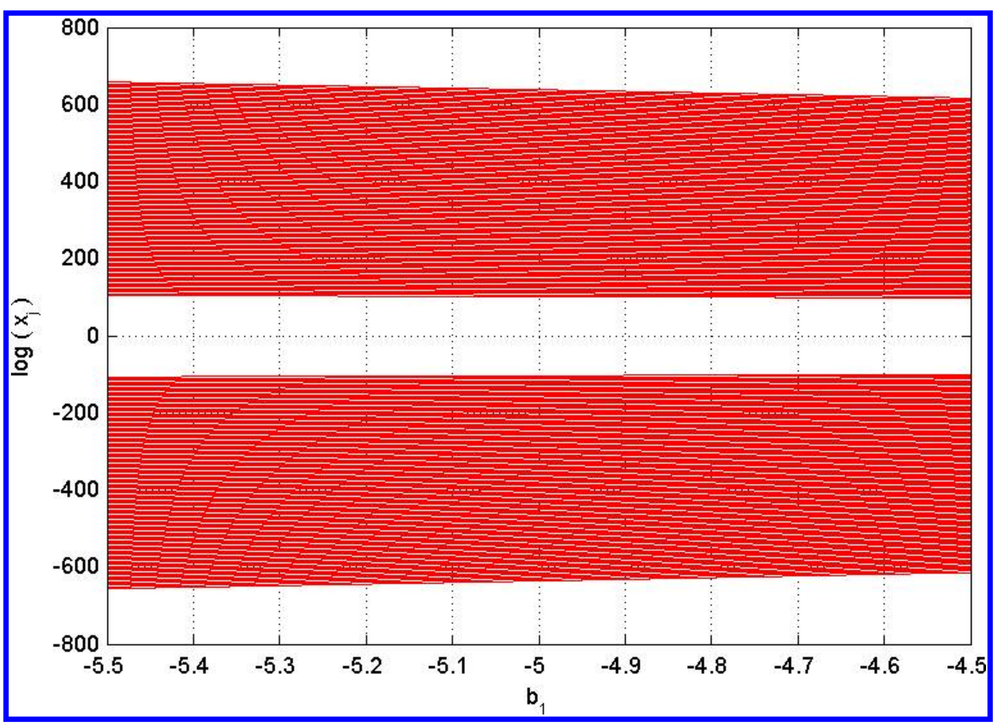
Next, their amplitude-space plots of y_j with a shift of time steps are given in Figs. 8 and 9, respectively. In (4), we have the equality $j = i + n$ for $c = 1$. Hence, y_j has the same height, provided that the index j is the same. In other words, if the sum of i and n is equal then these y_j have the same height for $i, n \in \mathbb{Z}$ (see Figs. 8 and 9). Here, we mention that the value of y_j at the corresponding i, n is shifted to 1.0.

We conclude this paper with the following remarks.



40
30
20
10
0
−10
−20
−30
−40
−0.15
−0.1
−0.05
0
0.05
0.1
0.15
0.2
log (x_j)
b_1

Fig. 3.



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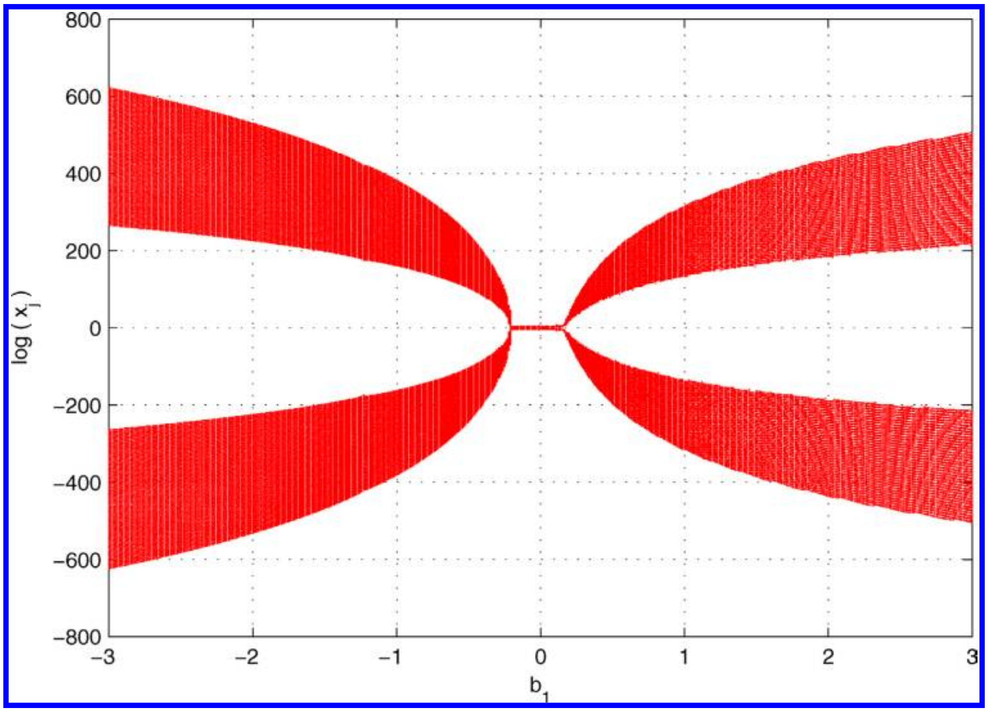
600
400
200
0
-200
-400
-600
-800
-5.5
-5.4
-5.3
-5.2
-5.1
-5
-4.9
-4.8
-4.7
-4.6
-4.5
log (x_j)
b_1

Fig. 4.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600
400
200
0
−200
−400
−600
−800
log (x_j)
−3
−2
−1
0
1
2
3
b_1

Fig. 5.

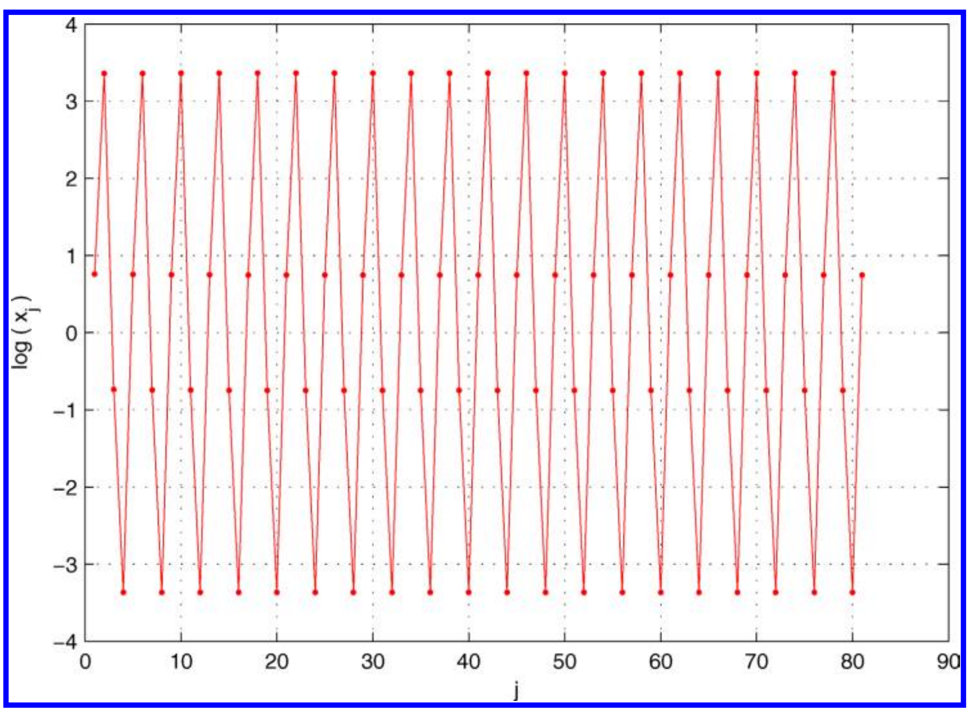
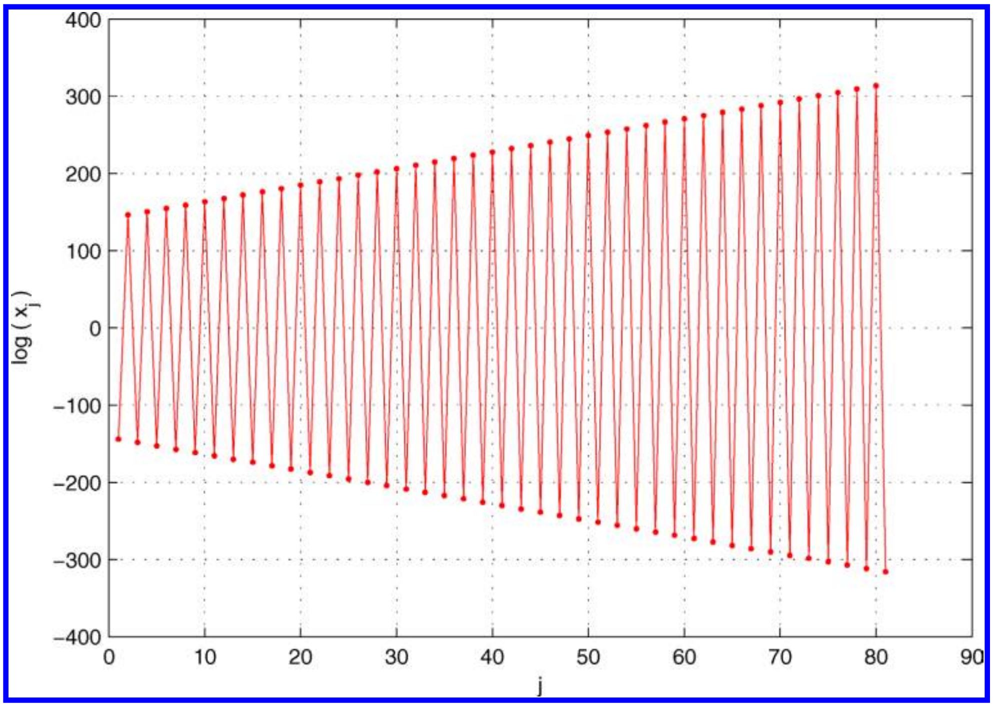

4
3
2
1
0
−1
−2
−3
−4
log (x_j)
0
10
20
30
40
50
60
70
80
90
j

Fig. 6.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300
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100
0
−100
−200
−300
−400
log (x_j)
0
10
20
30
40
50
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70
80
90
j

Fig. 7.

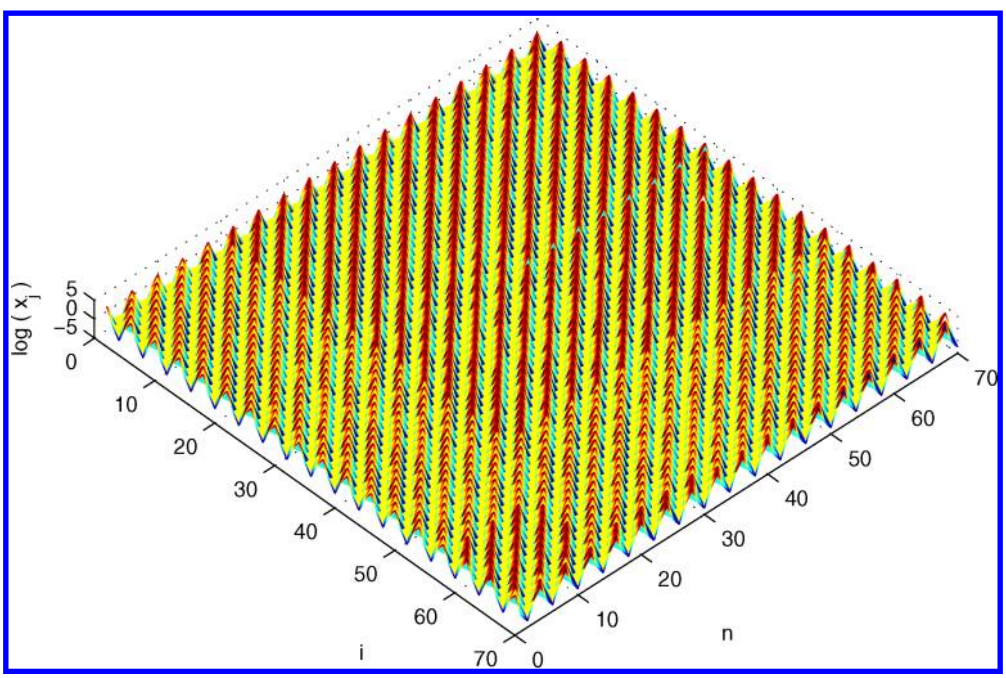
log (x_j)
5
0
−5
0
10
20
30
40
50
60
70
i
0
10
20
30
40
50
60
70
n

Fig. 8.

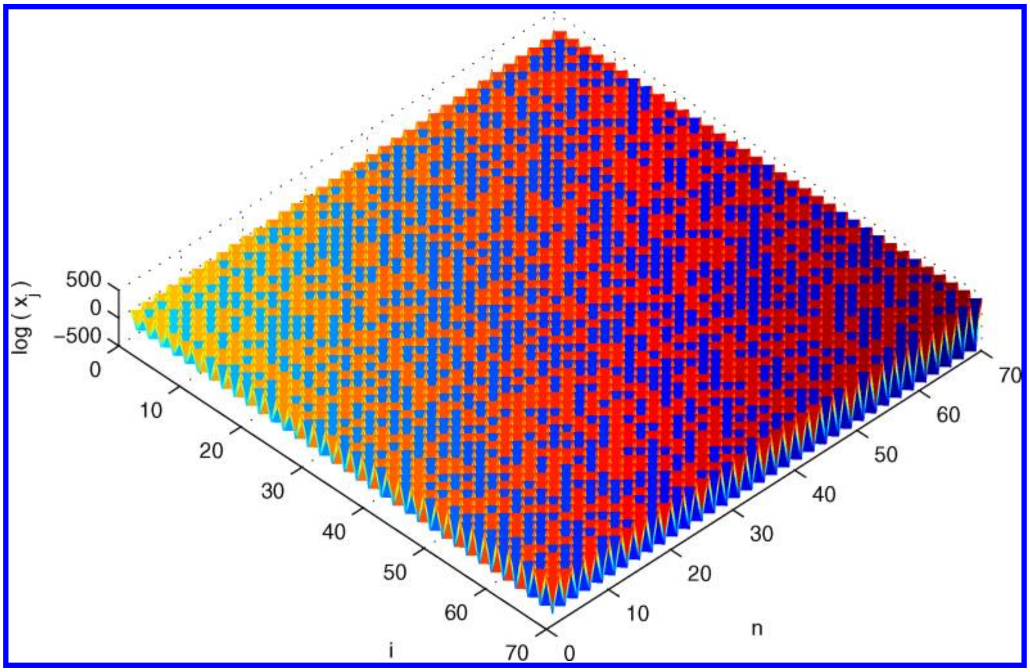

Fig. 9.

(1) Let $\triangle t$ be small, let r and $\overline{\alpha}$, as given in (1), be such that $r > 0$ is small and $\overline{\alpha}$ is negative. Then (31) is satisfied under some mild compatibility conditions of other parameters.

(2) Under similar parameters conditions, $\overline{\mathbf{x}}_{-1}$ is also a snap-back repeller. Moreover, $\overline{\mathbf{x}}_0$ can be made a snap-back repeller by a proper choice of the parameters.

(3) It is also of interest to study the chaotic dynamics of the backward map $\mathbf{B}$ for the other combinations of g_1 and g_2, as well as those of the forward map $\mathbf{F}$.

References

Afraimovich, V. S., Glebsky, L. Y. & Nekorkin, V. I. [1994] "Stability of stationary states and toplogical spatial chaos in multidimensional lattice dynamical systems," *Rand. Comput. Dyn.* **2**, 287–303.

Afraimovich, V. S. & Nekorkin, V. I. [1994] "Chaos of traveling waves in a discrete chain of diffusively coupled maps," *Int. J. Bifurcation and Chaos* **4**, 631–637.

Ban, J.-C., Chien, K.-P., Lin, S.-S. & Hsu, C.-H. [2001] "Spatial disorder of CNN-with asymmetric output function," *Int. J. Bifurcation and Chaos* **11**, 2085–2095.

Ban, J.-C., Lin, S.-S. & Hsu, C.-H. [2002] "Spatial disorder of cellular neural networks-with biased term," *Int. J. Bifurcation and Chaos* **12**, 525–534.

Chen, G., Hsu, S.-B. & Zhou, J. [1998] "Snap-back repellers as a cause of chaotic vibration of the wave equation with a van der Pol boundary condition and energy injection at the middle of the span," *J. Math. Phys.* **39**, 6459–6489.

Chow, S.-N., Mallet-Paret, J. & Shen, W. [1998] "Traveling waves in lattice dynamical systems," *J. Diff. Eqns.* **149**, 248–291.

Chua, L.-O. [1998] *CNN: A Paradigm for Complexity* (World Scientific, Singapore).

Chua, L.-O. & Yang, L. [1998a] "Cellular neural networks: Theory," *IEEE Trans. Circuits Syst.* **35**, 1257–1272.

Chua, L.-O. & Yang, L. [1998b] "Cellular neural networks: Application," *IEEE Trans. Circuits Syst.* **35**, 1273–1290.

Devaney, R. [1989] *An Introduction to Chaotic Dynamical Systems*, 2nd edition (Addison-Wesley, NY).

Erneux, T. & Nicolis, G. [1993] "Propagation waves in discrete bistable reaction diffusion systems," *Physica D* **67**, 273–244.

Fáth, G. [1998] "Propagation failure of traveling waves in discrete bistable medium," *Physica D* **116**, 176–190.

Hänggi, M. & Chua, L. O. [2000] *Simulation of RTD-based CNN Cells*, Memorandum UCB/ERL M00/51 (Electronic Research Laboratory University of California, Berkeley).

Hsu, C.-H., Lin, S.-S. & Shen, W. [1999] "Traveling waves in cellular neural networks," *Int. J. Bifurcation and Chaos* **9**, 1307–1319.

Hsu, C.-H. & Lin, S.-S. [2000] "Existence and multiplicity of traveling waves in a lattice dynamical system," *J. Diff. Eqns.* **164**, 431–450.

Hsu, C.-H. & Yang, S.-Y [2004] "On camel-like traveling wave solutions in cellular neural networks," *J. Diff. Eqns.* **196**, 481–514.

Hudson, H. & Zinner, B. [1994] "Existence of traveling waves for a generalized discrete Fisher's equation," *Commun. Appl. Nonlin. Anal.* **1**, 23–46.

Itoh, M., Julián, P. & Chua, L. O. [2001] "RTD-based cellular neural networks with multiple steady states," *Int. J. Bifurcation and Chaos* **11**, 2913–2959.

Juang, J. & Lin, S.-S. [2000] "Cellular neural networks: Mosaic pattern and spatial chaos," *SIAM J. Appl. Math.* **47**, 891–915.

Keener, J. P. [1987] "Propagation and its failure in coupled systems of discrete excitable cells," *SIAM J. Appl. Math.* **47**, 556–572.

Li, C. & Chen, G. [2003] "An improved version of the Marotto theorem," *Chaos Solit. Fract.* **18**, 69–77.

Li, T.-Y. & Yorke, J. A. [1975] "Period three implies chaos," *Amer. Math. Monthly* **82**, 985–992.

Lin, W., Ruan, J. & Zhao, W. R. [2002] "On the mathematical clarification of the snap-back repeller in high-dimensional systems and chaos in a discrete neural network model," *Int. J. Bifurcation and Chaos* **12**, 1129–1139.

Mallet-Paret, J. [1999a] "The Fredhole alternative for functional differential equations of mixed type," *J. Dyn. Diff. Eqns.* **11**, 1–48.

Mallet-Paret, J. [1999b] "The global structure of traveling waves in spatial discrete dynamical systems," *J. Dyn. Diff. Eqns.* **11**, 49–127.

Marotto, F.-R. [1978] "Snap-back repellers imply chaos in $\mathbb{R}^n$," *J. Math. Anal. Appl.* **63**, 199–223.

Ortega, J. & Rheinbaldt, W. C. [1970] *Iterative Solution of Nonlinear Equations in Several Variables* (Academic Press, NY).

Robinson, C. [1995] *Dynamical Systems: Stability, Symbolic Dynamics, and Chaos* (CRC Press, Inc.).

Thiran, P. [1997] *Dynamics and Self-organization of Locally Coupled Neural Networks* (Presses Polytechniques et Universitaries Romandes, Lausanne, Switzerland).

Wu, J. & Zou, X. [1997] "Asymptotial and periodic boundary value problems of mixed PDEs a wave solutions of lattice differential equations," *J. Diff. Eqns.* **135**, 315–357.

Zinner, B. [1992] "Existence of traveling wavefronts for the discrete Nagumo equation," *J. Diff. Eqns.* **96**, 1–27.

Zinner, B., Harris, G. & Hudson, W. [1993] "Traveling wavefronts for the discrete Fisher's equation," *J. Diff. Eqns.* **105**, 46–62.

Zou, B. & Wu, J. [1998] "Local existence and stability of periodic traveling waves of lattice differential equations," *Canadian Appl. Math. Quart.* **6**, 397–418.