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Hamiltonian circuit and linear array embeddings in faulty k-ary n-cubes

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Abstract

In this paper, we investigate the fault-tolerant capabilities of the k-ary n-cubes for even integer k with respect to the hamiltonian and hamiltonian-connected properties. The k-ary n-cube is a bipartite graph if and only if k is an even integer. Let F be a faulty set with nodes and/or links, and let $k \geqslant 3$ be an odd integer. When $|F| \leqslant 2n-2$, we show that there exists a hamiltonian cycle in a wounded k-ary n-cube. In addition, when $|F| \leqslant 2n-3$, we prove that, for two arbitrary nodes, there exists a hamiltonian path connecting these two nodes in a wounded k-ary n-cube. Since the k-ary n-cube is regular of degree $2n$, the degrees of fault-tolerance $2n-3$ and $2n-2$ respectively, are optimal in the worst case.

Keywords: Cycle embeddings; Hamiltonian; k-ary n-cube; Fault tolerance; Linear array embeddings

1. Introduction

In many parallel computer systems, processors are connected based on an interconnection network. Such networks usually have a regular degree, i.e., every node is incident with the same number of links. Popular instances of interconnection networks include hypercubes, star graphs, meshes, the k-ary n-cubes, etc.

The k-ary n-cube, denoted by Q_n^k, is regular of degree $2n$, edge symmetric, and vertex symmetric. Several properties of it has been studied in the literature. For example, in [3,4], meshes and hamiltonian cycles are embedded into healthy k-ary n-cubes, and the connectivity of Q_n^k is shown to be $2n$, which equals the degree of each vertex. Furthermore, message routing and single-node broadcasting algorithms are given in [4]. The problem of conditional node connectivity on Q_n^k is investigated in [6]. Cycles are said to be disjoint if they share no edges. In [2], n edge disjoint hamiltonian cycles are found in Q_n^k. In [1], Ashir and Stewart studied the problem of hamiltonian cycle embeddings in Q_n^k with a possibility of link failures.

Hamiltonian circuit and linear array embeddings are desired properties in an interconnection network [5,9,14]. Many works related to embeddings of longest cycles and paths in various interconnection networks have been studied previously, including hypercubes [5,11], k-ary n-cubes [1], stars [8,14], arrangement graphs [9,12], etc.

Ashir and Stewart [1] showed that, with only edge faults and under the condition that every node is incident with at least two fault-free edges, a wounded k-ary n-cube still has a hamiltonian circuit, provided that there are no more than $4n-5$ faulty edges. The situation of having both faulty nodes and faulty links remains unanswered, and the hamiltonian linear array embeddings in Q_n^k have not been discussed yet even in a healthy Q_n^k.

Since failures are inevitable, fault-tolerance is an important issue in multiprocessor systems. In this paper, we consider a possibility of both node and link failures, and discuss the fault-tolerant capabilities of the k-ary n-cubes with respect to the hamiltonian and hamiltonian-connected properties. Let F be a faulty set with nodes and/or links. We observe that Q_n^k is bipartite if and only if k is even. When k is even and there is a faulty node, there exists neither a hamiltonian cycle nor a hamiltonian path between two vertices in different partite sets in a wounded Q_n^k. Therefore, throughout this paper, we suppose that k is an odd integer with $k \geqslant 3$. Then, a ring of maximum length, or a hamiltonian cycle, in a wounded Q_n^k can be constructed, provided that $|F| \leqslant 2n-2$ for $n \geqslant 2$. On the other hand, if $|F| \leqslant 2n-3$ for $n \geqslant 2$, we provide a construction of a linear array of maximum length, or a hamiltonian path, connecting two arbitrary vertices in a wounded Q_n^k. In both cases, we have achieved optimal solutions.

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The reason is as follows. First, any hamiltonian cycles cannot be found in a wounded Q_n^k when there are $2n-1$ faulty edges incident to a single node. Second, suppose that there are $2n-2$ edge faults incident to a node x. Let y and z be two nodes of Q_n^k incident to x. Then, there is no hamiltonian path connecting y and z when all the edges incident to x are faulty except (x, y) and (x, z).

The rest of this paper is organized as follows. We give some definitions, notation, and terminology in Section 2. Using the recursive structure of the k-ary n-cubes, we construct rings and linear arrays, respectively, traversing all the nodes in wounded k-ary n-cubes in Section 3. Finally, in Section 4, we present the conclusion.

2. Preliminaries

Throughout this paper, an interconnection network is represented by an undirected simple graph G. Given a graph G, we denote the *vertex set* and the *edge set* as $V(G)$ and $E(G)$, respectively. A *path*, denoted by $\langle v_1, v_2, \ldots, v_k\rangle$, is a sequence of adjacent vertices where all the vertices are distinct except possibly $v_1 = v_k$. We say that a path is a *hamiltonian path* if it traverses all the vertices of G exactly once. A *cycle* is a path that begins and ends with the same vertex. A *hamiltonian cycle* is a cycle which includes all the vertices of G. A graph is *hamiltonian* if it has a hamiltonian cycle. A graph G is *hamiltonian connected* if, for any two arbitrary vertices x and y in G, there is a hamiltonian path connecting x and y.

We consider the fault-tolerance of a graph G in the following. Let F be a faulty set which may contain both vertices and edges. Let $F_v = F \bigcap V(G)$ and $F_e = F \bigcap E(G)$. $G - F$ denotes the subgraph of $G - F_e$ induced by $V(G) - F_v$. Let k be a positive integer. A graph G is *k-fault-tolerant hamiltonian* (abbreviated as *k-hamiltonian*) if $G - F$ is hamiltonian for every F with $|F| \leqslant k$. A graph G is *k-fault-tolerant hamiltonian connected* (abbreviated as *k-hamiltonian connected*) if $G - F$ is hamiltonian connected for every F with $|F| \leqslant k$.

The k-ary n-cube Q_n^k is a graph consisting of k^n vertices labeled by the integers from 0 to $k^n - 1$ for $k \geqslant 3$ and $n \geqslant 1$. Two vertices are adjacent if and only if the representations of their labels in base k differ by one (modulo k) in exactly one position. We refer to $(x, y) \in E(Q_n^k)$ where x differs from y in the dth position, for $0 \leqslant d \leqslant n-1$, as an *edge of dimension d*. We say that Q_n^k is divided into $Q_n^k[0], Q_n^k[1], \ldots, Q_n^k[k-1]$ (abbreviated as $Q[0], Q[1], \ldots, Q[k-1]$, if there are no ambiguities) along dimension d for some $0 \leqslant d \leqslant n-1$ if $Q[l]$, for every $0 \leqslant l \leqslant k-1$, is a subgraph of Q_n^k induced by the vertices labeled by $x_{n-1} \ldots x_{d+1} l x_{d-1} \ldots x_0$ (see Fig. 1). It is clear that each $Q[l]$ is isomorphic to Q_{n-1}^k for $0 \leqslant l \leqslant k-1$. Note that Q_n^k can be divided into k copies of Q_{n-1}^k along n different dimensions. For $0 \leqslant i, j \leqslant k-1$, we use $[i, j]$ to denote a set of integers: $[i, j] = \{l \mid i \leqslant l \leqslant j\}$ if $i \leqslant j$, and $[i, j] = \{l \mid i \leqslant l \leqslant k-1 \text{ or } 0 \leqslant l \leqslant j\}$ if $i > j$. $Q_n^k[i, j]$ (abbreviated as $Q[i, j]$ if there is no ambiguity) denotes the subgraph of Q_n^k which is induced by $\{u \mid u \in V(Q[l]); l \in [i, j]\}$.

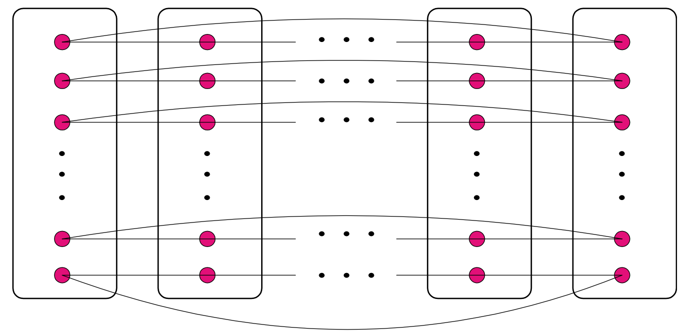

Fig. 1. Q_n^k is divided into $Q[0], Q[1], \ldots, Q[k-1]$.

3. Hamiltonian path and cycle embeddings

Let k be an odd integer with $k \geqslant 3$, and let $n \geqslant 2$ be an integer. Let $F \subseteq V(Q_n^k) \bigcup E(Q_n^k)$ be the set of faulty vertices and/or edges in Q_n^k. Let Q_n^k be divided into $Q[0], Q[1], \ldots, Q[k-1]$ along some dimension, and let $F^l = F \bigcap (V(Q[l]) \bigcup E(Q[l]))$ for every $0 \leqslant l \leqslant k-1$. We refer to an edge $(x, y) \in E(Q_n^k)$ where all of x, y, and (x, y) are fault-free, as a *safe crossing-edge*.

In the following lemmas, namely Lemmas 1–3, we shall construct hamiltonian paths in faulty $Q[i, j]$ for every $i, j \in [0, k-1]$ when each faulty $Q[l]$ is hamiltonian connected for $l \in [i, j]$. These preliminaries will be useful for further discussions.

As a first step, we shall construct a hamiltonian path between two arbitrary vertices belonging to $Q[i]$ in a faulty $Q[i, j]$ (see Fig. 2(a)).

Lemma 1. *Let $i, j \in [0, k-1]$, and let $F \subseteq V(Q[i, j]) \bigcup E(Q[i, j])$ be a faulty set with $|F| \leqslant 2n-3$. If $Q[l] - F^l$ is hamiltonian connected for every $l \in [i, j]$, there exists a hamiltonian path connecting every two vertices u_i and $v_i \in V(Q[i] - F^i)$ in $Q[i, j] - F$ for every $n \geqslant 3$ and odd $k \geqslant 3$.*

Proof. If $i = j$, this lemma holds. So we suppose that $i \neq j$. We may assume without loss of generality that $i = 0$ in the following discussion. Since $Q[l] - F^l$ is hamiltonian connected for every $l \in [0, j]$, there is a hamiltonian path, say $P_0(u_0, v_0)$ ($u_0 = u_i$ and $v_0 = v_i$), in $Q[0] - F^0$ (see Fig. 3(a)). The length of $P_0(u_0, v_0) = |V(Q[0] - F^0)| - 1 \geqslant k^{n-1} - |F^0| - 1$, and the number of faults outside $Q[0]$ is at most $(2n-3) - |F^0|$. When n and $k \geqslant 3$, $\lceil \frac{k^{n-1} - |F^0| - 1}{2} \rceil \geqslant \frac{3^{n-1} - |F^0| - 1}{2} > (2n-3) - |F^0|$. Hence, we can find two consecutive vertices, say w_0 and z_0, on $P_0(u_0, v_0)$ such that (w_0, w_1) and (z_0, z_1) are safe crossing-edges where w_1 and z_1 are the neighbors of w_0 and z_0 in $Q[1]$, respectively. Let $\langle u_0, P_{0,1}(u_0, w_0), w_0, z_0, P_{0,2}(z_0, v_0), v_0\rangle = P_0(u_0, v_0)$, and let $P_1(w_1, z_1)$ be a hamiltonian path in $Q[1] - F^1$. $\langle u_0, P_{0,1}(u_0, w_0), w_0, w_1, P_1(w_1, z_1), z_1, z_0, P_{0,2}(z_0, v_0), v_0\rangle$ forms a hamiltonian path in $Q[0, 1] - F$. Repeating the above construction, we have a hamiltonian path in $Q[0, j] - F$. $\square$

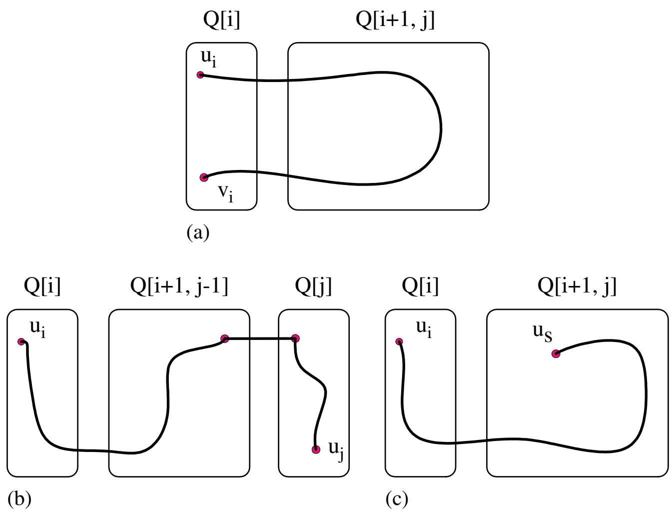

Fig. 2. Hamiltonian paths in faulty $Q[i, j]$. (a) Lemma 1; (b) Lemma 2; (c) Lemma 3.

In the following lemma, we shall construct a hamiltonian path between two arbitrary vertices $u_i \in V(Q[i] - F^i)$ and $u_j \in V(Q[j] - F^j)$ in a faulty $Q[i, j]$ (see Fig. 2(b)). Note that $Q[i, j]$ can tolerate $2n - 2$ faults in this lemma, which is the maximum degree of the fault-tolerance of hamiltonian cycle embeddings. In addition, we want all the vertices in $Q[j] - F^j$ to form a subpath on this hamiltonian path for proving Lemma 3.

Lemma 2. *Let $i, j \in [0, k-1]$, and let $F \subseteq V(Q[i, j]) \bigcup E(Q[i, j])$ be a faulty set with $|F| \leqslant 2n-2$. If $Q[l]-F^l$ is hamiltonian connected for every $l \in [i, j]$, there exists a hamiltonian path connecting two arbitrary vertices $u_i \in V(Q[i]-F^i)$ and $u_j \in V(Q[j]-F^j)$ in $Q[i, j]-F$ such that all the vertices in $Q[j]-F^j$ form a subpath on this hamiltonian path for every $n \geqslant 3$ and odd $k \geqslant 3$.*

Proof. If $i = j$, the statement follows. Hence, we suppose that $i \neq j$. Without loss of generality, we may assume that $i = 0$ (see Fig. 3(b)). Note that $|F| = (2n-2)$ and $|V(Q[0])| = k^{n-1}$. Since $k^{n-1} - (2n-2) \geqslant 9 - 4 = 5$ for every $n \geqslant 3$ and odd $k \geqslant 3$, there exists a safe crossing-edge, say (v_0, v_1), where $v_0 \neq u_0$, $v_0 \in V(Q[0] - F^0)$, $v_1 \neq u_j$, and $v_1 \in V(Q[1] - F^1)$. By assumption, $Q[l] - F^l$ is hamiltonian connected for every $l \in [0, j]$, so we have a hamiltonian path, say $P_0(u_0, v_0)$, in $Q[0] - F^0$. Continuing this process, we can join all hamiltonian paths in $Q[l] - F^l$, for all $l \in [0, j-1]$, to form a hamiltonian path, namely $R(u_0, v_{j-1})$, in $Q[0, j-1] - F$ such that (v_{j-1}, v_j) is a safe crossing-edge where $v_{j-1} \neq u_0$, $v_{j-1} \in V(Q(j-1) - F^{j-1})$, $v_j \neq u_j$, and $v_j \in V(Q[j] - F^j)$. Let $S(v_j, u_j)$ be a hamiltonian path in $Q[j]$. $\langle u_0, R(u_0, v_{j-1}), v_{j-1}, v_j, S(v_j, u_j), u_j\rangle$ is a hamiltonian path in $Q[0, j] - F$, and $S(v_j, u_j)$ contains all vertices in $Q[j] - F^j$. $\square$

In the following lemma, we construct a hamiltonian path between two arbitrary vertices $u_i \in V(Q[i] - F^i)$ and $u_s \in V(Q[s] - F^s)$ with $s \in [i, j]$ in a faulty $Q[i, j]$ (see Fig. 2(c)). Note that $Q[i, j]$ can tolerate $2n-3$ faults in this lemma, which is the maximum degree of the fault-tolerance of hamiltonian path embeddings.

Lemma 3. *Let $i, j \in [0, k-1]$, and let $F \subseteq V(Q[i, j]) \bigcup E(Q[i, j])$ be a faulty set with $|F| \leqslant 2n-3$. If $Q[l]-F^l$ is hamiltonian connected for every $l \in [i, j]$, there exists a hamiltonian path connecting every two vertices $u_i \in V(Q[i] - F^i)$ and $u_s \in V(Q[s] - F^s)$ in $Q[i, j] - F$ with $s \in [i, j]$ for every $n \geqslant 3$ and odd $k \geqslant 3$.*

Proof. If $i = j$, the statement is true. Therefore, we assume that $i \neq j$. By Lemma 2, there exists a hamiltonian path, say $R(u_i, u_s)$, in $Q[i, s] - F$ such that all the vertices in $Q[s] - F^s$ form a subpath on $R(u_i, u_s)$. Using the counting argument in the proof of Lemma 1, we can find two consecutive vertices, say u_s and $v_s \in V(Q[s])$, on $R(u_i, u_s)$ such that (u_s, u_{s+1}) and (v_s, v_{s+1}) are safe crossing-edges where u_{s+1} and $v_{s+1} \in V(Q[s+1])$. By Lemma 1, there is a hamiltonian path, namely $S(u_{s+1}, v_{s+1})$, in $Q[s+1, j] - F$. Let $\langle u_i, R_1(u_i, u_s), u_s, v_s,$

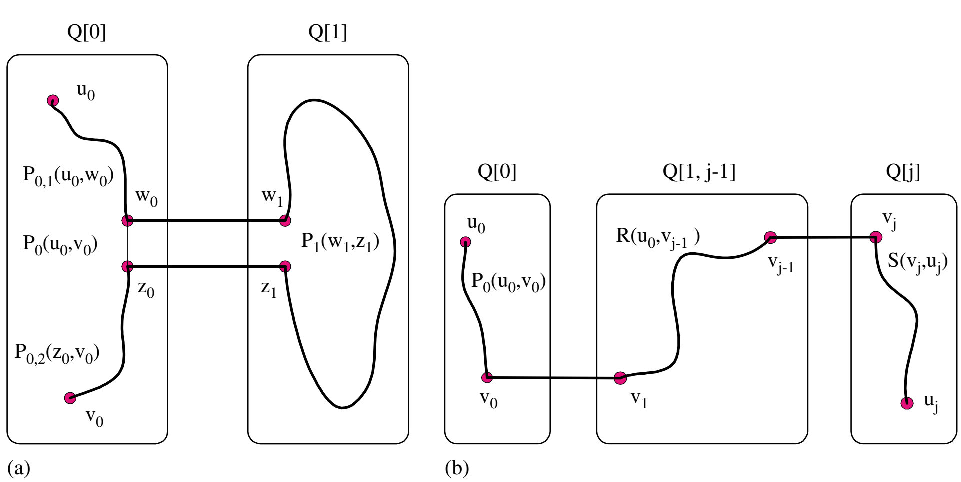

Fig. 3. (a,b) The proofs of Lemmas 1 and 2.

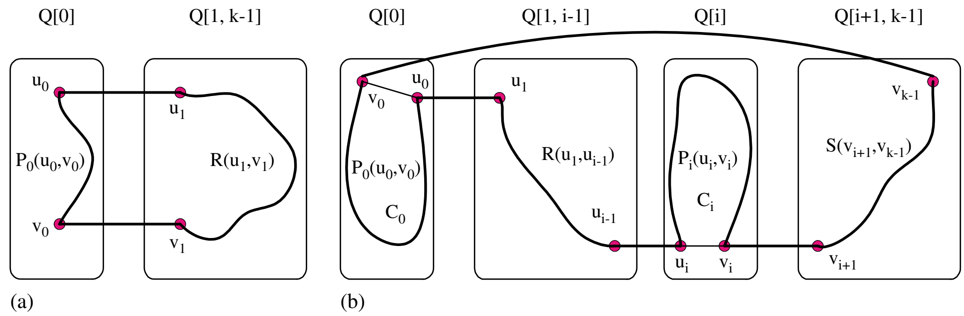

Fig. 4. Cases of Theorem 6. (a) Case 1; (b) Case 2 (when $Q[i]$-F is not hamiltonian connected).

$R_2(v_s, v_i),\ v_i\rangle = R(u_i, u_s)$. Then, $\langle u_i,\ R_1(u_i, u_s),\ u_s, u_{s+1},\ S(u_{s+1}, v_{s+1}),\ v_{s+1}, v_s,\ R_2(v_s, v_i), v_i\rangle$ forms a hamiltonian path in $Q[j] - F$. $\square$

The $m \times n$ torus is a graph of mn vertices labeled as ab where a and b are integers with $0 \leqslant a \leqslant m-1$ and $0 \leqslant b \leqslant n-1$. Two vertices ab and cd are adjacent if and only if either $a = c$ and $b = d \pm 1_{(mod\ n)}$ or $b = d$ and $a = c \pm 1_{(mod\ m)}$. Therefore, Q_2^k is a $k \times k$ torus for every $k \geqslant 3$ by the definition. The following theorem related to the fault-tolerant hamiltonicity of the $m \times n$ torus is proved in [10].

Theorem 4 (*Kim and Park [10]*). *If $m \geqslant 3$, $n \geqslant 3$, and n is odd, the $m \times n$ torus is 2-hamiltonian and 1-hamiltonian connected.*

The following corollary immediately follows by Theorem 4.

Corollary 5. *If k is odd with $k \geqslant 3$, Q_2^k is 2-hamiltonian and 1-hamiltonian connected.*

Using the fault-tolerant hamiltonian and hamiltonian connected properties of Q_{n-1}^k, we shall show the fault-tolerant hamiltonian property of Q_n^k.

Theorem 6. *Let k be an odd integer with $k \geqslant 3$. If Q_{n-1}^k is $(2n-4)$-hamiltonian and $(2n-5)$-hamiltonian connected for some $n \geqslant 3$, then Q_n^k is $(2n-2)$-hamiltonian.*

Proof. Let $F \subseteq V(Q_n^k) \bigcup E(Q_n^k)$ be the set of faulty vertices and/or edges in Q_n^k with $|F| \leqslant 2n-2$. We claim that we can divide Q_n^k into $Q[0], Q[1], \ldots, Q[k-1]$ along some dimension such that $|F^l| \leqslant 2n-3$ for every $0 \leqslant l \leqslant k-1$. If $|F| \leqslant 2n-3$, it is done. So we assume that $|F| = 2n-2$. Then, if there is a faulty edge, we can divide Q_n^k along the dimension of this faulty edge. On the other hand, suppose that $F \subseteq V(Q_n^k)$. Since $|F| \geqslant 4$, for every $n \geqslant 3$, picking arbitrarily two faulty vertices in Q_n^k, we can divide Q_n^k along some dimension such that these two faulty vertices are in different Q_{n-1}^k's. Hence, the claim follows. Furthermore, without loss of generality, we may assume that $|F^0| \geqslant |F^l|$ for every $l \in [0, k-1]$. We discuss the existence of a hamiltonian cycle in the following three cases.

Case 1: $|F^0| = 2n-3$ (see Fig. 4(a)).

By assumption, Q_{n-1}^k is $(2n-4)$-hamiltonian. Therefore, there is a hamiltonian path, namely $P_0(u_0, v_0)$, in $Q[0] - F^0$. Let u_1 and v_1 be the neighbors of u_0 and v_0 in $Q[1]$, respectively, and let u_{k-1} and v_{k-1} be the neighbors of u_0 and v_0 in $Q[k-1]$, respectively. Since there is at most one fault outside $Q[0]$, either the two edges (u_0, u_1) and (v_0, v_1) are safe crossing-edges or the two edges (u_0, u_{k-1}) and (v_0, v_{k-1}) are safe crossing-edges. Without loss of generality, we may assume that (u_0, u_1) and (v_0, v_1) are safe crossing-edges. By assumption, Q_{n-1}^k is $(2n-5)$-hamiltonian connected and $2n-5 \geqslant 1$ for $n \geqslant 3$, so $Q[l] - F^l$ is hamiltonian connected for every $l \in [1, k-1]$ and $n \geqslant 3$. Since $1 < 2n-3$ for $n \geqslant 3$, by Lemma 3, there is a hamiltonian path, namely $R(u_1, v_1)$, in $Q[1, k-1] - F$. Therefore, $\langle u_0, P_0(u_0, v_0), v_0, v_1, R(v_1, u_1), u_1, u_0\rangle$ forms a hamiltonian cycle in $Q_n^k - F$.

Case 2: $|F^0| = 2n-4$.

By assumption, Q_{n-1}^k is $(2n-4)$-hamiltonian. Therefore, there is a hamiltonian cycle, say C_0, in $Q[0] - F^0$. Since there are at most two faults outside $Q[0]$, we can find two consecutive vertices, namely u_0 and v_0, on C_0 for $n \geqslant 3$ such that (u_0, u_1) and (v_0, v_1) are safe crossing-edges, where u_1 and v_1 are the neighbors of u_0 and v_0 in $Q[1]$ respectively. Note that $Q[l] - F^l$ is hamiltonian-connected for every $l \in [1, k-1]$ and $n \geqslant 4$. In this situation, the proof is similar to Case 1.

When $n = 3$, it is possible that in addition to $Q[0]$, there exists another copy of Q_{n-1}^k, say $Q[i]$, which contains two faults (if all other copies contain at most 1 fault then by proceeding as above we are done). Hence, both of $Q[0] - F^0$ and $Q[i] - F^i$ are not necessarily hamiltonian connected, but both are hamiltonian. There is a hamiltonian cycle, say C_i, in $Q[i]$ (see Fig. 4(b)). Note that there is no fault outside $Q[0]$ and $Q[i]$, and $Q[l] - F^l$ is hamiltonian connected for every $l \notin \{0, i\}$. We may assume without loss of generality that $i \neq k-1$. We can find a safe crossing-edge, say (u_{i-1}, u_i), where $u_{i-1} \in Q[i-1]$ and $u_i \in Q[i]$. By Lemma 3, there is a hamiltonian path, namely $R(u_1, u_{i-1})$, in $Q[1, i-1]$ (if $i = 1$, then $(u_{i-1}, u_i) = (u_0, u_1)$, and there is no $R(u_1, u_{i-1})$). Let $v_{k-1} \in V(Q[k-1])$ be a neighbor of v_1. Let v_i be adjacent to u_i on C_i such that v_{i+1}, the neighbor of v_i in $Q[i+1]$, $\neq v_{k-1}$. By Lemma 3, there exists a hamiltonian path, namely $S(v_{i+1}, v_{k-1})$, in $Q[i+1, k-1]$. Furthermore, let $\langle u_0, P_0(u_0, v_0), v_0\rangle = C_0$ and

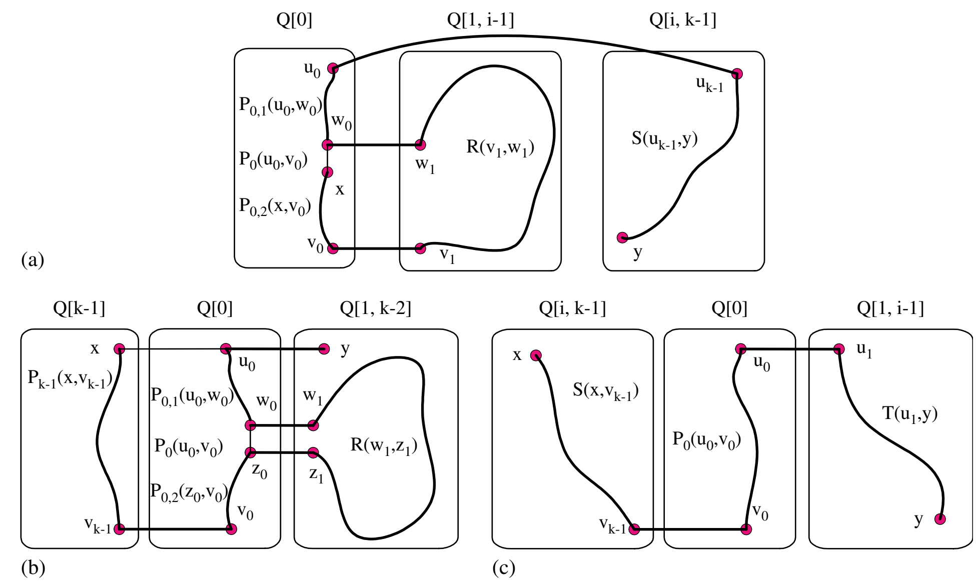

Fig. 5. Case 1 of Theorem 7.

$\langle u_i, P_i(u_i, v_i), v_i\rangle = C_i$. Then, $\langle u_0, u_1, R(u_1, u_{i-1}), u_{i-1}, u_i, P_i(u_i, v_i), v_i, v_{i+1}, S(v_{i+1}, v_{k-1}), v_{k-1}, v_0, P_0(v_0, u_0), u_0\rangle$ is a hamiltonian cycle in $Q_3^k - F$.

Case 3: $|F^0| \leqslant 2n-5$.

Since $k^{n-1} > 2n-2$ for $k \geqslant 3$ and $n \geqslant 3$, we can find a safe crossing-edge, say (u_0, u_{k-1}), where $u_0 \in Q[1]$ and $u_{k-1} \in Q[k-1]$. $|F^0| \leqslant 2n-5$, and, by assumption, Q_{n-1}^k is $(2n-5)$-hamiltonian connected. Therefore, $Q[l] - F^l$ is hamiltonian connected for every $0 \leqslant l \leqslant k-1$. By Lemma 2, there is a hamiltonian path, namely $P(u_0, u_{k-1})$, in $Q[0, k-1]$. Therefore, $\langle u_0, P(u_0, u_{k-1}), u_{k-1}, u_0\rangle$ is a hamiltonian cycle in $Q_n^k - F$. $\square$

Using the fault-tolerant hamiltonian and hamiltonian connected properties of Q_{n-1}^k again, we shall prove the fault-tolerant hamiltonian connected property of Q_n^k as follows.

Theorem 7. *Let k be an odd integer with $k \geqslant 3$. If Q_{n-1}^k is $(2n-4)$-hamiltonian and $(2n-5)$-hamiltonian connected for some $n \geqslant 3$, Q_n^k is $(2n-3)$-hamiltonian connected.*

Proof. We want to prove that there exists a hamiltonian path connecting every two vertices x and y in $Q_n^k - F$ for every F with $|F| \leqslant 2n-3$. Since $x \neq y$, we can divide Q_n^k into $Q[0], Q[1] \ldots, Q[k-1]$ along some dimension such that x and y are in different Q_{n-1}^k's. Furthermore, without loss of generality, we may assume that $|F^0| \geqslant |F^l|$ for every $0 \leqslant l \leqslant k-1$. We discuss the existence of a hamiltonian path connecting x and y in the following three cases. $\square$

Case 1: $|F^0| = 2n-3$.

By assumption, Q_{n-1}^k is $(2n-4)$-hamiltonian. Hence, there is a hamiltonian path, namely $P_0(u_0, v_0)$, in $Q[0] - F^0$. Note that there is no fault outside $Q[0]$. So $Q[l]$ is hamiltonian connected for every $l \in [1, k-1]$. We divide this case further into two subcases, Case 1.1 and Case 1.2, as follows.

Case 1.1: $x \in V(Q[0] - F^0)$ and $y \in V(Q[i] - F^i)$ where $i \neq 0$ (see Fig. 5(a)).

We may assume that the distance from x to u_0 is at least as far as the distance from x to v_0 on $P_0(u_0, v_0)$. Let $\langle u_0, P_{0,1}(u_0, w_0), w_0, x, P_{0,2}(x, v_0), v_0\rangle = P_0(u_0, v_0)$. $|V(P_0(u_0, v_0))| \geqslant k^{n-1} - (2n-3) \geqslant 3^2 - 3 = 6$ for k and $n \geqslant 3$, so $w_0 \neq u_0$ and $w_0 \neq x$. Without loss of generality, we may assume that $i \neq 1$. Then, let v_1 and w_1 be the neighbors of v_0 and w_0 in $Q[1]$, respectively. Furthermore, let u_{k-1} be the neighbor of u_0 in $Q[k-1]$. First, we consider the case $y \neq u_{k-1}$. By Lemma 3, there is a hamiltonian path $R(v_1, w_1)$ in $Q[1, i-1]$. By Lemma 3, there exists a hamiltonian path $S(u_{k-1}, y)$ in $Q[i, k-1]$. Then, $\langle x, P_{0,2}(x, v_0), v_0, v_1, R(v_1, w_1), w_1, w_0, P_{0,1}(w_0, u_0), u_0, u_{k-1}, S(u_{k-1}, y), y\rangle$ forms a hamiltonian path in $Q_n^k - F$. Next, we consider the case $y = u_{k-1}$. Since $n \geqslant 3$, by Lemma 3, there is a hamiltonian path $R(v_1, w_1)$ in $Q[1, k-1] - y$. Then, $\langle x, P_{0,2}(x, v_0), v_0, v_1, R(v_1, w_1), w_1, w_0, P_{0,1}(w_0, u_0), u_0, y\rangle$ forms a hamiltonian path in $Q_n^k - F$.

Case 1.2: $x \in V(Q[i] - F^i)$ and $y \in V(Q[j] - F^j)$ where $i, j \neq 0$.

We may assume that $i > j$. Suppose that both of x and y are neighbors of u_0 (or v_0). So, $x \in Q[k-1]$ and $y \in Q[1]$ (see Fig. 5(b)). Let v_{k-1} be the neighbor of v_0 in $Q[k-1]$. Since there is no fault in $Q[k-1]$, by assumption, there exists a hamiltonian path, say $P_{k-1}(x, v_{k-1})$, in $Q[k-1]$. Let w_0 and z_0 be two consecutive vertices on $P_0(u_0, v_0)$. Also, let w_1 and z_1 be the neighbors of w_0 and z_0 in $Q[1]$, respectively. By Lemma 3, there is a hamiltonian path, namely $R(w_1, z_1)$, in $Q[1, k-2] - y$. Let $\langle u_0, P_{0,1}(u_0, w_0), w_0, z_0, P_{0,2}(z_0, v_0), v_0\rangle = P_0(u_0, v_0)$. $\langle y, u_0, P_{0,1}(u_0, w_0), w_0, w_1, R(w_1, z_1), z_1, z_0, P_{0,2}(z_0, v_0), v_0, v_{k-1}, P_{k-1}(v_{k-1}, x), x\rangle$ is a hamiltonian path connecting x and y in $Q_n^k - F$. Otherwise, suppose that

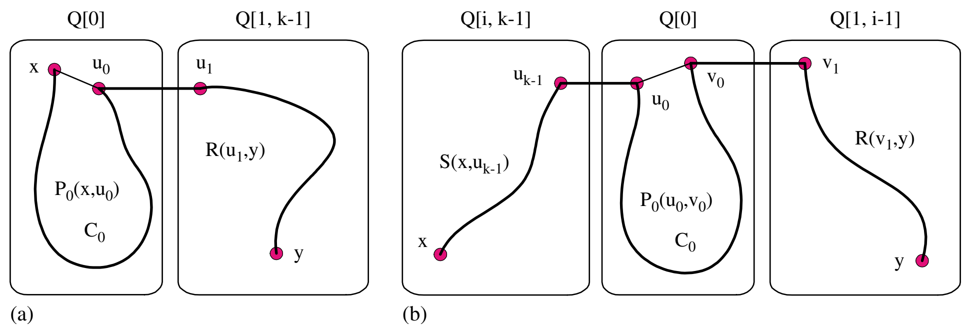

Fig. 6. Case 2 of Theorem 7. (a) Case 2.1; (b) Case 2.2.

either x or y is not a neighbor of u_0(or v_0). Let $u_1 \in Q[1]$ and $v_{k-1} \in Q[k-1]$ be neighbors of u_0 and v_0, respectively (see Fig. 5(c)). We may assume without loss of generality that $u_1 \neq y$ and $v_{k-1} \neq x$. By Lemma 3, there exist hamiltonian paths, say $S(x, v_{k-1})$ and $T(u_1, y)$, in $Q[i, k-1]$ and $Q[1, i-1]$, respectively. As a result, $\langle x,\ S(x, v_{k-1}),\ v_{k-1},\ v_0,\ P_0(v_0, u_0),\ u_0, u_1,\ T(u_1, y),\ y\rangle$ is a hamiltonian path connecting x and y in $Q_n^k - F$.

Case 2: $|F^0| = 2n - 4$.

By assumption, $Q[0]$ is $(2n-4)$-hamiltonian. So there is a hamiltonian cycle, namely C_0, in $Q[0]-F^0$. Note that there is at most one fault outside $Q[0]$. Therefore, $Q[l]-F^l$ is hamiltonian connected for every $l \in [1, k-1]$. We divide this case further into two subcases Case 2.1 and Case 2.2 as follows.

Case 2.1: $x \in V(Q[0] - F^0)$ and $y \in V(Q[i] - F^i)$ where $i \neq 0$ (see Fig. 6(a)).

Let $u_0 \in V(C_0)$ be adjacent to x on C_0 such that u_0 is not a neighbor of y. Let $u_1 \in V(Q[1] - F^1)$ be a neighbor of u_0. Since there is at most one fault outside $Q[0]$, we may assume without loss of generality that (u_0, u_1) is a safe crossing-edge. By Lemma 3, there is a hamiltonian path, namely $R(u_1, y)$, in $Q[1, k-1] - F$. Let $\langle x, P_0(x, u_0), u_0, x\rangle = C_0$. $\langle x, P_0(x, u_0), u_0, u_1, R(u_1, y), y\rangle$ forms a hamiltonian path connecting x and y in $Q_n^k - F$.

Case 2.2: $x \in V(Q[i] - F^i)$ and $y \in V(Q[j] - F^j)$ where $i, j \neq 0$ (see Fig. 6(b)).

We may assume that $i > j$. Since there is at most one fault outside $Q[0]$, we can choose two adjacent vertices, say u_0 and v_0, on C_0 such that (u_0, u_{k-1}) and (v_0, v_1) are safe crossing-edges, $u_{k-1} \neq x$, and $v_1 \neq y$ where $u_{k-1} \in Q[k-1]$ and $v_1 \in Q[1]$ are neighbors of u_0 and v_0, respectively. By Lemma 3, there exists a hamiltonian path, namely $R(v_1, y)$, in $Q[1, i-1] - F$, and also, a hamiltonian path, namely $S(x, u_{k-1})$, in $Q[i, k-1] - F$. Let $\langle u_0, P_0(u_0, v_0), v_0, u_0\rangle = C_0$. Then, $\langle x, S(x, u_{k-1}), u_{k-1}, u_0, P_0(u_0, v_0), v_0, v_1, R(v_1, y), y\rangle$ is a hamiltonian path in $Q_n^k - F$.

Case 3: $|F^0| \leqslant 2n - 5$.

As a result, $Q[l] - F^l$ is hamiltonian connected for every $l \in [0, k-1]$. We may assume without loss of generality that $x \in V(Q[0] - F^0)$. Since $|F| \leqslant 2n - 3$, by Lemma 3, there is a hamiltonian path connecting x and y in $Q[0, k-1] - F$. Hence there exists a hamiltonian path connecting x and y in $Q_n^k - F$.

In conclusion, the fault-tolerant hamiltonicity of Q_n^k is given in the following theorem.

Theorem 8. *If k is odd with $k \geqslant 3$ and $n \geqslant 2$, Q_n^k is $(2n-2)$-hamiltonian and $(2n-3)$-hamiltonian connected.*

Proof. By Corollary 5, Theorems 6, 7, and a simple mathematical induction, this theorem is proved. □

4. Conclusion

We have shown how to find a hamiltonian cycle and a hamiltonian path joining two arbitrary vertices in a wounded k-ary n-cube. When k is an odd integer, Q_n^k is $(2n-2)$-hamiltonian and $(2n-3)$-hamiltonian connected. Furthermore, our results are optimal (explained in Section 1). For even integer k, Q_n^k is a bipartite graph. It is easy to see that Q_n^k contains a hamiltonian cycle. However, with one single vertex fault, the remaining network does not contain any hamiltonian cycle. Therefore, for the fault-tolerant hamiltonian and hamiltonian-connected properties of Q_n^k, with k even, we can only consider edge faults. Let $F_e \subseteq E(Q_n^k)$ be the set of faulty edges in Q_n^k with $|F_e| \leqslant 2n-2$ (not $2n-3$). For even integer k, we intend in future to show that $Q_n^k - F_e$ has a hamiltonian path connecting two arbitrary vertices belonging to different partite sets and a path of maximum length, $k^n - 2$, connecting two arbitrary vertices in the same partite set for every $n \geqslant 2$ and even $k \geqslant 4$. This problem has not yet been resolved.

The fault-tolerant hamiltonian and hamiltonian-connected properties are fundamental tools for exploring further properties concerning cycle or path embedding problems. For example, a graph G is pancyclic if a cycle of length l can be embedded into G for $4 \leqslant l \leqslant |V(G)|$. In [7], fault-tolerant pancyclicity of Möbius cubes was studied by using the fault-tolerant hamiltonian and hamiltonian-connected properties of Möbius cubes. In addition, by employing hamiltonian cycles and paths in faulty hypercubes, linear array and cycle embeddings in conditional faulty hypercubes were investigated [13].

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