Available online at www.sciencedirect.com
ScienceDirect
Journal of Systems Architecture 53 (2007) 227–232
JOURNAL OF SYSTEMS ARCHITECTURE
www.elsevier.com/locate/sysarc

Highly fault-tolerant cycle embeddings of hypercubes ☆

Ming-Chien Yang [a], Jimmy J.M. Tan [a,*], Lih-Hsing Hsu [b]

[a] Department of Computer and Information Science, National Chiao Tung University, 1001 Ta Hsueh Road, 300, Hsinchu 30050, Taiwan, ROC

[b] Department of Computer Science and Information Engineering, Providence University, 43301, Taiwan, ROC

Received 8 December 2004; received in revised form 27 April 2006; accepted 12 October 2006
Available online 1 December 2006

Abstract

The hypercube Q_n is one of the most popular networks. In this paper, we first prove that the n-dimensional hypercube is $2n-5$ conditional fault-bipancyclic. That is, an injured hypercube with up to $2n-5$ faulty links has a cycle of length l for every even $4 \leqslant l \leqslant 2^n$ when each node of the hypercube is incident with at least two healthy links. In addition, if a certain node is incident with less than two healthy links, we show that an injured hypercube contains cycles of all even lengths except hamiltonian cycles with up to $2n-3$ faulty links. Furthermore, the above two results are optimal. In conclusion, we find cycles of all possible lengths in injured hypercubes with up to $2n-5$ faulty links under all possible fault distributions.

Keywords: Cycle embedding; Hypercube; Bipancyclic; Conditional; Fault tolerance

1. Introduction

Processors of a multiprocessor system are connected according to a given interconnection network. Many interconnection networks have been proposed with their superb merits demonstrated. Among them, the hypercube network is one of the most popular candidates when choosing an interconnection network. Newly proposed properties or measures with respect to interconnection networks are usually studied first on the hypercube because of its symmetric structure and popularity.

In order to speed up computations, a number of processors are grouped together to run a given parallel algorithm. A cycle is a preferred structure for a group of processors to carry out an algorithm because it is branch-free and has low degree. In addition, a ring structure can be used as a control or data flow structure for distributed computations. For more benefits and applications of cycles, refer to [4,10,15]. Many researchers have studied the existence of cycle structures in various interconnection networks, for example, [2,4,8,10,12,17,18].

Failures of interconnection network components are inevitable. Accordingly, various fault-tolerant measures have been proposed in the literature,

☆ This work was supported in part by the National Science Council of the Republic of China under Contract NSC 93-2213-E-009-091.
* Corresponding author.
E-mail address: jmtan@cis.nctu.edu.tw (J.J.M. Tan).

1383-7621/$ - see front matter
doi:10.1016/j.sysarc.2006.10.008

including fault diameter [14], fault hamiltonicity [8], fault pancyclicity [7,18], fault bipancyclicity [12], and fault hamiltonian laceability [16]. Specifically, there have been many works on pancyclicity conducted in recent years. They aim to find cycles of as many lengths as possible in a variety of interconnection networks. Precisely, a network N is pancyclic if it contains cycles of all lengths from 3 to the number of vertices in N. Researches related to pancyclicity can be found in [1,3,5–7,9,18]. However, since the hypercube is bipartite, it has no odd cycles. Another definition, bipancyclicity, was revealed accordingly. A network or a graph G is bipancyclic if G has cycles of all even lengths ranging from 4 to the number of vertices in G. Tsai et al. [12] studied the fault-tolerant bipancyclic property on hypercube. They found that an injured hypercube Q_n with up to $n-2$ faulty links is bipancyclic. However, this measure underestimates the fault-tolerant capability of an interconnection network. Although there is no hamiltonian cycle in an injured hypercube if there are $n-1$ faulty links incident to a single node, this is the unique case.

For making sure the usability of a particular interconnection network, it is good to know that this network can tolerate many faults. In this paper, we show that the degree of fault-tolerance of the hypercube is almost twice as many as the degree of the hypercube while almost preserving the bipancyclicity property. This goal is achieved by going through two steps. Firstly, we study a kind of fault-tolerant measure, the conditional fault-tolerant bipancyclicity, on the hypercube. By restricting fault distributions, an injured hypercube is still bipancyclic with a large amount of faulty links. We show that an injured hypercube is bipancyclic with up to $2n-5$ faulty links under the condition that every node is incident with at least two healthy links. Some other conditional properties concerning with connectivity [11], diameter [13], and hamiltonian cycle embeddings [2] have been studied. These networks come out to tolerate more faults than expected while preserving the desired properties. Secondly, as mentioned above, there is no hamiltonian cycle with $n-1$ faulty links in the worst case. When the condition is not satisfied, i.e., a certain node is incident with less than two healthy links, an injured hypercube has cycles of all even lengths except hamiltonian cycles with up to $2n-3$ faulty links. The above two results are optimal, and for details, refer to Section 3. Finally, based on these results, we conclude that we can find cycles of all possible lengths in an injured hypercube with up to arbitrary $2n-5$ faulty links.

The rest of this paper is organized as follows. Section 2 introduces definitions and notation. In Section 3, the highly fault-tolerant bipancyclic property is discussed on hypercube. Section 4 concludes our result.

2. Definitions and notation

In this paper, we represent an interconnection network as an undirected simple graph G. We denote the *vertex set* and the *edge set* of a graph G as $V(G)$ and $E(G)$, respectively. The hypercube Q_n is a graph with $|V(Q_n)| = 2^n$ and $|E(Q_n)| = n2^{n-1}$. Vertices are assigned binary strings of length n ranging from 0 to $2^n - 1$. Two vertices are adjacent if they differ only in one bit position.

A *path*, denoted by $\langle v_1, v_2, \ldots, v_k \rangle$, is a sequence of adjacent vertices where all the vertices are distinct except possibly $v_1 = v_k$. We say that a path is a *hamiltonian path* if it traverses all the vertices of G exactly once. A *cycle* is a path that begins and ends with the same vertex. A *hamiltonian cycle* is a cycle which walks through all the vertices of G. A graph is *hamiltonian* if it has a hamiltonian cycle. A bipartite graph is *hamiltonian laceable* if, for two arbitrary vertices x and y in different partite sets, there is a hamiltonian path connecting x and y. A bipartite graph G is *bipancyclic* if, for every even integer l with $4 \leqslant l \leqslant |V(G)|$, G has a cycle of length l.

Let $F \subseteq E(G)$ be a faulty set containing edges of G. $G - F$ denotes the subgraph of G obtained by deleting the edges in F from G. Let k be a positive integer. A bipartite graph G is *k edge fault-tolerant hamiltonian laceable* (abbreviated as *k fault-hamiltonian laceable* in this paper) if $G - F$ is hamiltonian laceable for every F with $|F| \leqslant k$. A bipartite graph G is k edge fault-tolerant bipancyclic (abbreviated as *k fault-bipancyclic*) if $G - F$ is bipancyclic for every F with $|F| \leqslant k$. A bipartite graph G is *k conditional edge fault-tolerant bipancyclic* (abbreviated as *k conditional fault-bipancyclic*) if $G - F$ is bipancyclic for every F with $|F| \leqslant k$ under the condition that every vertex is incident with at least two non-faulty edges.

3. Main result

The following lemma is proved in [16].

Theorem 1 [16]. *Q_n is $(n-2)$ edge fault-tolerant hamiltonian laceable for $n \geqslant 2$.*

For convenience of further discussion, we say that Q_n is divided into Q^0_{n-1} and Q^1_{n-1} along dimension k for $0 \leqslant k \leqslant n-1$ if Q^i_{n-1} is an $(n-1)$-dimensional hypercube which is a subgraph of Q_n induced by the vertices labeled by $x_{n-1},\ldots,x_{k+1}ix_{k-1},\ldots,x_0$. We say that $(x,y) \in E(Q_n)$ is a *k-dimensional* edge if x differs from y in the kth position for $0 \leqslant k \leqslant n-1$. In addition, let $F \subset E(Q_n)$ be the set of faulty edges, $F_0 = F \bigcap E(Q^0_{n-1})$, and $F_1 = F \bigcap E(Q^1_{n-1})$.

The following theorem states that, under the condition that each node of Q_n is incident with at least two healthy links, an injured Q_n is still bipancyclic with $F \leqslant 2n-5$ for $n \geqslant 3$. We note that this condition implies that the number of faulty edges incident to any vertex is at most $n-2$.

Theorem 2. *The hypercube Q_n is $(2n-5)$ conditional fault-bipancyclic for $n \geqslant 3$.*

Proof. We prove this by induction on n. It is straightforward to see that Q_3 is 1 edge fault-tolerant bipancyclic. Since $2\times 3-5=1$, the theorem holds for $n=3$. Assume that Q_{n-1} is $2(n-1)-5=2n-7$ conditional fault-bipancyclic for some $n \geqslant 4$. We shall prove that Q_n is $(2n-5)$ conditional fault-bipancyclic. There are three possible fault distributions:

(1) There is only one vertex incident with $n-2$ faulty edges. Without loss of generality, we may assume that one of these $n-2$ faulty edges is an $(n-1)$-dimensional edge.
(2) There are two vertices which share a faulty edge and are both incident with $n-2$ faulty edges. Without loss of generality, we may assume that the faulty edge they share is an $(n-1)$-dimensional edge.
(3) Every vertex is incident with less than $n-2$ faulty edges. We may assume without loss of generality that one of them is an $(n-1)$-dimensional edge.

Note that there cannot be more than two vertices which are incident with $n-2$ faulty edges for $n \geqslant 3$. Then, we can divide Q_n into Q^0_{n-1} and Q^0_{n-1} along dimension $n-1$. So both of Q^0_{n-1} and Q^1_{n-1} satisfy the condition that every vertex is incident with at least two non-faulty edges. Furthermore, we may assume without loss of generality that $|F^0| \geqslant |F^1|$. Since $\frac{(2n-5)-1}{2} = n-3$, $|F^1| \leqslant n-3$. We discuss the existence of cycles of all even lengths from 4 to 2^n in the following two cases.

Case 1. Cycles of even lengths from 4 to 2^{n-1}. Note that $|F^1| \leqslant n-3 \leqslant 2n-7$ for $n \geqslant 4$. By induction hypothesis, Q^1_{n-1} is $(2n-7)$ conditional fault-bipancyclic, so $Q^1 - F^1$ contains cycles of all even lengths from 4 to 2^{n-1}.

Case 2. Cycles of even lengths from $2^{n-1}+2$ to 2^n. We divide this case further into two subcases.

Case 2.1. $|F^0| = 2n-6$ (see Fig. 1(a)). Hence, there is only one $(n-1)$-dimensional faulty edge, say e. Let $x \in V(Q^0_{n-1})$ be the vertex incident with e. Notice that there are at most $n-3$ faulty edges incident with x in Q^0_{n-1}. Since $2n-6 > n-3$ for $n \geqslant 4$, there must be a faulty edge in Q^0_{n-1}, say e', such that it is not incident to x. Let $F' = F^0 - e'$. Clearly, $|F'| = 2n-7$. By induction hypothesis, Q^0_{n-1} is $(2n-7)$ conditional fault-bipancyclic, so $Q^0_{n-1} - F'$ contains a hamiltonian cycle, say C. Then, $Q^0_{n-1} - F^0$ contains a hamiltonian path on C, say $P = \langle u_1, u_2, \ldots, u_{2^{n-1}} \rangle$, such that $u_1 \neq x$ and $u_{2^{n-1}} \neq x$. Let $2 \leqslant l \leqslant 2^{n-1}$ be an even integer. We construct a cycle of length $2^{n-1}+l$ as follows.Since the edge e is the only faulty edge in $(n-1)$-dimension, there must exist two vertices u_i and u_j such that the two $(n-1)$-dimensional edges incident to u_i and u_j, respectively, are non-faulty and $j-i = l-1$. Let v_i and v_j be the neighbors of u_i and u_j in Q^1_{n-1}, respectively. Since $j-i$ is odd, u_i and u_j are in different partite sets, and then v_i and v_j are also in different partite sets. By Theorem 1, Q^1_{n-1} is $(n-3)$ fault-hamiltonian laceable. Since $|F^1| \leqslant n-3$, $Q^1_{n-1} - F^1$ contains a hamiltonian path, say Q. Then, $\langle u_i, u_{i+1}, \ldots, u_j, v_j, Q, v_i, u_i \rangle$ is a cycle of length $2^{n-1}+l$ in $Q_n - F$.

Case 2.2. $|F^0| \leqslant 2n-7$ (see Fig. 1(b)). By induction hypothesis, Q^0_{n-1} is $(2n-7)$ conditional fault-bipancyclic. Therefore, $Q^0_{n-1} - F^0$ contains a hamiltonian cycle, say $C = \langle u_1, u_2, \ldots, u_{2^{n-1}}, u_1 \rangle$. Let l be an even integer for $2 \leqslant l \leqslant 2^{n-1}$. We construct a cycle of length $2^{n-1}+l$ as follows: First, we claim that there exist two vertices u_i and u_j on C such that the two $(n-1)$-dimensional edges incident to u_i and u_j, respectively are non-faulty, and $(j-i) = l-1 (\mathrm{mod}\ 2^{n-1})$. Suppose on the contrary that there do not exist such u_i and u_j. Then there are at least $\frac{2^{n-1}}{2} = 2^{n-2}$ $(n-1)$-dimensional faulty edges. However, $2^{n-2} > 2n-5$ for $n \geqslant 4$. We obtain a contradiction. Thus, there exist such u_i and u_j. By Theorem 1, Q^1_{n-1} is $(n-3)$ fault-hamiltonian laceable. Since $|F^1| \leqslant n-3$, $Q^1_{n-1} - F^1$ is still hamiltonian laceable. Let v_i and v_j be the neighbors of u_i and u_j in Q^1_{n-1}, respectively. Since u_i and u_j are in different partite

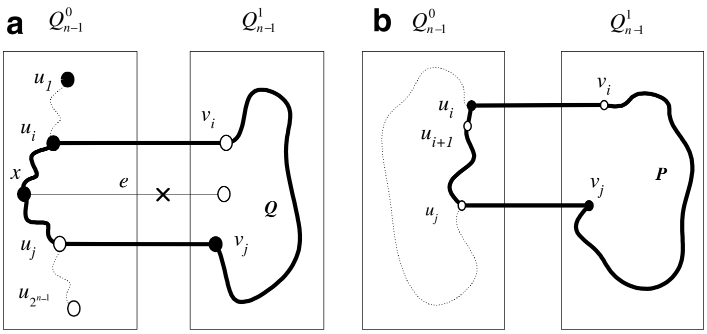

Fig. 1. *Case 2.1* and *Case 2.2* of Theorem 2.

sets, v_i and v_j are also in different partite sets. There is a hamiltonian path P in $Q^1_{n-1} - F^1$ between v_i and v_j. Then $\langle u_i, u_{i+1}, \ldots, u_j, v_j, P, v_i, u_i \rangle$ forms a cycle of length $2^{n-1} + l$. □

The above result is optimal in the sense that if there are more than $2n-5$ faulty edges, there is no guarantee to have a fault-free cycle in an injured hypercube. For example, let $\langle a, b, c, d, a \rangle$ be a 4-cycle in Q_n (see Fig. 2). Assume that all the edges incident to b are faulty except (a,b) and (b,c), and all the edges incident to d are faulty except (a,d) and (d,c). Then, there are $(n-2)+(n-2) = 2n-4$ faulty edges, and there is no hamiltonian cycle in the injured hypercube.

In addition, if the condition is not satisfied, i.e., there are more than $n-2$ faulty edges incident to a certain vertex, there cannot be a hamiltonian cycle in an injured hypercube. But what about the other cycles of even lengths? The following two theorems address this problem. Our finding is that the hamiltonian cycle is the only missing cycle.

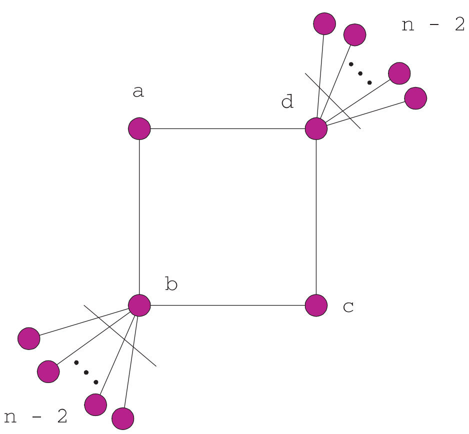

Fig. 2. There are $n-2$ fault edges incident to b and d, respectively, and the injured hypercube has no hamiltonian cycle.

Theorem 3. *If there are 2 or 3 faulty edges incident to some vertex in Q_3, then $Q_3 - F$ contains cycles of lengths 4 and 6 with $|F| \leqslant 3$.*

Proof. Since Q_3 is vertex symmetric (see Fig. 3), we may assume that 000 is incident with 2 or 3 faulty edges. First, suppose that there are two faulty edges incident to 000, and the other faulty edge, say e, is not. We observe that e is not on one of the 4-cycles (see Fig. 3(a)–(c)): $\langle 100, 101, 111, 110, 100 \rangle$, $\langle 001, 101, 111, 011, 001 \rangle$, and $\langle 010, 011, 111, 110, 010 \rangle$. Also, e is not on one of the 6-cycles (see Fig. 3(d)–(f)): $\langle 010, 011, 001, 101, 111, 110, 010 \rangle$, $\langle 001, 101, 100, 110, 111, 011, 001 \rangle$, and $\langle 010, 011, 111, 101, 100, 110, 010 \rangle$. Therefore, we have cycles of lengths 4 and 6 in $Q_3 - F$. Second, suppose that there are three faulty edges incident to 000. Since 000 is not on all the above cycles, $Q_3 - F$ contains all the above cycles. The proof is complete. □

Theorem 4. *If there are more than $n-2$ faulty edges incident to some vertex in Q_n, then $Q_n - F$ contains cycles of all even lengths from 4 to $2^n - 2$ with $|F| \leqslant 2n-3$ for $n \geqslant 3$.*

Proof. This theorem is proved by induction on n. By Theorem 3, the theorem holds for $n = 3$. Assume that theorem is true for Q_{n-1} with some $n \geqslant 4$. We shall prove that $Q_n - F$ contains cycles of all even lengths from 4 to $2^n - 2$ with $|F| \leqslant 2n-3$. Note that we only need to consider the case $|F| = 2n-3$. There is only one vertex, say a which is incident with at least $n-1$ faulty edges. Since $(2n-3) - n = n-3 \geqslant 1$ for $n \geqslant 4$, there is a faulty edge which is not incident to a. Without loss of generality, we may assume that it is an $(n-1)$-dimensional edge. Then, we divide Q_n into Q^0_{n-1} and Q^0_{n-1} along dimension $n-1$. Without loss of generality, assume that $a \in V(Q^0_{n-1})$. Then, $|F^1| \leqslant (2n-3) - (n-1) - 1 =$

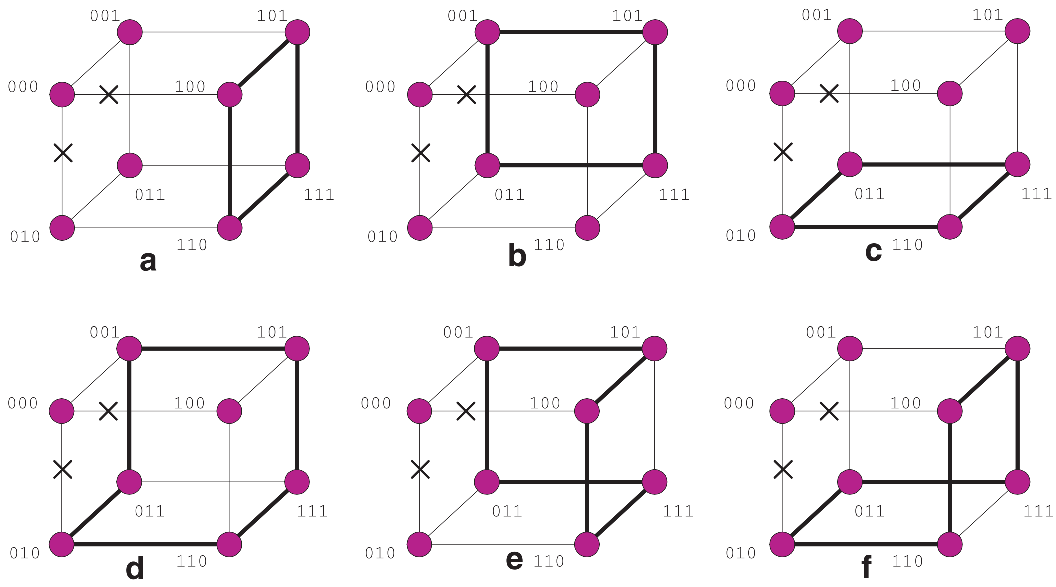

Fig. 3. 4-Cycles and 6-cycles in $Q_3 - F$.

$n-3$. By Theorem 1, Q^1_{n-1} is $(n-3)$ fault-hamiltonian laceable, so $Q^1_{n-1} - F^1$ is still hamiltonian laceable. We discuss the existence of cycles of all even lengths from 4 to $2^n - 2$ as follows.

Case 1. Cycles of even lengths from 4 to 2^{n-1}. By Theorem 2, Q^1_{n-1} is $(2n-7)$ conditional fault-bipancyclic, Since $|F^1| \leqslant n-3$, we have cycles of all even lengths from 4 to 2^{n-1} in $Q^1 - F^1$.

Case 2. Cycles of even lengths from $2^{n-1}+2$ to $2^n - 2$. The proof is similar to the one in *Case* 2.2 of Theorem 2. First, if $|F^0| \leqslant 2n-5$, by induction hypothesis, $Q^0_{n-1} - F^0$ contains a cycle of length $2^{n-1}-2$. Second, if $|F^0| = 2n-4$, there is only one $(n-1)$-dimensional faulty edge, and the $(n-1)$-dimensional edge incident to a is non-faulty. Hence, all the $n-1$ edges incident to a in Q^0_{n-1} is faulty. Let e' be one of this $n-1$ faulty edges. Let $F' = F^0 - e'$. Then, $|F'| = 2n-5$. By inductionhypothesis, $Q^0_{n-1} - F'$ contains a cycle of length $2^{n-1}-2$. Since a is not on this cycle, $Q^0_{n-1} - F^0$ also contains this cycle. Let l be an even integer for $2 \leqslant l \leqslant 2^{n-1}-2$. Repeating the argument in the proof of *Case* 2.2 of Theorem 2, if $(n-1)$-dimensional faulty edges are less than $\frac{2^{n-1}-2}{2} = 2^{n-2}-1$, we have a cycle of length $2^{n-1}+l$ in $Q_n - F$. $2^{n-2}-1 \geqslant (2n-3)-(n-2)$ (a is incident with at least $n-2$ faulty edges in Q^0_{n-1}) for $n \geqslant 4$, and the equality holds only when $n=4$, i.e., Q_4 contains three $(n-1)$-dimensional faulty edges. In this situation, there are three faulty edges incident to a, and the other two faulty edges, say e_1 and e_2, are $(n-1)$-dimensional. We can divide Q_4 into Q^0_3 and Q^1_3 along a dimension k, $0 \leqslant k \leqslant 3$, such that e_1 and e_2 are in different $(n-1)$-dimensional subcubes. Hence, $|F^1| = 1$, and there are at most 2 k-dimensional faulty edges. And either $|F^0| \leqslant 3$, or a is incident with 3 faulty edges in Q^0_3. Hence, this theorem is proved. □

Let $x, y \in V(Q_n)$ be two vertices in the same partite set. Suppose that there are $n-1$ faulty edges incident to x and y, respectively. There cannot be a cycle of length 2^n-2 in $Q_n - F$. Therefore, the number of faulty edges, $2n-3$, provided in the above theorem is maximum.

In fact, we have found cycles of all possible lengths in $Q_n - F$ with $|F| \leqslant 2n-5$ under all possible fault distributions. By Theorems 2 and 4, the following theorem follows.

Theorem 5. *Suppose that* $|F| \leqslant 2n-5$ *and* $n \geqslant 4$. *If the condition that every vertex in* Q_n *has at least two non-faulty edges is satisfied,* $Q_n - F$ *contains cycles of all even lengths from* 4 *to* 2^n. *Otherwise,* $Q_n - F$ *contains cycles of all even lengths from* 4 *to* $2^n - 2$.

4. Conclusion

In this paper, we extend the result of [12] by restricting fault distributions to increase the degree of fault tolerance, and we prove that the hypercube is $2n-5$ conditional fault-bipancyclic. Therefore, the degree of fault tolerance doubles that of [12].

Then, we show that with up to $2n - 3$ faulty edges if a certain vertex is incident with less than two non-faulty edges, an injured Q_n has a cycle of length l for every even l, $4 \leqslant l \leqslant 2^n - 2$.

References

[1] Toru Araki, Yukio Shibata, Pancyclicity of recursive circulant graphs, Inform. Process. Lett. 81 (4) (2002) 187–190.

[2] M.Y. Chan, S.-J. Lee, On the existence of Hamiltonian circuits in faulty hypercubes, SIAM J. Discrete Math. 4 (1991) 511–527.

[3] G.H. Chen, J.S. Fu, J.F. Fang, Hypercomplete: a pancyclic recursive topology for large-scale distributed multicomputer systems, Networks 35 (1) (2000) 56–69.

[4] K. Day, A. Tripathi, Embedding of cycles in arrangement graphs, IEEE Trans. Comput. 42 (8) (1993) 1002–1006.

[5] J. Fan, Hamilton-connectivity and cycle-embedding of the Möbius cubes, Inform. Process. Lett. 82 (2002) 113–117.

[6] A. Germa, M.C. Heydemann, D. Sotteau, Cycles in the cube-connected cycles graph, Discrete Appl. Math. 83 (1998) 135–155.

[7] S.Y. Hsieh, C.H. Chen, Pancyclicity on Möbius cubes with maximal edge faults, Parallel Comput. 30 (2004) 407–421.

[8] H.C. Hsu, T.K. Li, J.M. Tan, L.H. Hsu, Fault Hamiltonicity and fault Hamiltonian connectivity of the arrangement graphs, IEEE Trans. Comput. 53 (1) (2004) 39–53.

[9] C.H. Huang, J.Y. Hsiao, R.C.T. Lee, An optimal embedding of cycles into incomplete hypercubes, Inf. Process. Lett. 72 (1999) 213–218.

[10] A. Kanevsky, C. Feng, On the embedding of cycles in pancake graphs, Parallel Comput. 21 (1995) 923–936.

[11] S. Latifi, M. Hegde, M. Naraghi-Pour, Conditional connectivity measures for large multiprocessor systems, IEEE Trans. Comput. 43 (2) (1994) 218–222.

[12] T.K. Li, C.H. Tsai, Jimmy J.M. Tan, L.H. Hsu, Bipanconnectivity and edge-fault-tolerant bipancyclicity of hypercubes, Inf. Process. Lett. 87 (2003) 107–110.

[13] Y. Rouskov, S. Latifi, P.K. Srimani, Conditional fault diameter of star graph networks, J. Parallel Distrib. Comput. 33 (1996) 91–97.

[14] Y. Saad, M.H. Shultz, Topological properties of hypercubes, IEEE Trans. Comput. 37 (7) (1998) 867–872.

[15] G. Tel, Topics in distributed algorithmsCambridge International Series on Parallel Computation, vol. 1, Cambridge University Press, 1991.

[16] C.H. Tsai, Jimmy J.M. Tan, T. Liang, L.H. Hsu, Fault-tolerant hamiltonian laceability of hypercubes, Inf. Process. Lett. 83 (2002) 301–306.

[17] Y.C. Tseng, S.H. Chang, J.P. Sheu, Fault-Tolerant Ring Embedding in a Star Graph with Both Link and Node Failures, IEEE Trans. Parallel Distrib. Syst. 8 (12) (1997) 1185–1195.

[18] M.C. Yang, T.K. Li, Jimmy J.M. Tan, L.H. Hsu, Fault-tolerant cycle-embedding of crossed cubes, Inf. Process. Lett. 88 (4) (2003) 149–154.