A Local Diagnosability Measure for Multiprocessor Systems

Guo-Huang Hsu and Jimmy J.M. Tan

Abstract—The problem of fault diagnosis has been discussed widely and the diagnosability of many well-known networks has been explored. Under the PMC model, we introduce a new measure of diagnosability, called local diagnosability, and derive some structures for determining whether a vertex of a system is locally t-diagnosable. For a hypercube, we prove that the local diagnosability of each vertex is equal to its degree under the PMC model. Then, we propose a concept for system diagnosis, called the strong local diagnosability property. A system $G(V, E)$ is said to have a strong local diagnosability property if the local diagnosability of each vertex is equal to its degree. We show that an n-dimensional hypercube Q_n has this strong property, $n \geq 3$. Next, we study the local diagnosability of a faulty hypercube. We prove that Q_n keeps this strong property even if it has up to $n-2$ faulty edges. Assuming that each vertex of a faulty hypercube Q_n is incident with at least two fault-free edges, we prove Q_n keeps this strong property even if it has up to $3(n-2)-1$ faulty edges. Furthermore, we prove that Q_n keeps this strong property no matter how many edges are faulty, provided that each vertex of a faulty hypercube Q_n is incident with at least three fault-free edges. Our bounds on the number of faulty edges are all tight.

Index Terms—PMC model, local diagnosability, strong local diagnosability property.

1 Introduction

The problem of identifying faulty processors in a multiprocessor system has been widely studied in the literature [9], [16], [18]. The bases of this area and the original diagnostic model were established by Preparata et al. [16]. This model, known as the PMC model, has been extensively studied [1], [2], [3], [4], [10], [11], [12], [13], [14], [16]. In [10], Hakimi and Amin proved that a system is t-diagnosable if it is t-connected with at least $2t+1$ vertices. They also gave a necessary and sufficient condition for verifying if a system is t-diagnosable under the PMC model.

The hypercube structure [17] is a popular topology for multiprocessor systems. An n-dimensional hypercube is denoted by Q_n and the diagnosability of Q_n is shown to be n [13] under the PMC model, $n \geq 3$. In [15], Lai et al. introduced a measure of diagnosability called conditional diagnosability by restricting that a faulty set cannot contain all the neighbors of any vertex. Based on this restriction, the conditional diagnosability of the n-dimensional hypercube is shown to be $4(n-2)+1$. Besides, Lai et al. introduced a concept called a *strongly* t-diagnosable system and proved that the n-dimensional hypercube is strongly n-diagnosable. Essentially, it means that an n-dimensional hypercube is almost $(n+1)$-diagnosable except for the case where all the neighbors of some vertex are faulty simultaneously. In [19], Wang proved that the diagnosability of an incomplete hypercube under some conditions can be determined by simply checking the degree of each vertex under the PMC model. An incomplete hypercube is a hypercube with some missing edges. It is also called a faulty hypercube. There are some results concerning the diagnosability of several variations of the hypercube [1], [5], [7], [8], [10], [13], [19]. In classical measures of system-level diagnosability for multiprocessor systems, it has generally been assumed that any subset of processors can potentially fail at the same time. As a consequence, the diagnosability of a system is upper bounded by its minimum degree.

We observe that the discussions in previous literature about the diagnosability of a system consider the global sense but ignore some local information. A system is t-diagnosable if all the faulty processors can be uniquely identified, provided that the number of faulty processors does not exceed t. However, it is possible to correctly indicate all the faulty processors in a t-diagnosable system when the number of faulty processors is greater than t. For example, consider a multiprocessor system generated by integrating two arbitrary subsystems with a few communication links in some way, where the two subsystems are m-diagnosable and n-diagnosable, respectively, and $m >> n$. The diagnosability of this system is limited by n, but it is possible to correctly point out all the faulty processors even if the number of the faulty ones is between m and n. Therefore, if we only consider the global faulty/fault-free status, we lose some local systematic details.

In this paper, we propose a new measure of diagnosability, called local diagnosability, and study the local diagnosability of each processor of a system. We can identify the diagnosability of a system by computing the local diagnosability of each processor. This measure of the local diagnosability leads us to study the local diagnosability of each processor instead of the whole system. We propose a necessary and sufficient condition, Theorem 3, to determine the local diagnosability of a processor. We also provide two useful structures, called the Type I structure and the Type II structure, to determine the local diagnosability of a processor under the PMC model. Based on these

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structures, the local diagnosability of each vertex in a hypercube is shown to be equal to its own degree. Then, we propose a concept for system diagnosis, called the strong local diagnosability property. A system $G(V, E)$ is said to have a strong local diagnosability property if the local diagnosability of each vertex is equal to its degree. We show that an n-dimensional hypercube Q_n has this strong property. Then, we study the local diagnosability of an incomplete hypercube. First, we show that Q_n keeps this strong property even if it has up to $n-2$ faulty edges. Second, assuming that each vertex of an incomplete hypercube Q_n is incident with at least two fault-free edges, we show that Q_n keeps this strong property even if it has up to $3(n-2)-1$ faulty edges. Finally, we show that Q_n keeps this strong property no matter how many edges are faulty, provided that each vertex of an incomplete hypercube Q_n is incident with at least three fault-free edges.

The rest of this paper is organized as follows: Section 2 provides preliminaries and previous results for diagnosing a system. Section 3 introduces the concept of local diagnosability and proposes a necessary and sufficient condition for verifying if it is locally t-diagnosable at a given processor in a system. In Section 4, we define a strong local diagnosability property for a system and study the strong property in a faulty hypercube. In Section 5, we study the strong property in a conditional faulty hypercube. Finally, our conclusions are given in Section 6.

2 Preliminaries and Previous Results

A multiprocessor system can be represented by a graph $G(V, E)$, where the set of vertices $V(G)$ represents processors and the set of edges $E(G)$ represents communication links between processors. Throughout this paper, we focus on an undirected graph without loops and follow [20] for graph theoretical definitions and notations.

Let $G(V, E)$ be a graph and $v \in V(G)$ be a vertex. We use the notation $E_G(v)$ to denote the set of edges incident with v. The cardinality $|E_G(v)|$ is called the degree of v, denoted by $deg_G(v)$ or simply $deg(v)$. G is d-regular if $deg(v) = d$ for every $v \in V(G)$. The neighborhood $N_G(v)$ of a vertex v in G is the set of all vertices that are adjacent to v in G. For a set of edges (respectively, vertices) S, we use the notation $G - S$ to denote the graph obtained from G by removing all the edges (respectively, vertices) in S. The components of a graph G are its maximal connected subgraphs. A component is trivial if it has no edges; otherwise, it is nontrivial. The connectivity $\kappa(G)$ of a graph $G(V, E)$ is the minimum number of vertices whose removal results in a disconnected or a trivial graph. Letting G_1 be a subgraph of G, we shall write the vertex set of G_1 as $V(G_1)$. The neighborhood set of $V(G_1)$ is defined as $N(V(G_1)) = \{u \in V(G) - V(G_1) |$ there exists a vertex $v \in V(G_1)$ such that $(u, v) \in E(G)\}$. Let $S_1, S_2 \subseteq V(G)$ be two distinct sets. The symmetric difference of the two sets S_1 and S_2 is defined as the set $S_1 \Delta S_2 = (S_1 - S_2) \bigcup (S_2 - S_1)$.

The PMC diagnosis model is presented by Preparata et al. [16]. In this model, a self-diagnosable system is often represented by a *directed graph* $T(V, E)$ in which an edge directed from vertex u to vertex v means that u can test v. In this situation, u is called the tester and v is called the tested vertex. The outcome of a test (u, v) is 1 (respectively, 0) if u evaluates v as faulty (respectively, fault-free). We assume that the testing results of fault-free vertices are always reliable and the testing results of faulty vertices are unreliable. The collection of all testing results is called a *syndrome*. Formally, a syndrome is a function $\sigma : E \to \{0, 1\}$. The set of all faulty processors in the system is called a *faulty set*. This can be any subset of $V(T)$. For a given syndrome σ, a subset of vertices $F \subset V(T)$ is compatible with σ if the syndrome σ can be produced from the situation that all vertices in F are faulty and all vertices in $V - F$ are fault-free. A syndrome σ is said to be compatible with a faulty set $F \subseteq V(T)$ if, for a $(u, v) \in E(T)$, such that $u \in V - F$, $\sigma(u, v) = 1$ if and only if $v \in F$. This corresponds to the assumption that fault-free testers always give correct testing results. Since faulty testers can give arbitrary testing results, any syndrome compatible with a faulty set F can occur when faulty processors in the system are exactly those in F. A system G is called t-diagnosable if, given the test outcomes obtained by the testing link, all the faulty vertices can be uniquely identified without replacement, provided that the number of faulty vertices does not exceed t. The maximum number of faulty vertices that the system G can guarantee to identify is called the *diagnosability* of G, written as $t(G)$. Let σ_F be the set of all syndromes which could be produced if F is the set of faulty vertices. Two distinct sets $F_1, F_2 \subseteq V(G)$ are said to be *distinguishable* if $\sigma_{F_1} \bigcap \sigma_{F_2} = \phi$; otherwise, F_1, F_2 are said to be *indistinguishable*. We say (F_1, F_2) is a *distinguishable pair* if $\sigma_{F_1} \bigcap \sigma_{F_2} = \phi$; otherwise, (F_1, F_2) is an *indistinguishable pair*. We need some previous results concerning the t-diagnosable systems.

Lemma 1 [6]. *A system $G(V, E)$ is t-diagnosable if and only if, for any two distinct sets $F_1, F_2 \subset V$ with $|F_1| \le t$ and $|F_2| \le t$, (F_1, F_2) is a distinguishable pair.*

Lemma 2 [6]. *Let $G(V, E)$ be a graph. For any two distinct sets $F_1, F_2 \subset V$, (F_1, F_2) is a distinguishable pair if and only if there exists a vertex $u \in V - (F_1 \bigcup F_2)$ and a vertex $v \in F_1 \Delta F_2$ such that $(u, v) \in E$.*

The following Lemma 3 is equivalent to Lemma 1:

Lemma 3 [6]. *A system $G(V, E)$ is t-diagnosable if and only if, for each indistinguishable pair $F_1, F_2 \subset V$, it implies that $|F_1| > t$ or $|F_2| > t$.*

The following two lemmas related to t-diagnosable systems are proposed by Hakimi and Amin [10] and Preparata et al. [16], respectively:

Lemma 4 [16]. *Let $G(V, E)$ be a graph and $|V| = N$. The following two conditions are necessary for G to be t-diagnosable:*

1. *$N \ge 2t + 1$, and*
2. *each processor in G is tested by at least t other processors.*

Lemma 5 [10]. *Let $G(V, E)$ be a graph and $|V| = N$. G is t-diagnosable if*

1. *$N \ge 2t + 1$, and*
2. *$\kappa(G) \ge t$.*

For our discussion later, a useful result presented by Lai et al. [15] is stated below:

Theorem 1 [15]. *Let $G(V,E)$ be a graph. G is t-diagnosable if and only if, for each set of vertices $S \subset V$ with $|S| = p$, $0 \le p \le t-1$, each connected component of $G - S$ has at least $2(t-p)+1$ vertices.*

3 Local Diagnosability

We first review some related results on system diagnosability of some well-known networks under the PMC model. In [13], Kavianpour and Kim proved that the diagnosability of an n-dimensional hypercube Q_n is n. In [7] and [8], Fan proved that an n-dimensional Crossed cube and an n-dimensional Möbius cube have diagnosability n under the PMC model. In [19], Wang proved that the diagnosability of a faulty hypercube can be determined by checking the degree of each vertex under the PMC model, provided that the minimum degree of the faulty hypercube is at least three.

We observe that the traditional diagnosability discussed in most literatures describes the global status of a system. In this paper, we study the local status of each processor instead of the global status of a system. For example, for any two positive integers m and n with $m >> n \ge 3$, the diagnosability of two hypercube systems Q_m and Q_n is m and n, respectively. Combining Q_m and Q_n with a few edges in some way may cause the diagnosability of the new system to become n. In this situation, the strong diagnosability of Q_m is disregarded. For this reason, we are motivated to study the local status of each processor. Given a single vertex, we require only identifying the status of this particular processor correctly. We now propose the following concept:

Definition 1. *Let $G(V,E)$ be a graph and $v \in V$ be a vertex. G is locally t-diagnosable at vertex v if, given a syndrome σ_F produced by a set of faulty vertices $F \subseteq V$ containing vertex v with $|F| \le t$, every set of faulty vertices F' compatible with σ_F and $|F'| \le t$ must also contain vertex v.*

Definition 2. *Let $G(V,E)$ be a graph and $v \in V$ be a vertex. The local diagnosability of vertex v, written as $t_l(v)$, is defined to be the maximum value of t such that G is locally t-diagnosable at vertex v.*

The following result is another point of view for checking whether a vertex is locally t-diagnosable:

Lemma 6. *Let $G(V,E)$ be a graph and $v \in V$ be a vertex. G is locally t-diagnosable at vertex v if and only if, for any two distinct sets of vertices F_1, $F_2 \subset V$, $|F_1| \le t$, $|F_2| \le t$, $v \in F_1 \Delta F_2$, and (F_1, F_2) is a distinguishable pair.*

In the following, we study some properties of a system being locally t-diagnosable at a given vertex and its relationship between a system being t-diagnosable:

Proposition 1. *Let $G(V,E)$ be a graph and $v \in V(G)$ be a vertex. If G is locally t-diagnosable at vertex v, then $|V(G)| \ge 2t+1$.*

Proof. We show this by contradiction. Assume that $|V(G)| \le 2t$. We partition $V(G)$ into two disjoint subsets F_1, F_2 with $|F_1| \le t$, $|F_2| \le t$. The vertex v is either in F_1 or in F_2. Since $V - (F_1 \bigcup F_2) = \emptyset$, there is no edge between $V - (F_1 \bigcup F_2)$ and $F_1 \Delta F_2$. By Lemma 2, (F_1, F_2) is an indistinguishable pair, which contradicts the assumption that G is locally t-diagnosable at vertex v. So, the result follows. □

Proposition 2. *Let $G(V,E)$ be a graph and $v \in V$ be a vertex with $deg(v) = n$. The local diagnosability of vertex v is at most n.*

Proof. Let F_1 be the set of vertices adjacent to vertex v, $F_1 = N_G(v)$ and $|F_1| = n$. Let $F_2 = F_1 \bigcup \{v\}$ with $|F_2| = n+1$. It is a simple matter to check that there is no edge between $V - (F_1 \bigcup F_2)$ and $F_1 \Delta F_2$. By Lemma 2, (F_1, F_2) is an indistinguishable pair. Thus, G is not locally $(n+1)$-diagnosable at vertex v, so $t_l(v) \le n = deg(v)$. We have the stated result. □

Proposition 3. *Let $G(V,E)$ be a graph. G is t-diagnosable if and only if G is locally t-diagnosable at every vertex.*

Proof. To prove the necessity, we assume that G is t-diagnosable. If the result is not true, there exists a vertex $v \in V$ such that G is not locally t-diagnosable at vertex v. By Lemma 6, there exists a distinct pair of sets F_1, $F_2 \subset V$ with $|F_1| \le t$, $|F_2| \le t$ and $v \in F_1 \Delta F_2$, (F_1, F_2) is an indistinguishable pair. By Lemma 1, G is not t-diagnosable. This contradicts the assumption; hence, the necessary condition follows.

To prove the sufficiency, suppose on the contrary that, if G is not t-diagnosable, there exists a distinct pair of sets F_1, $F_2 \subset V$ with $|F_1| \le t$, $|F_2| \le t$; thus, (F_1, F_2) is an indistinguishable pair. Being distinct, using the set $F_1 \Delta F_2 \ne \emptyset$, we can find a vertex $v \in F_1 \Delta F_2$. By Lemma 6, G is not locally t-diagnosable at vertex v, which is a contradiction. This completes the proof. □

By Definition 2 and Proposition 3, we know that the diagnosability of a multiprocessor system is equal to the minimum local diagnosability of all vertices of the system. Thus, we have the following theorem:

Theorem 2. *Let $G(V,E)$ be a multiprocessor system. The diagnosability of G is t if and only if*

$$min\{t_l(v) \mid \text{ for every } v \in V\} = t.$$

From Theorem 2, we can identify the diagnosability of a system by computing the local diagnosability of each vertex. Because many well-known systems are vertex-symmetric, the diagnosability of these system can be easily identified by this effective method.

Before studying the local diagnosability of a vertex, we need some definitions for further discussion. Let S be a set of vertices and v be a vertex not in S. After deleting the vertices in S from G, we use C_v to denote the connected component that vertex v belongs to. Now, we propose a necessary and sufficient condition for verifying if a system is locally t-diagnosable at a given vertex v.

Theorem 3. *Let $G(V,E)$ be a graph and $v \in V$ be a vertex. G is locally t-diagnosable at vertex v if and only if, for each set of vertices $S \subset V$ with $|S| = p$, $0 \le p \le t-1$, and $v \notin S$, the connected component, which v belongs to in $G - S$, has at least $2(t-p)+1$ vertices.*

Proof. To prove the necessity, we assume that G is locally t-diagnosable at vertex v. If the result does not hold, there

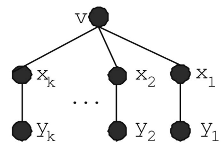


Fig. 1. A Type I structure $T_1(v;k)$ consists of $2k+1$ vertices and $2k$ edges.

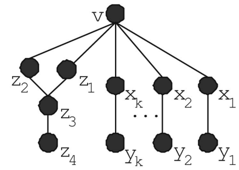


Fig. 2. A Type II structure $T_2(v;k,2)$ consists of $2k+5$ vertices and $2k+5$ edges.

exists a set of vertices $S \subset V$ with $|S| = p$, $0 \le p \le t-1$, $v \notin S$ such that the connected component C_v has strictly less than $2(t-p)+1$ vertices, $|V(C_v)| \le 2(t-p)$. We then arbitrarily partition $V(C_v)$ into two disjoint subsets, $V(C_v) = S_1 \bigcup S_2$ with $|S_1| \le t-p$, $|S_2| \le t-p$. Let $F_1 = S_1 \bigcup S$ and $F_2 = S_2 \bigcup S$. It is clear that

$$|F_1| \le (t-p)+p = t,$$

$|F_2| \le (t-p)+p = t$, the vertex $v \in F_1 \Delta F_2$ and there is no edge between $V - (F_1 \bigcup F_2)$ and $F_1 \Delta F_2$. By Lemma 6, (F_1, F_2) is an indistinguishable pair. This contradicts the assumption that G is locally t-diagnosable at vertex v.

We now prove the sufficiency by contradiction. Suppose G is not locally t-diagnosable at vertex v, then, there exists an indistinguishable pair (F_1, F_2) with $|F_1| \le t$, $|F_2| \le t$ and $v \in F_1 \Delta F_2$. By Lemma 2, there is no edge between $V - (F_1 \bigcup F_2)$ and $F_1 \Delta F_2$. Let $S = F_1 \bigcap F_2$ with $|S| = p$, $0 \le p \le t-1$ and $v \notin S$. $F_1 \Delta F_2$ is disconnected from other parts after removing all the vertices in S from G. We observe that $|F_1 \Delta F_2| \le 2(t-p)$. Thus, the connected component C_v has at most $2(t-p)$ vertices and $|V(C_v)| \le 2(t-p)$. This contradicts the assumption that the connected component C_v has to satisfy $|V(C_v)| \ge 2(t-p)+1$. Hence, the theorem holds. □

We now propose two special subgraphs called Type I structure and Type II structure. They provide us with an efficient and simple method to identify the local diagnosability of each vertex of a system under the PMC diagnosis model.

Definition 3. *Letting $G(V,E)$ be a graph, $v \in V$ be a vertex, and k be an integer, $k \ge 1$, a Type I structure $T_1(v;k)$ of order k at vertex v is defined to be the following graph:*

$$T_1(v;k) = [V(v;k), E(v;k)],$$

which is composed of $2k+1$ vertices and of $2k$ edges as illustrated in Fig. 1, where

- $V(v;k) = \{v\} \cup \{x_i, y_i | 1 \le i \le k\}$, *and*
- $E(v;k) = \{(v,x_i), (x_i,y_i) | 1 \le i \le k\}$.

Following Theorem 3 and Definition 3, we propose a sufficient condition for verifying if it is locally t-diagnosable at a given processor in a system.

Theorem 4. *Let $G(V,E)$ be a graph and $v \in V$ be a vertex. G is locally t-diagnosable at vertex v if G contains a Type I structure $T_1(v;t)$ of order t at vertex v as a subgraph.*

Proof. We use Theorem 3 to prove this result. Assume that G contains a subgraph $T_1(v;t)$ at vertex v. Let $e_i = (x_i, y_i)$ be the edge for each i, $1 \le i \le t$, with respect to $T_1(v;t)$. The number of vertices of the connected component including vertex v is at least $2t+1$. Let $S \subset V(G)$ be a set of vertices with $|S| = p$, $0 \le p \le t-1$, and $v \notin S$. After deleting S from $V(G)$, there are at least $(t-p)$ complete e_is still remaining in $T_1(v;t)$. Therefore, the number of vertices of the connected component C_v is at least $2(t-p)+1$. By Theorem 3, G is locally t-diagnosable at vertex v. The proof is complete. □

A Type II structure $T_2(v;k,2)$ at a vertex v is defined as follows:

Definition 4. *Letting $G(V,E)$ be a graph, $v \in V$ be a vertex, and k be an integer, $k \ge 1$, a Type II structure $T_2(v;k,2)$ of order $k+2$ at vertex v is defined to be the following graph:*

$$T_2(v;k,2) = [V(v;k,2), E(v;k,2)],$$

which is composed of $2k+5$ vertices and of $2k+5$ edges as illustrated in Fig. 2, where

- $V(v;k,2) = \{v\} \cup \{x_i, y_i | 1 \le i \le k\} \cup \{z_1, z_2, z_3, z_4\}$, *and*
- $E(v;k,2) = \{(v,x_i), (x_i,y_i) | 1 \le i \le k\}$ $\cup \{(v,z_1), (v,z_2), (z_1,z_3), (z_2,z_3), (z_3,z_4)\}$.

In the following, we propose another sufficient condition for verifying if it is locally t-diagnosable at a given processor in a system:

Theorem 5. *Let $G(V,E)$ be a graph and $v \in V$ be a vertex. G is locally t-diagnosable at vertex v if G contains a Type II structure $T_2(v;k,2)$ of order $k+2$ at vertex v as a subgraph, where $t = k+2$.*

Proof. We use Theorem 3 to prove this result. Assume that G contains a subgraph $T_2(v;k,2)$ of order $t = k+2$ at vertex v. The number of vertices of the connected component including vertex v is at least $2k+5 = 2t+1$. Letting $S \subset V$ be a set of vertices with $|S| = p$, $0 \le p \le t-1$, and $v \notin S$, the number of vertices of C_v is at least $(2k+5) - 2*1$ after removing one vertex in S, the number of vertices of C_v is at least $(2k+5) - 2*2$ after removing two vertices in S, and so on. Thus, the connected component C_v satisfies $|V(C_v)| \ge (2k+5) - 2p = 2(t-p)+1$. By Theorem 3, G is locally t-diagnosable at vertex v. This proves the theorem. □

In the following, we give some examples:

Example 1. Let us consider a cycle of length four as shown in Fig. 3a. We can find a Type I structure $T_1(v;1)$ of order 1 at vertex v as shown in Fig. 3b; hence, vertex v is locally 1-diagnosable.

Example 2. Consider examples as shown in Fig. 4a, 4b, and 4c. It is a routine work to check that there is a subgraph $T_1(v_1;2)$, $T_1(v_2;2)$, and $T_2(v_3;1,2)$ at vertex v_1, v_2, and v_3, respectively. Hence, it is locally 2-diagnosable, 2-diagnosable, and 3-diagnosable at vertex v_1, v_2, and v_3,

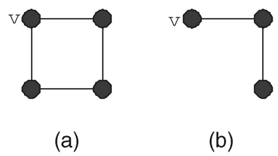


Fig. 3. A cycle of length four and a Type I structure $T_1(v;1)$ of order 1 at v.

respectively.

By Theorem 4 and Theorem 5, we have the following result:

Theorem 6. *Let $G(V,E)$ be a graph and $v \in V$ be a vertex with $deg(v) = n$. The local diagnosability of vertex v is n if G contains a subgraph which is either a Type I structure $T_1(v;n)$ of order n or a Type II structure $T_2(v;n-2,2)$ of order n at vertex v.*

4 Strong Local Diagnosability Property

We use a hypercube as an example to introduce our concept of the strong local diagnosability property. An n-dimensional hypercube can be modeled as a graph Q_n, with the vertex set $V(Q_n)$ and the edge set $E(Q_n)$. There are 2^n vertices in Q_n, and each vertex has degree n. Each vertex v of Q_n can be distinctly labeled by a binary n-bit string, $v = v_{n-1}v_{n-2}\dots v_1v_0$. There is an edge between two vertices if and only if their binary labels differ in exactly one bit position. Let u and v be two adjacent vertices. If the binary labels of u and v differ in the ith position, then the edge between them is said to be in the ith dimension and the edge (u,v) is called an ith dimensional edge. Letting i be a fixed position, we use Q^0_{n-1} to denote the subgraph of Q_n induced by $\{v \in V(Q_n) | v_i = 0\}$ and Q^1_{n-1} to denote the subgraph of Q_n induced by $\{v \in V(Q_n) | v_i = 1\}$. Consequently, Q_n is decomposed to Q^0_{n-1} and Q^1_{n-1} by dimension i, and Q^0_{n-1} and Q^1_{n-1} are $(n-1)$-dimensional subcubes of Q_n induced by the vertices with the ith bit position being 0 and 1, respectively. Q^0_{n-1} and Q^1_{n-1} are isomorphic to Q_{n-1}. For each vertex $v \in V(Q^0_{n-1})$, there is exactly one vertex in Q^1_{n-1}, denoted by $v^{(1)}$, such that $(v, v^{(1)}) \in E(Q_n)$. Conversely, for each vertex $v \in V(Q^1_{n-1})$, there is exactly one vertex in Q^0_{n-1}, denoted by $v^{(0)}$, such that $(v, v^{(0)}) \in E(Q_n)$. Let D_i be the set of all edges with one end in Q^0_{n-1} and the other in Q^1_{n-1}. These edges are called crossing edges in the ith dimension between Q^0_{n-1} and Q^1_{n-1}. We also call D_i the set of all ith dimensional edges.

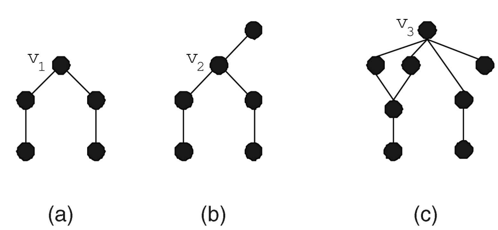


Fig. 4. Some examples of local diagnosability.

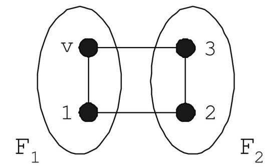


Fig. 5. An indistinguishable pair (F_1, F_2) in Q_2.

In the previous section, we presented two sufficient conditions for identifying the local diagnosability of a vertex. It seems that identifying the local diagnosability of a vertex is the same as counting its degree. We give an example to show that this is not true in general. As shown in Fig. 5, we take a vertex v in 2-dimensional hypercube Q_2; let $F_1 = \{v, 1\}$ and $F_2 = \{2, 3\}$ with $|F_1| = 2$ and $|F_2| = 2$. It is a simple matter to check that (F_1, F_2) is an indistinguishable pair. Hence, $t_l(v) \neq deg(v) = 2$. We then propose the following two concepts:

Definition 5. *Let $G(V,E)$ be a graph and $v \in V$ be a vertex. Vertex v has the strong local diagnosability property if the local diagnosability of vertex v is equal to its degree.*

Definition 6. *Let $G(V,E)$ be a graph. G has the strong local diagnosability property if every vertex in the graph G has the strong local diagnosability property.*

By Definition 5 and Definition 6, we have the following theorem:

Theorem 7. *Let Q_n be an n-dimensional hypercube, $n \geq 3$. Q_n has the strong local diagnosability property.*

Proof. We use Theorem 6 to prove this result, and we shall construct a Type I structure of order n at each vertex for $n \geq 3$. We prove this by induction on n. Since an n-dimensional hypercube Q_n is vertex-symmetric, we can concentrate on the construction of Type I structure at a given vertex v. For $n = 3$, $deg(v) = 3$ and it is clear that Q_3 contains a Type I structure $T_1(v;3)$ of order 3 at vertex v (see Fig. 6a and 6b). As the inductive hypothesis, we assume that Q_{n-1} contains a Type I structure $T_1(v;n-1)$ of order $n-1$ at each vertex for some $n \geq 4$. Now, we consider that Q_n, Q_n can be decomposed into two subcubes Q^0_{n-1} and Q^1_{n-1} by some dimension. Without loss of generality, we may assume that the vertex $v \in Q^0_{n-1}$. By the inductive hypothesis, Q^0_{n-1} contains a Type I structure $T_1(v;n-1)$ of order $n-1$ at vertex v. Consider the vertex $v^{(1)}$ in Q^1_{n-1}. Vertex $v^{(1)}$ has an adjacent neighbor that is in Q^1_{n-1} due to $deg(v^{(1)}) = n$, where $n \geq 3$. Thus, Q_n contains a Type I structure $T_1(v;n)$ of order n at vertex v. By Theorem 6,

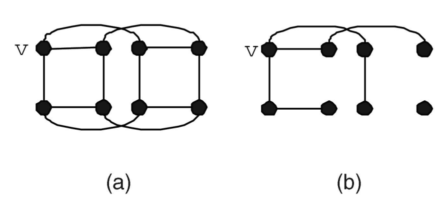
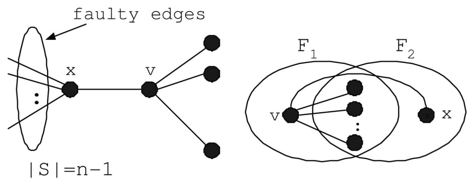


Fig. 6. A Q_3 and a Type I structure $T_1(v;3)$ of order 3 at vertex v.



(a) (b)

Fig. 7. An indistinguishable pair (F_1, F_2), where $|F_1| = |F_2| = n$.

Definition 5, and Definition 6, Q_n has the strong local diagnosability property. □

We now consider a system which is not vertex-symmetric. Let $G(V, E)$ be a graph and $S \subset E(G)$ be a set of edges. Removing the edges in S from G, the degree of each vertex in the resulting graph $G - S$ is called the remaining degree of v and is denoted by $deg_{G-S}(v)$. We consider a faulty hypercube Q_n with a faulty set $S \subset E(Q_n)$, $n \geq 3$. We shall prove that Q_n has the strong local diagnosability property even if it has up to $(n-2)$ faulty edges. The number $n-2$ is optimal in the sense that a faulty hypercube Q_n cannot be guaranteed to have this strong property if there are $n-1$ faulty edges. As shown in Fig. 7a and 7b, we take a vertex $v \in V(Q_n)$ and a vertex x which is an adjacent neighbor of v. Letting $S = \{(y, x) \in E(Q_n) \mid \text{vertex } y \text{ is directly adjacent to } x\} - \{(v, x)\}$, then $|S| = n-1$ and the remaining degree of v in $Q_n - S$ is n. Let $F_1 = (N_{Q_n - S}(v) - \{x\}) \bigcup \{v\}$ and $F_2 = N_{Q_n - S}(v)$, then $|F_1| = |F_2| = n$ and $v \in F_1 \Delta F_2$. It is clear that there is no edge between $V - (F_1 \bigcup F_2)$ and $F_1 \Delta F_2$. By Lemma 2, (F_1, F_2) is an indistinguishable pair; hence, $t_l(v) \neq deg_{Q_n - S}(v) = n$. Therefore, $Q_n - S$ may not have this strong property if $|S| \geq n-1$.

Theorem 8. *Let Q_n be an n-dimensional hypercube with $n \geq 3$ and $S \subset E(Q_n)$ be a set of edges, $0 \leq |S| \leq n-2$. Removing all the edges in S from Q_n, the local diagnosability of each vertex is still equal to its remaining degree.*

Proof. We use Theorem 6 to prove this result, and we shall construct a Type I structure at each vertex. We prove this by induction on n. For $n = 3$, $0 \leq |S| \leq 1$, if $|S| = 0$, it is clear that Q_3 contains a Type I structure $T_1(v; 3)$ of order 3 at every vertex. If $|S| = 1$, a 3-dimensional hypercube Q_3 with one missing edge is shown in Fig. 8. It is a routine work to see that every vertex has a Type I structure $T_1(v; k)$ of order k at it, where k is the remaining degree of the vertex. As the inductive hypothesis, we assume that the result is true for Q_{n-1}, $0 \leq |S| \leq (n-1) - 2$, for some $n \geq 4$. Now, we consider Q_n, $0 \leq |S| \leq n-2$. If $|S| = 0$, referring to the proof of Theorem 7, Q_n contains a Type I structure $T_1(v; n)$ of order n at every vertex. If $1 \leq |S| \leq n-2$, we choose an edge in S, the edge is in some dimension, decomposing Q_n into two subcubes Q^0_{n-1} and Q^1_{n-1} by this dimension, such that the edge is a crossing edge. Consider a vertex $v \in V(Q_n)$. Let $S_0 = S \bigcap E(Q^0_{n-1})$, $0 \leq |S_0| \leq (n-3)$, and $S_1 = S \bigcap E(Q^1_{n-1})$, $0 \leq |S_1| \leq (n-3)$. Without loss of generality, we may assume that the vertex v is in Q^0_{n-1} and $deg_{Q^0_{n-1} - S_0}(v) = k$. By the inductive hypothesis, $Q^0_{n-1} - S_0$ contains a Type I structure $T_1(v; k)$ at v. Consider the crossing edge $(v, v^{(1)})$. If $(v, v^{(1)}) \in S$, $Q_n - S$ contains a Type I structure $T_1(v; k)$ of order k at vertex v. If $(v, v^{(1)}) \notin S$, the remaining degree of v in $Q_n - S$ is $k+1$ and the vertex $v^{(1)}$ has at least an adjacent neighbor in Q^1_{n-1} due to $0 \leq |S_1| \leq (n-1) - 2$. Therefore, $Q_n - S$ contains a Type I structure $T_1(v; k+1)$ of order $k+1$ at vertex v. By Theorem 6, removing all the edges in S from Q_n, the local diagnosability of each vertex is still equal to its remaining degree. □

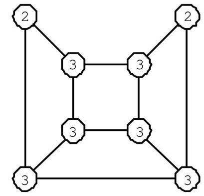


Fig. 8. Q_3 with one missing edge. The number labeled on each vertex represents its local diagnosability.

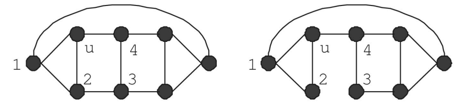
We have the following corollary:

Corollary 1. *Let Q_n be an n-dimensional hypercube with $n \geq 3$, and $S \subset E(Q_n)$ be a set of edges, $0 \leq |S| \leq n-2$. Then, $Q_n - S$ has the strong local diagnosability property.*

We give an example to show that an n-regular graph $G(V, E)$ has the strong local diagnosability property, but it may not keep this strong property after removing $n-2$ edges from G. For example, a 3-regular graph is shown in Fig. 9a. The degree of each vertex is 3 and there exists a Type I structure $T_1(v; 3)$ of order 3 at each vertex. By Theorem 6, Definition 5, and Definition 6, this graph has the strong local diagnosability property. Letting $S = \{(2, 3)\}$ be a set of one single edge, $G - S$ is shown in Fig. 9b. The vertex u does not have the strong local diagnosability property. The reason is as follows: Let $F_1 = \{u, 1, 4\}$ and $F_2 = \{1, 2, 4\}$ with $|F_1| \leq 3$, $|F_2| \leq 3$. Since there is no edge between $V(G) - (F_1 \bigcup F_2)$ and $F_1 \Delta F_2$, by Lemma 2, (F_1, F_2) is an indistinguishable pair. Therefore, the local diagnosability of vertex u is at most 2 which is smaller than its degree.

5 Conditional Fault Local Diagnosability

In the previous section, we know that Q_n does not have the strong local diagnosability property if there are $n-1$ faulty edges, all these faulty edges are incident with a single vertex and this vertex is incident with only one fault-free edge. Therefore, we are led to the following question: How



(a) (b)

Fig. 9. A 3-regular graph without the strong local diagnosability property after removing one edge.

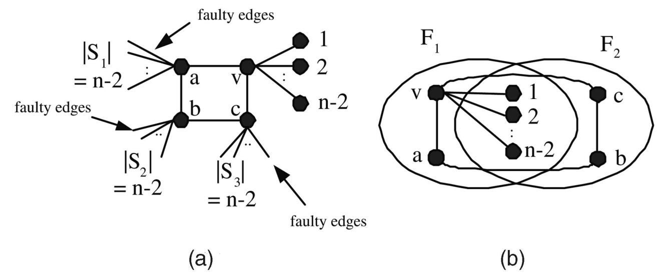


Fig. 10. An indistinguishable pair (F_1, F_2), where $|F_1| = |F_2| = n$.

many edges can be removed from Q_n such that Q_n keeps the strong local diagnosability property under the condition that each vertex of the faulty hypercube Q_n is incident with at least two fault-free edges? First, we give an example to show that a faulty hypercube Q_n with $3(n-2)$ faulty edges may not have the strong local diagnosability property, even if each vertex of the faulty hypercube Q_n is incident with at least two fault-free edges. As shown in Fig. 10a, we take a cycle of length four in Q_n, $n \geq 3$. Let $\{v, a, b, c\}$ be the four consecutive vertices on this cycle, and $S \subset E(Q_n)$ be a set of edges, $S = S_1 \bigcup S_2 \bigcup S_3$, where S_1 is the set of all edges incident with a except (v, a) and (b, a), S_2 is the set of all edges incident with b except (a, b) and (c, b), and S_3 is the set of all edges incident with c except (v, c) and (b, c), then $|S_1| = |S_2| = |S_3| = n - 2$. The remaining degree of vertex v in $Q_n - S$ is n, $deg_{Q_n-S}(v) = n$. As shown in Fig. 10b, let $F_1 = (N_{Q_n-S}(v) - \{c\}) \bigcup \{v\}$ and $F_2 = (N_{Q_n-S}(v) - \{a\}) \bigcup \{b\}$, then $|F_1| = |F_2| = n$ and $v \in F_1 \Delta F_2$. It is clear that there is no edge between $V(Q_n) - (F_1 \bigcup F_2)$ and $F_1 \Delta F_2$. By Lemma 2, (F_1, F_2) is an indistinguishable pair, hence, $t_l(v) \neq deg_{Q_n-S}(v) = n$. So, some vertex of $Q_n - S$ may not have this strong property if $|S| \geq 3(n-2)$. Then, we shall show that $Q_n - S$ has the strong local diagnosability property if each vertex of $Q_n - S$ is incident with at least two fault-free edges and $|S| \leq 3(n-2) - 1$. We need the following results to construct a Type I structure or a Type II structure at a vertex of a faulty hypercube.

Theorem 9 [20]. *Let $G(V, E)$ be a bipartite graph with bipartition (X, Y). Then, G has a matching that saturates every vertex in X if and only if*

$$|N(S)| \leq |S|, \text{ for all} \subseteq X.$$

Theorem 10 [20]. *Let $G(V, E)$ be a bipartite graph. The maximum size of a matching in G equals the minimum size of a vertex cover of G.*

Lemma 7. *An n-dimensional hypercube Q_n has no cycle of length 3 and any two vertices have at most two common neighbors.*

For our discussion later, we need some definitions. Let Q_n be an n-dimensional hypercube and $S \subseteq E(Q_n)$ be a set of edges. Removing the edges in S from Q_n, for a vertex v in the resulting graph $Q_n - S$, we define $BG(v) = (L_1(v) \bigcup L_2(v), E)$ to be the bipartite graph under v with bipartition $(L_1(v), L_2(v))$, where $L_1(v) = \{x \in V(Q_n) \mid$ vertex x is adjacent to vertex v in $Q_n - S\}$, $L_2(v) = \{y \in V(Q_n) \mid$ there exists a vertex $x \in L_1(v)$ such that $(x, y) \in E(Q_n)$ in $Q_n - S\} - \{v\}$, and $E(BG(v)) = \{(x, y) \in E(Q_n) \mid$ vertex $x \in L_1(v)$ and vertex $y \in L_2(v)\}$. $L_1(v)$ ($L_2(v)$, respectively) is called the level one (level two, respectively) vertex under v (see Fig. 11).

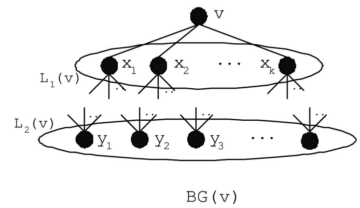


Fig. 11. The bipartite graph $BG(v)$.

Theorem 11. *Let Q_n be an n-dimensional hypercube with $n \geq 3$ and $S \subset E(Q_n)$ be a set of edges, $0 \leq |S| \leq 3(n-2) - 1$. Assume that each vertex of $Q_n - S$ is incident with at least two fault-free edges. Removing all the edges in S from Q_n, the local diagnosability of each vertex is still equal to its remaining degree.*

Proof. According to Theorem 6, we can concentrate on the construction of Type I structure or Type II structure at each vertex. Consider a vertex v in $Q_n - S$ with $deg_{Q_n-S}(v) = k$. As shown in Fig. 11, let $BG(v) = (L_1(v) \bigcup L_2(v), E)$ be the bipartite graph under v. Then, $|L_1(v)| = k$. Let $M \subset E(BG(v))$ be a maximum matching from $L_1(v)$ to $L_2(v)$. In the following proof, we consider three cases by the size of M: 1) $|M| = k$, 2) $|M| = k - 1$, and 3) $|M| \leq k - 2$.

Case 1: $|M| = k$. Since $|M| = k$ and $|L_1(v)| = k$, there exists a Type I structure $T_1(v; k)$ of order k at vertex v. By Theorem 6, the local diagnosability of vertex v is equal to k.

Case 2: $|M| = k - 1$. We shall show that there is a Type II structure of order k at vertex v. As shown in Fig. 12, let $L_1(v) = \{x_1, x_2, \ldots, x_k\}$ and let $ML_2(v) \subset L_2(v)$ be the set of vertices matched under M, $ML_2(v) = \{y \in L_2(v) \mid$ there exists a vertex $x \in L_1(v)$ such that $(x, y) \in M\}$. So, $|ML_2(v)| = k - 1$. Let $ML_2(v) = \{y_1, y_2, \ldots, y_{k-1}\}$ and assume vertex x_i is matched with vertex y_i for each i, $1 \leq i \leq k - 1$. Then, there exists a vertex $x_k \in L_1(v)$ and x_k is unmatched by M. Since each vertex of $Q_n - S$ is incident with at least two fault-free edges, there exists a vertex $y_i \in ML_2(v)$, $i \in \{1, 2, \ldots, k-1\}$, such that $(x_k, y_i) \in E(BG(v))$. Without loss of generality, let $(x_k, y_1) \in E(BG(v))$. If the remaining degree of y_1 is at least three, as shown in Fig. 13, there exists a Type II structure $T_2(v; k-2, 2)$ of order k at vertex v. By Theorem 6, the local

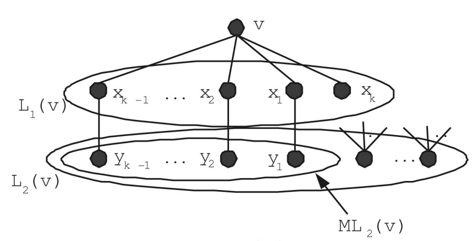


Fig. 12. An illustration for Case 2 of Theorem 11 and Theorem 12.

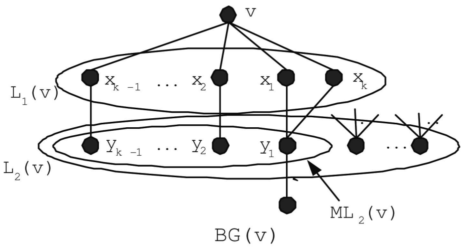


Fig. 13. A Type II structure $T_2(v; k-2, 2)$ of order k at vertex v.

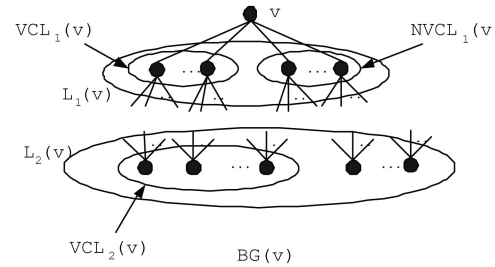


Fig. 14. An illustration for Case 3 of Theorem 11 and Theorem 12.

diagnosability of vertex v is equal to k and the result follows. If the remaining degree of y_1 is two, the number of faulty edges incident with y_1 is $n-2$. Next, we divide the case into two subcases: Subcase 2.1, where both x_k and x_1 have remaining degree two, and Subcase 2.2, where one of x_k and x_1 has remaining degree at least three and the other has at least two.

Subcase 2.1: Both x_k and x_1 have remaining degree two. This is an impossible case. Since the number of faulty edges incident with x_k and x_1 is $2(n-2)$, the total number of faulty edges is at least $3(n-2)$ which is greater than $3(n-2)-1$, a contradiction.

Subcase 2.2: One of x_k and x_1 has remaining degree at least three and the other has at least two. Without loss of generality, assume x_k has remaining degree at least three and x_1 has remaining degree at least two. Since $deg_{Q_n-S}(x_k) \geq 3$, there exist at least two vertices in $ML_2(v)$ that are the neighbors of vertex x_k. Then, we can find a vertex $y_i \in ML_2(v)$ and $y_i \neq y_1$, $i \in \{2, 3, \ldots, k-1\}$, such that $(x_k, y_i) \in E(BG(v))$. Without loss of generality, let $(x_k, y_2) \in E(BG(v))$. If the remaining degree of y_2 is at least three, there exists a Type II structure $T_2(v; k-2, 2)$ of order k at vertex v. By Theorem 6, the local diagnosability of vertex v is equal to k and the result follows. If the remaining degree of y_2 is two, the number of faulty edges incident with y_2 is $n-2$. We then consider two further cases:

Subcase 2.2.1: Vertex x_1 has remaining degree two. This is an impossible case. Since the number of faulty edges incident with x_1 is $n-2$, the total number of faulty edges is at least $3(n-2)$ which is greater than $3(n-2)-1$, a contradiction.

Subcase 2.2.2: Vertex x_1 has remaining degree at least three. Since $deg_{Q_n-S}(x_1) \geq 3$, there exist at least two vertices in $ML_2(v)$ that are the neighbors of vertex x_1. By Lemma 7, any two vertices of Q_n have at most two common neighbors. We can find a vertex $y_i \in ML_2(v)$, $y_i \neq y_1$ and $y_i \neq y_2$, $i \in \{3, 4, \ldots, k-1\}$, such that $(x_1, y_i) \in E(BG(v))$. Without loss of generality, let $(x_1, y_3) \in E(BG(v))$. If the remaining degree of y_3 is at least three, there exists a Type II structure $T_2(v; k-2, 2)$ of order k at vertex v. By Theorem 6, the local diagnosability of vertex v is equal to k and the result follows. If the remaining degree of y_3 is two, then the number of faulty edges incident with y_3 is $n-2$, and the total number of faulty edges is at least $3(n-2)$ which is greater than $3(n-2)-1$, a contradiction.

Case 3: $|M| \leq k-2$. We shall see that this is an impossible case. By Theorem 10, the minimum size of a vertex cover of the bipartite graph $BG(v)$ is no greater than $k-2$. We take a vertex cover with the minimum size and let $VCL_1(v) \subset L_1(v)$, $VCL_2(v) \subset L_2(v)$, and $VCL_1(v) \bigcup VCL_2(v)$ be the vertex cover as shown in Fig. 14. $VCL_1(v)$ and $VCL_2(v)$ can cover all the edges of $BG(v)$. Let $NVCL_1(v) = L_1(v) - VCL_1(v)$. We claim that the total number of faulty edges is at least $(n-1)|NVCL_1(v)| - 2|VCL_2(v)|$, and this number is greater than $3(n-2)$, which is a contradiction. With this claim, the case is impossible.

Now, we prove the claim. First, for each vertex $x \in NVCL_1(v)$, the edges connecting x except (x, v) must be incident with the vertices in $VCL_2(v)$. For each vertex $y \in VCL_2(v)$, by Lemma 7, at most two edges connecting y are incident with the vertices in $NVCL_1(v)$. Then, the total number of faulty edges is at least $(n-1)|NVCL_1(v)| - 2|VCL_2(v)|$. Since $VCL_1(v) \bigcup VCL_2(v)$ is a minimum vertex cover, $|VCL_1(v)| + |VCL_2(v)| \leq k-2$. Since $|L_1(v)| = k$ and each vertex of $Q_n - S$ is incident with at least two fault-free edges, there exists a vertex in $L_1(v) - VCL_1(v)$ such that the vertex has at least one neighbor in $VCL_2(v)$. Thus, $|VCL_2(v)| \geq 1$. Now, we show that the number $(n-1)|NVCL_1(v)| - 2|VCL_2(v)|$ is greater than $3(n-2)$. With $|VCL_1(v)| + |VCL_2(v)| \leq k-2$ and $|VCL_2(v)| \geq 1$, we have the following:

$$
\begin{aligned}
&[(n-1)|NVCL_1(v)| - 2|VCL_2(v)|] - [3(n-2)] \\
&= [(n-1)(k - |VCL_1(v)|) - 2|VCL_2(v)|] - [3(n-2)] \\
&\geq [(n-1)(|VCL_2(v)| + 2) - 2|VCL_2(v)|] - [3(n-2)] \\
&= (|VCL_2(v)| - 1)(n-3) + 1 \\
&> 0, \text{ for all } n \geq 3.
\end{aligned}
$$

Thus, our claim holds.

In summary, aside from those impossible cases, we showed that $Q_n - S$ contains either a Type I structure $T_1(v; k)$ or a Type II structure $T_2(v; k-2, 2)$ of order k at vertex v. By Theorem 6, removing all the edges in S from Q_n, the local diagnosability of each vertex is still equal to its remaining degree. □

By Theorem 11, we have the following corollary:

Corollary 2. *Let Q_n be an n-dimensional hypercube with $n \geq 3$ and $S \subset E(Q_n)$ be a set of edges, $0 \leq |S| \leq 3(n-2)-1$. $Q_n - S$ has the strong local diagnosability property, provided that each vertex of $Q_n - S$ is incident with at least two fault-free edges.*

Finally, we consider another condition: Each vertex of a faulty hypercube Q_n is incident with at least three fault-free edges. Based on this condition, we prove that Q_n keeps the strong local diagnosability property no matter how many edges are faulty.

Theorem 12. *Let Q_n be an n-dimensional hypercube with $n \geq 3$ and $S \subset E(Q_n)$ be a set of edges. Assume that each vertex of $Q_n - S$ is incident with at least three fault-free edges. Removing all the edges in S from Q_n, the local diagnosability of each vertex is still equal to its remaining degree.*

Proof. According to Theorem 6, we can concentrate on the construction of Type I structure or Type II structure at each vertex. Consider a vertex v in $Q_n - S$ with $deg_{Q_n-S}(v) = k$. Let $BG(v) = (L_1(v) \bigcup L_2(v), E)$ be the bipartite graph under v. Then, $|L_1(v)| = k$. Let $M \subset E(BG(v))$ be a maximum matching from $L_1(v)$ to $L_2(v)$. In the following proof, we consider three cases by the size of M: 1) $|M| = k$, 2) $|M| = k-1$, and 3) $|M| \leq k-2$.

Case 1: $|M| = k$. Since $|M| = k$ and $|L_1(v)| = k$, there exists a Type I structure $T_1(v;k)$ of order k at vertex v. By Theorem 6, the local diagnosability of vertex v is equal to k.

Case 2: $|M| = k-1$. We will show that there is a Type II structure of order k at vertex v. As shown in Fig. 12, let $L_1(v) = \{x_1, x_2, \ldots, x_k\}$ and let $ML_2(v) \subset L_2(v)$ be the set of vertices matched under M, $ML_2(v) = \{y \in L_2(v) \mid$ there exists a vertex $x \in L_1(v)$ such that $(x,y) \in M\}$. So, $|ML_2(v)| = k-1$. Let $ML_2(v) = \{y_1, y_2, \ldots, y_{k-1}\}$ and assume vertex x_i is matched with vertex y_i for each i, $1 \leq i \leq k-1$. Then, there exists a vertex $x_k \in L_1(v)$ and x_k is unmatched by M. Since each vertex of $Q_n - S$ is incident with at least three fault-free edges, there exists a vertex $y_i \in ML_2(v)$, $i \in \{1, 2, \ldots, k-1\}$, such that $(x_k, y_i) \in E(BG(v))$. Without loss of generality, let $(x_k, y_1) \in E(BG(v))$. Since the remaining degree of y_1 is at least three, as shown in Fig. 13, there exists a Type II structure $T_2(v; k-2, 2)$ of order k at vertex v. By Theorem 6, the local diagnosability of vertex v is equal to k and the result follows.

Case 3: $|M| \leq k-2$. We will see that this is an impossible case. By Theorem 10, the minimum size of a vertex cover of the bipartite graph $BG(v)$ is no greater than $k-2$. However, we claim that any $k-2$ vertices of $BG(v)$ can not cover all the edges of $BG(v)$. With this claim, the case is impossible.

Now, we prove this claim. Suppose we take a vertex cover with the minimum size and let $VCL_1(v) \subset L_1(v)$, $VCL_2(v) \subset L_2(v)$, and $VCL_1(v) \bigcup VCL_2(v)$ be the vertex cover as shown in Fig. 14. $VCL_1(v)$ and $VCL_2(v)$ can cover all the edges of $BG(v)$. Since

$$|VCL_1(v)| + |VCL_2(v)| \leq k-2,$$

we rewrite this inequality into the following equivalent form:

$$2(k - |VCL_1(v)|) \geq 2(|VCL_2(v)| + 2).$$

Let $NVCL_1(v) = L_1(v) - VCL_1(v)$. Since each vertex of $Q_n - S$ is incident with at least three fault-free edges, for each vertex $x \in NVCL_1(v)$, aside from the edge (x,v), at least two edges connecting x must be incident with the vertices in $VCL_2(v)$. So, the total number of edges incident with the vertices in $VCL_2(v)$ is at least $2|NVCL_1(v)|$. For each vertex $y \in VCL_2(v)$, by Lemma 7, at most two edges connecting y are incident with the vertices in $NVCL_1(v)$. So, the total number of edges incident with the vertices in $NVCL_1(v)$ is at most $2|VCL_2(v)|$. Compare the lower bound $2|NVCL_1(v)|$ and the upper bound $2|VCL_2(v)|$. We have the following inequality:

$$2|NVCL_1(v)| = 2(k - |VCL_1(v)|)$$
$$\geq 2(|VCL_2(v)| + 2) > 2|VCL_2(v)|.$$

The lower bound $2|NVCL_1(v)|$ is greater than the upper bound $2|VCL_2(v)|$. It means that some edges are not covered by $VCL_1(v)$ or $VCL_2(v)$ in $BG(v)$. Thus, our claim follows.

In Case 1, $Q_n - S$ contains a Type I structure $T_1(v;k)$ of order k at vertex v. In Case 2, $Q_n - S$ contains a Type II structure $T_2(v; k-2, 2)$ of order k at vertex v. We also proved that Case 3 is impossible. By Theorem 6, removing all the edges in S from Q_n, the local diagnosability of each vertex is still equal to its remaining degree. □

By Theorem 12, the following corollary holds:

Corollary 3. *Let Q_n be an n-dimensional hypercube with $n \geq 3$ and $S \subset E(Q_n)$ be a set of edges. Q_n keeps the strong local diagnosability property no matter how many edges are faulty, provided that each vertex of $Q_n - S$ is incident with at least three fault-free edges.*

6 Conclusion

In this paper, we propose a new concept called local diagnosability for a system and derive some structures for determining whether a system is locally t-diagnosable at a given vertex. Through this concept, the diagnosability of a system can be determined by computing the local diagnosability of each vertex. We also introduce a concept for system diagnosis, called the strong local diagnosability property. A system has this strong property if the local diagnosability of every vertex is equal to its degree. We prove that the hypercube has this strong property. Then, we consider an n-dimensional faulty hypercube Q_n with a set of faulty edges $S \subset E(Q_n)$, $0 \leq |S| \leq n-2$, $n \geq 3$. We prove that a faulty hypercube $Q_n - S$ keeps this strong property. According to Theorem 2, the global diagnosability of $Q_n - S$ is equal to the minimum local diagnosability of all vertices. A conditional local diagnosability measure for systems is also introduced in this paper. Assume that each vertex of a faulty hypercube Q_n is incident with at least two fault-free edges, we prove that Q_n has this strong property even if it has up to $3(n-2)-1$ faulty edges. Finally, we prove that Q_n keeps this strong property no matter how many edges are faulty, provided that each vertex of a faulty hypercube Q_n is incident with at least three fault-free edges.

We use the hypercube as an example to introduce the concepts of the local diagnosability, the local structures, and the strong local diagnosability property. In fact, many well-known systems also have these local structures and this strong property. Furthermore, there is a close relationship between its local structure and its local syndrome. We are currently studying on these issues. There are several different fault diagnosis models in the area of diagnosability. It is worth investigating, under various models, whether a system has this strong local diagnosability property after removing some edges. It is also an attractive work to develop more different measures of diagnosability based on network reliability, network topology, application environment, and statistics related to fault patterns.

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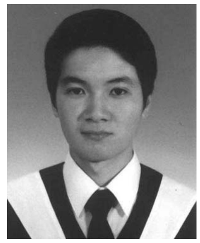

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