# Stability analysis of the Mur's absorbing boundary condition in the alternating direction implicit finite-difference method

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**Abstract:** The stability analysis of the Mur's first order absorbing boundary condition (ABC) in the alternating direction implicit finite-difference time-domain (ADI-FDTD) method is presented. The theoretical stability analysis of this scheme is studied by deriving the amplification matrix. The effect of wave propagation direction on the stability of this scheme is investigated. From the stability analysis, it is found that the Mur's first order ABC in the ADI-FDTD method will be unstable. The instability of the scheme is validated by means of actual numerical simulations.

## 1 Introduction

Finite-difference time-domain (FDTD) method has been widely used to analyse the electromagnetic problems [1, 2]. The time step size is restricted by the Courant, Friedrichs and Lewy (CFL) stability condition due to the explicit nature of this method. Recently, an alternating direction implicit (ADI) scheme was introduced for the FDTD method. The newly proposed ADI-FDTD method is an attractive method due to its unconditional stability with large CFL numbers [3–6]. When the ADI-FDTD method is used to simulate unbounded region problems, efficient absorbing boundary conditions (ABCs) must be employed. The commonly used ABCs are Mur's first order ABC and perfectly matched layer (PML) medium. In [7, 8], the Mur's first order ABC was implemented in the ADI-FDTD method to simulate microstrip circuits. A split-field PML [9] was employed for the ADI-FDTD method [10, 11]. However, the implementation of ABCs in the ADI-FDTD method can affect the stability of this scheme. For analytical ABCs, it is found that the implementation of the third order Higdon's ABC in the ADI-FDTD method will cause instability in the simulation results [12]. In [13, 14], it was found that the ADI-FDTD method with split-field PML will lead to late-time instability from numerical simulations. Therefore it is important to analyse the stability of the ABC for the ADI-FDTD method.

In this paper, the stability analysis of the Mur's first order ABC in the ADI-FDTD method is demonstrated. The theoretical stability analysis of this scheme is studied by deriving the amplification matrix. The effect of the wave propagation direction on the stability of this scheme is investigated. From the stability analysis, it is found that the ADI scheme of the Mur's first order ABC is unstable. Since we focus on analysing the stability of the Mur's ABC at the boundary and do not consider the stability of the total computation domain, the proposed stability analysis is approximate. The stability analysis of the total computational domain can be accomplished by numerical simulation with a large number of time steps. In this work, numerical tests of the ADI-FDTD method with Mur's ABC are performed. Numerical results of this scheme with different time step size will be demonstrated to validate the instability of this scheme.

## 2 Stability analysis of the Mur's first order ABC in the ADI-FDTD method

The stability of the Mur's ABC in the ADI-FDTD method is studied. For simplicity, we consider the 2-D TM ADI-FDTD. This scheme at the $y = j_{\max-1}$ grid boundary is illustrated. Based on [5], the formulations of $H_x$, $H_y$ and $E_z$ components in the first updating step are

$$H^{n+1/2}_{xi,j_{\max-1/2}} = H^{n}_{xi,j_{\max-1/2}} - \frac{\Delta t}{2\mu\Delta y}\left(E^{n}_{zi,j_{\max}} - E^{n}_{zi,j_{\max-1}}\right) \quad (1a)$$

$$H^{n+1/2}_{yi+1/2,j_{\max-1}} = H^{n}_{yi+1/2,j_{\max-1}} + \frac{\Delta t}{2\mu\Delta x} \times \left(E^{n+1/2}_{zi+1,j_{\max-1}} - E^{n+1/2}_{zi,j_{\max-1}}\right) \quad (1b)$$

$$E^{n+1/2}_{zi,j_{\max-1}} = E^{n}_{zi,j_{\max-1}} + \frac{\Delta t}{2\varepsilon\Delta x} \times \left(H^{n+1/2}_{yi+1/2,j_{\max-1}} - H^{n+1/2}_{yi-1/2,j_{\max-1}}\right) - \frac{\Delta t}{2\varepsilon\Delta y} \times \left(H^{n}_{xi,j_{\max-1/2}} - H^{n}_{xi,j_{\max-3/2}}\right) \quad (1c)$$


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and in the second updating step

$$H_{xi,j_{\max}-1/2}^{n+1} = H_{xi,j_{\max}-1/2}^{n+1/2} - \frac{\Delta t}{2\mu\Delta y}\left(E_{zi,j_{\max}}^{n+1} - E_{zi,j_{\max}-1}^{n+1}\right) \quad (2a)$$

$$H_{yi+1/2,j_{\max}-1}^{n+1} = H_{yi+1/2,j_{\max}-1}^{n+1/2} + \frac{\Delta t}{2\mu\Delta x} \times \left(E_{zi+1,j_{\max}-1}^{n+1/2} - E_{zi,j_{\max}-1}^{n+1/2}\right) \quad (2b)$$

$$E_{zi,j_{\max}-1}^{n+1} = E_{zi,j_{\max}-1}^{n+1/2} + \frac{\Delta t}{2\varepsilon\Delta x} \times \left(H_{yi+1/2,j_{\max}-1}^{n+1/2} - H_{yi-1/2,j_{\max}-1}^{n+1/2}\right) - \frac{\Delta t}{2\varepsilon\Delta y} \times \left(H_{xi,j_{\max}-1/2}^{n+1} - H_{xi,j_{\max}-3/2}^{n+1}\right) \quad (2c)$$

The Mur's ABC is implemented at the boundary $y = j_{\max}$. The ADI schemes of the Mur's ABC are based on the formulations in [7]. As an explicit direction in the first updating step, the wave equation is written as

$$\frac{\partial}{\partial t}E_{zi,j_{\max}-1/2}^{n+1/4} = v_{\max}\frac{\partial}{\partial y}E_{zi,j_{\max}-1/2}^{n+1/4} \quad (3)$$

From (3), the field component $E_z$ at the boundary $y = j_{\max}$ can be written as

$$E_{zi,j_{\max}}^{n+1/2} = E_{zi,j_{\max}-1}^{n} + \left(\frac{v_{\max}\Delta t - 2\Delta y}{v_{\max}\Delta t + 2\Delta y}\right) \times \left(E_{zi,j_{\max}-1}^{n+1/2} - E_{zi,j_{\max}}^{n}\right) \quad (4)$$

On the other hand, as an implicit direction in the second updating step, the wave equation is written as

$$\frac{\partial}{\partial t}E_{zi,j_{\max}-1/2}^{n+3/4} = v_{\max}\frac{\partial}{\partial y}E_{zi,j_{\max}-1/2}^{n+1} \quad (5)$$

From (5), the implementation of the Mur's first order ABC for the ADI-FDTD should be applied inside the tridiagonal matrix, the field component $E_z$ at the boundary $y = j_{\max}$ becomes

$$E_{zi,j_{\max}}^{n+1}\left(1 + \frac{v_{\max}\Delta t}{\Delta y}\right) + E_{zi,j_{\max}-1}^{n+1}\left(1 - \frac{v_{\max}\Delta t}{\Delta y}\right) = \left(E_{zi,j_{\max}-1}^{n+1/2} + E_{zi,j_{\max}}^{n+1/2}\right) \quad (6)$$

Due to the adoption of the Mur's first order ABC at the boundary, the $E_{zi,j_{\max}}$ expression in (6) is substituted into (1a) and (2a), respectively. The $H_x$ components become

$$H_{xi,j_{\max}-1/2}^{n+1/2} = H_{xi,j_{\max}-1/2}^{n} - \frac{\Delta t}{2\mu\Delta y} \times \left(-E_{zi,j_{\max}-1}^{n}\left(\frac{2\Delta y}{\Delta y + v_{\max}\Delta t}\right) + \left(E_{zi,j_{\max}}^{n-1/2} + E_{zi,j_{\max}-1}^{n-1/2}\right)\left(\frac{\Delta y}{\Delta y + v_{\max}\Delta t}\right)\right) \quad (7a)$$

$$H_{xi,j_{\max}-1/2}^{n+1} = H_{xi,j_{\max}-1/2}^{n+1/2} - \frac{\Delta t}{2\mu\Delta y} \times \left(-E_{zi,j_{\max}-1}^{n+1}\left(\frac{2\Delta y}{\Delta y + v_{\max}\Delta t}\right) + \left(E_{zi,j_{\max}}^{n+1/2} + E_{zi,j_{\max}-1}^{n+1/2}\right)\left(\frac{\Delta y}{\Delta y + v_{\max}\Delta t}\right)\right) \quad (7b)$$

As shown in [5], the stability analysis of the ADI-FDTD method is studied from deriving the amplification matrix or the amplification factor for the two updating steps of this scheme. To derive the amplification matrix for the first updating step, the relation of field components at $n$th time step and $(n + 1/2)$th time step in the system of first updating equations are employed. However, the $H_x$ at $(n + 1/2)$th time step is calculated from the $E_z$ components at $(n - 1/2)$th time step and $n$th time step, as shown in (7a). To write this equation into matrix form from $n$th time step to $(n + 1/2)$th time step, we need to introduce the amplification factor $\xi$ for $E_z$ components at $(n - 1/2)$th time step and rewrite (7a) to be

$$H_{xi,j_{\max}-1/2}^{n+1/2} = H_{xi,j_{\max}-1/2}^{n} - \frac{\Delta t}{2\mu\Delta y} \times \left(-E_{zi,j_{\max}-1}^{n}\left(\frac{2\Delta y}{\Delta y + v_{\max}\Delta t}\right) + \frac{1}{\xi}\left(E_{zi,j_{\max}}^{n} + E_{zi,j_{\max}-1}^{n}\right)\left(\frac{\Delta y}{\Delta y + v_{\max}\Delta t}\right)\right) \quad (8)$$

Since $\xi$ is the amplification factor from $(n - 1/2)$th time step to $n$th time step, it is identical to the amplification factor of the second updating step. As a result, the first updating equations (1b), (1c) and (8) can be formulated in the matrix form.

The numerical stability of this scheme was determined by the Fourier method described in [15]. We assume the spatial frequencies to be $k_x$, $k_y$ and $k_z$ along the $x$, $y$ and $z$ directions, respectively, and the field components in the spatial spectral domain can be written as

$$E_{z\,i,j}^{n} = E_z^n \mathrm{e}^{j(k_x i\Delta x + k_y j\Delta y)} \quad (9a)$$

$$H_{x_{i,j+(1/2)}}^{n} = H_x^n \mathrm{e}^{j(k_x i\Delta x + k_y (j+(1/2))\Delta y)} \quad (9b)$$

$$H_{y_{i+(1/2),j}}^{n} = H_y^n \mathrm{e}^{j(k_x (i+(1/2))\Delta x + k_y j\Delta y)} \quad (9c)$$

After substituting (9) for the first updating equations, we can obtain

$$H_x^{n+1/2} = H_x^n + \left(\frac{\Delta t}{2\mu\Delta y}\left(\frac{2\Delta y}{\Delta y + v_{\max}\Delta t}\right)\exp\left\{-j\left(\frac{k_y\Delta y}{2}\right)\right\} - \frac{(\Delta t/\mu\Delta y)\cos(k_y\Delta y/2)(\Delta y/(\Delta y + v_{\max}\Delta t))}{\xi}\right)E_z^n \quad (10a)$$

$$H_y^{n+1/2} = H_y^n + j\frac{\Delta t}{\mu\Delta x}\sin\left(\frac{k_x\Delta x}{2}\right)E_z^{n+1/2} \quad (10b)$$

$$E_z^{n+1/2} = E_z^n + j\frac{\Delta t}{\varepsilon\Delta x}\sin\left(\frac{k_x\Delta x}{2}\right)H_y^{n+1/2} - j\frac{\Delta t}{\varepsilon\Delta y}\sin\left(\frac{k_y\Delta y}{2}\right)H_x^n \quad (10c)$$

Denote the field vector in the spatial spectral domain as

$$\boldsymbol{X}^n = [E_z^n \quad H_x^n \quad H_y^n]^{\mathrm{T}} \quad (11)$$

The field components for the first updating equations from $n$th time step to $(n + 1/2)$th time step can be written in the matrix form

$$\boldsymbol{X}^{n+(1/2)} = \boldsymbol{M}_1^{-1}\boldsymbol{P}_1\boldsymbol{X}^n = \boldsymbol{\Lambda}_1\boldsymbol{X}^n \quad (12)$$

where

$$\boldsymbol{M}_1 = \begin{bmatrix} 1 & 0 & -j\frac{\Delta t}{\varepsilon \Delta x}\sin\left(\frac{k_x \Delta x}{2}\right) \\ 0 & 1 & 0 \\ -j\frac{\Delta t}{\mu \Delta x}\sin\left(\frac{k_x \Delta x}{2}\right) & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{P}_1 = \begin{bmatrix} 1 & -j\frac{\Delta t}{\varepsilon \Delta y}\sin\left(\frac{k_y \Delta y}{2}\right) & 0 \\ \frac{\Delta t}{2\mu \Delta y}\left(\frac{2\Delta y}{\Delta y + v_{\max}\Delta t}\right)\mathrm{e}^{-jk_y\Delta y/2} + \frac{-(\Delta t/\mu\Delta y)\cos(k_y\Delta y/2)(\Delta y/(\Delta y + v_{\max}\Delta t))}{\xi} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The growth factor for the first updating step is the eigenvalues of $\boldsymbol{\Lambda}_1$. It can be found that $\xi$ is one element of the matrix $\boldsymbol{P}_1$.

Similar procedure can be applied to the second updating equations 2(b), 2(c) and 7(b), the field components for the second updating equations from $(n+1/2)$th time step to $(n+1)$th time step can also be written in the matrix form

$$\boldsymbol{X}^{n+1} = \boldsymbol{M}_2^{-1}\boldsymbol{P}_2\boldsymbol{X}^{n+(1/2)} = \boldsymbol{\Lambda}_2\boldsymbol{X}^{n+(1/2)} \tag{13}$$

where

$$\boldsymbol{M}_2 = \begin{bmatrix} 1 & j\frac{\Delta t}{\varepsilon \Delta y}\sin\left(\frac{k_y \Delta y}{2}\right) & 0 \\ -\frac{\Delta t}{2\mu \Delta y}\left(\frac{2\Delta y}{\Delta y + v_{\max}\Delta t}\right)\mathrm{e}^{-jk_y\Delta y/2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{P}_2 = \begin{bmatrix} 1 & 0 & j\frac{\Delta t}{\varepsilon \Delta x}\sin\left(\frac{k_x \Delta x}{2}\right) \\ -\frac{\Delta t}{\mu \Delta y}\cos\left(\frac{k_y \Delta y}{2}\right)\left(\frac{\Delta y}{\Delta y + v_{\max}\Delta t}\right) & 1 & 0 \\ j\frac{\Delta t}{\mu \Delta x}\sin\left(\frac{k_x \Delta x}{2}\right) & 0 & 1 \end{bmatrix}$$

The growth factor $\xi$ for the second updating step is the eigenvalues of $\boldsymbol{\Lambda}_2$.

Combining the two half time steps can lead to one time step

$$\boldsymbol{X}^{n+1} = \boldsymbol{\Lambda}_1\boldsymbol{\Lambda}_2\boldsymbol{X}^{n} = \boldsymbol{\Lambda}\boldsymbol{X}^{n} \tag{14}$$

Therefore to obtain the amplification factors for the two updating steps, one can solve the eigenvalues of $\boldsymbol{\Lambda}_2$ to obtain the amplification factor $\xi$ first. After the $\xi$ value is obtained, the amplification factor for the first updating step can be solved. In this work, the $\xi$ value is solved

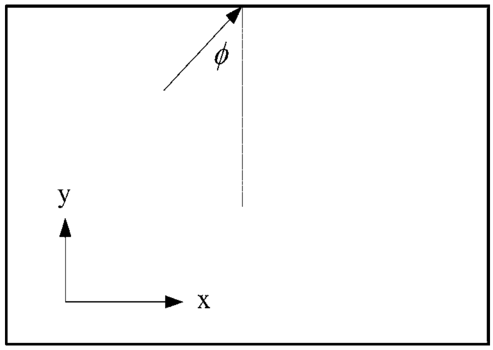


**Fig. 1** *Wave propagates upon the Mur's ABC at $y = j_{max}$*

to obtain the matrix $\boldsymbol{\Lambda}$. When the matrix $\boldsymbol{\Lambda}$ is obtained, the amplification factor for the total updating step can be found. The stability of this scheme requires that the eigenvalues of $\boldsymbol{\Lambda}$ lie within or on the unit circle that is $|\lambda_{\boldsymbol{\Lambda}}| \leq 1$. Due to the complexity of the amplification matrix $\boldsymbol{\Lambda}$, it is difficult to obtain a simplified analytical expression for the eigenvalues. The eigenvalues are numerically calculated by Matlab®. The amplification matrix $\boldsymbol{\Lambda}$ is a function of the discrete wavenumber. All propagation directions are considered to study the stability of this scheme. Let $kx = k\sin\phi$, $ky = k\cos\phi$ and $k = \sqrt{(k_x^2 + k_y^2)}$; angle $\phi$ is incident angle with respective to $y$-axis, as shown in Fig. 1.

A 2-D computation domain is studied and the ratio of $\Delta t/\Delta t_{\max}$ is defined as the CFL number (CFLN). The cell size with $\Delta x = \Delta y = 1.0$ mm and FDTD time step size limit $\Delta t_{\max} = 2.35$ ps are used. The effects of the time steps and the wave propagation direction on the stability of this scheme are investigated. The calculated maximum eigenvalues of $\boldsymbol{\Lambda}$ for different time step size and wave propagation direction are shown in Fig. 2. This scheme will be stable only when the propagation directions are at $\phi = 0°$, 45° and 90° and will become unstable at other propagation directions. It can also be found that the eigenvalues are larger than unity even when CFLN = 1 is used. In a practical ADI-FDTD simulation, the electromagnetic wave will not propagate at specific direction when it reaches the ABC. Since the ADI scheme of the Mur's ABC is unstable, the field components at the boundary will become unstable. Due to the unstable field components at the boundary, the stability of the total computation domain will be affected.

## 3 Numerical simulation

In this paper, we use the Fourier method to study the stability of the Mur's ABC in ADI scheme. When the

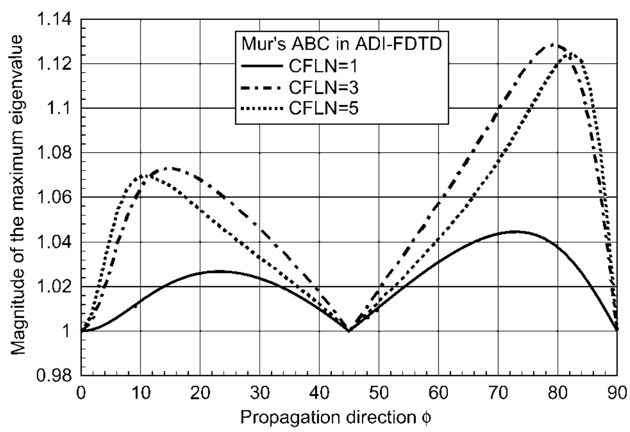


**Fig. 2** *Maximum eigenvalue of $\boldsymbol{\Lambda}$ for different propagation directions*

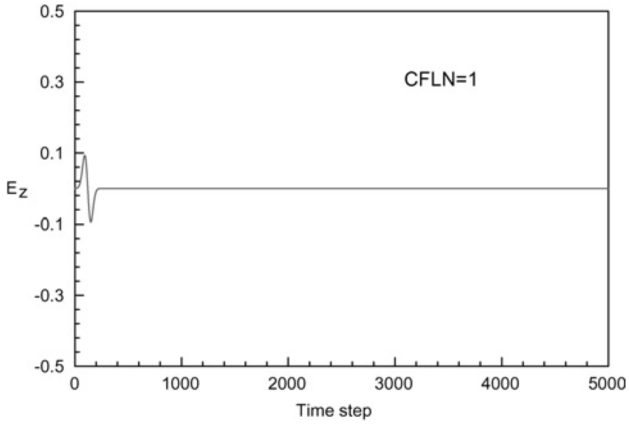


**Fig. 3** *Numerical simulation of 3-D ADI-FDTD with Mur's ABC (CFLN = 1)*

eigenvalues of this scheme are larger than unity, it means the ADI scheme of the Mur's ABC will be unstable. Since the field components will be unstable at the boundary, the ADI-FDTD method with Mur's ABC will become unstable. To illustrate the instability of this scheme, numerical verification of instability is performed by 3-D ADI-FDTD with Mur's first order ABC. In this study, a uniform mesh with cell size $\Delta x = \Delta y = \Delta z = 1.0$ mm and the maximum FDTD time step $\Delta t_{\max} = 1.92 \times 10^{-12}$ s are used. The computation domain is $42 \times 42 \times 42$. The Mur's first order ABCs are applied on the six sides of the computation domain. A differential Gaussian pulse is launched for $E_z$ component. The source is excited at the centre position (21, 21, 21) and the observation point is positioned at (21, 20, 21).

Numerical simulations of the ADI-FDTD with Mur's ABC for different CFLN are demonstrated. First, numerical simulation of this scheme with CFLN = 1 is studied. From the theoretical stability analysis, we find that the Mur's ABC in the ADI-FDTD will be unstable. However, it is found that numerical simulation of this scheme for the total computational domain can still be stable after running 5000 time step, as shown in Fig. 3. We have extended the time steps to 100 000 time steps and no instability is observed. This is because the maximum eigenvalues of the Mur's ABC in ADI-FDTD are slightly larger than unity when CFLN = 1, and it will require a large number of time steps to make this scheme unstable. The ADI-FDTD method can be efficient only when large CFLN is used. This scheme with

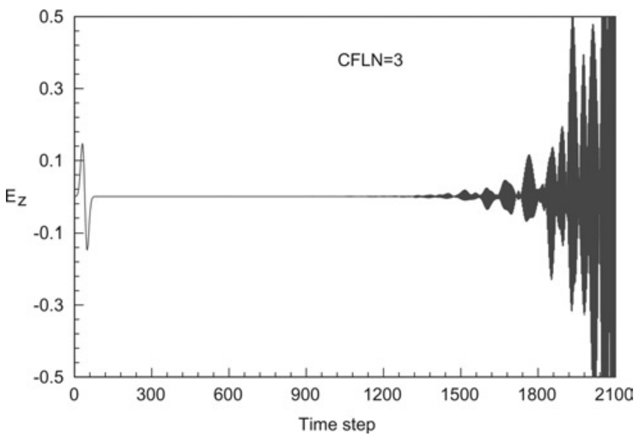


**Fig. 4** *Numerical simulation of 3-D ADI-FDTD with Mur's ABC (CFLN = 3)*

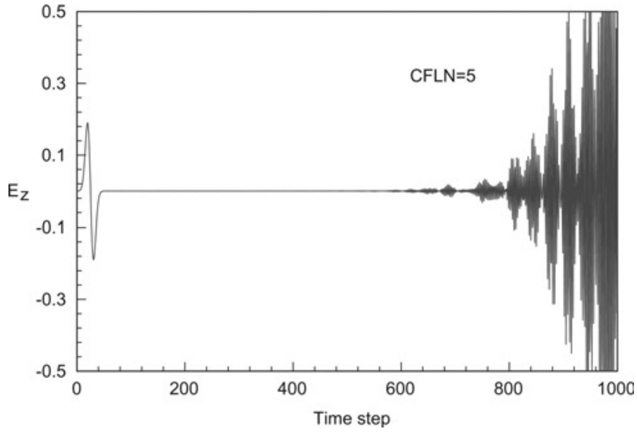


**Fig. 5** *Numerical simulation of 3-D ADI-FDTD with Mur's ABC (CFLN = 5)*

CFLN = 3 and CFLN = 5 are studied, as shown in Figs. 4 and 5, respectively. It can be found that instability of this scheme will appear after running 1300 time steps and 600 time steps for CFLN = 3 and CFLN = 5, respectively. With the implementation of the Mur's ABC in the ADI-FDTD method, this scheme will become unstable with less time steps when larger CFLN is used.

## 4 Conclusion

In this work, the stability analysis of the Mur's ABC in the ADI-FDTD is studied. The stability analysis is performed by deriving the amplification matrix of this scheme. The effect of the propagation direction on the stability is investigated. From the stability analysis, we find this scheme will be stable only when the propagation directions are at $\phi = 0^\circ$, $45^\circ$ and $90^\circ$ and will become unstable at other propagation directions. Since the ADI scheme of the Mur's ABC is unstable, the field components at the boundary will become unstable. Due to the unstable field components at the boundary, the stability of the total computation domain will be affected. Numerical simulations of 3-D ADI-FDTD method with Mur's ABC are demonstrated to validate the instability of this scheme.

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