A closed-form solution for a confined flow into a tunnel during progressive drilling in a multi-layer groundwater flow system

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[1] A mathematical model is developed to describe the groundwater inflow into a tunnel in a multi-layer aquifer system. Based on the model, the closed-form solution is derived to estimate the groundwater flow rate entering the multi-layer tunnel during progressive drilling. The solution has an integrand not only consisting of the product and square of the Bessel functions but also having a singularity at the origin. A unified numerical approach is proposed to evaluate the solution with accuracy to five decimal places. This approach includes a singularity removal scheme, the Gaussian quadrature, and the Shanks method. For a multi-layer formation, the results obtained from the solution based on the equivalent hydraulic conductivity and the newly derived solution differ significantly. This solution is capable of estimating the maximum flow rate inside the horizontal tunnel, and thus can be used as a tool for designing the drainage tunnel system in a multi-layer formation.

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1. Introduction

[2] Groundwater flow entering a tunnel is a geotechnical problem commonly encountered during a progressive drilling in saturated geological formations. Some of engineering hazards occurring in a tunnel under excavation might be attributed to the problem of large flows entering the tunnel from highly fractured, water-saturated rocks. Since this heavy flow makes the operation of drilling a tunnel difficult and dangerous, the design and construction must be carefully exercised to stem the groundwater flow. The amount and rate of groundwater flow entering a tunnel needed to be accurately estimated in advance and a suitable drainage system can then be designed to draw off the water and avoid engineering disasters.

[3] To monitor drilling status, the types of the instant and progressive drilling must be considered in predicting the groundwater flow rate entering a tunnel. An install drilling type assumes that the tunnel is installed instantaneously through a whole length. However, this assumption results in the initial inflow rate being unrealistically high. In reality, the construction of a tunnel is drilled progressively and the flow rate into a tunnel increases from zero to maximum and is then followed by a period of decay. *Perrochet* [2005a] developed a simple analytical formula to evaluate the transient flow rate into a tunnel or well under constant drawdown. By using a straight-forward function, an alternative solution is suggested to replace the well function $G(\tau)$ of *Jacob and Lohman* [1952] by $\ln\,(1+\sqrt{\pi\tau})^{-1}$. In engineering applications, that solution can be used with great computational benefit and can avoid directly evaluating the integration in Jacob and Lohman's solution. *Perrochet* [2005b] further developed an analytical solution via a convolution integral to evaluate the transient, drilling speed-dependent discharge rate into a tunnel gradually excavated in a homogeneous, infinite, and confined aquifer. Based on these assumptions, he provided the type curves to estimate total discharge sensitivity during the drilling time and predict the maximum flow rates.

[4] There are several models seen in the literature for calculating the flows into a tunnel in a fully saturated, homogeneous, isotropic, and infinite aquifer system. Yet, those existing models all assume that the aquifer is a single permeable layer, whereas the excavation of a tunnel may in fact go through the formation with several different geological materials. Under such circumstances, the single-layer models cannot correctly calculate the groundwater flow entering a tunnel during the drilling. This paper presents a multi-layer model for predicting the transient inflow rate into a horizontal tunnel during progressive drilling. In addition, a unified numerical approach is provided to evaluate the newly derived closed-form solution of a multi-layer model. This approach includes a singularity removal scheme, the Gaussian quadrature, and the Shanks method.

2. Theoretical Development

2.1. Formulas of Flow Rate Across the Wellbore

[5] Employing Darcy's law, the formula representing the flow rate $Q(t)$ across the wellbore originally given by *Jacob and Lohman* [1952] was expressed as

$$Q(t) = 2\pi K L s_0 \left\{ \frac{4\tau}{\pi} \int_0^\infty u e^{-\tau u^2} \left\{ \frac{\pi}{2} + \tan^{-1}\left(\frac{Y_0(u)}{J_0(u)} \right) \right\} du \right\},$$
$$\tau = \frac{Kt}{S_s r_w^2} \tag{1}$$

where u is a dummy variable, K is the hydraulic conductivity, S_s is the specific storage, L is the tunnel length through a subvertical aquifer, s_0 is the specified drawdown at a tunnel, r_w is the tunnel radius, τ is the dimensionless time, t is the test time, and $J_0(u)$ and $Y_0(u)$ are the Bessel functions of the first and second kinds of order zero, respectively. Actually, *Jaeger* [1942] also presented a

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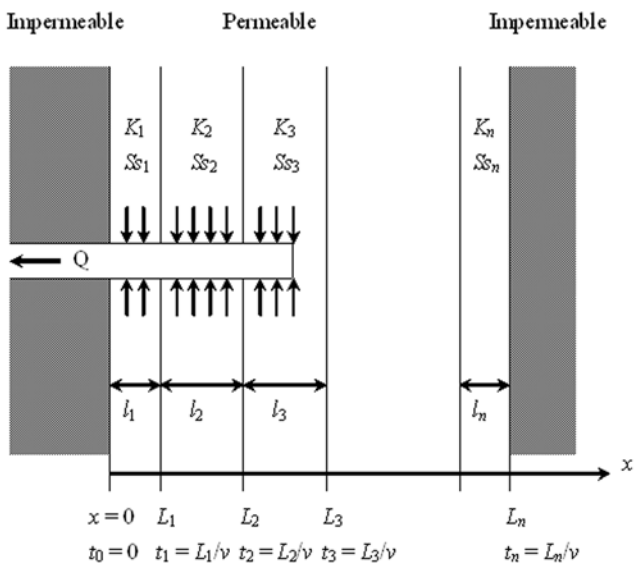

Figure 1. A tunnel through a multi-layer subvertical aquifer.

solution for the problem of a well under constant drawdown, which was analogous to a heat flow problem and written as

$$Q(t) = 2\pi K L s_0 \left\{ \frac{4}{\pi^2} \int_0^\infty \frac{e^{-\tau u^2}}{\left[J_0^2(u) + Y_0^2(u)\right] u} du \right\} \tag{2}$$

[6] Based on Darcy's law and the solution of drawdown, a formula of flow rate across the wellbore was also derived as [*Peng et al.*, 2002]

$$Q(t) = 2\pi K L s_0 G(\tau) = 2\pi K L s_0 \cdot \left\{ \frac{2}{\pi} \int_0^\infty e^{-\tau u^2} \frac{[J_1(u)Y_0(u) - J_0(u)Y_1(u)]}{J_0^2(u) + Y_0^2(u)} du \right\} \tag{3}$$

where $J_1(u)$ and $Y_1(u)$ are the Bessel functions of the first and second kinds of order one, respectively. *Peng et al.* [2002] proved that these three formulas, equations (1)–(3), are mathematically equal. Considering an instantaneously drilling in a permeable zone, the flow rate into a tunnel can be estimated based on equation (3).

2.2. Temporal Evolution of Flow Rate During Progressive Drilling

[7] Recently, Taiwan opened Asia's longest road tunnel, Hsuehshan (or Snow Mountain) Tunnel, which is 12.9 km long and links the capital Taipei to the northeastern county of Ilan. The geological conditions along this tunnel are very complex. The major geological elements of Hsuehshan mountain ridge are Eocene, Oligocene and minor Miocene folded sedimentary rock formations. There are six major faults, numerous fracture zones, and high-pressure groundwater everywhere. The rock formations traversed by the tunnel are highly fractured and contain huge reserve of groundwater, which made the engineering very difficult. There were many serious collapses of the wall inside the tunnel due to the problem of huge groundwater flow, leading to serious delays in the construction project. Therefore, intensive investigation on the geology and groundwater network for the site before and during excavating the tunnel was crucial and necessary. Accordingly, development of a multi-layer model for the estimation of transient inflow rate into a horizontal tunnel during progressive drilling is important and essential in engineering practices.

[8] A confined flow rate of groundwater into a horizontal multi-layer tunnel as shown in Figure 1 is considered. It assumes the drilling speed (v) through a subvertical formation is uniform and the aquifer is not significantly perturbed beyond the drilling front. In addition, the flow rate entering the tunnel is preserved as a radial symmetry. The flow rate at any drilling location $x < vt$ along the tunnel axis may be expressed as

$$q_i(x,t) = 2\pi K_i l_i s_0 \left\{ \frac{2}{\pi l_i} \int_0^\infty \exp\left[\frac{-K_i}{S_{s_i} r_w^2} \left(t - t_{i-1} - \frac{x}{v} \right) u^2 \right] \cdot \frac{[J_1(u)Y_0(u) - J_0(u)Y_1(u)]}{J_0^2(u) + Y_0^2(u)} du \right\} \tag{4}$$

Since the groundwater flow equation for a confined aquifer is linear, the superposition principle is applied to account for a permeable formation system with several subvertical layers. The total flow rate entering the tunnel during and after excavation through the permeable zone is expressed as

$$Q(t) = \sum_{i=1}^{n} \int_0^{v(t-t_{i-1})} q_i(x,t) H(l_i - x) U(t - t_{i-1}) dx \tag{5}$$

where n is the total number of layers, $H(l_i - x)$ is the Heaviside step unction, and $U(t - t_{i-1})$ is the unit step function. Letting $w = x/v$, equations (4) and (5) can be respectively rewritten as

$$q_i(w,t) = 2\pi K_i l_i s_0 \left\{ \frac{2}{\pi l_i} \int_0^\infty \exp\left[\frac{-K_i}{S_{s_i} r_w^2} (t - t_{i-1} - w) u^2 \right] \cdot \frac{[J_1(u)Y_0(u) - J_0(u)Y_1(u)]}{J_0^2(u) + Y_0^2(u)} du \right\} \tag{6}$$

and

$$Q(t) = \sum_{i=1}^{n} \int_0^{t-t_{i-1}} v q_i(w,t) H\left(\frac{l_i}{v} - w \right) U(t - t_{i-1}) dw \tag{7}$$

3. Dimensionless Equations

[9] Define the associated variables in dimensionless form as

$$\zeta_i = \frac{K_i / S_{s_i}}{K_1 / S_{s_1}}, \tau = \frac{K_1 t}{S_{s_1} r_w^2}, \tau_i = \frac{K_i t_{i-1}}{S_{s_i} r_w^2}, \tau_{Li} = \frac{K_i}{S_{s_i} r_w^2} \frac{l_i}{v}, w_i = \frac{K_i}{S_{s_i} r_w^2} \frac{x}{v} \tag{8}$$

where ζ_i represents the ratio of hydraulic diffusivity, τ is the dimensionless drilling time, τ_i is the dimensionless drilling time starting from the i-th zone, $i = 2, 3, \ldots$, τ_{Li} is the dimensionless drilling time at the i-th zone, and w_i is the dimensionless time at the drilling location x. Then, equation (7) expressed in dimensionless form is

$$Q(\tau) = \sum_{i=1}^{n} 2\pi K_i l_i s_0 \int_0^{\tau-\tau_i/\zeta i} H(\tau_{L_i} - w_i) U(\tau - \tau_i/\zeta_i) \cdot \left\{ \frac{2}{\pi \tau_{L_i}} \int_0^{\infty} e^{-(\tau-\tau_i/\zeta_i - w_i)u^2} \cdot \frac{[J_1(u)Y_0(u) - J_0(u)Y_1(u)]}{J_0^2(u) + Y_0^2(u)} du \right\} dw_i \quad (9)$$

Applying the convolution to the Heaviside step function and dimensionless flow rate, equation (9) gives

$$Q(\tau) = \frac{1}{\tau_{L_1}} F_1(\tau), \tau \le \tau_{L_1} \quad (10)$$

$$Q(\tau) = \frac{1}{\tau_{L_1}} (F_1(\tau) - F_1(\tau - \tau_{L_1})) + \frac{1}{\tau_{L_2}} F_2(\tau - \tau_{L_1}), \tau_{L_1} \le \tau \le \tau_{L_2}/\zeta_2 \quad (11)$$

$$Q(\tau) = \frac{1}{\tau_{L_1}} (F_1(\tau) - F_1(\tau - \tau_{L_1})) + \frac{1}{\tau_{L_2}} (F_2(\tau - \tau_{L_1}) - F_2(\tau - \tau_{L_2}/\zeta_2)) + \frac{1}{\tau_{L_3}} F_3(\tau - \tau_{L_3}/\zeta_3), \tau_{L_2}/\zeta_2 \le \tau \le \tau_{L_3}/\zeta_3 \quad (12)$$

...

$$Q(\tau) = \frac{1}{\tau_{L_1}} (F_1(\tau) - F_1(\tau - \tau_{L_1})) + \frac{1}{\tau_{L_2}} (F_2(\tau - \tau_{L_1}) - F_2(\tau - \tau_{L_2}/\zeta_2)) + \cdots + \frac{1}{\tau_{L_n}} (F_n(\tau - \tau_{L_{n-1}}/\zeta_{n-1}) - F_n(\tau - \tau_{L_n}/\zeta_n)), \tau \ge \tau_{L_n}/\zeta_n \quad (13)$$

where

$$F(\tau - \tau_{Li}/\zeta_i) = 2\pi K_i l_i s_0 \int_0^{\tau} G(w_i) dw_i = (2\pi K_i l_i s_0) \frac{4}{\pi^2} \int_0^{\infty} \left(1 - e^{-(\tau-\tau_i/\zeta_i)u^2}\right) \cdot \frac{1}{\left[J_0^2(u) + Y_0^2(u)\right] u^3} du \quad (14)$$

[10] For the case of a homogeneous aquifer with instant drilling, the hydrogeological properties are constant and the tunnel is assumed to instantaneously penetrate the whole aquifer. Then equation (9) divided by $2\pi K_1 L s_0$ becomes

$$q_D(\tau) = \frac{1}{\tau} \int_0^{\tau} G(w) dw = \frac{4}{\tau \pi^2} \int_0^{\infty} \frac{1 - e^{-\tau u^2}}{\left[J_0^2(u) + Y_0^2(u)\right] u^3} du \quad (15)$$

which indeed is the time-domain solution for dimensionless flow rate presented in *Carslaw and Jaeger* [1940].

4. Numerical Evaluation of Closed-Form Solution

[11] A unified numerical method is presented to estimate the values of the closed-form solution for dimensionless flow rate at various dimensionless times. The method initially adopts an approach of infinite series expansion given by *Harvard University Computation Laboratory* [1950] to remove the singularity of the integrand at $u = 0$ so that the numerical integration for equation (14) with integration limit from zero is possible. The Gaussian quadrature is then chosen to perform the numerical integrations. Finally, the Shanks method [*Shanks*, 1955; *Yang and Yeh*, 2002] is applied to accelerate the convergence when evaluating the related Bessel functions.

4.1. Removal of Singularity of Integrand at an Origin

[12] One efficient way of evaluating the integral in equation (14) is to transform it to an alternating infinite series. When a is a small value, the integral over the half-domain may be expressed by piecewise integrations as

$$F(\tau) = \frac{4}{\pi^2} \left\{ \int_0^{a} \frac{1 - e^{-\tau u^2}}{\left[J_0^2(u) + Y_0^2(u)\right] u^3} du + \int_a^{\infty} \frac{1 - e^{-\tau u^2}}{\left[J_0^2(u) + Y_0^2(u)\right] u^3} du \right\} \quad (16)$$

[13] The value of the first term on the right-hand-side (RHS) of equation (16) approaches infinity as $u \to 0$ because its numerator and denominator approach zero. The arctangent in the interval $0 \le u \le a$ is a continuous function, so the second term of equation (16) has a finite value. The integral in the first term on the RHS of equation (16) can be evaluated if the singularity at the origin is removed. The numerator of the first term may be approximated by an infinite series as

$$1 - e^{-\tau u^2} = \tau u^2 - \frac{1}{2!}\left(\tau u^2\right)^2 + \frac{1}{3!}\left(\tau u^2\right)^3 - \frac{1}{4!}\left(\tau u^2\right)^4 + \cdots \quad (17)$$

In addition, based on the differentiation formula for an arctangent function [*Abramowitz and Stegun*, 1964, p. 79, equation (4.4.3)], one can write

$$-\frac{d}{du} \tan^{-1}\left[\frac{J_0(u)}{Y_0(u)}\right] = \frac{2}{\pi u} \frac{1}{J_0^2(u) + Y_0^2(u)} \quad (18)$$

Taking the integration of the RHS of equation (18) from zero to a yields

$$\frac{2}{\pi} \int_0^{a} \frac{du}{\left[J_0^2(u) + Y_0^2(u)\right] u} = -\tan^{-1}\left[\frac{J_0(a)}{Y_0(a)}\right] \quad (19)$$

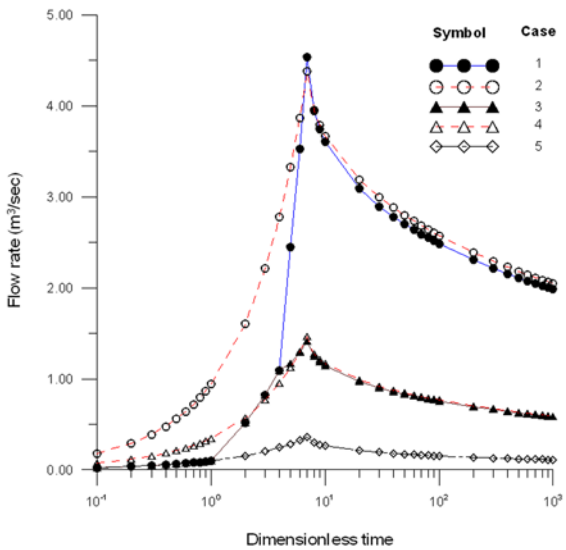

Figure 2. The transient flow-rate in a horizontal three-layer tunnel.

Note that $J_0(0)/Y_0(0) = 0$ when $u = 0$, thus $\tan^{-1}[J_0(0)/Y_0(0)] = 0$. Based on equations (17)–(19), equation (16) can be rewritten as

$$F(\tau) = \frac{4}{\pi^2}\left\{\begin{array}{l} \dfrac{\pi\tau}{2}\tan^{-1}\left[\dfrac{J_0(a)}{Y_0(a)}\right] + \displaystyle\int_0^a \dfrac{-\frac{1}{2!}\tau^2 u + \frac{1}{3!}\tau^3 u^3 - \frac{1}{4!}\tau^4 u^5 + \cdots + (-1)^{n+1}\frac{1}{n!}\tau^n u^{2n-3}}{\left[J_0^2(u) + Y_0^2(u)\right]} du \\ + \displaystyle\int_a^\infty \left(1 - e^{-\tau u^2}\right) \dfrac{1}{\left[J_0^2(u) + Y_0^2(u)\right] u^3} du \end{array}\right\} \quad (20)$$

The variable u in the denominator of the first term on the RHS of equation (16), which poses the problem of a singularity at $u = 0$, can be cancelled out. Similarly, the arctangent is a continuous function in the interval $0 \le u \le a$ because $J_0(a)/Y_0(a)$ is a finite value as discussed before. Thus, the formula for dimensionless flow rate, equation (20), not contain a singular point. Because the integrand of the third term on the RHS of equation (20) is a monotonically decreasing function, equation (20) can be easily evaluated by the Gaussian quadrature [*Yang and Yeh*, 2002].

4.2. Numerical Integration

[14] The Shanks transform is employed to accelerate the calculation of the Bessel functions of $J_0(u)$, $J_1(u)$, $Y_0(u)$, and $Y_1(u)$ [*Shanks*, 1955; *Yang and Yeh*, 2002]. Both the six-point and ten-point formulas of the Gaussian quadrature are used at the same time to carry out the numerical integration for each RHS term of equation (14). If the difference of these two results for any interval between two consecutive roots is greater than the prescribed criterion, then the interval will be divided into two portions. The same integration procedure is repeatedly applied to each portion until the convergence criteria are met to ensure that the result bears the desired accuracy. The result of the integration within the interval, considered as a term of infinite series, is equal to the sum of the areas obtained from total divided portions.

[15] The interval for the numerical integration of dimensionless flow rate in equation (14) is chosen as 10^{-10}. Then, both the six-point and ten-point formulas of the Gaussian quadrature are also used at the same time to carry out the integration of equation (14). If the difference of these two integration results is greater than the prescribed criterion, say 10^{-7}, then, the interval will be divided into two portions, and the same integration procedure is again applied to each portion until the integration result for each portion is less than 10^{-7}. Finally, the numerical integration result for dimensionless flow rate can be obtained by simply adding all the results from each interval or portion.

5. Numerical Evaluations and Discussions

[16] A heterogeneous hydraulic conductivity field may be formed by geologic processes that do not yield uniform characteristics of aquifer formation over appreciable areas. In engineering practice, a multi-layer formation may be represented by an equivalent homogeneous medium. An ideal groundwater flow entering a tunnel assumes that the speed of a progressive drilling is uniform and the radial symmetry of flow is preserved at all times. In this study five cases for the flow rate into a horizontal tunnel in a three-layer aquifer system and single-layer aquifer are considered and examined. Cases 1 and 3 give examples for a three-layer aquifer system and case 5 shows an example for a single-layer system. In addition, cases 2 and 4 represent formation systems with an equivalent hydraulic conductivity to cases 1 and 3, respectively. Note that the equivalent hydraulic conductivity for a three-layer system is defined as $\overline{K} = \sum K_i l_i / L$, $i = 1, 2, 3$. Assume that the specific storage Ss is 10^{-2}/m, the specified drawdown at a tunnel (s_0) and the tunnel radius (r_w) are both 5 m, and the length of progressive drilling in a fully saturated formation (L) is 140 m. The curves of flow rate versus dimensionless time (τ) for those five cases are plotted in Figure 2 to investigate the impacts of the layered formation properties on the flow rate for different formations with the hydrogeological parameters listed in Table 1. The hydraulic conductivities are $K_1 = 10^{-4}$ m/s and $K_2 = 10^{-3}$ m/s for both cases 1 and 3, and $K_3 = 5 \times 10^{-3}$ m/s for case 1 and 5×10^{-4} m/s for case 3. Accordingly, the equivalent hydraulic conductivity is 2.59×10^{-3} m/s for case 2 and 6.57×10^{-4} m/s for case 4. For case 5, the hydraulic conductivity is $K = 10^{-4}$ m/s. The temporal evolutions of tunnel flow rate for cases 1 to 5 are presented in Figure 2. Figure 2 indicates that the flow rates quickly increase with penetration distance and reach maximums when the tunnel penetration is completed. In addition, the flow rates tend to decrease with increasing dimensionless time after the complete penetration and stabilize when the dimensionless time becomes very large. The flow rates in cases 1 and 3 are significantly different

Table 1. Parameters for a Multi-Layer Aquifer System

Parameter	Layer 1	Layer 2	Layer 3	Equivalent K	Singe Layer
			Case 1		
K, m/s	10^{-4}	10^{-3}	5×10^{-3}		
L, m	20	60	60		
ζ	1	10	50		
τ_L	1	30	150		
			Case 2		
K, m/s				2.59×10^{-3}	
l, m				140	
ζ				25.9	
τ_L				181.3	
			Case 3		
K, m/s	10^{-4}	10^{-3}	5×10^{-4}		
l, m	20	60	60		
ζ	1	10	5		
τ_L	1	30	15		
			Case 4		
K, m/s				6.57×10^{-4}	
l, m				140	
ζ				6.57	
τ_L				45.99	
			Case 5		
K, m/s					10^{-4}
l, m					140
ζ					1
τ_L					7

when the layer 1, 2, and 3 are respectively penetrated at τ = 1, 4 and 7. The estimated flow rate in case 1 is apparently smaller than that of case 2 when $\tau \leq 6$. The estimated flow rate in case 2 is 4.38 m^3/s, which is smaller than 4.54 m^3/s of case 1, when the permeable aquifer is completely penetrated at τ = 7. However, the flow rate in case 2 is slightly larger than that of case 1 from the beginning of excavation toward the complete penetration. The flow rate in case 4 is smaller than that of case 3 except at the second layer before the penetration of the permeable aquifer and slightly larger while the permeable formation is completely penetrated; that is, 1.41 m^3/s for case 3 and 1.46 m^3/s for case 4. The flow rate in case 4 is slightly larger than that of case 3 after the full penetration of the permeable aquifer and tends to equal case 3 at a large dimensionless time. Therefore, the difference of the flow rates between the case with equivalent hydraulic conductivity and our solution increases with increasing variability of the permeable layers. The flow rate of a single layer in case 5 is significantly smaller than those of cases 1–4 at all dimensionless times of progressive drilling. Those results show that the proposed model can efficiently estimate the flow rate of groundwater entering a tunnel during the progressive drilling in a multi-layer aquifer system.

6. Conclusions

[17] A closed-form solution is derived to estimate the flow rate of the ground water into a tunnel during the progressive drilling in a multi-layer formation system. We provide a unified numerical approach including a scheme for singularity removal, the Gaussian quadrature, and the Shanks method to evaluate the solution with accuracy to five decimal places. For a multi-layer formation, the results evaluated by assuming an equivalent hydraulic conductivity are over-/under-estimated compared to those predicted by our solution. The multi-layer model we present can estimate the flow rate into a horizontal tunnel during over progressive drilling time and thus can be used to design the drainage system for various underground excavation projects.

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