Available online at www.sciencedirect.com
ScienceDirect
Applied Mathematics and Computation 188 (2007) 1848–1855
APPLIED MATHEMATICS AND COMPUTATION
www.elsevier.com/locate/amc

Restricted connectivity for three families of interconnection networks

Y-Chuang Chen [a,*], Jimmy J.M. Tan [b]

[a] *Ming Hsin University of Science and Technology, Hsin Feng, Hsinchu 304, Taiwan, ROC*
[b] *Department of Computer and Information Science, National Chiao Tung University, Hsinchu 300, Taiwan, ROC*

Abstract

Vertex connectivity and edge connectivity are two important parameters in interconnection networks. Even though they reflect the fault tolerance correctly, they undervalue the resilience of large networks. By the concept of conditional connectivity and super-connectivity, the concept of restricted vertex connectivity and restricted edge connectivity of graphs was proposed by Esfahanian [A.H. Esfahanian, Generalized measures of fault tolerance with application to N-cube networks, IEEE Transactions on Computers 38 (1989) 1586–1591]. Such measures take the resilience of large networks into consideration. In this paper, we propose three families of interconnection networks and discuss their restricted vertex connectivity and restricted edge connectivity. In particular, the hypercubes, twisted-cubes, crossed-cubes, möbius cubes, star graphs, pancake graphs, recursive circulant graphs, and k-ary n-cubes are special cases of these families.

Keywords: Connectivity; Super-connectivity; Restricted vertex connectivity; Restricted edge connectivity

1. Introduction

For the interconnection network topology, it is usually represented by a graph $G = (V, E)$, while vertices represent processors and edges represent links between processors. For the purpose of connecting hundreds or thousands of processing elements, many interconnection network topologies have been proposed in the literature. Graph theory can be used to analyze the network reliability and most of the graph definitions we use are standard (see, e.g. [2]). We use terms graphs and networks interchangeably. $G = (V, E)$ is a simple graph if V is a finite set and E is a subset of $\{(a, b) \mid (a, b)$ is an unordered pair of $V\}$. We say that V is the vertex set and E is the edge set. The neighborhood of a vertex v in graph G, $N_G(v)$, is $\{x \mid (v, x) \in E\}$. The neighbor-edge set of v in graph G, $NE_G(v)$, is $\{(v, x) \mid (v, x) \in E\}$. The degree of v in G, denoted by $\deg_G(v)$, is the number of vertices in $N_G(v)$. G is k-regular if $\deg_G(v) = k$, for every vertex $v \in V$. A perfect matching of a graph G is a set M of edges such that (1) no two edges are incident with a common vertex, and (2) each vertex of G is incident to some edge in M.

* Corresponding author.
E-mail address: cardy@must.edu.tw (Y-C. Chen).

0096-3003/$ - see front matter © 2006 Elsevier Inc. All rights reserved.
doi:10.1016/j.amc.2006.11.085

A *vertex cut* of a graph G is a set $S \subseteq V(G)$ such that $G - S$ has more than one connected component. It is a known fact that only complete graphs do not have vertex cuts. The *vertex connectivity* of G, written $\kappa(G)$, is defined as the minimum size of a vertex cut if G is not a complete graph, and $\kappa(G) =| V(G) | -1$ otherwise. A graph G is k-vertex-connected if $\kappa(G) \geqslant k$. Assume that graph G is k-regular with vertex connectivity κ. We say that G is *maximum vertex connected* if $\kappa = k$; and G is *super-vertex-connected* if it is a complete graph, or it is maximum vertex connected and every minimum vertex cut is $N_G(v)$ for some vertex v. An *edge disconnecting set* is a set $F \subseteq E(G)$ such that $G - F$ has more than one connected component. A graph is *k-edge-connected* if every disconnecting set has at least k edges. The *edge connectivity* of G, written $\lambda(G)$, is the minimum size of an edge disconnecting set. A graph G is k-edge-connected if $\lambda(G) \geqslant k$. Assume that G is a k-regular graph with edge connectivity λ. We recall that G is *maximum edge connected* if $\lambda = k$; and G is *super-edge-connected* if it is maximum edge connected and every minimum edge disconnecting cut is $NE_G(v)$ for some vertex v. Results concerning the super-vertex-connectivity and super-edge-connectivity of the hypercubes, twisted-cubes, crossed-cubes, and möbius cubes can be found in [5,6].

The fault tolerance of a network with respect to processor (respectively link) failures is directly related to the vertex (respectively edge) connectivity of the corresponding graph. Even though κ and λ reflect the fault tolerance correctly, they undervalue the resilience of large networks [14]. In interconnection networks, there are many parameters to evaluate the performance of the network topologies. Super-vertex-connectivity and super-edge-connectivity are extensions of κ and λ [5]. The concept of conditional connectivity was proposed by Harary [12], and the extension restricted edge connectivity was given in [16,17]. Let $G = (V, E)$ be a connected graph. A set of vertices S, $S \subset V$, is a 1-*cut* if $|S \cap N_G(v)| \leqslant \deg(v) - 1$ for every vertex $v \in V(G)$ and $G - S$ is disconnected. The *restricted vertex connectivity* κ_1 is defined to be the size of minimize 1-cut if such 1-cut exists and undefined otherwise. In other words, every vertex $v \in V(G)$ has at least 1 non-faulty neighbor vertex, even though vertex v is faulty. The definition of *restricted edge connectivity* λ_1 is defined similarly to κ_1. A set of edges S, $S \subset E$, is a *1-edge-cut* if $|S \cap NE_G(v)| \leqslant \deg(v) - 1$ for every vertex $v \in V(G)$ and $G - S$ is disconnected. The restricted edge connectivity λ_1 is defined to be the size of minimize 1-edge-cut if such 1-edge-cut exists and undefined otherwise. It has been shown that the hypercube Q_n can tolerate upto $2n - 3$ faulty vertices [10] without disconnecting under these conditions. The restricted vertex connectivity κ_1 and restricted edge connectivity λ_1 provide more accurate measures of fault-tolerance of interconnection networks than super-vertex-connectivity and super-edge-connectivity, respectively.

In this paper, we propose three families of interconnection networks: $G(G_0, G_1; M)$, SP_n, and $G(G_0, G_1, \ldots, G_{r-1}; \mathscr{M})$ to discuss their κ_1 and λ_1 properties. We note however, many popular networks belong to some groups of these three families, such as hypercubes [3,10], twisted-cubes [4], crossed-cubes [9], möbius cubes [7], star graphs [1], pancake graphs [11], recursive circulant graphs [15], and k-ary n-cubes [8]. Additionally, rings, meshes, tori, hypercubes, and Omega networks are isomorphic to k-ary n-cube [8].

There are many useful topologies proposed in interconnection networks. Among them, the binary hypercube, Q_n, is one of the most popular topology. However, a hypercube does not make the best use of its hardware since it is possible to fashion networks with lower diameters than that of Q_n. One such topology is the crossed-cube CQ_n [9,13]. It has a diameter of $\lceil (n+1)/2 \rceil$, an improvement of approximately a factor of 2 as compared with Q_n. Additionally, there are some other popular graphs, such as the twisted-cubes, möbius cubes, star graphs, pancake graphs, recursive circulant graphs, and k-ary n-cubes each of them has some recursively construction schemes similar to that of the hypercubes and crossed-cubes, and each has useful topological properties. It has been shown that if a network possesses the restricted vertex connectivity (respectively restricted edge connectivity) property, it is more reliable and has the smaller vertex (respectively edge) failure rate [10,14].

The outline of this paper is as follows. In the next section, we give three families of interconnection networks which satisfies some of the κ_1 and λ_1 properties. In Section 3, we state and show the values of κ_1 and λ_1 for these families and discuss the κ_1 and λ_1 for some popular networks. Section 4 concludes our results.

2. Three families of interconnection networks

Many popular networks, such as the hypercubes, twisted-cubes, crossed-cubes, möbius cubes, star graphs, pancake graphs, recursive circulant graphs, and k-ary n-cubes, are composed of some lower dimension

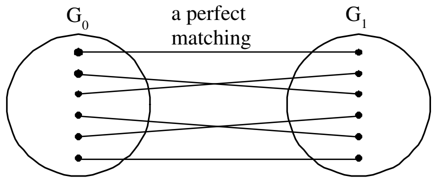

Fig. 1. Graph $G(G_0, G_1; M)$.

components and some specific links between components. For example, (1) the crossed-cube CQ_n is composed of two CQ_{n-1}'s and a perfect matching between the two CQ_{n-1}; (2) the star graph S_n is composed of n S_{n-1}'s and some additional links with a specific rule; and (3) the recursive circulant graph $RC(c,d,r)$ is composed of d $RC(c,d,r-1)$'s and some links with a specific rule. In this section, we give three families of networks, and show that they satisfy some properties of restricted vertex connectivity and restricted edge connectivity.

2.1. The first family $G(G_0, G_1; M)$ of networks

Let G_0 and G_1 be two graphs with the same number of vertices. We define a new graph $G(G_0, G_1; M)$, which has vertex set $V(G_0) \cup V(G_1)$ and edge set $E(G_0) \cup E(G_1) \cup M$, where M is an arbitrary perfect matching between the vertices of G_0 and G_1; i.e., a set of $|V(G_0)|$ cross edges with one endpoint in G_0, and the other endpoint in G_1 (see Fig. 1). We observe that the hypercubes are constructed in this way, so are many variations of the hypercubes, such as the twisted-cubes, crossed-cubes, möbius cubes etc. We also note that these cubes do not contain any triangle, i.e., cycles of length three.

2.2. The second family SP_n of networks

We define the graph SP_n for $n \geqslant 3$. SP_3 is a cycle of length 6 which is isomorphic to the star graph S_3 and pancake graph P_3. For $n > 3$, SP_n consists of n disjoint SP_{n-1}'s, say $SP^1_{n-1}, SP^2_{n-1}, \ldots, SP^n_{n-1}$. The vertex set of each SP^i_{n-1} for $1 \leqslant i \leqslant n$ is divided arbitrarily into $n-1$ disjoint vertex sets equally, say $S^{i,1}, S^{i,2}, \ldots, S^{i,n-1}$. For every SP^i_{n-1} and SP^j_{n-1}, $i \neq j$, there exists a perfect matching between $S^{i,x}$ and $S^{j,y}$ for some x and y, so that SP_n is $(n-1)$-regular. Examples of SP_3, SP_4, and SP_5 are shown in Fig. 2. Note that the star graph S_n and pancake graph P_n are special cases of SP_n and they contain no triangle.

2.3. The third family $G(G_0, G_1, \ldots, G_{r-1}; \mathscr{M})$ of networks

Let $G_0, G_1, \ldots, G_{r-1}$ be graphs with $|V(G_i)| = t$ for $i = 0, 1, \ldots, r-1$. Let r and t be positive integers with $r \geqslant 3$. We denote $H = G(G_0, G_1, \ldots, G_{r-1}; \mathscr{M})$ with $V(H) = V(G_0) \cup V(G_1) \cup \cdots \cup V(G_{r-1})$ and $E(H) = \bigcup_{i=0}^{r-1} M_{i,i+1(\mathbf{mod}\, r)}$, where $M_{i,i+1(\mathbf{mod}\, r)}$ is an arbitrary perfect matching between $V(G_i)$ and $V(G_{i+1(\mathbf{mod}\, r)})$. An example of graph H is illustrated in Fig. 3. Recursive circulant graphs [15] and k-ary n-cubes [8] are special cases of this family.

3. Restricted vertex connectivity and restricted edge connectivity

We show two lemmas before showing the restricted vertex connectivity of graph $G(G_0, G_1; M)$.

Lemma 1. *Given an n-regular graph G which contains no triangle. Then,* $|V(G)| \geqslant 2n$.

Proof. Let (a,b) be an edge in G. Since $\deg_G(a) = \deg_G(b) = n$ and there is no triangle, we conclude that $N_G(a) \cap N_G(b) = \emptyset$. Therefore, $|N_G(a) \cup N_G(b)| = 2n$, and the lemma follows. □

Let $K_{n,n}$ be a bipartite complete graph with $V(K_{n,n}) = \{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\}$ and $E(K_{n,n}) = \{(x_i, y_j) |$ for every i and every $j\}$. It is clear that $K_{n,n}$ is an n-regular graph containing no triangle with $|V(G)| = 2n$. Thus, the result of Lemma 1 is a tight bound. The following lemma concerning the vertex connectivity of $G(G_0, G_1; M)$ which was discussed in [5].

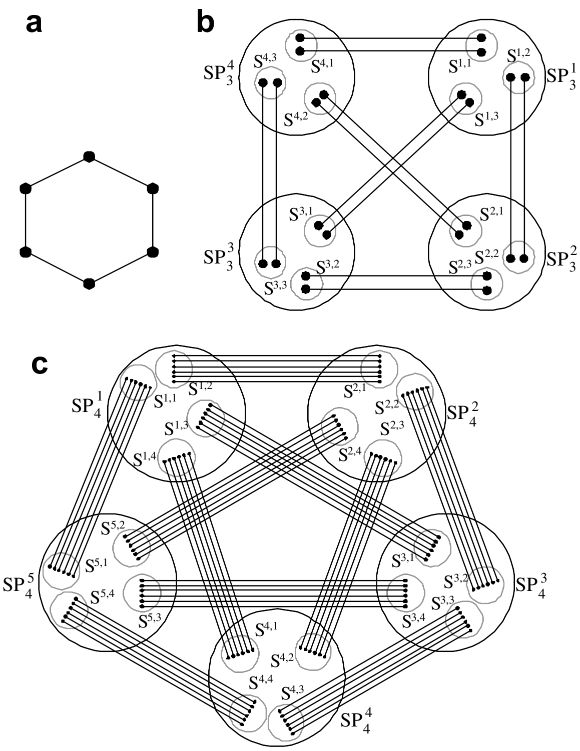

Fig. 2. Graphs (a) SP_3, (b) SP_4, and (c) SP_5.

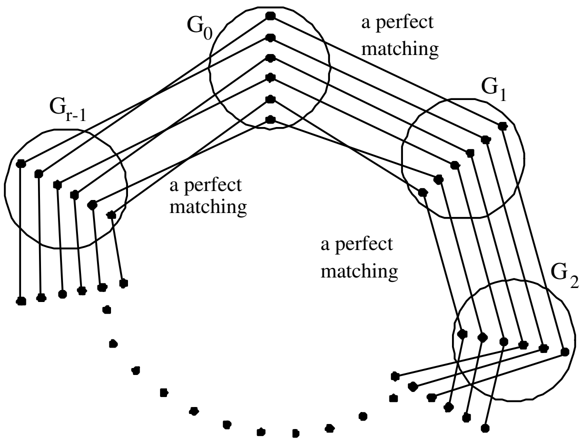

Fig. 3. Graph $G(G_0, G_1, \ldots, G_{r-1}; \mathscr{M})$.

Lemma 2. *Let graphs G_0 and G_1 be two n-regular graphs with the same number of vertices with $n \geqslant 1$. Assume that $\kappa(G_0) = \kappa(G_1) = n$. Then, $\kappa(G(G_0, G_1; M)) = n + 1$.*

Let G_0 and G_1 be n-regular graphs with the same number of vertices, containing no triangle, and with connectivity $\kappa = n$. It is trivial that $G(G_0, G_1; M)$ is $(n+1)$-regular. We shall write $G(G_0, G_1; M)$ simply as G, if there is no ambiguity. We show that not only $\kappa(G)$ is increasing to $n+1$, but also $\kappa_1(G)$ is increasing to $2(n+1) - 2$.

Theorem 1. *Let n and t be two positive integers with $n \geqslant 2$ and $t > n+1$. Let G_0 and G_1 be two n-regular graphs, $|V(G_0)| = |V(G_1)| = t$, containing no triangle. Assume that $\kappa(G_0) = \kappa(G_1) = n$. Then, $G = G(G_0, G_1; M)$ is $(n+1)$-regular, containing no triangle, $\kappa(G) = n+1$, and $\kappa_1(G) = 2n$.*

Proof. By definition, G is $(n+1)$-regular and contains no triangle. By Lemma 2, $\kappa(G) = n+1$. To prove $\kappa_1(G)$ is equal to $2n$, we shall prove that both $\kappa_1(G) \leqslant 2n$ and $\kappa_1(G) \geqslant 2n$. Since G contains no triangle, for every edge $(u, v) \in E(G)$, $|N_G(u) \cup N_G(v) - \{u, v\}| = 2n$ and $G - (N_G(u) \cup N_G(v) - \{u, v\})$ is disconnected. Therefore, $\kappa_1(G) \leqslant 2n$.

To prove that $\kappa_1(G) \geqslant 2n$, we shall check that $G - F$ is connected for every vertex subset F of G with $|F| = 2n-1$ and $F \not\supset N_G(v)$ for any vertex $v \in V(G)$. Let $F_{G_0} = G_0 \cap F$ and $F_{G_1} = G_1 \cap F$. Since $|F| = 2n-1$ and $F_{G_0} \cap F_{G_1} = \emptyset$, $|F_{G_0}| \leqslant n-1$ or $|F_{G_1}| \leqslant n-1$. We may without loss of generality assume that $|F_{G_0}| \leqslant n-1$. Since $\kappa(G_0) = n$, $G_0 - F_{G_0}$ is connected. A *cross edge* of G is an edge such that one endpoint is in G_0 and the other one is in G_1.

Case 1: $G_1 - F_{G_1}$ is also connected.By Lemma 1, $|V(G_0)| = |V(G_1)| \geqslant 2n$. The number of cross edges between G_0 and G_1 is at least $2n$, which is great than $|F|$. So there exists a cross edge (a_0, a_1) such that $a_0 \in V(G_0 - F_{G_0})$ and $a_1 \in V(G_1 - F_{G_1})$. Thus, $G - F$ is connected.

Case 2: $G_1 - F_{G_1}$ is disconnected.We may without loss of generality assume that $G_1 - F_{G_1}$ is divided into k disjoint connected components, say $H_1, H_2, \ldots, H_k$, where $k \geqslant 2$. That is, $V(H_i) \cap V(H_j) = \emptyset$ for every $i \neq j$ and $H_1 \cup H_2 \cup \cdots \cup H_k = G_1 - F_{G_1}$. Now, we shall prove that there exists an edge (a_i, b_i) such that $a_i \in G_0$, $b_i \in H_i$, and $a_i \notin F$ for $i = 1, 2, \ldots, k$. In the following two subcases, we consider the number of vertices of H_i with $i \in 1, 2, \ldots, k$.

Subcase 2.1: $|V(H_i)| = 1$.
Let u_h be the vertex in H_i and (u_h, u_g) be its corresponding cross edge between H_i and G_0. By definition, $N_G(u_h) \not\subset F$. So $u_g \notin F$. Then, u_h is connected with every vertex in $G_0 - F$.

Subcase 2.2: $|V(H_i)| \geqslant 2$.
Let (u_h, v_h) be an edge in H_i. Since there is no triangle in G, $|N_{G_1}(u_h) \cup N_{G_1}(v_h)| = n + n = 2n$. Let $|V(H_i)| = m \geqslant 2$. We have the following inequality:

$$\left|\bigcup_{v \in V(H_i)} N_{G_1}(v)\right| - |V(H_i)| \geqslant 2n - m.$$

Suppose each cross edge between G_0 and H_i has at least one faulty vertex, then $|F| \geqslant (2n-m) + m = 2n > 2n-1$. It's a contradiction to our assumption that $|F| = 2n-1$. So there exists a cross edge (a_i, b_i) such that $a_i \in V(G_0)$, $b_i \in V(H_i)$, and $a_i \notin F$. Therefore, $G - F$ is connected. Thus, $\kappa_1(G) = 2(n+1) - 2 = 2n$ and we complete the proof of this theorem. □

With a similar argument as above, we have the following result.

Theorem 2. *Let n and t be two positive integers with $n \geqslant 1$ and $t > n+1$. Let G_0 and G_1 be two n-regular graphs, $|V(G_0)| = |V(G_1)| = t$, containing no triangle. Assume that $\lambda(G_0) = \lambda(G_1) = n$. Then, $G = G(G_0, G_1; M)$ is $(n+1)$-regular, containing no triangle, $\lambda(G) = n+1$, and $\lambda_1(G) = 2n$.*

Since graph $G(G_0, G_1; M)$ is a generalization of the hypercubes Q_n, twisted-cubes TQ_n, crossed-cubes CQ_n, and möbius cubes MQ_n, we have the following corollary.

Corollary 1. *$\kappa_1(Q_n) = \kappa_1(TQ_n) = \kappa_1(CQ_n) = \kappa_1(MQ_n) = 2n-2$ for $n \geqslant 3$ and $\lambda_1(Q_n) = \lambda_1(TQ_n) = \lambda_1(CQ_n) = \lambda_1(MQ_n) = 2n-2$ for $n \geqslant 2$.*

For the connectivity of SP_n for $n \geqslant 3$, $\kappa(SP_n) = \deg_{SP_n}(v) = n-1$ for every vertex $v \in V(SP_n)$. Since the proof is very similar to that of the connectivity of $G(G_0, G_1; M)$ in [5], we omit the proof here.

Lemma 3. *$\kappa(SP_n) = n-1$ for $n \geqslant 3$.*

For the restricted vertex connectivity κ_1 of SP_n, we show that $\kappa_1(SP_n) = 2(n-1) - 2 = 2n-4$.

Theorem 3. *$\kappa_1(SP_n) = 2n-4$ for $n \geqslant 3$.*

Proof. To prove that $\kappa_1(SP_n) = 2n-4$ for $n \geqslant 3$, we shall prove that $\kappa_1(SP_n) \leqslant 2n-4$ and $\kappa_1(SP_n) \geqslant 2n-4$. Since SP_n contains no triangle, for every edge $(u,v) \in E(SP_n)$, $|N_{SP_n}(u) \cup N_{SP_n}(v) - \{u,v\}| = 2n-4$ and $SP_n - (N_{SP_n}(u) \cup N_{SP_n}(v) - \{u,v\})$ is disconnected. Therefore, $\kappa_1(SP_n) \leqslant 2n-4$ for $n \geqslant 3$.

Now, to show that $\kappa_1(SP_n) \geqslant 2n-4$ for $n \geqslant 3$, we need to check that $SP_n - F$ is connected for every vertex subset F of SP_n such that $|F| = 2n-5$ and $F \not\supset N_{SP_n}(v)$ for any vertex $v \in V(SP_n)$. Let $F_i = SP^i_{n-1} \cap F$ for $1 \leqslant i \leqslant n$.

Case 1: $SP^i_{n-1} - F_i$ is connected for $1 \leqslant i \leqslant n$.$|V(SP^i_{n-1})| = (n-1)!$ and $|F| = 2n-5$. For every $1 \leqslant i \neq j \leqslant n$, there are $(n-2)!$ cross edges between $SP^i_{n-1} - F_i$ and $SP^j_{n-1} - F_j$. Because $(n-2)! > |F| = 2n-5$ for every $n \geqslant 5$. Therefore, for $n \geqslant 5$ and every $1 \leqslant i \neq j \leqslant n$, there exists a cross edge (u,v) between $SP^i_{n-1} - F_i$ and $SP^j_{n-1} - F_j$. As for $n = 3,4$, it can be proved by brute force that $SP_n - F$ is connected. So $SP_n - F$ is connected.

Case 2: Some of $SP^1_{n-1} - F_1, SP^2_{n-1} - F_2, \ldots, SP^n_{n-1} - F_n$ are disconnected.We may without loss of generality assume that $SP^1_n - F_1$ is disconnected. Thus, $|F_1| \geqslant n-2$ and $|F - F_1| \leqslant n-3$. So $SP^2_{n-1} - F_2$, $SP^3_{n-1} - F_3, \ldots, SP^n_{n-1} - F_n$ are all connected. Similarly to the proof of Case 1, $SP_n - (SP^1_{n-1} \cup F)$ is connected.Now, we may without loss of generality assume that $SP^1_{n-1} - F_1$ is divided into k disjoint connected components, say $H_1, H_2, \ldots, H_k$, where $k \geqslant 2$. That is, $V(H_i) \cap V(H_j) = \emptyset$ for every $i \neq j$ and $H_1 \cup H_2 \cup \cdots \cup H_k = SP^1_{n-1} - F_1$. Now, we shall prove that there exists a cross edge (a_i, b_i) such that $a_i \in V(H_i)$, $b_i \in V(SP_n - SP^1_{n-1})$, and $b_i \notin F$ for $i = 1,2,\ldots,k$. In the following two subcases, we consider the number of vertices of H_i with $i = 1,2,\ldots,k$.

Subcase 2.1: $|V(H_i)| = 1$.
Let u_h be the only vertex in H_i and (u_h, u) be its corresponding cross edge between H_i and $SP_n - SP^1_{n-1}$. By definition, $N_{SP_n}(u_h) \not\subset F$. So $u \notin F$. Then, $SP_n - F$ is connected.

Subcase 2.2: $|V(H_i)| \geqslant 2$.Let (u_h, v_h) be an edge in H_i. Thus, $|N_{SP^1_{n-1}}(u_h) \cup N_{SP^1_{n-1}}(v_h)| = (n-2) + (n-2) = 2n-4$. Let $|V(H_i)| = m \geqslant 2$. We have the following inequality:

$$\left| \bigcup_{v \in V(H_i)} N_{SP^1_{n-1}}(v) \right| - |V(H_i)| \geqslant (2n-4) - m.$$

Suppose each cross edge between H_i and $SP_n - SP^1_{n-1}$ has at least one faulty vertex, then $|F| \geqslant (2n-4-m) + m = 2n-4 > 2n-5$. It is a contradiction to our assumption that $|F| = 2n-5$. So there exists a cross edge (w_h, w) such that $w_h \in V(H_i)$, $w \in V(SP_n - SP^1_{n-1})$, and $w \notin F$. Therefore, $SP_n - F$ is connected. So $\kappa_1(SP_n) = 2n-4$ for $n \geqslant 3$. Thus, we complete the proof of this theorem. □

With a similar argument as above, we have the following theorem.

Theorem 4. *$\lambda_1(SP_n) = 2n-4$ for $n \geqslant 3$.*

It is easy to check that $\kappa_1(S_3)$ and $\kappa_1(P_3)$ is exactly 2. As a consequence of the above two theorems, we have the following corollary about the restricted vertex connectivity of the star graphs S_n and pancake graphs P_n.

Corollary 2. *$\kappa_1(S_n) = \kappa_1(P_n) = 2n-4$ and $\lambda_1(S_n) = \lambda_1(P_n) = 2n-4$ for $n \geqslant 3$.*

For the third family, the degree of every vertex in $G(G_0, G_1, \ldots, G_{r-1}; \mathscr{M})$ is increasing by 2 as compared with that of G_i's, and the connectivity is also increasing by 2. Moreover, the restricted vertex connectivity is $2(n+2) - 2 = 2n+2$.

Theorem 5. *Let n, r, and t be positive integers with $r \geqslant 3$ and $t > n+1$. Assume that G_i is an n-regular maximum vertex connected graph with no triangle and $|V(G_i)| = t$ for $0 \leqslant i \leqslant r-1$. Let $H = G(G_0, G_1, \ldots, G_{r-1}; \mathscr{M})$,* *where* $\mathscr{M} = \bigcup_{i=0}^{r-1} M_{i,i+1(\mathbf{mod}r)}$ *and* $M_{i,i+1(\mathbf{mod}r)}$ *is any arbitrary perfect matching between* $V(G_i)$ *and* $V(G_{i+1(\mathbf{mod}r)})$*. Then, H is* $(n+2)$*-regular, contain no triangle,* $\kappa(H) = n+2$*, and* $\kappa_1(H) = 2n+2$*.*

Proof. By definition, H is $(n+2)$-regular and contains no triangle. It is proved in [5] that $\kappa(H) = n+2$. Now, we shall prove that $\kappa_1(H) = 2(n+2) - 2 = 2n+2$. To prove that $\kappa_1(H) = 2n+2$, we shall prove that $\kappa_1(H) \leqslant 2n+2$ and $\kappa_1(H) \geqslant 2n+2$. Since graph H contains no triangle, for every edge $(u,v) \in E(H)$, $|N_H(u) \cup N_H(v) - \{u,v\}| = 2n+2$ and $H - (N_H(u) \cup N_H(v) - \{u,v\})$ is disconnected. Therefore, $\kappa_1(H) \leqslant 2n+2$.

To show that $\kappa_1(H) \geqslant 2n+2$, we need to check that $H - F$ is disconnected for every vertex subset F of H such that $|F| = 2n+1$ and $F \not\supset N_H(v)$ for every vertex $v \in V(H)$. Let $F_i = G_i \cap F$ for $0 \leqslant i \leqslant r-1$.

Case 1: $G(G_i, G_{i+1(\mathbf{mod}r)}; M) - F$ is connected for $0 \leqslant i \leqslant r-1$.
Thereby, $H - F$ is also connected.

Case 2: $G(G_i, G_{i+1(\mathbf{mod}r)}; M) - F$ is disconnected for some $i \in \{0, 1, \ldots, r-1\}$.
We may without loss of generality assume that (1) $G(G_{r-1}, G_0; M)$ contains the most faulty vertices among $G(G_i, G_{i+1(\mathbf{mod}r)}; M)$ for $0 \leqslant i \leqslant r-1$; and (2) $G(G_{r-1}, G_0; M) - F$ is disconnected. Hence, $|F_0 \cup F_{r-1}| \geqslant n+1$ and $H - (G(G_{r-1}, G_0; M) \cup F)$ is connected. Now, we shall show that for every non-faulty vertex v in $G(G_{r-1}, G_0; M)$, v is connected to $H - (G(G_{r-1}, G_0; M) \cup F)$. We may assume that $G(G_{r-1}, G_0; M) - F$ is divided into k disjoint connected components, say $H_1, H_2, \ldots, H_k$, where $k \geqslant 2$. Now, we shall prove that there exists a cross edge (a_i, b_i) such that $a_i \in H - G(G_{r-1}, G_0; M)$, $b_i \in H_i$, and $a_i \notin F$ for $i = 1, 2, \ldots, k$. In the following two subcases, we consider the number of vertices of H_i with $i \in 1, 2, \ldots, k$.

Subcase 2.1: $|V(H_i)| = 1$.
Let u_h be the vertex in H_i and (u_h, u_g) be its corresponding cross edge between H_i and $H - G(G_{r-1}, G_0; M)$. By definition, $N_H(u_h) \not\subset F$. So $u_g \notin F$. Then, u_h is connected with every vertex in $H - (G(G_{r-1}, G_0; M) \cup F)$.

Subcase 2.2: $|V(H_i)| \geqslant 2$.
Let (u_h, v_h) be an edge in H_i. $|N_{G(G_{r-1},G_0;M)}(u_h) \cup N_{G(G_{r-1},G_0;M)}(v_h)| = (n+1) + (n+1) = 2n+2$ since there is no triangle in H. Let $|V(H_i)| = m \geqslant 2$. We have the following inequality:

$$\left| \bigcup_{v \in V(H_i)} N_{G(G_{r-1},G_0;M)}(v) \right| - |V(H_i)| \geqslant (2n+2) - m.$$

Suppose each cross edge between $H - G(G_{r-1}, G_0; M)$ and H_i has at least one faulty vertex, then $|F| \geqslant (2n+2-m) + m = 2n+2$. It is a contradiction to that $|F| = 2n+1$. So there exists a cross edge (w_h, w_g) such that $w_h \in V(H_i)$, $w_g \in V(H - G(G_{r-1}, G_0; M))$, and $w_g \notin F$. Hence, $H - F$ is connected. That is, $\kappa_1(H) = 2n+2$ and we complete the proof of this theorem. □

With a similar argument as above, we have the following theorem.

Theorem 6. *Let n, r, and t be positive integers with* $r \geqslant 3$ *and* $t > n+1$*. Assume that* G_i *is an n-regular maximum edge connected graph with no triangle and* $|V(G_i)| = t$ *for* $0 \leqslant i \leqslant r-1$*. Let* $H = G(G_0, G_1, \ldots, G_{r-1}; \mathscr{M})$*, where* $\mathscr{M} = \bigcup_{i=0}^{r-1} M_{i,i+1(\mathbf{mod}r)}$ *and* $M_{i,i+1(\mathbf{mod}r)}$ *is any arbitrary perfect matching between* $V(G_i)$ *and* $V(G_{i+1(\mathbf{mod}r)})$*. Then, H is* $(n+2)$*-regular, contain no triangle,* $\lambda(H) = n+2$*, and* $\lambda_1(H) = 2n+2$*.*

Most of the recursive circulant graphs $RC(c,d,r)$ are special cases of $G(G_0, G_1, \ldots, G_{r-1}; \mathscr{M})$. Let $RC(c,d,r)$ be k-regular, we have the following corollary.

Corollary 3. $\kappa_1(RC(c,d,r)) = 2k-2$ *and* $\lambda_1(RC(c,d,r)) = 2k-2$ *with* $r \geqslant 0$, $d > 2$, *and* $1 \leqslant c < d$.

The k-ary n-cube is also a special case of $G(G_0, G_1, \ldots, G_{r-1}; \mathscr{M})$ for $k \geqslant 3$. In this paper, we do not consider the 3-ary n-cube since it contains triangles. By definition, the k-ary n-cube is $2n$-regular for $k \geqslant 3$. We denote k-ary n-cube as $G_{r,n}$. Along the way, we establish some results on the k-ary n-cube for $k \geqslant 4$.

Corollary 4. $\kappa_1(G_{r,n}) = 4n - 2$ *for* $k = 4, 5$, $n \geqslant 2$, *or* $k \geqslant 6$, $n \geqslant 1$. $\lambda_1(G_{r,n}) = 4n - 2$ *for* $k \geqslant 4$ *and* $n \geqslant 1$.

4. Conclusion and discussion

The vertex connectivity and edge connectivity of many popular networks have been already established. In this paper, we address the restricted vertex connectivity κ_1 and restricted edge connectivity λ_1 on the three families of interconnection networks: $G(G_0, G_1; M)$, SP_n, and $G(G_0, G_1, \ldots, G_{r-1}; \mathscr{M})$. In particular, hypercubes, twisted-cubes, crossed-cubes, möbius cubes, star graphs, pancake graphs, recursive circulant graphs, and k-ary n-cubes are special cases of these families. As a corollary, we have obtained the restricted vertex connectivity κ_1 and restricted edge connectivity λ_1 of these graphs.

Finally, we raise a few questions. Are there other families of graphs worth discussing their restricted vertex connectivity and restricted edge connectivity? In addition, if we restrict the fault condition that every vertex in graph G has at least i non-faulty neighbor vertices (respectively edges) for $i > 1$, what is the restricted vertex connectivity (respectively restricted edge connectivity) in each of these graphs?

References

[1] S.B. Akers, D. Harel, B. Krishnamurthy, The star graph: an attractive alternative to the *n*-cube, in: Proceedings of Internet Conference on Parallel Processing, St. Charles, IL, 1987, pp. 393–400.

[2] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, North Holland, New York, 1980.

[3] L. Bhuyan, D.P. Agrawal, Generalized hypercube and hyperbus structures for a computer network, IEEE Transactions on Computers 33 (1984) 323–333.

[4] C.P. Chang, J.N. Wang, L.H. Hsu, Topological properties of twisted cubes, Information Sciences 113 (1999) 147–167.

[5] Y.C. Chen, Jimmy J.M. Tan, L.H. Hsu, Super-connectivity and super-edge-connectivity for some interconnection networks, Applied Mathematics and Computation 140 (2003) 245–254.

[6] W.S. Chiue, B.S. Shieh, On connectivity of the Cartesian product of two graphs, Applied Mathematics and Computation 102 (1999) 129–137.

[7] P. Cull, S.M. Larson, The möbius cubes, Distributed Memory Computing Conference Proceedings The Sixth 44 (1991) 699–702.

[8] W.J. Dally, Performance analysis of *k*-ary *n*-cube interconnection networks, IEEE Transactions on Computers 39 (1990) 775–785.

[9] K. Efe, The crossed cube for architecture for parallel computing, Parallel Distributed Systems 3 (5) (1992) 513–524.

[10] A.H. Esfahanian, Generalized measures of fault tolerance with application to *N*-cube networks, IEEE Transactions on Computers 38 (1989) 1586–1591.

[11] L. Gargano, U. Vaccaro, A. Vozella, Fault tolerant routing in the star and pancake interconnection networks, Information Processing Letters 45 (1993) 315–320.

[12] F. Harary, Conditional connectivity, Networks 13 (1983) 346–357.

[13] P.D. Kulasinghe, Connectivity of the crossed cube, Information Processing Letters 61 (1997) 221–226.

[14] S. Latifi, M. Hegde, M. Naraghi-Pour, Conditional connectivity measures for large multiprocessor systems, IEEE Transactions on Computers 43 (1994) 218–222.

[15] J.H. Lee, S.M. Park, K.Y. Chwa, Recursive circulant: a new topology for multicomputer networks, Proceedings of Internet Symposium Parallel Architectures, Algorithms and Networks (ISPAN'94) Japan, IEEE Press, New York, 1994, pp. 73–80.

[16] J. Meng, Y. Ji, On a kind of restricted edge connectivity of graphs, Discrete Mathematics 117 (2002) 183–193.

[17] J.M. Xu, K.L. Xu, On the restricted edge-connectivity of graphs, Discrete Mathematics 243 (2002) 291–298.