



Chaos in a nonlinear damped Mathieu system, in a nano resonator system and in its fractional order systems

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Accepted 25 October 2005

Communicated by Prof. Ji-Huan He

Abstract

In this paper, the chaotic behaviors of a nonlinear damped Mathieu system and of a nonlinear nano resonator system with integral orders and with fractional orders are studied. By applying numerical analyses such as phase portraits, Poincaré maps and bifurcation diagrams, the periodic and chaotic motions are observed. It is found that chaos exists both in the nonlinear damped Mathieu system and in the integral order and fractional order nano resonator systems. © 2005 Elsevier Ltd. All rights reserved.

1. Introduction

Chaos and chaotic systems have received a flurry of research effort in the past few decades. Such systems are nonlinear by nature, can occur in various natural and man-made systems, and are characterized by great sensitivity to initial conditions [1]. Besides the theoretical interest in the analysis of such nonlinear systems, there is another dimension to that interest; namely, utilizing such systems for useful practical applications [2–9]. Many researchers have devoted themselves to finding new ways to control chaos more efficiently [10–13]. Chaotic phenomena are quite useful in many applications such as fluid mixing [14], human brain dynamics [15], and heart beat regulation [16], information processing, etc. Therefore, making a periodic dynamical system chaotic, or preserving chaos of a chaotic dynamical system, is very meaningful and worthy to be investigated [17,18].

Fractional calculus is a 300-year-old mathematical topic [19–22]. Although it has a long history, for many years it was not used in physics and engineering. However, during the last 10 years or so, fractional calculus starts to attract increasing attention of physicists and engineers from an application point of view [23,24]. It was found that many systems in interdisciplinary fields can be elegantly described with the help of fractional derivatives. Many systems are known to display fractional-order dynamics, such as viscoelastic systems [25], dielectric polarization [26], electrode–electrolyte polarization [27], electromagnetic waves [28], quantitative finance [29], and quantum evolution of complex systems [30].

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It is well known that chaos cannot occur in autonomous continuous time systems of integer-order less than three according to the Poincaré–Bendixon theorem [31,32]. A recent example of a continuous time third order system that exhibits chaos is the Chen system [33]. The order of a system can be defined as the sum of the orders of all involved derivatives. However, in autonomous fractional order systems, it is not the case. For example, it has been shown that the fractional order Chua’s circuit with an appropriate cubic nonlinearity and with order as low as 2.7 can produce a chaotic attractor [34]. In [35,36], the bifurcation and the chaotic dynamics of fractional order cellular neural networks are studied. In [37], chaotic behaviors of a fractional order “jerk” model is studied, in which a chaotic attractor can be

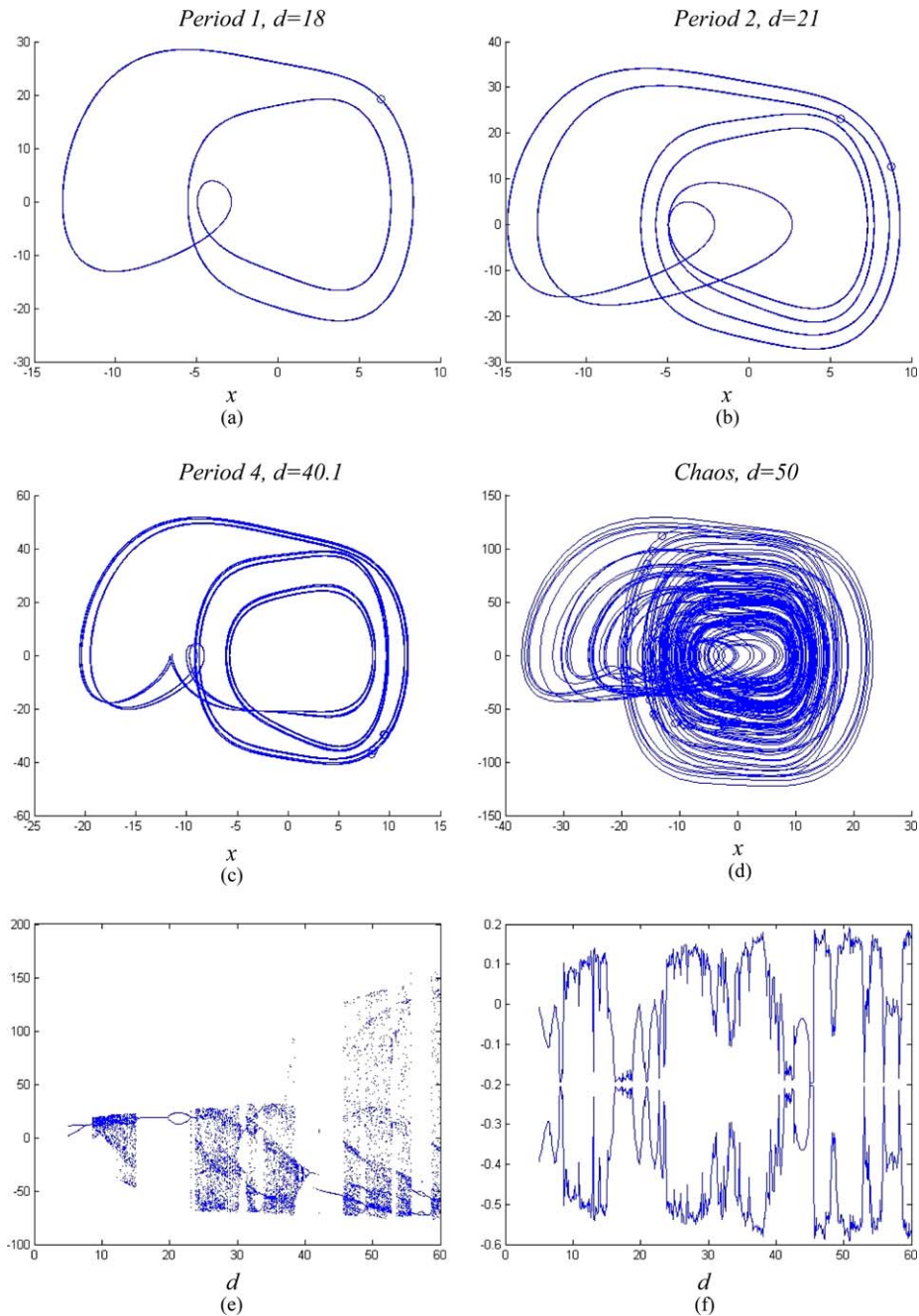


Fig. 1. The phase portraits, bifurcation diagram and the Lyapunov exponent for the nonlinear damped Mathieu system.

generated with the system order as low as 2.1 and a conjecture is presented that third order chaotic systems can still produce chaotic behavior with a total system order of $2 + e$, $0 < e < 1$. In [38], chaotic behavior of the fractional order Lorenz system is studied, but unfortunately, the results presented in this paper are not correct as pointed out by [39]. Also in [39], chaos and hyperchaos in fractional order Rössler equations are discussed, in which, it is shown that chaos can exist in the fractional order Rössler equation with order as low as 2.4, and hyperchaos can also exist in the fractional order Rössler hyperchaotic system with order as low as 3.8. In [40–43], chaotic behaviors in the fractional order

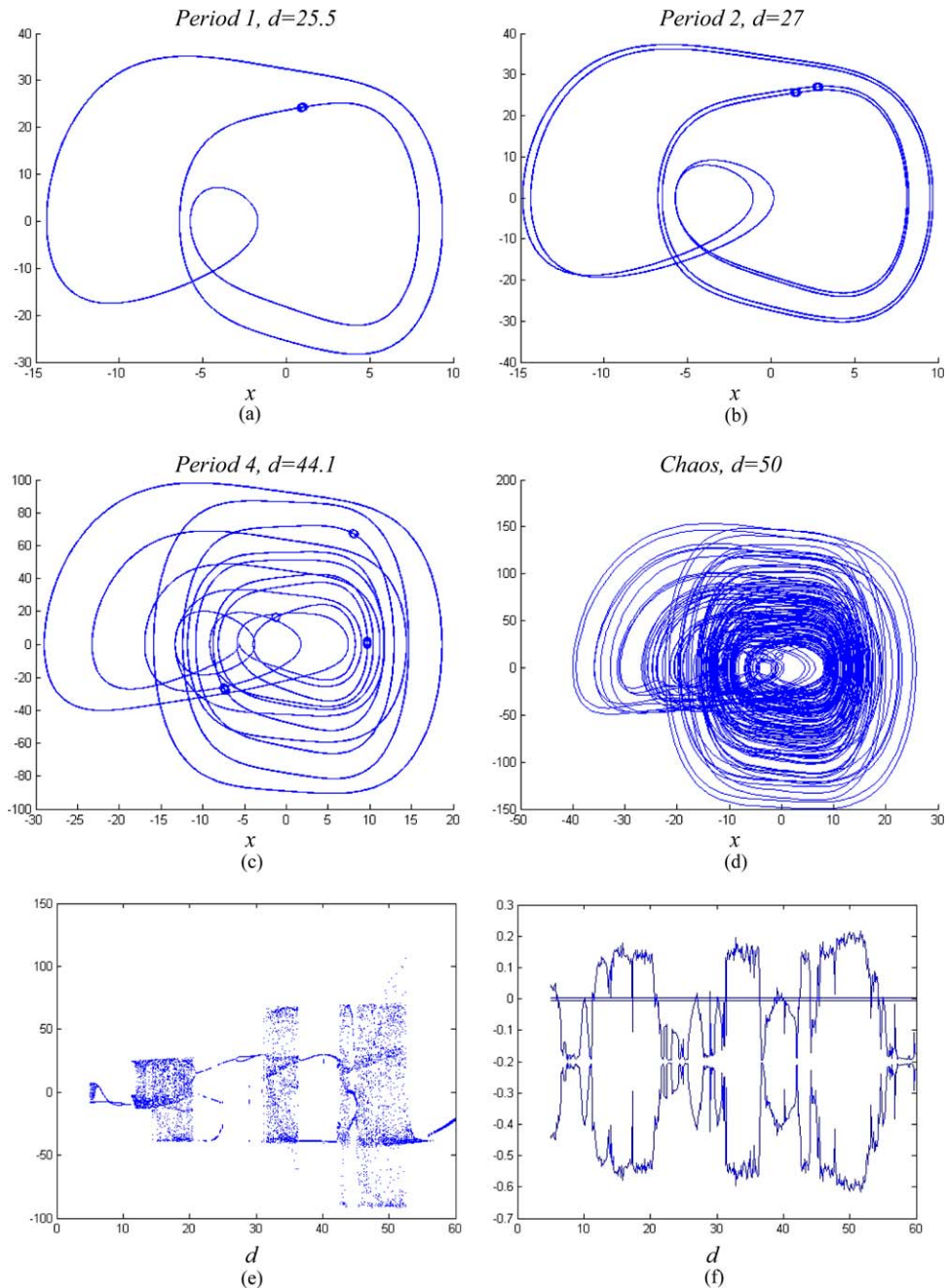


Fig. 2. The phase portraits, bifurcation diagram and the Lyapunov exponent for the nano resonator system with order $\alpha = 1$ and $\beta = 1$.

Table 1
Relation between orders of derivatives and existence of chaos

Total order 2				Total order 1.9			
Cases	Orders	Existence of chaos	Fig. no.	Cases	Orders	Existence of chaos	Fig. no.
1	$\alpha = 1.1, \beta = 0.9$	Yes	3	20	$\alpha = 1.1, \beta = 0.8$	No	
2	$\alpha = 0.9, \beta = 1.1$	Yes	4	21	$\alpha = 0.8, \beta = 1.1$	No	
3	$\alpha = 1.2, \beta = 0.8$	Yes	5	22	$\alpha = 1.2, \beta = 0.7$	No	
4	$\alpha = 0.8, \beta = 1.2$	Yes	6	23	$\alpha = 0.7, \beta = 1.2$	Yes	16
5	$\alpha = 1.3, \beta = 0.7$	No		24	$\alpha = 1.3, \beta = 0.6$	No	
6	$\alpha = 0.7, \beta = 1.3$	No		25	$\alpha = 0.6, \beta = 1.3$	Yes	17
7	$\alpha = 1.4, \beta = 0.6$	No		26	$\alpha = 1.4, \beta = 0.5$	No	
8	$\alpha = 0.6, \beta = 1.4$	No		27	$\alpha = 0.5, \beta = 1.4$	No	
9	$\alpha = 1.5, \beta = 0.5$	No		28	$\alpha = 1.5, \beta = 0.4$	No	
10	$\alpha = 0.5, \beta = 1.5$	Yes	10	29	$\alpha = 0.4, \beta = 1.5$	No	
11	$\alpha = 1.6, \beta = 0.4$	Yes	7	30	$\alpha = 1.6, \beta = 0.3$	No	
12	$\alpha = 0.4, \beta = 1.6$	Yes	8	31	$\alpha = 0.3, \beta = 1.6$	Yes	18
13	$\alpha = 1.7, \beta = 0.3$	Yes	9	32	$\alpha = 1.7, \beta = 0.2$	No	
14	$\alpha = 0.3, \beta = 1.7$	No		33	$\alpha = 0.2, \beta = 1.7$	Yes	19
15	$\alpha = 1.8, \beta = 0.2$	Yes	11	34	$\alpha = 1.8, \beta = 0.1$	Yes	15
16	$\alpha = 0.2, \beta = 1.8$	Yes	12	35	$\alpha = 0.1, \beta = 1.8$	No	
17	$\alpha = 1.9, \beta = 0.1$	Yes	13				
18	$\alpha = 0.1, \beta = 1.9$	Yes	14				
19	$\alpha = 1, \beta = 1$	Yes	2				

Chen system are studied and the lowest order to have chaos in this fractional order Chen system is shown to be 2.1 and 2.92, respectively.

This paper is organized as follows. In Section 2, a method for the approximation of the fractional derivative is given. In Section 3, the nano resonator system and its fractional order form are presented. In Section 4, numerical simulations, phase portraits and bifurcation diagrams, for various different fractional order nano resonator systems are described. In Section 5, conclusions are drawn.

2. Method for the approximation of the fractional derivative

The idea of fractional integrals and derivatives has been known since the development of the regular calculus, with the first reference probably being associated with Leibniz in 1695 [44].

Two commonly used definitions for the general fractional differintegral are the Grunwald definition and the Riemann–Liouville definition. The latter is given here

$$\frac{d^q f(t)}{dt^q} = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_0^t \frac{f(\tau)}{(t-\tau)^{q-n+1}} d\tau \tag{1}$$

where $n - 1 \leq q < n$ and $\Gamma(\cdot)$ is an Euler’s gamma function.

The Laplace transformation of the Riemann–Liouville fractional derivative (1) is

$$L\left\{\frac{d^q f(t)}{dt^q}\right\} = s^q L\{f(t)\} - \sum_{k=0}^{n-1} s^k \left[\frac{d^{q-1-k} f(t)}{dt^{q-1-k}}\right]_{t=0} \quad \text{for } n - 1 \leq q < n \tag{2}$$

By considering the initial conditions to be zero, this formula reduces to the more expected and comforting form

$$L\left\{\frac{d^q f(t)}{dt^q}\right\} = s^q L\{f(t)\} \tag{3}$$

and the fractional integral of order q can be described as $F(s) = \frac{1}{s^q}$ in the frequency domain.

The standard definitions of the fractional differintegral do not allow direct implementation of the operator in time domain simulations of complicated systems with fractional elements. Using the standard integer order operators to approximate the fractional operators is an effective method to analyze such systems.

The approximation approach taken here is to approximate the system behavior in the frequency domain [45]. By utilizing frequency domain techniques based in Bode diagrams, one can obtain a linear approximation of a fractional order integrator. Thus an approximation of any desired accuracy over any frequency band can be achieved.

Table 1 of Ref. [34] gives approximations for $\frac{1}{s^q}$ with $q = 0.1-0.9$ in steps of 0.1 with errors of approximately 2 dB from $\omega = 10^{-2}$ to 10^2 rad/s. These approximations will be used in the following numerical simulations.

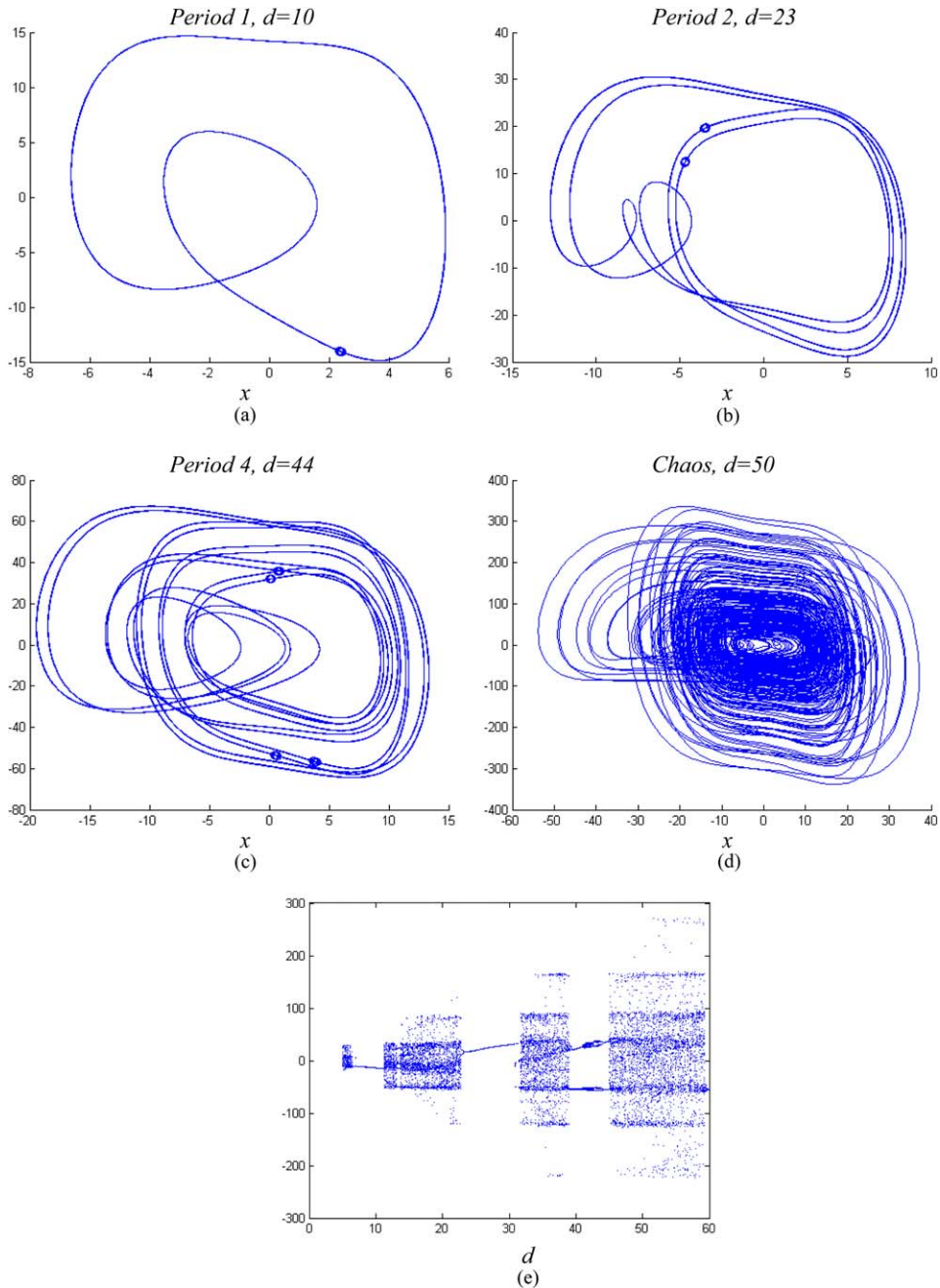


Fig. 3. The phase portraits and the bifurcation diagram for the nano resonator system with order $\alpha = 1.1$ and $\beta = 0.9$.

3. The chaos of the nonlinear damped Mathieu system and of the nano resonator system with its fractional order form

Mechanical resonance is widely applied in high-precision oscillators for a multitude of time-keeping and frequency reference applications. In all such cases, the high-precision resonating element consists of an off-chip passive component, such as a quartz crystal. Major drawback of these off-chip resonator technologies is that they are bulky and must interface with transistor chips at the boards, posing a bottleneck against the ultimate miniaturization of e.g., wireless

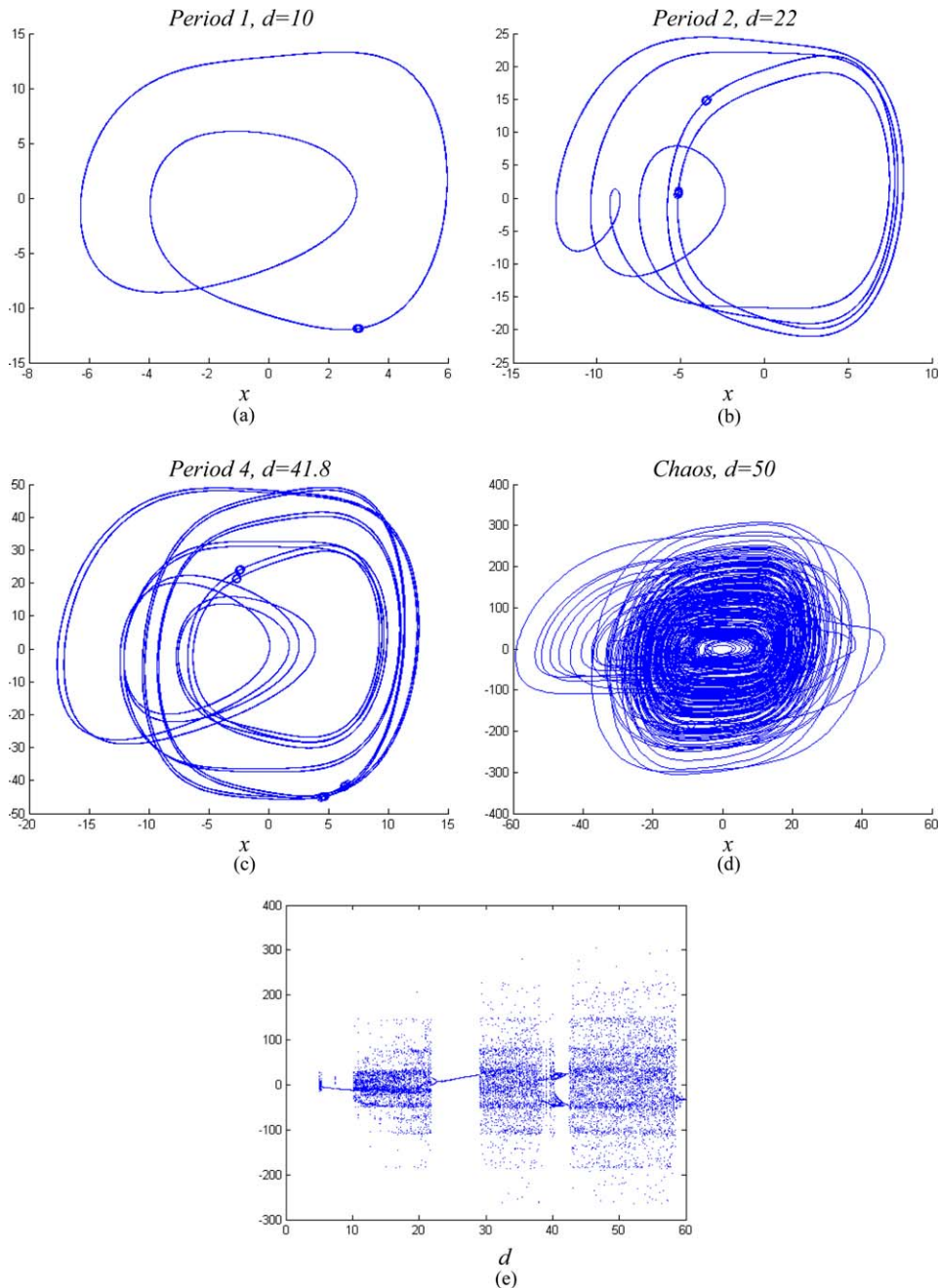


Fig. 4. The phase portraits and the bifurcation diagram for the nano resonator system with order $\alpha = 0.9$ and $\beta = 1.1$.

devices. The extraordinary small size and high level of integration that can be achieved with nano resonators appear to open exceptional possibilities for creating miniature-scale precision oscillators to be used in e.g., mobile communication and navigation devices.

Nano resonator system studied in this paper is a modified form of nonlinear damped Mathieu system. The nonlinear damped Mathieu system is a nonautonomous system with two states x and y :

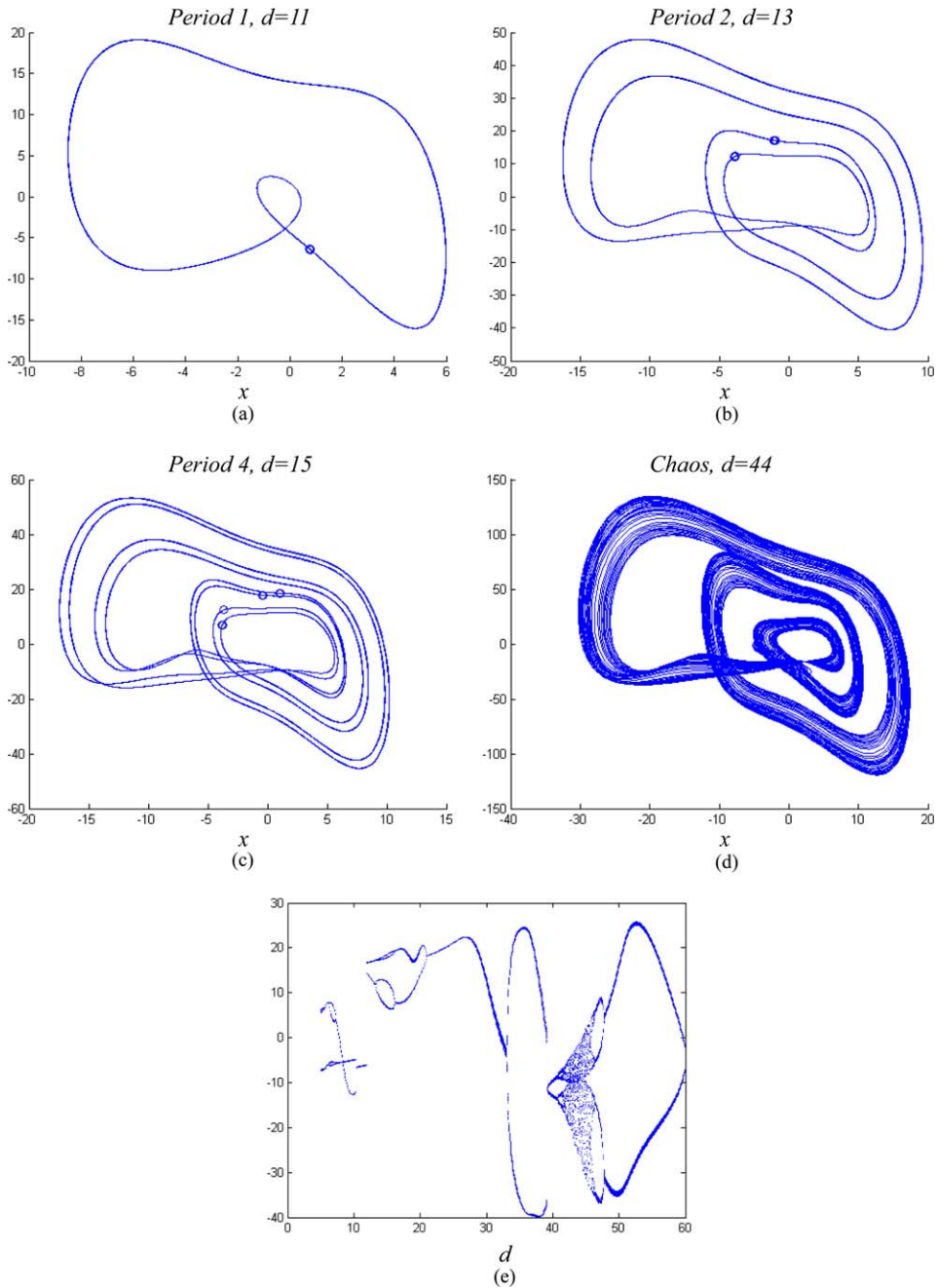


Fig. 5. The phase portraits and the bifurcation diagram for the nano resonator system with order $\alpha = 1.2$ and $\beta = 0.8$.

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -(a + b \sin \omega_1 t)x - (a + b \sin \omega_1 t)x^3 - cy + d \sin \omega_2 t \end{cases} \quad (4)$$

where a, b, c, d are constant parameters, and ω_1, ω_2 are circular frequencies. The phase portraits, Poincaré maps, bifurcation diagram and the Lyapunov exponent for system (4) are showed in Fig. 1 where $a = 0.2, b = 0.2, c = 0.4,$

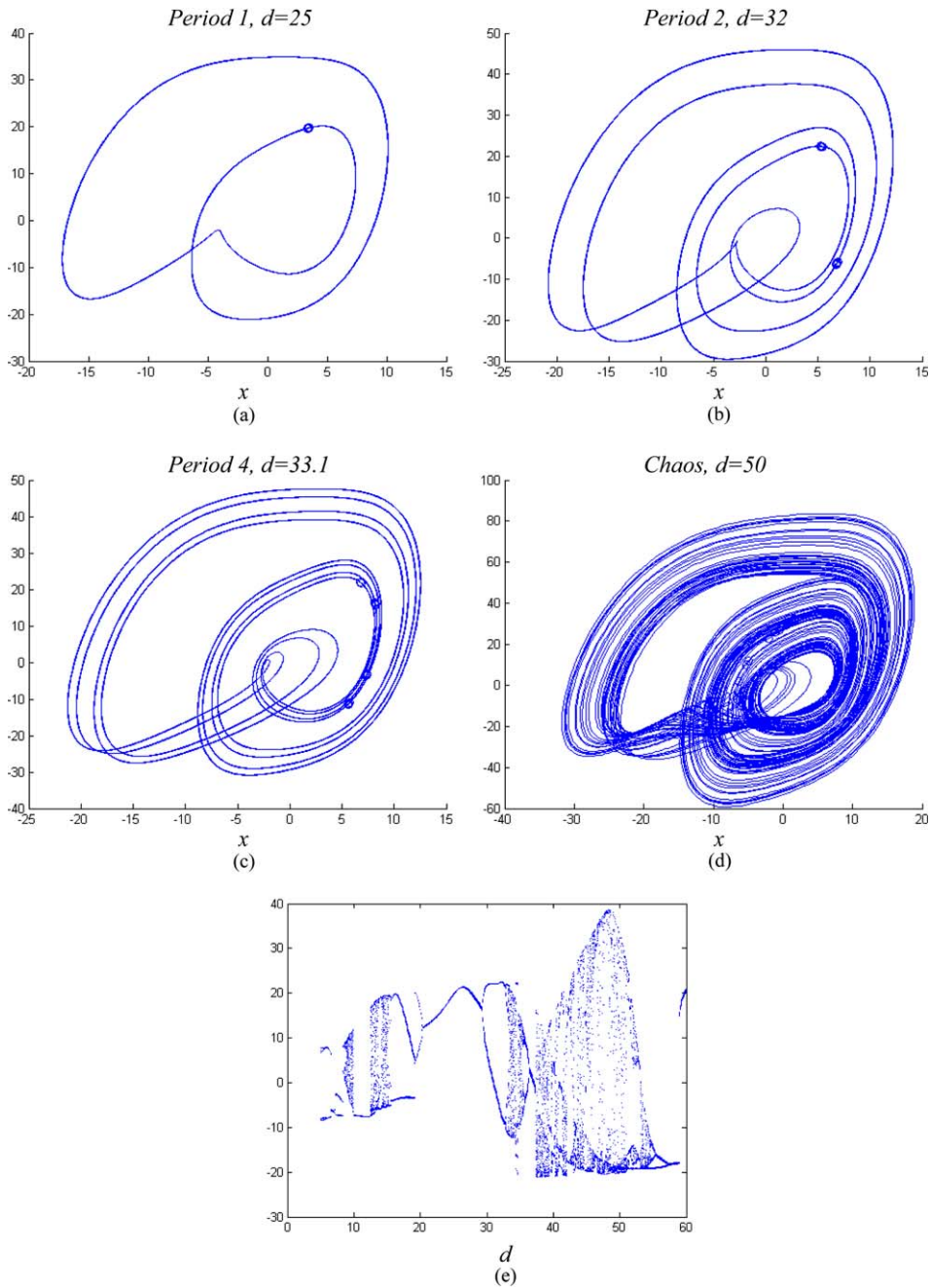


Fig. 6. The phase portraits and the bifurcation diagram for the nano resonator system with order $\alpha = 0.8$ and $\beta = 1.2$.

$\omega_1 = \omega_2 = \omega = 1$. Let $\omega_1 = \omega_2 = \omega$, and replace $\sin \omega t$ by z which is the periodic time function solution of the nonlinear oscillator

$$\begin{cases} \frac{dz}{dt} = w \\ \frac{dw}{dt} = -ez - fz^3 \end{cases} \quad (5)$$

where e, f are constant parameters. Then we have the modified nonlinear damped Mathieu system:

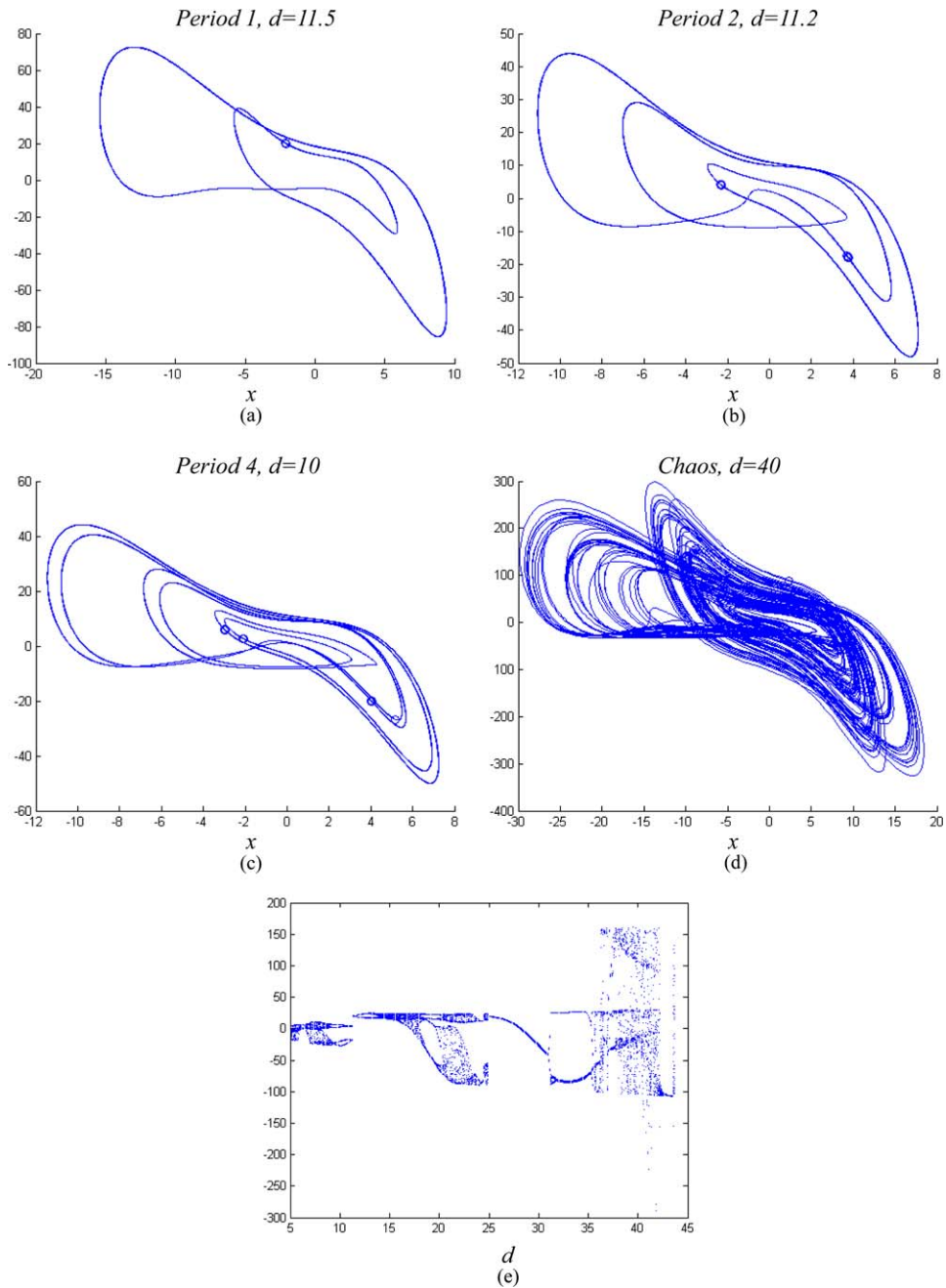


Fig. 7. The phase portraits and the bifurcation diagram for the nano resonator system with order $\alpha = 1.6$ and $\beta = 0.4$.

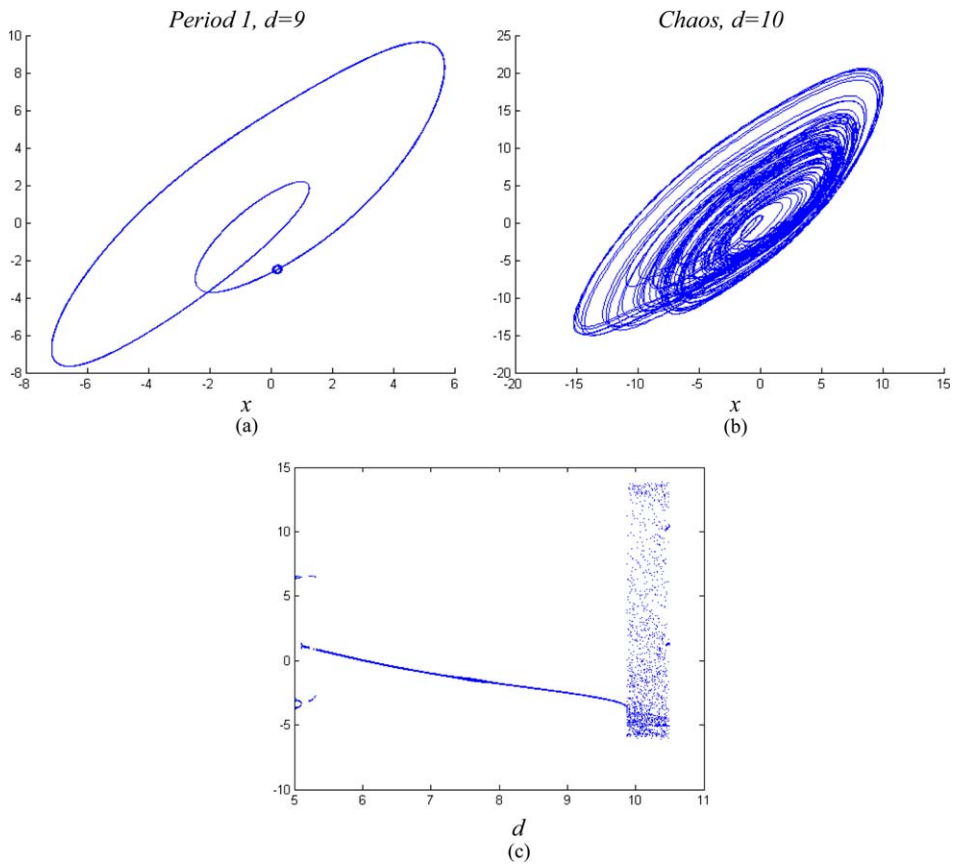


Fig. 8. The phase portraits and the bifurcation diagram for the nano resonator system with order $\alpha = 0.4$ and $\beta = 1.6$.

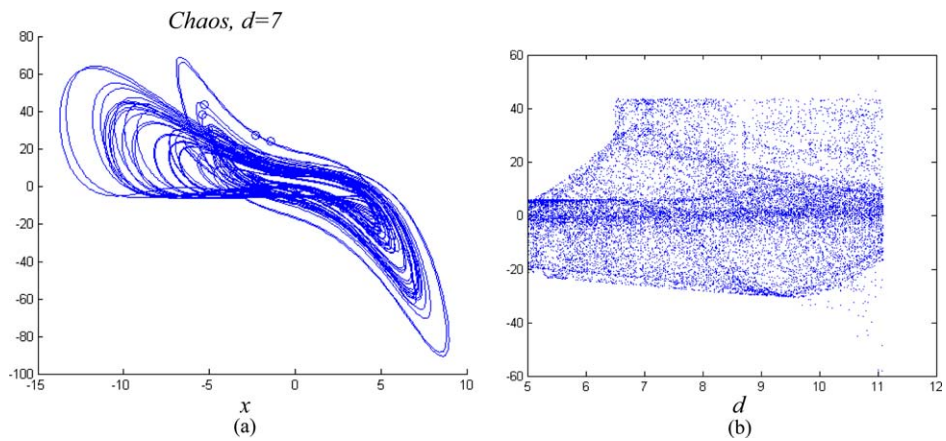


Fig. 9. The phase portraits and the bifurcation diagram for the nano resonator system with order $\alpha = 1.7$ and $\beta = 0.3$.

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -(a + bz)x - (a + bz)x^3 - cy + dz \\ \frac{dz}{dt} = w \\ \frac{dw}{dt} = -ez - fz^3 \end{cases} \quad (6)$$

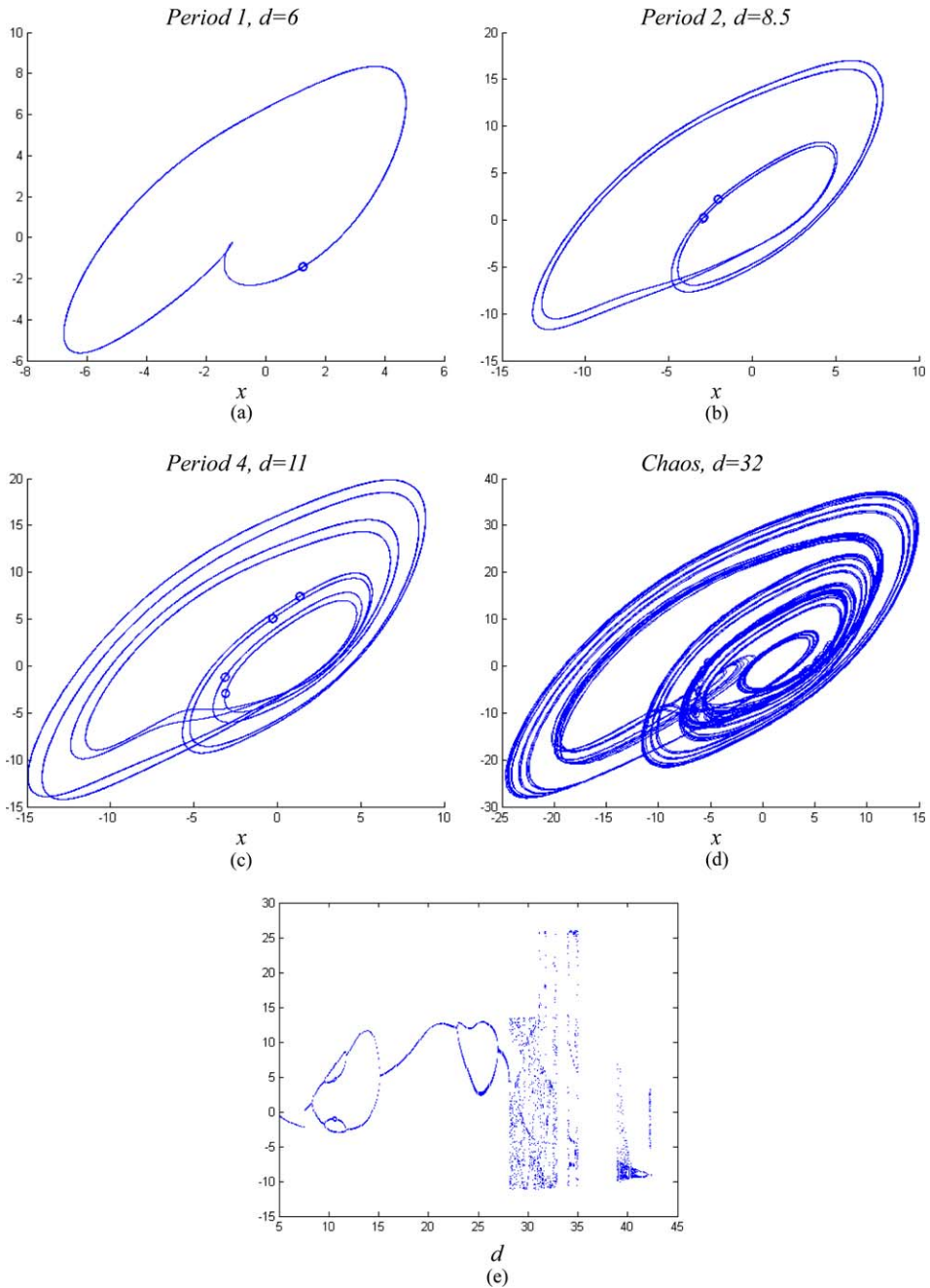


Fig. 10. The phase portraits and the bifurcation diagram for the nano resonator system with order $\alpha = 0.5$ and $\beta = 1.5$.

It becomes an autonomous system with four states where a, b, c, d, e and f are constant parameters of the system. System (4) consists of two parts:

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -(a + bz)x - (a + bz)x^3 - cy + dz \end{cases} \quad (7)$$

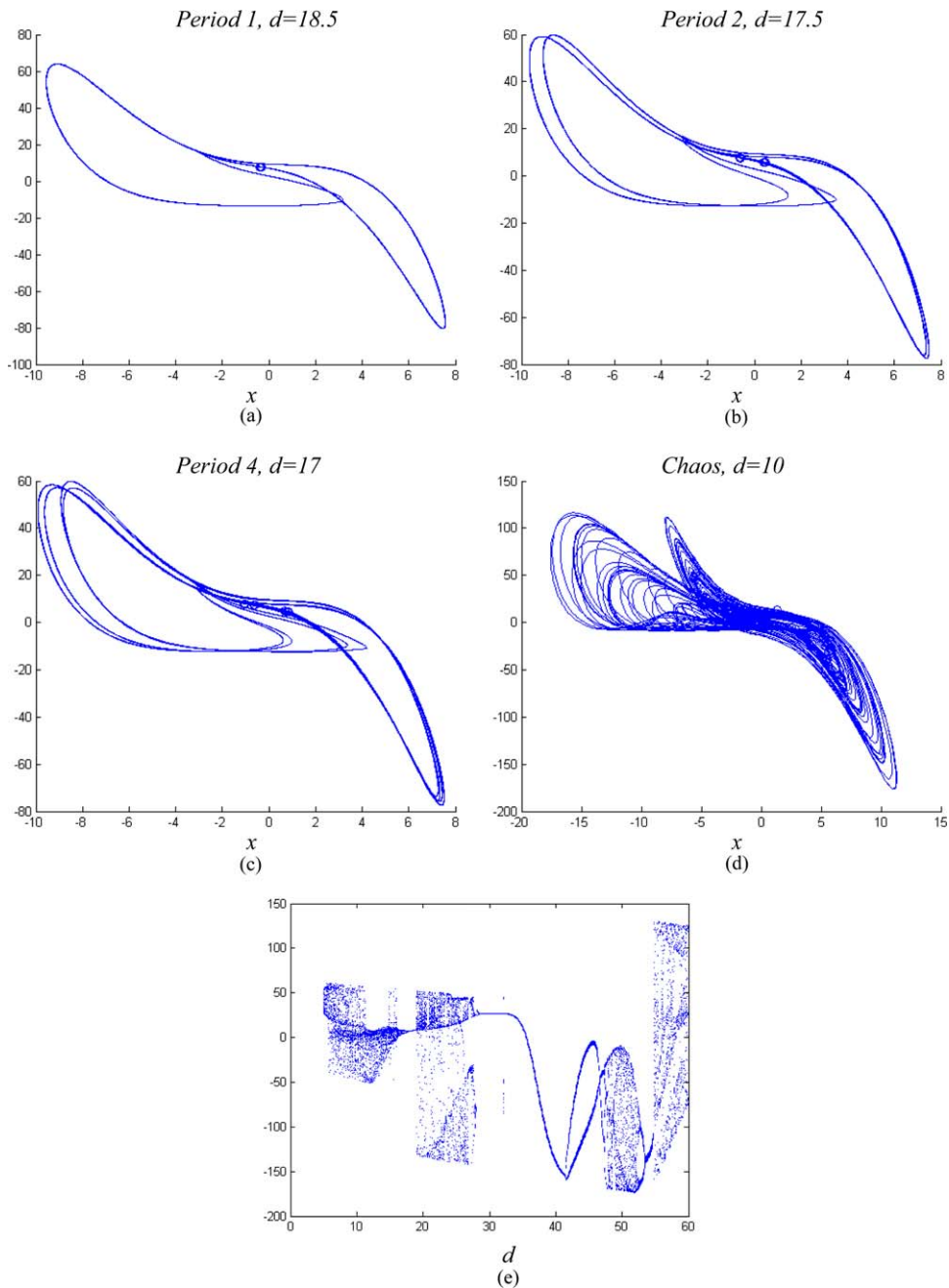


Fig. 11. The phase portraits and the bifurcation diagram for the nano resonator system with order $\alpha = 1.8$ and $\beta = 0.2$.

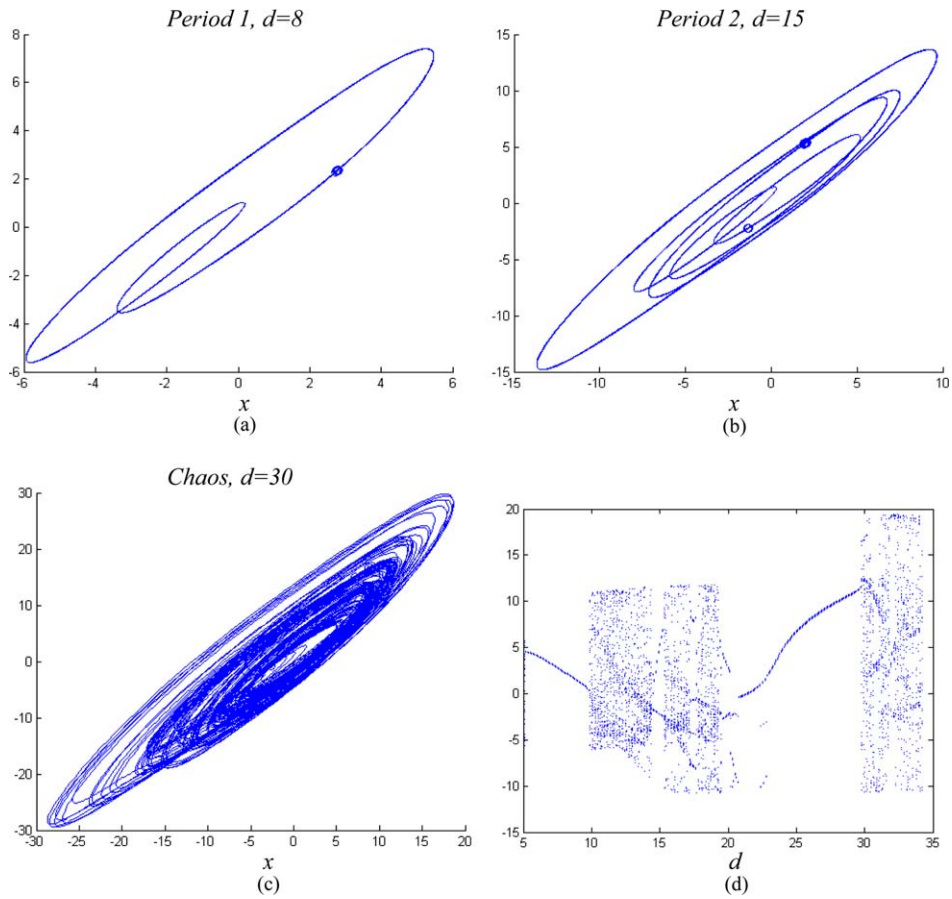


Fig. 12. The phase portraits and the bifurcation diagram for the nano resonator system with order $\alpha = 0.2$ and $\beta = 1.8$.

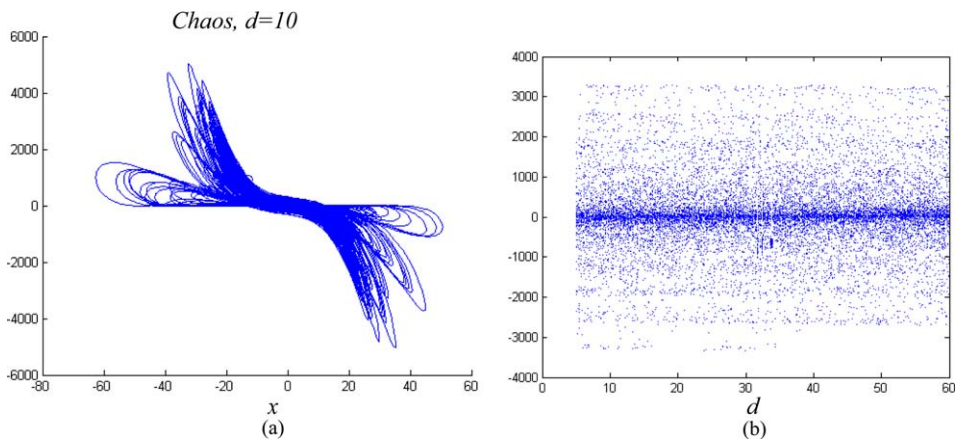


Fig. 13. The phase portraits and the bifurcation diagram for the nano resonator system with order $\alpha = 1.9$ and $\beta = 0.1$.

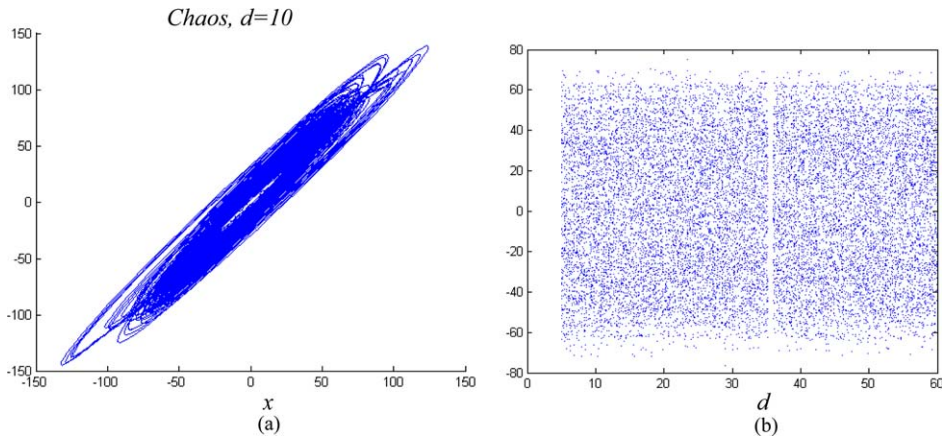


Fig. 14. The phase portraits and the bifurcation diagram for the nano resonator system with order $\alpha = 0.1$ and $\beta = 1.9$.

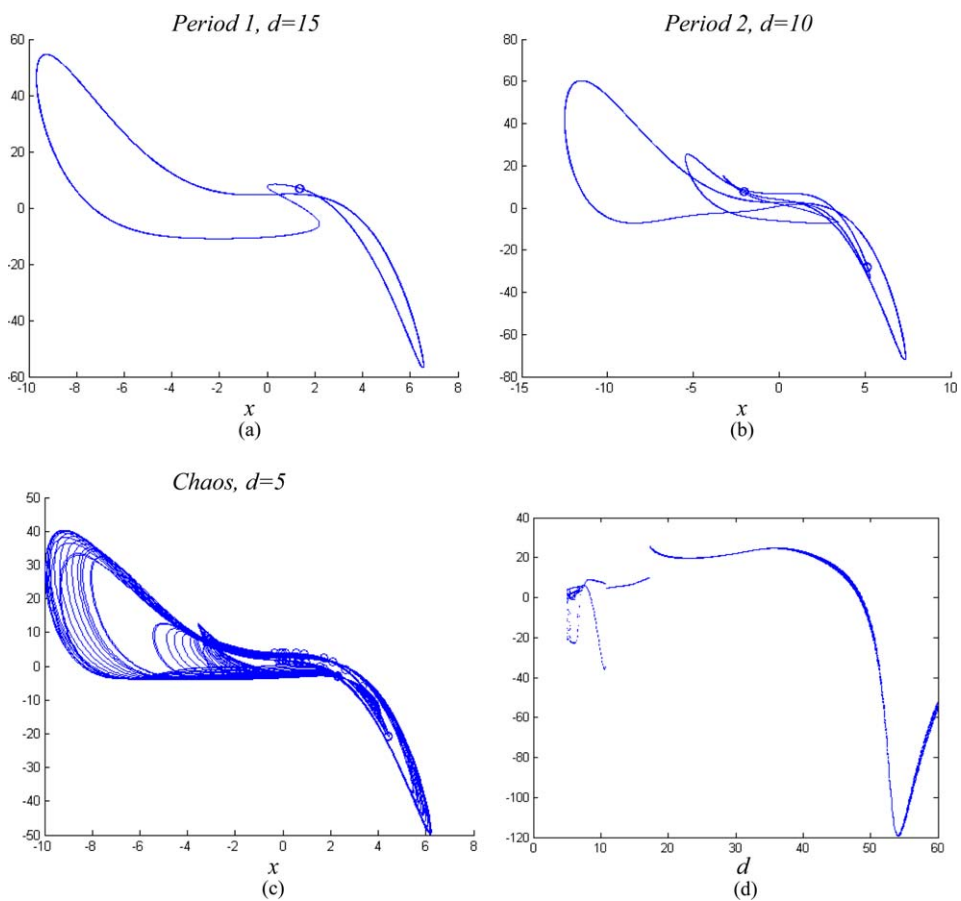


Fig. 15. The phase portraits and the bifurcation diagram for the nano resonator system with order $\alpha = 1.8$ and $\beta = 0.1$.

and

$$\begin{cases} \frac{dz}{dt} = w \\ \frac{dw}{dt} = -ez - fz^3 \end{cases} \quad (8)$$

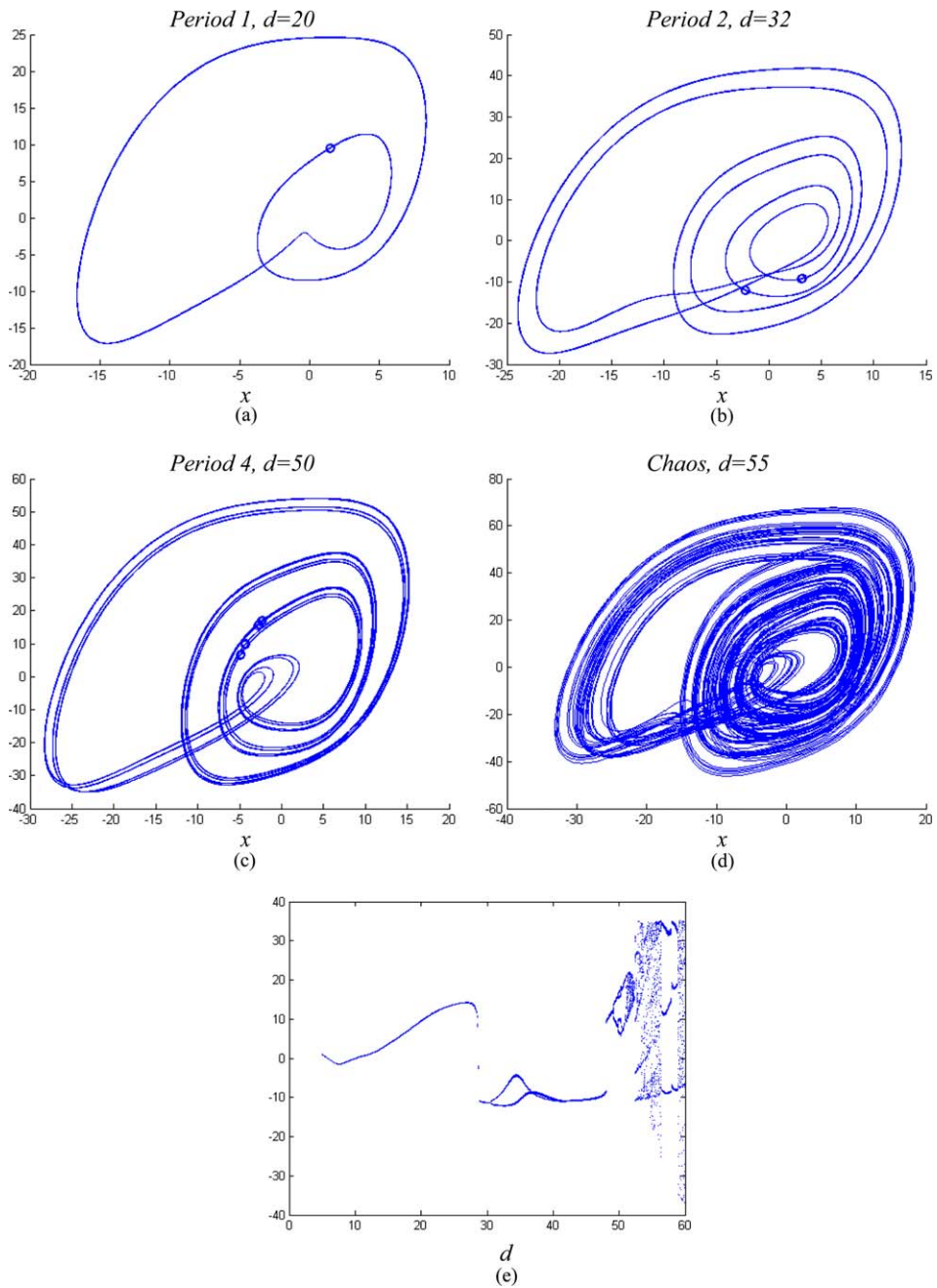


Fig. 16. The phase portraits and the bifurcation diagram for the nano resonator system with order $\alpha = 0.7$ and $\beta = 1.2$.

Eq. (8) affords the periodic time function solution to system (7) as an excitation which induces the chaos in system (7). As a result, Eq. (7) can be considered as a nonautonomous system with two states, while Eqs. (7) and (8) together can be considered as an autonomous system with four states. Our main interest devotes to Eq. (7), while Eq. (8) remains an integral order system. The phase portraits, Poincaré maps, bifurcation diagram and the Lyapunov exponent for Eq. (6) are showed in Fig. 2. Obviously, phase portraits of the nonlinear damped Mathieu system and the nano resonator system are similar, but chaos in the damped Mathieu system is more than that in the nano resonator system.

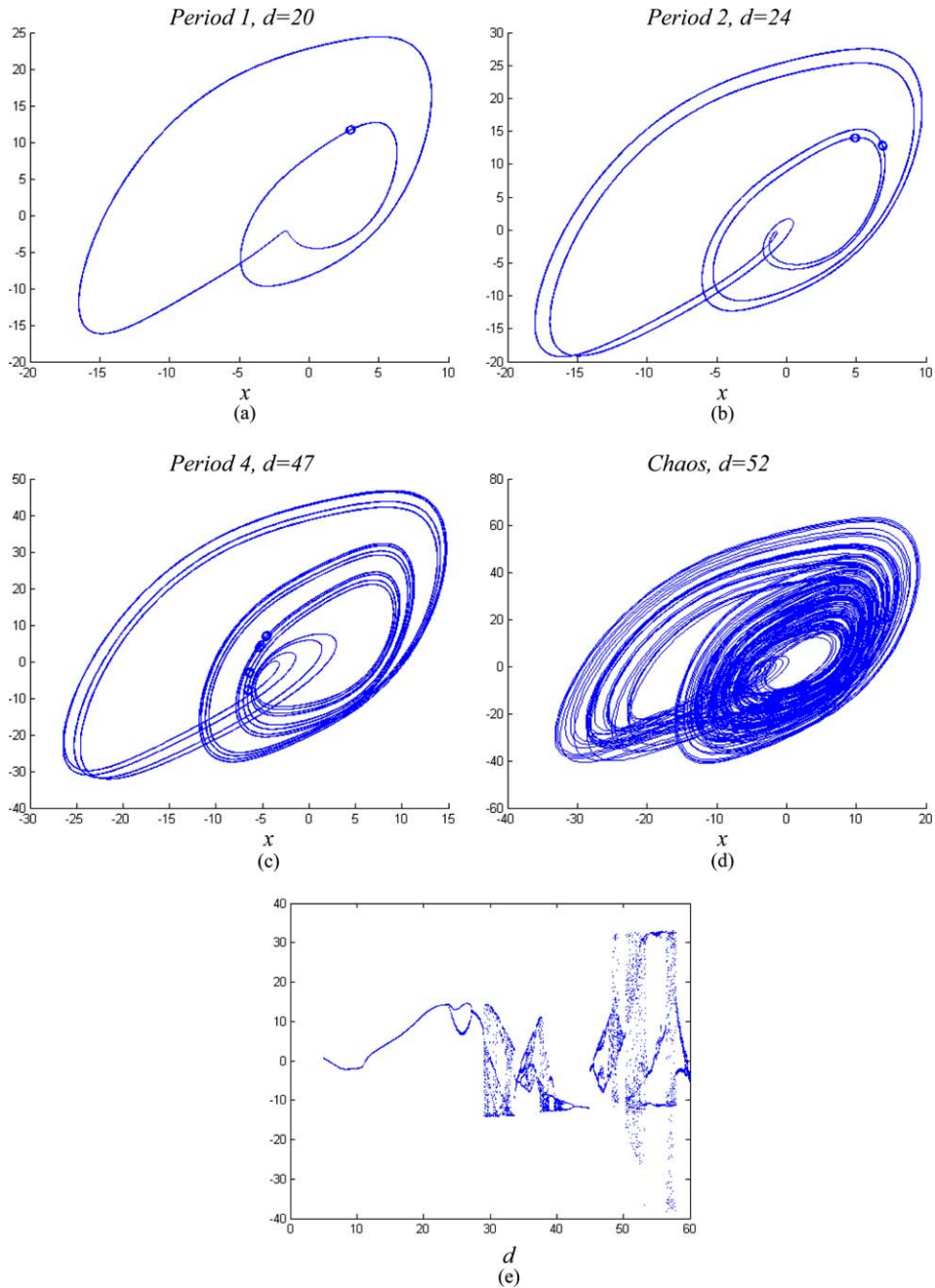


Fig. 17. The phase portraits and the bifurcation diagram for the nano resonator system with order $\alpha = 0.6$ and $\beta = 1.3$.

The corresponding modified nonlinear fractional order damped Mathieu system, the fractional order nano resonator system, is

$$\begin{cases} \frac{d^\alpha x}{dt^\alpha} = y \\ \frac{d^\beta y}{dt^\beta} = -(a + bz)x - (a + bz)x^3 - cy + dz \\ \frac{dz}{dt} = w \\ \frac{dw}{dt} = -ez - fz \end{cases} \quad (9)$$

where α and β are the fractional orders.

4. Numerical simulations for the fractional order systems

We vary the derivative orders α , β and the system parameter d , the other system parameters are fixed. Simulations are performed under $\alpha + \beta = 2$, $\alpha + \beta = 1.9$ where α , β are not integers. In our numerical simulations, five parameters $a = 0.2$, $b = 0.2$, $c = 0.4$, $e = 1$ and $f = 0.3$ are fixed and d is varied. The initial states of the nano resonator system are $x(0) = 3$, $y(0) = 4$, $z(0) = 1$ and $w(0) = 0$. The numerical simulations are carried out by MATLAB, and are summarized in Table 1.

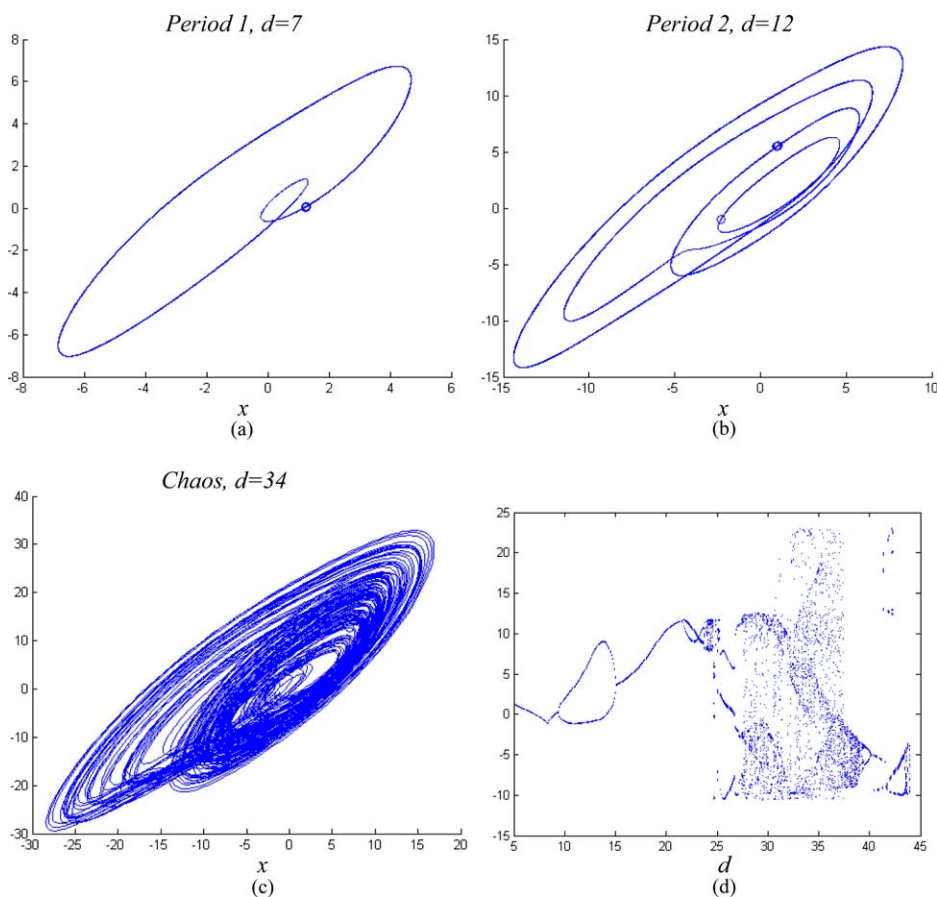


Fig. 18. The phase portraits and the bifurcation diagram for the nano resonator system with order $\alpha = 0.3$ and $\beta = 1.6$.

The phase portraits, Poincaré maps and the bifurcation diagrams of Case 1, 2, 3, 4, 10, 11, 12, 13, 15, 16, 17, 18, 19, 23, 25, 31, 33 and 34 for nano resonator system are showed in Figs. 3–6, 10, 7–9, 11–14, 2, 16–19 and 15, respectively. Case 1, 2, 3 and 19 have similar shapes in their phase portraits and Poincaré maps, chaos in Case 3 is only distributed over the parameter $d = 40 \sim 50$, relatively, chaos in Case 1, 2 and 19 are distributed more wide than that of Case 3. Case 4, 23 and 25 have similar shapes in their phase portraits. Case 11, 13, 15, 17 and 34 have similar shapes in their phase portraits. Case 17 has the largest range of γ among Case 11, 13, 15, 17 and 34, even beyond 3000, chaos in Case 13 and 17 are distributed over all varied parameter region, but in Case 15, chaos is distributed over about the parameter $d > 50$. Case 10, 12, 16, 18, 31 and 33 have similar shapes in their phase portraits.

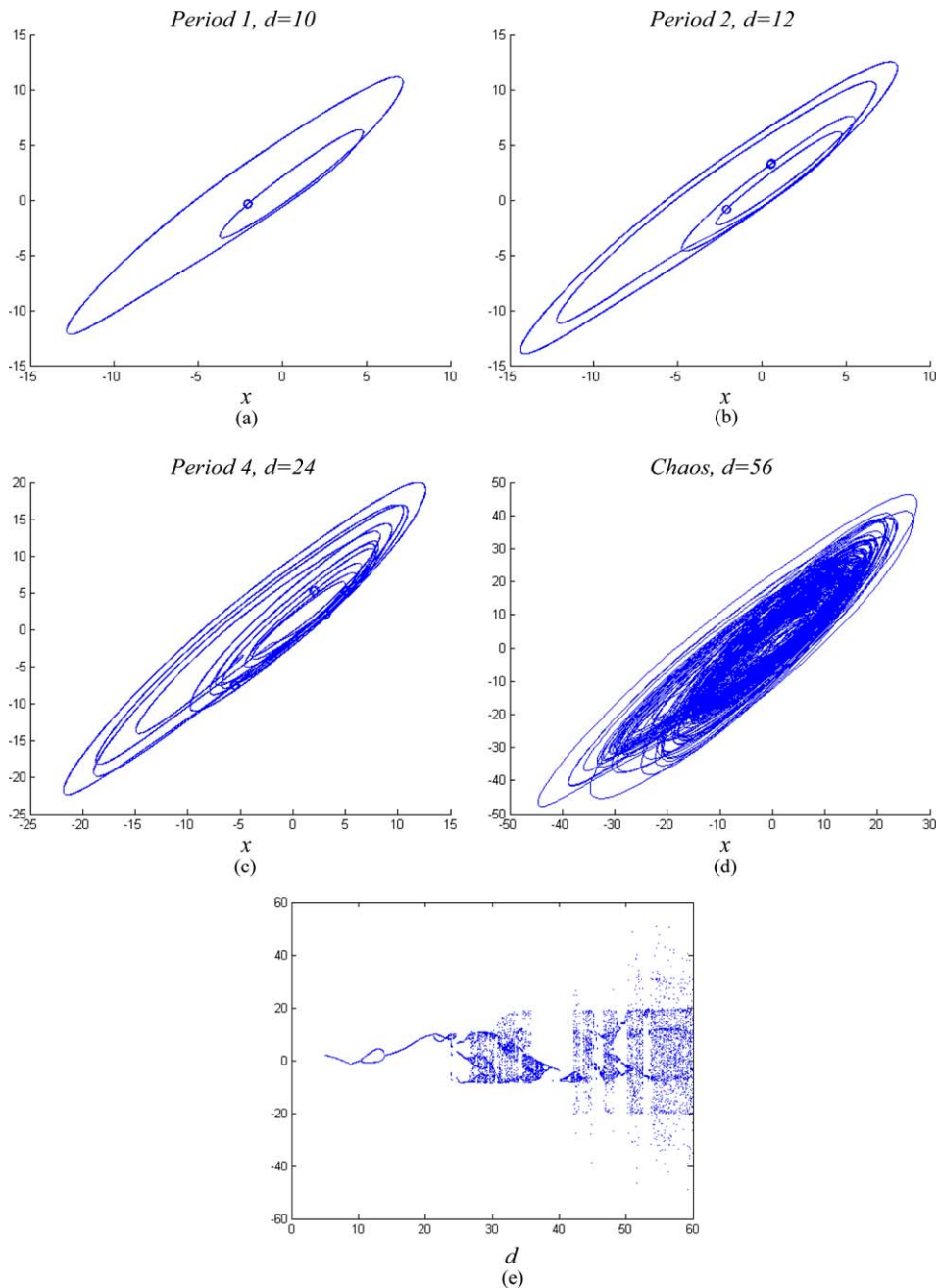


Fig. 19. The phase portraits and the bifurcation diagram for the nano resonator system with order $\alpha = 0.2$ and $\beta = 1.7$.

The results from simulation verified that chaos indeed exists in the system with total fractional orders $\alpha + \beta = 2$ and $\alpha + \beta = 1.9$, which are summarized in Table 1.

5. Conclusions

In this paper, the chaotic behaviors of a nonlinear damped Mathieu system and of nano resonator systems with integral order and with fractional order are investigated by means of phase portraits, Poincaré maps and bifurcation diagrams.

In Section 2, the approximation method of the fractional derivative with the Riemann–Liouville algorithm is given. In Section 3, the chaos of the nonlinear damped Mathieu system and of the nano resonator system with its fractional order form are obtained. In Section 4, the phase portraits, Poincaré maps and bifurcation diagrams for various different fractional order nano resonator systems are represented.

Acknowledgements

This research was supported by the National Science Council, Republic of China under grant number NSC 94-2212-E-009-013.

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