

# Node-pancyclicity and edge-pancyclicity of hypercube variants

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## Abstract

Twisted cubes, crossed cubes, Möbius cubes, and locally twisted cubes are some of the widely studied hypercube variants. The 4-pancyclicity of twisted cubes, crossed cubes, Möbius cubes, locally twisted cubes and the 4-edge-pancyclicity of twisted cubes, crossed cubes, Möbius cubes are proven in [C.P. Chang, J.N. Wang, L.H. Hsu, Topological properties of twisted cube, *Inform. Sci.* 113 (1999) 147–167; C.P. Chang, T.Y. Sung, L.H. Hsu, Edge congestion and topological properties of crossed cubes, *IEEE Trans. Parall. Distr.* 11 (1) (2000) 64–80; J. Fan, Hamilton-connectivity and cycle embedding of the Möbius cubes, *Inform. Process. Lett.* 82 (2002) 113–117; X. Yang, G.M. Megson, D.J. Evans, Locally twisted cubes are 4-pancyclic, *Appl. Math. Lett.* 17 (2004) 919–925; J. Fan, N. Yu, X. Jia, X. Lin, Embedding of cycles in twisted cubes with edge-pancyclic, *Algorithmica*, submitted for publication; J. Fan, X. Lin, X. Jia, Node-pancyclic and edge-pancyclic of crossed cubes, *Inform. Process. Lett.* 93 (2005) 133–138; M. Xu, J.M. Xu, Edge-pancyclicity of Möbius cubes, *Inform. Process. Lett.* 96 (2005) 136–140], respectively. It should be noted that 4-edge-pancyclicity implies 4-node-pancyclicity which further implies 4-pancyclicity. In this paper, we outline an approach to prove the 4-edge-pancyclicity of some hypercube variants and we prove in particular that Möbius cubes and locally twisted cubes are 4-edge-pancyclic.

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**Keywords:** Interconnection networks; Hypercube; Crossed cube; Möbius cube; Locally twisted cube; Pancyclicity

## 1. Introduction

Interconnection networks are essential for parallel and distributed computing. The hypercube is one of the most popular interconnection networks since it has simple structure and is easy to implement. An interconnection network can be represented by a graph  $G = (V, E)$ ,

where  $V$  is the set of nodes and  $E$  is the set of edges of the network. In this paper, we will use graphs and interconnection networks interchangeably.

It has been shown that hypercubes do not achieve the smallest possible diameter for its resources. Therefore, many variants were proposed. The most well-known variants are twisted cubes [9], crossed cubes [4], and Möbius cubes [3]; they have diameters about half of that of a hypercube. Generally, the drawback of these variants is that the labels of some neighboring nodes may differ in as many as  $n/2$  bits, where  $n$  is the dimension of these hypercube variants (see [11] for details). For example, in the 10-dimensional crossed cube, nodes

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0001010101 and 1011111111 are adjacent and they differ in 5 bits. Based on this observation, Yang et al. [11] proposed the locally twisted cubes with diameters about half of that of a hypercube, of which the labels of any two neighboring nodes differ in at most two successive bits.

The following terminologies will be used throughout this paper. An  $\ell$ -cycle is a cycle of length  $\ell$ . Let  $G = (V, E)$  be a graph and  $L \leq |V|$  be a positive integer.  $G$  is  $L$ -pancyclic if for every integer  $\ell \in \{L, L+1, \dots, |V|\}$ ,  $G$  contains an  $\ell$ -cycle.  $G$  is  $L$ -node-pancyclic if for every node  $x \in V$  and every integer  $\ell \in \{L, L+1, \dots, |V|\}$ ,  $G$  contains an  $\ell$ -cycle  $C$  such that  $x$  is in  $C$ .  $G$  is  $L$ -edge-pancyclic if for every edge  $(x, y) \in E$  and every integer  $\ell \in \{L, L+1, \dots, |V|\}$ ,  $G$  contains an  $\ell$ -cycle  $C$  such that  $(x, y)$  is in  $C$ .

One way to evaluate an interconnection network (a host graph) is to see how well other existing networks (the guest graphs) can be embedded into it. Graph embedding can be formally defined as follows: Given two graphs  $G = (V, E)$  and  $H = (V', E')$ , an embedding from  $G$  to  $H$  is a mapping  $\psi: V \rightarrow V'$ . An important benefit of graph embedding is that we can apply existing algorithms for the guest graphs to the host graph. Cycles (i.e., rings) and trees are commonly used guest graphs. This paper will discuss the cycle-embedding properties of Möbius cubes and locally twisted cubes (these cubes will be defined later).

Twisted cubes, crossed cubes, Möbius cubes, and locally twisted cubes are superior to hypercubes when the cycle-embedding capability is considered. The 4-pancyclicity of twisted cubes, crossed cubes, Möbius cubes, and locally twisted cubes are proven in [2,1,5,12], respectively. Recently, Fan et al. [6] proved that crossed cubes are not only 4-node-pancyclic but also 4-edge-pancyclic. It should be noted that 4-edge-pancyclicity implies 4-node-pancyclicity (thus the proof in [6] for the 4-node-pancyclicity of crossed cubes is actually redundant) which further implies 4-pancyclic. Lately, the 4-edge-pancyclicity of twisted cubes and Möbius cubes are proven in [7,10], respectively (see also [8]).

In this paper, we outline an approach to prove the 4-edge-pancyclicity of some hypercube variants and we prove, in particular, that Möbius cubes and locally twisted cubes are 4-edge-pancyclic. We also show how to use our approach to prove that crossed cubes are 4-edge-pancyclic.

This paper is organized as follows. In Section 2, we give some definitions and notations. In Section 3, we outline an approach to prove 4-edge-pancyclicity. In Sections 4–6, we prove that locally twisted cubes,

crossed cubes, and Möbius cubes are 4-edge-pancyclic. The final section concludes this paper.

## 2. Preliminaries

Let  $G = (V, E)$  be a graph and let  $L \leq |V| - 1$  be a positive integer.  $G$  is  $L$ -path-connected if  $G$  contains a path of length  $L$  between any two distinct nodes.  $G$  is *Hamiltonian-connected* if  $G$  is  $(|V| - 1)$ -path-connected.

The  $n$ -dimensional hypercube  $Q_n$  is a graph with  $2^n$  nodes and  $n \cdot (2^{n-1})$  edges such that its nodes are  $n$ -tuples with entries in  $\{0, 1\}$  and its edges are the pairs of  $n$ -tuples that differ in exactly one position. Thus  $Q_1$  is the complete graph with two nodes 0 and 1, and  $Q_n$  ( $n \geq 2$ ) is built from two copies of  $Q_{n-1}$  as follows: Let  $k \in \{0, 1\}$  and let  $kQ_{n-1}$  denote the graph obtained by prefixing the label of each node of one copy of  $Q_{n-1}$  with  $k$ ; connect each node  $0x_{n-1} \dots x_2x_1$  of  $0Q_{n-1}$  with the node  $1x_{n-1} \dots x_2x_1$  of  $1Q_{n-1}$  by an edge.

We now define a generalization of  $Q_n$ . The  $n$ -dimensional general cube  $GQ_n$  is defined recursively as follows (see Fig. 1).  $GQ_1$  is  $Q_1$ , and  $GQ_n$  ( $n \geq 2$ ) is built from two  $GQ_{n-1}$ 's (not necessarily identical) as follows: Let  $k \in \{0, 1\}$  and let  $kGQ_{n-1}$  denote the graph obtained by prefixing the label of each node of one of the two  $GQ_{n-1}$ 's with  $k$ ; add a perfect matching between  $0GQ_{n-1}$  and  $1GQ_{n-1}$ , i.e., each node in  $0GQ_{n-1}$  is adjacent to exactly one node in  $1GQ_{n-1}$ .

We assume conventionality of the node prefixing method  $kGQ_{n-1}$  which will be used repeatedly in the definitions of specific hypercube variants late in this paper unless otherwise specified. We will see in the following sections that crossed cubes, Möbius cubes, and locally twisted cubes are the examples of  $GQ_n$ . Note that the two  $GQ_{n-1}$ 's in  $GQ_n$  are not necessarily identical. For instance, for crossed cubes and locally twisted cubes, the two  $GQ_{n-1}$ 's are identical; but for Möbius cubes, they are not.

For clarity, let  $V(G)$  and  $E(G)$  denote the set of nodes and the set of edges of  $G$ , respectively. We

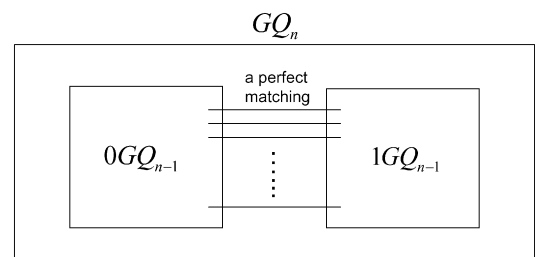


Fig. 1. The  $n$ -dimensional general cube  $GQ_n$ .

say that  $(x, y)$  is a *matching edge* in  $GQ_n$  if  $x \in V(0GQ_{n-1})$ ,  $y \in V(1GQ_{n-1})$ , and  $x$  is matched with  $y$ . If  $(x, y)$  is a matching edge, then we write  $m(x)$  for  $y$  and  $m(y)$  for  $x$ . We say that  $GQ_n$  has the *4-cycle property* if for every matching edge  $(x, y)$ , there exists a matching edge  $(u, v)$  such that  $(x, u, v, y, x)$  form a 4-cycle in  $GQ_n$ . We say that  $GQ_n$  has the *5-cycle property* if for every matching edge  $(x, y)$ , there exist a matching edge  $(s, t)$  and a node  $r \in V(0GQ_{n-1})$  such that  $(x, r, s, t, y, x)$  form a 5-cycle in  $GQ_n$ .

### 3. 4-edge-pancyclicity of general cubes

In this section, we outline an approach to prove 4-edge-pancyclicity. We first give two lemmas.

**Lemma 1.** *For  $n \geq 4$ , if both  $0GQ_{n-1}$  and  $1GQ_{n-1}$  are Hamiltonian-connected, then  $GQ_n$  is Hamiltonian-connected.*

**Proof.** Let  $x$  and  $y$  be two arbitrary distinct nodes of  $GQ_n$ . Then there are four cases.

*Case 1.*  $x \in V(0GQ_{n-1})$  and  $y \in V(0GQ_{n-1})$ . Since  $0GQ_{n-1}$  is Hamiltonian-connected, it has a Hamiltonian path  $(p_1, p_2, \dots, p_{2^{n-1}})$  such that  $p_1 = x$  and  $p_{2^{n-1}} = y$ . Since  $1GQ_{n-1}$  is Hamiltonian-connected, it has a Hamiltonian path  $(q_1, q_2, \dots, q_{2^{n-1}})$  such that  $q_1 = m(p_1)$  and  $q_{2^{n-1}} = m(p_{2^{n-1}})$ . Hence  $(x, q_1, q_2, \dots, q_{2^{n-1}}, p_2, p_3, \dots, p_{2^{n-1}-1}, y)$  is a Hamiltonian path between  $x$  and  $y$  in  $GQ_n$ .

*Case 2.*  $x \in V(1GQ_{n-1})$  and  $y \in V(1GQ_{n-1})$ . The argument is similar to that of Case 1.

*Case 3.*  $x \in V(0GQ_{n-1})$  and  $y \in V(1GQ_{n-1})$ . Let  $z \in V(0GQ_{n-1})$  such that  $z \neq x$ . Since  $0GQ_{n-1}$  is Hamiltonian-connected, it has a Hamiltonian path  $(p_1, p_2, \dots, p_{2^{n-1}})$  such that  $p_1 = x$  and  $p_{2^{n-1}} = z$ . Since  $1GQ_{n-1}$  is Hamiltonian-connected, it has a Hamiltonian path  $(q_1, q_2, \dots, q_{2^{n-1}})$  such that  $q_1 = m(z)$  and  $q_{2^{n-1}} = y$ . Hence  $(x, p_2, \dots, p_{2^{n-1}-1}, q_1, q_2, \dots, q_{2^{n-1}-1}, y)$  is a Hamiltonian path between  $x$  and  $y$  in  $GQ_n$ .

*Case 4.*  $x \in V(1GQ_{n-1})$  and  $y \in V(0GQ_{n-1})$ . The argument is similar to that of Case 3.  $\square$

**Lemma 2.** *For  $n \geq 4$ , if both  $0GQ_{n-1}$  and  $1GQ_{n-1}$  are Hamiltonian-connected and  $(2^{n-1} - 2)$ -path-connected, then  $GQ_n$  is  $(2^n - 2)$ -path-connected.*

**Proof.** Let  $x$  and  $y$  be two arbitrary distinct nodes of  $GQ_n$ . Then there are four cases.

*Case 1.*  $x \in V(0GQ_{n-1})$  and  $y \in V(0GQ_{n-1})$ . Since  $0GQ_{n-1}$  is Hamiltonian-connected, it has a Hamil-

tonian path  $(p_1, p_2, \dots, p_{2^{n-1}})$  such that  $p_1 = x$  and  $p_{2^{n-1}} = y$ . Since  $1GQ_{n-1}$  is  $(2^{n-1} - 2)$ -path-connected, it has a path  $(q_1, q_2, \dots, q_{2^{n-1}-1})$  of length  $2^{n-1} - 2$  such that  $q_1 = m(p_1)$  and  $q_{2^{n-1}-1} = m(p_{2^{n-1}})$ . Hence  $(x, q_1, q_2, \dots, q_{2^{n-1}-1}, p_2, p_3, \dots, p_{2^{n-1}-1}, y)$  is a path of length  $2^n - 2$  between  $x$  and  $y$  in  $GQ_n$ .

*Case 2.*  $x \in V(1GQ_{n-1})$  and  $y \in V(1GQ_{n-1})$ . The argument is similar to that of Case 1.

*Case 3.*  $x \in V(0GQ_{n-1})$  and  $y \in V(1GQ_{n-1})$ . Let  $z \in V(0GQ_{n-1})$  such that  $z \neq x$ . Since  $0GQ_{n-1}$  is Hamiltonian-connected, it has a Hamiltonian path  $(p_1, p_2, \dots, p_{2^{n-1}})$  such that  $p_1 = x$  and  $p_{2^{n-1}} = z$ . Since  $1GQ_{n-1}$  is  $(2^{n-1} - 2)$ -path-connected, it has a path  $(q_1, q_2, \dots, q_{2^{n-1}-1})$  of length  $2^{n-1} - 2$  such that  $q_1 = m(z)$  and  $q_{2^{n-1}-1} = y$ . Hence  $(x, p_2, \dots, p_{2^{n-1}-1}, q_1, q_2, \dots, q_{2^{n-1}-2}, y)$  is a path of length  $2^n - 2$  between  $x$  and  $y$  in  $GQ_n$ .

*Case 4.*  $x \in V(1GQ_{n-1})$  and  $y \in V(0GQ_{n-1})$ . The argument is similar to that of Case 3.  $\square$

We now outline an approach to prove the 4-edge-pancyclicity of  $GQ_n$ .

**Theorem 3.** *For  $n \geq 4$ , if all the  $GQ_3$ 's in  $GQ_n$  are 4-edge-pancyclic, Hamiltonian-connected, and  $(2^3 - 2)$ -path-connected, and if  $GQ_n$  has both the 4-cycle and the 5-cycle properties, then  $GQ_n$  is 4-edge-pancyclic.*

**Proof.** This theorem follows from Lemmas 1, 2, and the following claim.

**Claim.** *For  $n \geq 4$ , if both  $0GQ_{n-1}$  and  $1GQ_{n-1}$  are 4-edge-pancyclic, Hamiltonian-connected, and  $(2^{n-1} - 2)$ -path-connected, and if  $GQ_n$  has both the 4-cycle property and the 5-cycle property, then  $GQ_n$  is 4-edge-pancyclic.*

We now prove the claim. Let  $(x, y)$  be an arbitrary edge of  $E(GQ_n)$  and let  $\ell \in \{4, 5, \dots, 2^n\}$ . There are four cases.

*Case 1.*  $x \in V(0GQ_{n-1})$  and  $y \in V(0GQ_{n-1})$ . Then there are three subcases.

*Subcase 1.1.*  $4 \leq \ell \leq 2^{n-1}$ . Since  $0GQ_{n-1}$  is 4-edge-pancyclic, there exists an  $\ell$ -cycle that contains  $(x, y)$  in  $0GQ_{n-1}$ , hence in  $GQ_n$ .

*Subcase 1.2.*  $\ell = 2^{n-1} + 1$ . Let  $u = m(x)$  and  $v = m(y)$ . Since  $1GQ_{n-1}$  is  $(2^{n-1} - 2)$ -path-connected, it has a path  $(p_1, p_2, \dots, p_{2^{n-1}-1})$  of length  $2^{n-1} - 2$  such that  $p_1 = v$  and  $p_{2^{n-1}-1} = u$ . Thus  $(x, y, p_1, p_2, \dots, p_{2^{n-1}-1}, x)$  is a  $(2^{n-1} + 1)$ -cycle in  $GQ_n$  that contains  $(x, y)$ .

*Subcase 1.3.*  $2^{n-1} + 2 \leq \ell \leq 2^n$ . Since  $0GQ_{n-1}$  is 4-edge-pancyclic and  $(x, y)$  is an edge in  $0GQ_{n-1}$ , there exists a  $2^{n-1}$ -cycle  $C = (p_1, p_2, \dots, p_{2^{n-1}}, p_1)$  in  $0GQ_{n-1}$  such that  $p_1 = x$  and  $p_2 = y$ . Note that  $1 \leq \ell - 2^{n-1} - 1 \leq 2^{n-1} - 1$ . Let  $(p_1, p_2, \dots, p_{\ell-2^{n-1}})$  be the path of length  $\ell - 2^{n-1} - 1$  in  $C$ . Set  $w = p_{\ell-2^{n-1}}$  for easy writing. Let  $u = m(x)$  and  $v = m(w)$ . Then  $u, v \in V(1GQ_{n-1})$ . Since  $1GQ_{n-1}$  is Hamiltonian-connected, there is a path  $(q_1, q_2, \dots, q_{2^{n-1}})$  of length  $2^{n-1} - 1$  in  $1GQ_{n-1}$  such that  $q_1 = v$  and  $q_{2^{n-1}} = u$ . Thus  $(p_1, p_2, \dots, p_{\ell-2^{n-1}}, q_1, q_2, \dots, q_{2^{n-1}}, p_1)$  is a cycle of length  $(\ell - 2^{n-1} - 1) + 1 + (2^{n-1} - 1) + 1 = \ell$  in  $GQ_n$  that contains  $(x, y)$ .

*Case 2.*  $x \in V(1GQ_{n-1})$  and  $y \in V(1GQ_{n-1})$ . The argument is similar to that of Case 1.

*Case 3.*  $x \in V(0GQ_{n-1})$  and  $y \in V(1GQ_{n-1})$ . Then there are four subcases.

*Subcase 3.1.*  $\ell \in \{4, 5\}$ . Since  $GQ_n$  has the 4-cycle property and the 5-cycle property, there exists a cycle of length  $\ell$  in  $GQ_n$  that contains  $(x, y)$ .

*Subcase 3.2.*  $6 \leq \ell \leq 2^{n-1} + 2$ . Since  $GQ_n$  has the 4-cycle property, there exist  $u \in V(0GQ_{n-1})$  and  $v \in V(1GQ_{n-1})$  such that  $(x, u, v, x)$  form a 4-cycle in  $GQ_n$ . Let  $m = \ell - 2$ . Then  $4 \leq m \leq 2^{n-1}$ . Since  $0GQ_{n-1}$  is 4-edge-pancyclic, there exists a  $m$ -cycle  $(p_1, p_2, \dots, p_m, p_1)$  in  $0GQ_{n-1}$  such that  $p_1 = x$  and  $p_m = u$ . Thus  $(x, p_2, \dots, p_m, v, y, x)$  is an  $(m + 2)$ -cycle (i.e., an  $\ell$ -cycle) in  $GQ_n$  that contains  $(x, y)$ .

*Subcase 3.3.*  $\ell = 2^{n-1} + 3$ . Since  $GQ_n$  has the 4-cycle property, there exist  $u \in V(0GQ_{n-1})$  and  $v \in V(1GQ_{n-1})$  such that  $(x, u, v, x)$  form a 4-cycle in  $GQ_n$ . Since  $0GQ_{n-1}$  is 4-edge-pancyclic, there exists a  $(2^{n-1} - 1)$ -cycle  $(p_1, p_2, \dots, p_{2^{n-1}-1}, p_1)$  in  $0GQ_{n-1}$  such that  $p_1 = x$  and  $p_{2^{n-1}-1} = u$ . Since  $1GQ_{n-1}$  is 4-edge-pancyclic, there exists a 4-cycle  $(q_1, q_2, q_3, q_4, q_1)$  in  $1GQ_{n-1}$  such that  $q_1 = v$  and  $q_4 = y$ . Thus  $(p_1, p_2, \dots, p_{2^{n-1}-1}, q_1, q_2, q_3, q_4, p_1)$  is a  $(2^{n-1} + 3)$ -cycle in  $GQ_n$  that contains  $(x, y)$ .

*Subcase 3.4.*  $2^{n-1} + 4 \leq \ell \leq 2^n$ . Since  $GQ_n$  has the 4-cycle property, there exist  $u \in V(0GQ_{n-1})$  and  $v \in V(1GQ_{n-1})$  such that  $(x, u, v, x)$  form a 4-cycle in  $GQ_n$ . Since  $0GQ_{n-1}$  is 4-edge-pancyclic, there exists a  $2^{n-1}$ -cycle  $(p_1, p_2, \dots, p_{2^{n-1}}, p_1)$  in  $0GQ_{n-1}$  such that  $p_1 = x$  and  $p_{2^{n-1}} = u$ . Let  $m = \ell - 2^{n-1}$ . Then  $4 \leq m \leq 2^{n-1}$ . Since  $1GQ_{n-1}$  is 4-edge-pancyclic, there exists a  $m$ -cycle  $(q_1, q_2, \dots, q_m, q_1)$  in  $1GQ_{n-1}$  such that  $q_1 = v$  and  $q_m = y$ . Thus  $(p_1, p_2, \dots, p_{2^{n-1}}, q_1, q_2, \dots, q_m)$  is a cycle of length  $(2^{n-1} - 1) + (m - 1) + 2 = m + 2^{n-1} = \ell$  in  $GQ_n$  that contains  $(x, y)$ .

*Case 4.*  $x \in V(1GQ_{n-1})$  and  $y \in V(0GQ_{n-1})$ . The argument is similar to that of Case 3.  $\square$

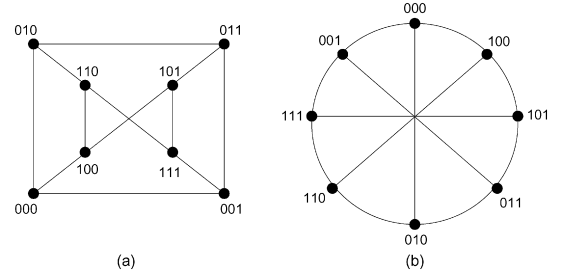


Fig. 2. (a)  $LTQ_3$ . (b) A symmetric drawing of  $LTQ_3$ .

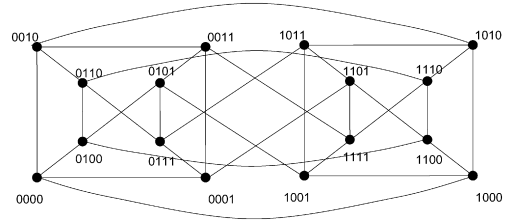


Fig. 3.  $LTQ_4$ .

#### 4. Pancyclicity of locally twisted cubes

The purpose of this section is to use Theorem 3 to prove that locally twisted cubes are 4-edge-pancyclic.

The  $n$ -dimensional locally twisted cube  $LTQ_n$  is defined recursively as follow.  $LTQ_1$  is  $Q_1$ , and  $LTQ_2$  is the graph consisting of four nodes labeled with 00, 01, 10, and 11, respectively, and connected by the four edges (00, 01) (00, 10), (01, 11), and (10, 11).  $LTQ_n$  ( $n \geq 3$ ) is built from two identical  $LTQ_{n-1}$ 's as follows: connect each node  $0x_{n-1}x_{n-2} \dots x_1$  of  $0LTQ_{n-1}$  with the node  $1(x_{n-1} + x_1)x_{n-2} \dots x_1$  of  $1LTQ_{n-1}$  by an edge, where '+' means the modulo 2 addition operation. See Figs. 2 and 3 for examples.

Before going any further, we work out the adjacency relation of  $LTQ_n$ . For convenience,  $\bar{x}_i$  denotes the complement of  $x_i$ .

**Lemma 4.** For every  $x = x_n x_{n-1} \dots x_1 \in V(LTQ_n)$ , the  $n$  nodes  $y_1, y_2, \dots, y_n$  adjacent to  $x$  are:

$$\begin{aligned} y_1 &= x_n x_{n-1} x_{n-2} \dots x_3 x_2 \bar{x}_1, \\ y_2 &= x_n x_{n-1} x_{n-2} \dots x_3 \bar{x}_2 x_1, \\ y_3 &= x_n x_{n-1} x_{n-2} \dots \bar{x}_3 (x_2 + x_1) x_1, \\ &\vdots \\ y_{n-1} &= x_n \bar{x}_{n-1} (x_{n-2} + x_1) \dots x_3 x_2 x_1, \\ y_n &= \bar{x}_n (x_{n-1} + x_1) x_{n-2} \dots x_3 x_2 x_1. \end{aligned}$$

**Proof.** By the definition of  $LTQ_n$ ,  $(x, y_n) \in E(LTQ_n)$ .  $(x, y_1) \in E(LTQ_n)$  because  $(x_1, \bar{x}_1) \in E(LTQ_1)$  and  $LTQ_n$  is built from  $LTQ_1$ . Similarly,  $(x, y_2) \in E(LTQ_n)$

because  $(x_2x_1, \bar{x}_2x_1) \in E(LTQ_2)$  and  $LTQ_n$  is built from  $LTQ_2$ . For  $3 \leq i \leq n-1$ ,  $(x, y_i) \in E(LTQ_n)$  because  $(x_i x_{i-1} x_{i-2} \dots x_1, \bar{x}_i (x_{i-1} + x_1) x_{i-2} \dots x_1) \in E(LTQ_i)$  and  $LTQ_n$  is built from  $LTQ_i$ .  $\square$

It is not difficult to see that: for each  $n$ , there is only one type of  $LTQ_n$ . Thus for  $n \geq 4$ , all the  $LTQ_3$ 's in  $LTQ_n$  are identical. We are now ready to prove that locally twisted cubes satisfy Theorem 3.

**Theorem 5.**  *$LTQ_3$  is 4-edge-pancyclic, Hamiltonian-connected, and  $(2^3 - 2)$ -path-connected. For  $n \geq 4$ ,  $LTQ_n$  has both the 4-cycle property and the 5-cycle property.*

**Proof.** In [12], it was proven that  $LTQ_n$  is Hamiltonian-connected and  $(2^n - 2)$ -path-connected for  $n \geq 3$ . Thus  $LTQ_3$  is Hamiltonian-connected and  $(2^3 - 2)$ -path-connected. We now prove that  $LTQ_3$  is 4-edge-pancyclic. Since  $LTQ_3$  is node-symmetric (see Fig. 2(b)), it suffices to consider the edge  $(x, y) \in \{(000, 001), (000, 010)\}$ . The cycles of lengths from 4 to 8 containing  $(000, 001)$  (underlined) are listed as follows:

- length 4: 000, 001, 011, 010, 000;
- length 5: 000, 001, 111, 101, 100, 000;
- length 6: 000, 001, 011, 010, 110, 100, 000;
- length 7: 000, 001, 011, 101, 111, 110, 100, 000;
- length 8: 000, 001, 111, 110, 010, 011, 101, 100, 000.

The cycles of lengths from 4 to 8 containing  $(000, 010)$  (underlined) are listed as follows:

- length 4: 000, 010, 110, 100, 000;
- length 5: 000, 010, 110, 111, 001, 000;
- length 6: 000, 010, 110, 111, 101, 100, 000;
- length 7: 000, 010, 110, 100, 101, 111, 001, 000;
- length 8: 000, 010, 110, 111, 001, 011, 101, 100, 000.

Thus  $LTQ_3$  is 4-edge-pancyclic.

We now prove that  $LTQ_n$  has the 4-cycle property and the 5-cycle property. Let  $(x, y)$  be an arbitrary matching edge of  $LTQ_n$  and let  $x = 0x_{n-1}x_{n-2} \dots x_2x_1$ . By the definition of  $LTQ_n$ ,  $y = 1(x_{n-1} + x_1)x_{n-2} \dots x_2x_1$ .

First consider the 4-cycle property. Let  $u = 0x_{n-1}x_{n-2} \dots \bar{x}_2x_1$  and  $v = 1(x_{n-1} + x_1)x_{n-2} \dots \bar{x}_2x_1$ . By Lemma 4,  $\{(x, u), (u, v), (v, y)\} \subseteq E(LTQ_n)$ . Hence  $(x, u, v, y, x)$  is a 4-cycle in  $LTQ_n$  that contains  $(x, y)$ . Now consider the 5-cycle property. If  $x_1 = 0$ , let  $r = 0\bar{x}_{n-1}x_{n-2} \dots x_20$ ,  $s = 0\bar{x}_{n-1}x_{n-2} \dots x_21$ , and  $t = 1x_{n-1}x_{n-2} \dots x_21$ ; otherwise, if  $x_1 = 1$ , let  $r = 0x_{n-1}x_{n-2} \dots x_20$ ,  $s = 0\bar{x}_{n-1}x_{n-2} \dots x_20$ , and  $t = 1\bar{x}_{n-1}x_{n-2} \dots x_20$ . By Lemma 4,  $\{(x, r), (r, s), (s, t), (t, y)\} \subseteq$

$E(LTQ_n)$ . Hence  $(x, r, s, t, y, x)$  is a 5-cycle in  $LTQ_n$  that contains  $(x, y)$ .  $\square$

It was proven in [12] that  $LTQ_n$  is 4-pancyclic. We now strengthen this result.

**Theorem 6.** *For  $n \geq 2$ ,  $LTQ_n$  is 4-edge-pancyclic.*

**Proof.** Clearly, this theorem holds when  $n = 2$ . By Theorem 5, this theorem holds when  $n = 3$ . For  $n \geq 4$ , this theorem follows from Theorems 3 and 5.  $\square$

The following corollary is obvious.

**Corollary 7.** *For  $n \geq 2$ ,  $LTQ_n$  is 4-node-pancyclic.*

### 5. Pancyclicity of crossed cubes

We first give the definition of crossed cubes. Two binary strings  $x = x_2x_1$  and  $y = y_2y_1$  of length two are said to be *pair related* (denoted by  $x \sim y$ ) if and only if  $(x, y) \in \{(00, 00), (10, 10), (01, 11), (11, 01)\}$ . The  $n$ -dimensional crossed cube  $CQ_n$  is defined recursively as follows.  $CQ_1$  is  $Q_1$ , and  $CQ_2$  is the graph consisting of four nodes labeled with 00, 01, 10 and 11, respectively, and connected by the four edges  $(00, 01)$ ,  $(00, 10)$ ,  $(01, 11)$ , and  $(10, 11)$ .  $CQ_n$  ( $n \geq 3$ ) is built from two identical  $CQ_{n-1}$ 's as follows: connect each node  $0x_{n-1} \dots x_2x_1$  of  $0CQ_{n-1}$  with the node  $1y_{n-1} \dots y_2y_1$  of  $1CQ_{n-1}$  by an edge if and only if

- (1)  $x_{n-1} = y_{n-1}$  if  $n$  is even, and
- (2)  $x_{2i}x_{2i-1} \sim y_{2i}y_{2i-1}$  for  $1 \leq i < \lceil n/2 \rceil$ .

In [6], Fan et al. have proven that crossed cubes are 4-edge-pancyclic. We now show how to use Theorem 3 to obtain this result. It is not difficult to see that: for each  $n$ , there is only one type of  $CQ_n$ . Thus for  $n \geq 4$ , all the  $CQ_3$ 's in  $CQ_n$  are identical. We are now ready to prove that crossed cubes satisfy Theorem 3.

**Theorem 8.**  *$CQ_3$  is 4-edge-pancyclic, Hamiltonian-connected, and  $(2^{3-1} - 2)$ -path-connected. For  $n \geq 4$ ,  $CQ_n$  has both the 4-cycle property and the 5-cycle property.*

Since the proof for each condition in this theorem can be found in [6], we omit the proof. We have the following theorem.

**Theorem 9.** [6] *For  $n \geq 2$ ,  $CQ_n$  is 4-edge-pancyclic.*

**Proof.** Clearly, this theorem holds when  $n = 2$ . By Theorem 8, this theorem holds when  $n = 3$ . For  $n \geq 4$ , this theorem follows from Theorems 3 and 8.  $\square$

By Theorem 9, it is obvious that for  $n \geq 2$ ,  $CQ_n$  is 4-node-pancyclic and 4-pancyclic.

### 6. Pancyclicity of Möbius cubes

In [10], Xu et al. have proven that Möbius cubes are 4-edge-pancyclic. In this section, we show how to use Theorem 3 to obtain this result.

The  $n$ -dimensional Möbius cube  $MQ_n$  is defined recursively as follow (see Figs. 4 and 5):

- (1)  $MQ_1$  is  $Q_1$ .
- (2) There are two types of  $MQ_2$ : one is named  $0-MQ_2$  and the other,  $1-MQ_2$ .  $0-MQ_2$  is the graph consisting of four nodes labeled with 00, 01, 10, and 11, respectively, and connected by the four edges (00, 01), (00, 10), (01, 11), and (10, 11).  $1-MQ_2$  has the same nodes as  $0-MQ_2$ , but connected by the four edges (00, 01), (00, 11), (01, 10), and (10, 11).
- (3) For  $n \geq 3$ , there are two types of  $MQ_n$ :  $0-MQ_n$  and  $1-MQ_n$ . Both  $0-MQ_n$  and  $1-MQ_n$  are built from

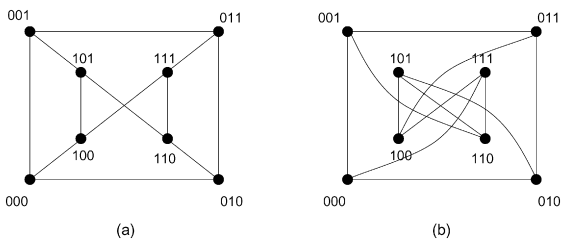


Fig. 4. (a)  $0-MQ_3$ . (b)  $1-MQ_3$ .

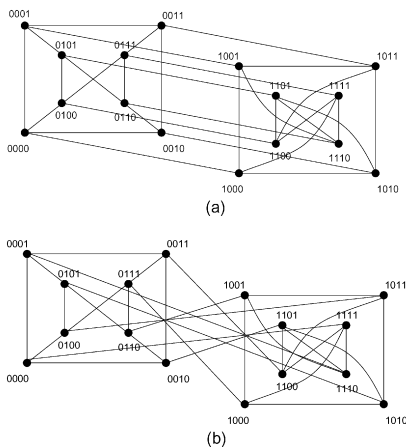


Fig. 5. (a)  $0-MQ_4$ . (b)  $1-MQ_4$ .

$0MQ_{n-1}$  and  $1MQ_{n-1}$  with the  $MQ_{n-1}$  in  $0MQ_{n-1}$  being  $0MQ_{n-1}$  and the  $MQ_{n-1}$  in  $1MQ_{n-1}$  being  $1MQ_{n-1}$ . In  $0-MQ_n$ , each node  $0x_{n-1}x_{n-2} \dots x_1$  of  $0MQ_{n-1}$  is connected with the node  $1x_{n-1}x_{n-2} \dots x_1$  of  $1MQ_{n-1}$ ; while in  $1-MQ_n$ , each node  $0x_{n-1}x_{n-2} \dots x_1$  of  $0MQ_{n-1}$  is connected with the node  $1\bar{x}_{n-1}\bar{x}_{n-2} \dots \bar{x}_1$  of  $1MQ_{n-1}$ .

Before going any further, we work out the adjacency relation of  $MQ_n$ .

**Lemma 10.** For every  $x = x_nx_{n-1} \dots x_2x_1 \in V(MQ_n)$ , the  $n$  nodes  $y_1, y_2, \dots, y_n$  adjacent to  $x$  are as follows. For  $1 \leq i \leq n - 1$ ,

$$y_i = \begin{cases} x_nx_{n-1} \dots x_{i+1}\bar{x}_ix_{i-1} \dots x_1 & \text{if } x_{i+1} = 0, \\ x_nx_{n-1} \dots x_{i+1}\bar{x}_i\bar{x}_{i-1} \dots \bar{x}_1 & \text{if } x_{i+1} = 1. \end{cases}$$

For  $0-MQ_n$ ,  $y_n = \bar{x}_nx_{n-1} \dots x_1$ ; for  $1-MQ_n$ ,  $y_n = \bar{x}_n\bar{x}_{n-1} \dots \bar{x}_1$ .

**Proof.** This lemma follows from the definition of Möbius cubes given in [3].  $\square$

It is not difficult to see that: for each  $n$ , there are two types of  $MQ_n$ : the  $0-MQ_n$  and the  $1-MQ_n$ . Thus for  $n \geq 4$ , all the  $MQ_3$ 's in  $MQ_n$  are either  $0-MQ_3$  or  $1-MQ_3$ . We are now ready to prove that Möbius cubes satisfy Theorem 3.

**Theorem 11.** Both the  $0-MQ_3$  and the  $1-MQ_3$  are 4-edge-pancyclic, Hamiltonian-connected, and  $(2^3 - 2)$ -path-connected. For  $n \geq 4$ ,  $MQ_n$  has both the 4-cycle property and the 5-cycle property.

**Proof.** From Figs. 2, 4, and 6, both  $0-MQ_3$  and  $1-MQ_3$  are isomorphic to  $LTQ_3$ . Thus by Theorem 5, both  $0-MQ_3$  and  $1-MQ_3$  are 4-edge-pancyclic, Hamiltonian-connected, and  $(2^3 - 2)$ -path-connected.

We now prove that  $MQ_n$  has the 4-cycle property and the 5-cycle property. Let  $(x, y)$  be an arbitrary matching

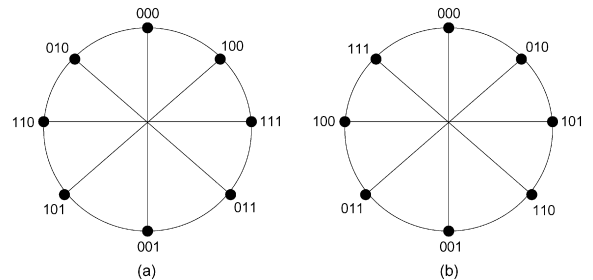


Fig. 6. (a) A symmetric drawing of  $0-MQ_3$ . (b) A symmetric drawing of  $1-MQ_3$ .

edge of  $MQ_n$  and let  $x = 0x_{n-1}x_{n-2} \dots x_2x_1$ . By the definition of  $MQ_n$ ,  $y = 1x_{n-1}x_{n-2} \dots x_2x_1$  if this  $MQ_n$  is  $0MQ_n$  and  $y = 1\bar{x}_{n-1}\bar{x}_{n-2} \dots \bar{x}_2\bar{x}_1$  if this  $MQ_n$  is  $1MQ_n$ .

First consider the 4-cycle property. Let  $u = x_nx_{n-1} \dots x_2\bar{x}_1$ . If this  $MQ_n$  is  $0MQ_n$ , then let  $v = \bar{x}_nx_{n-1} \dots x_2\bar{x}_1$ ; otherwise, if this  $MQ_n$  is  $1MQ_n$ , then let  $v = \bar{x}_n\bar{x}_{n-1} \dots \bar{x}_2x_1$ . By Lemma 10,  $\{(x, u), (u, v), (v, y)\} \subseteq E(MQ_n)$ . Hence  $(x, u, v, y, x)$  is a 4-cycle in  $MQ_n$  that contains  $(x, y)$ .

Now consider the 5-cycle property. Let  $s = 0\bar{x}_{n-1}\bar{x}_{n-2} \dots \bar{x}_2\bar{x}_1$  and choose  $r$  and  $t$  according to the following rules:

1. If this  $MQ_n$  is  $0MQ_n$  and  $x_{n-1} = 0$ , then let  $r = 0\bar{x}_{n-1}x_{n-2} \dots x_2x_1$  and  $t = 1\bar{x}_{n-1}\bar{x}_{n-2} \dots \bar{x}_2\bar{x}_1$ .
2. If this  $MQ_n$  is  $0MQ_n$  and  $x_{n-1} = 1$ , then let  $r = 0x_{n-1}\bar{x}_{n-2} \dots \bar{x}_2\bar{x}_1$  and  $t = 1\bar{x}_{n-1}\bar{x}_{n-2} \dots \bar{x}_2\bar{x}_1$ .
3. If this  $MQ_n$  is  $1MQ_n$  and  $x_{n-1} = 0$ , then let  $r = 0\bar{x}_{n-1}x_{n-2} \dots x_2x_1$  and  $t = 1x_{n-1}x_{n-2} \dots x_2x_1$ .
4. If this  $MQ_n$  is  $1MQ_n$  and  $x_{n-1} = 1$ , then let  $r = 0x_{n-1}\bar{x}_{n-2} \dots \bar{x}_2\bar{x}_1$  and  $t = 1x_{n-1}x_{n-2} \dots x_2x_1$ .

By Lemma 10,  $\{(x, r), (r, s), (s, t), (t, y)\} \subseteq E(MQ_n)$ . Hence  $(x, r, s, t, y, x)$  is a 5-cycle in  $MQ_n$  that contains  $(x, y)$ .  $\square$

It was proven in [5] that  $MQ_n$  is 4-pancyclic. We now strengthen this result (see also [10]).

**Theorem 12.** For  $n \geq 2$ ,  $MQ_n$  is 4-edge-pancyclic.

**Proof.** Clearly, this theorem holds when  $n = 2$ . By Theorem 11, this theorem holds when  $n = 3$ . For  $n \geq 4$ , this theorem follows from Theorems 3 and 11.  $\square$

The following corollary is obvious.

**Corollary 13.** For  $n \geq 2$ ,  $MQ_n$  is 4-node-pancyclic.

## 7. Concluding remarks

In this paper, we outline an approach to prove the 4-edge-pancyclic (hence 4-node-pancyclic and 4-

Table 1  
Pancyclicity of hypercube variations

cubes	4-pan	4-node-pan	4-edge-pan
twisted	[2]	[7,8]	[7,8]
crossed	[1]	[6]	[6]
Möbius	[5]	[10]	[10]
loc twisted	[12]	this paper	this paper

pancyclicity) of some hypercube variants. We prove in particular that Möbius cubes and locally twisted cubes are 4-edge-pancyclic. We now summarize known results on the pancyclicity properties of various hypercube variants in Table 1 (in this table, “pan” means pancyclic and “loc twisted” means locally twisted).

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