

The r -Neighborhood Graph: An Adjustable Structure for Topology Control in Wireless Ad Hoc Networks

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Abstract—In wireless ad hoc networks, constructing and maintaining a topology with lower node degrees is usually intended to mitigate excessive traffic load on wireless nodes. However, keeping lower node degrees often prevents nodes from choosing better routes that consume less energy. Therefore, the trade-off is between the node degree and the energy efficiency. In this paper, an adjustable structure, named the r -neighborhood graph, is proposed to control the topology. This structure has the flexibility to be adjusted between the two objectives through a parameter r , $0 \leq r \leq 1$. More explicitly, for any set of n nodes, the maximum node degree and power stretch factor can be bounded from above by some decreasing and increasing functions of r , respectively. Specifically, the bounds can be constants in some ranges of r . Even more, the r -neighborhood graph is a general structure of both RNG and GG , two well-known structures in topology control. Compared with YG_k , another famous adjustable structure, our method always results in a connected planar with symmetric edges. To construct this structure, we investigate a localized algorithm, named PLA , which consumes less transmitting power during construction and executes efficiently in $O(n \log n)$ time.

Index Terms—Wireless ad hoc networks, topology control, energy-efficient, localized algorithm.



1 INTRODUCTION

WIRELESS ad hoc networks enhance the conventional deployment of communicating environments for many applications, such as conferences, hospitals, battlefields, search and rescue teams, etc. In these environments, the performance of network operations heavily depends upon the underlying topology [4]. For instance, the delivery rate would be significantly lower as the underlying topology breaks. Therefore, appropriately controlling the topology is a crucial stage in communication. The topology control problem in wireless ad hoc networks has been widely studied in recent years [3], [15], [18], [19], [20], [23], [29], [32]. Generally speaking, the core of this problem is to determine a set of wireless links such that the composed topology is able to achieve certain goals [23]. These goals would be variant depending upon the circumstances and could be either qualitative features or quantitative objectives. Since wireless nodes usually struggle with limited bandwidth and computation power, a genius way should be able to simultaneously achieve several goals. In this paper, we aim to control the topology with the following goals, which are extremely desired in wireless environments:

1. *Symmetry*. The existence of asymmetric links may complicate many communication primitives. For instance, the MAC layer's ACK is hard to implement

when some links are not bidirectional [21]. Besides, asymmetric links in topology would also cause inconsistent routing qualities at two ends.

2. *Connectivity*. Connectivity is unquestionably the most essential prerequisite in any communicable topology [23]. Two nodes u and v are strongly connected if there is a directed path from u to v and vice versa. A directed topology is strongly connected if all pairs of nodes are strongly connected. If the links are symmetric, we should aim at the *connectivity* of an undirected topology instead.
3. *Energy efficiency*. Energy is the most crucial resource in wireless nodes. Due to the severe path loss in radio carriers, transmitting with large ranges would exponentially run out of nodes' energy. Therefore, relaying messages through multiple hops with shorter ranges could usually consume less energy [24]. How to choose the links between nodes for relaying is a critical point in this goal.
4. *Sparseness*. Numerous distributed and localized routing protocols are based on flooding [13]; however, this may burden networks with unavoidable redundant messages. Thus, keeping a *sparse* topology, consisting of linear number of links [15], would be an ingenious way to shrink the expenditure from network operations.
5. *Maximum node degree*. For some nodes with overly large degrees, the network flows will concentrate on them and rapidly draw out their energy. Besides, a larger node degree means tighter dependency among nodes, which is not expected when wireless nodes move frequently. Therefore, the maximum node degree over a topology should be bounded from above by some constant.

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6. *Planarity.* A graph is *planar* if it has no crossed links inside. It is helpful for many geometric problems: The shortest path (least energy unicast route) can be quickly found in linear time when the underlying topology is planar [12]. Besides, in many position-based routing algorithms, the successful delivery can be guaranteed only if the underlying topology is planar [2], [11].

Taking a further look, a topology having constant node degree must be sparse. So, we can be concerned with the fifth goal only. Unfortunately, keeping nodes with lower node degree would possibly sacrifice some potential links composing more energy efficient routes in topology. Therefore, empirically, a trade-off is between the node degree and energy efficiency [15]. For this reason, we aim to design an adjustable way so that the trade-off can be adjusted flexibly.

In wireless ad hoc networks, due to the absence of a central arbitrator and the limited sensing range, centralized approaches [3], [30] are rarely attainable. Therefore, a variety of distributed approaches were proposed [17], [19], [29]. A distributed protocol passes messages hop-by-hop. However, this may cause considerable overhead through the entire network. So, a localized approach is more preferred. According to Stojmenovic and Lin [27], a node using the localized topology control method requires information within constant hop(s). However, in some localized approaches [15], [16], [18], [27], the operations should *recursively* depend upon the computed status or partial results from nearby nodes, which may hurt their practicability. Therefore, in the following, we define a new type of methodology for more practicability:

Definition 1. An algorithm L is purely localized if it is localized and all operations depend upon only the information inherent¹ in nodes, available before any execution of L .

A purely localized topology control algorithm is more useful to large-scale and high mobility environments, since the operation of a node is completely isolated from any execution of other nodes. Further, we say that a structure is *purely localizable* if we can construct it by a purely localized algorithm. Our goal is to investigate a purely localizable structure so that all the desired goals listed above can be achieved.

The rest of this paper is organized as follows: Section 2 specifies the network model and formally describes the problem under study. In Section 3, we review and summarize the related works. The main structure, components, and their theoretical results are presented in Section 4. Some detailed derivations are given in the Appendix. In Section 5, we investigate an extended version of the main structure to comprehend our theoretical properties. In Section 6, a purely localized algorithm is investigated to construct our structure. Finally, concluding remarks and some directions for further research are given in the last section.

2 The MODEL AND PROBLEM

The wireless ad hoc network discussed in this paper consists of a set V of n wireless nodes distributed on a

two-dimensional plane \mathbb{R}^2 . Each node is equipped with an omnidirectional antenna and can change its transmission range by adjusting the transmitting power at any level. The maximum transmission ranges are equal among all nodes. In other words, we can normalize the maximum transmission ranges of all nodes to be 1 for simplicity. In addition, each node u can obtain its position $P(u)$ through a lower-power GPS or some other ways [14], and a unique $id(u)$ is also available to each node u .

This network can be modeled as a *unit disk graph*, $UDG(V)$. In this graph, an edge uv exists if and only if the euclidean distance between u and v , denoted as $\|uv\|$, is at most 1.

The least power required to transmit immediately between u and v is modeled as $\|uv\|^\alpha$, where α is typically taken on a value between 2 and 4, depending on the attenuation strength of the communication environment [5]. To measure the power efficiency of a topology, Li et al. [15] defined a well-formed measure, named *power stretch factor*. We reintroduce it below. Let $\pi(u, v) = v_0v_1 \dots v_{h-1}v_h$ be a unicast path connecting nodes u and v , where $v_0 = u$ and $v_h = v$. The *total transmission power* consumed by path $\pi(u, v)$ is defined as

$$p(\pi(u, v)) = \sum_{i=1}^h \|v_{i-1}v_i\|^\alpha.$$

Let $\pi_{G(V)}^*(u, v)$ be the *least-energy path* connecting u and v in graph $G(V)$. Given a subgraph $G'(V)$ in $UDG(V)$, the power stretch factor of $G'(V)$ with respect to $UDG(V)$ is defined as

$$\rho(G'(V)) = \max_{u, v \in V} \frac{p(\pi_{G'(V)}^*(u, v))}{p(\pi_{UDG(V)}^*(u, v))}.$$

On the other hand, the maximum node degree of graph $G(V)$ is defined as

$$d_{\max}(G(V)) = \max_{u \in V} d_{G(V)}(u),$$

where $d_{G(V)}(u)$ is the degree of node u in graph $G(V)$.

3 RELATED WORK

Many localizable structures, used to control the network topology, have been proposed in the literature [15], [16], [18], [26], while only a few of them are purely localizable. In the following, we list four well-known structures. Most of them or their extensions are purely localizable:

- The *constrained Relative Neighborhood Graph* [28], denoted by $RNG(V)$, has an edge uv if and only if $\|uv\| \leq 1$ and the intersection of two *open disks*² centered at u, v with radius $\|uv\|$ contains no node $w \in V$; see Fig. 1a.
- The *constrained Gabriel Graph* [6], denoted by $GG(V)$, has an edge uv if and only if $\|uv\| \leq 1$ and the open disk using $\|uv\|$ as diameter contains no node $w \in V$; see Fig. 1b.

1. The node's position and id are usually assumed to be inherited in nodes. See Section 2 for more explanation.

2. An open disk centered at point x with radius d is the collection of points with distance less than d from $P(x)$.

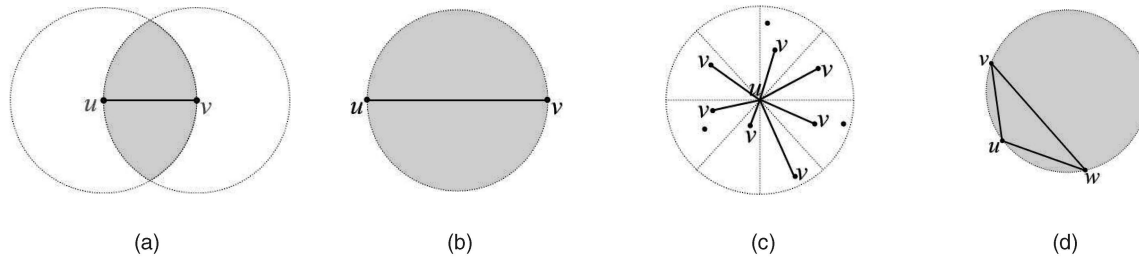


Fig. 1. (a) $RNG(V)$. (b) $GG(V)$. (c) $YG_k(V)$, $k = 8$. (d) $UDel(V)$.

- The *constrained Yao Graph* [33] with a parameter $k \geq 6$, denoted by $\overrightarrow{YG}_k(V)$, is constructed as follows: For each node u , define k equal cones by k equal-separated rays originated at u . At each cone, a directed edge uw exists if $\|uw\| \leq 1$ and the cone contains no vertex $w \in V$ such that $\|uw\| < \|uv\|$. Ties are broken arbitrarily. $YG_k(V)$ is denoted as the underlying undirected graph of $\overrightarrow{YG}_k(V)$; see Fig. 1c.
- A *Delaunay Triangulation*, denoted by $Del(V)$, is a triangulation of V in which the interior of the circumcircle of each Δuvw contains no node $w \in V$. The *unit Delaunay Triangulation*, denoted by $UDel(V)$, has all edges of $Del(V)$ except those longer than 1 [8], [18]; see Fig. 1d.

Let us discuss the properties of these structures and their extensions. We say an objective $f(\cdot)$ of a structure $S(V)$ is *bounded* if there is a constant C such that $f(S(V)) \leq C$, for any set V of n nodes. Li et al. [15] showed that $d_{\max}(RNG(V))$ is unbounded if there is a node $u \in V$ having an unbounded number of neighbors adjacent to u at exactly the same distance in the underlying $UDG(V)$. To overcome this problem, Wattenhofer and Zollinger [32] proposed an algorithm to find a structure, denoted by $XTC(V)$. They showed that $XTC(V)$ is a subgraph of $RNG(V)$ and the $d_{\max}(XTC(V))$ is at most 6. Especially, if there is no node having two or more neighbors at exactly the same distance in V , $XTC(V)$ is identical to $RNG(V)$ [24]. Their results infer the following theorem:

Theorem 1. *Given a set V of nodes on \mathbb{R}^2 , if there is no node having two or more neighbors at exactly the same distance, then $d_{\max}(RNG(V)) \leq 6$.*

We denote the condition in Theorem 1 as Assumption *AS*. That is,

Assumption AS. *There is no node in V having two or more neighbors at exactly the same distance.*

This theorem reveals that even $RNG(V)$ has no constant bound on its node degree. It is still useful since the distances of nodes in the real world are rarely exactly the same. The constrained Gabriel Graph $GG(V)$ has the least power stretch factor 1 in comparison with the unbounded power stretch factor $n - 1$ of $RNG(V)$ [15]. However, $d_{\max}(GG(V))$ could be as large as $n - 1$. An extended structure, *Enclosure graph* [16], [14], [24], denoted by $EG(V)$, is generalized from $GG(V)$. It can always result in a subgraph of $GG(V)$ [16]. Even so, its maximum node degree is still unbounded [20], [24].

To overcome the trade-off between the maximum node degree and the power stretch factor, an adjustable structure, having the flexibility to be adjusted between the two objectives, becomes more attractive. $\overrightarrow{YG}_k(V)$ is an adjustable structure. It can be adjusted through a parameter k such that, for any given k , the maximum *out-degree* is at most k , and the power stretch factor is at most $1/(1 - (2 \sin \pi/k)^\alpha)$ [15]. We say an objective $f(\cdot)$ of an adjustable structure $S_k(V)$ with parameter k is *partially bounded* if there is at least one k_0 such that $f(S_{k_0}(V))$ is *bounded*. According this definition, the maximum *out-degree* and power stretch factor of $\overrightarrow{YG}_k(V)$ are partially bounded since, for some ranges of k , k and $1/(1 - (2 \sin \pi/k)^\alpha)$ are constants. However, the asymmetric edges of $\overrightarrow{YG}_k(V)$ may lead to large *in-degrees* even when k is very small [15]. So, $d_{\max}(YG_k(V))$ can be neither bounded nor partially bounded. To improve this, an extension of $\overrightarrow{YG}_k(V)$, named *Yao and Sink*, was proposed [15], [17], [29]. It can limit the maximum node degree in $(k + 1)^2 - 1$ and result in symmetric edges. Unfortunately, in this structure, the neighbors of some node should be recursively determined by one another so that it cannot be purely localizable. The unit Delaunay triangulation $UDel(V)$ has a bounded power stretch factor. However, neither $Del(V)$ nor $UDel(V)$ can be computed locally. So, Li et al. [18] suggested a localized version of the Delaunay graph, denoted by $LDel^{(h)}(V)$, where h means that each node uses at most k -hop information. The power stretch factor of $LDel^{(k)}(V)$ is bounded for all $k \geq 1$. Even so, its maximum node degree is not bounded for any h .

The relations among these structures were studied in several papers [7], [10], [16], [22], [24], [33]. We summarize them in Fig. 2, where $EMST(V)$ is the euclidean minimum spanning tree of $UDG(V)$. With these relations, their connectivity and planarity can be easily inferred.

Regarding their connectivity, we know that $EMST(V)$ is connected if $UDG(V)$ is itself a connected component of V . Therefore, when $UDG(V)$ is connected, all graphs containing $EMST(V)$ are connected. That is, $RNG(V)$, $GG(V)$, $EG(V)$, $UDel(V)$, $LDel^{(k)}(V)$, and $YG_k(V)$ are all connected. The connectivity of $XTC(V)$ was proven in a different way [24].

Regarding their planarity, $LDel^{(k)}(V)$ is planar for any $k \geq 2$ [18]. Therefore, all subgraphs of $LDel^{(2)}(V)$ are planar. That is, $UDel(V)$, $GG(V)$, $EG(V)$, $RNG(V)$, $XTC(V)$, and $EMST(V)$ are all planar. On the contrary, $\overrightarrow{YG}_k(V)$ and $LDe^{(1)}(V)$ cannot avoid producing a crossed link, so they are not planar [15], [18]. Table 1 summarizes the above discussion.

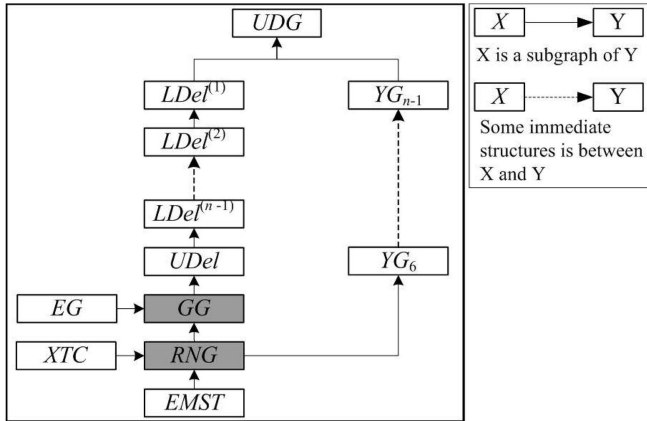


Fig. 2. The relations of the purely localizable structures and their extensions.

From Table 1, we can see that no presented structure can bound or even partially bound the two objectives. Besides, to the best of our knowledge, no other structure can be purely localizable and achieve this goal. Therefore, we will propose the first purely localizable structure, named r -Neighborhood Graph, to fill this gap. This structure is adjustable and can always result in a connected planar with symmetric edges. In addition, we can show that our structure is a generation of both $GG(V)$ and $RNG(V)$.

Apart from the purely localizable structures, several composite methods, based on combining two or more existent structures, were investigated in the last few years [17], [19], [25], [31]. Conceptually, the main idea is to use the virtue of one structure to patch up the fault in the other structures. For examples, the *ordered Yao structure*, denoted as $OrdYao(V)$ [1], is a variation of $YG_k^*(V)$. It has the partially bounded maximum node degree and length stretch factor. However, the planarity cannot be guaranteed. Therefore, Li and Wang [19], [31] applied $OrdYao(V)$ onto $LDel^{(2)}(V)$ to avoid the crossed edges produced by $OrdYao(V)$; Song et al. [25] improved it by applying the $OrdYao(V)$ on $GG(V)$ using only one-hop information. However, the construction of $OrdYao(Y)$ requires exchanging the computed status as well as partial results between nodes. Consequently, none of them is purely localized or purely localizable.

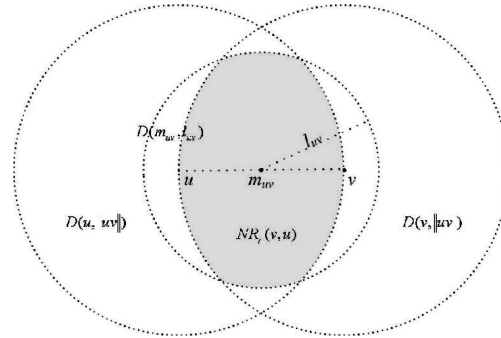


Fig. 3. The r -neighborhood region of nodes u and v .

4 THE r -NEIGHBORHOOD GRAPH

In this section, we introduce a new adjustable structure. First, we define a region on \mathbb{R}^2 . It will be used to compose our structure. Let x be any point on \mathbb{R}^2 and the *open disk* and *circle* centered at $P(x)$ with radius d are denoted as $D(x, d)$ and $C(x, d)$, respectively. The region is defined as follows:

Definition 2. Given a node pair (u, v) on \mathbb{R}^2 , the r -neighborhood region of (u, v) , denoted as $NR_r(u, v)$, is defined as

$$NR_r(u, v) = D(u, \|uv\|) \cap D(v, \|uv\|) \cap D(m_{uv}, l_{uv}),$$

where m_{uv} is the middle point on uv ,

$$l_{uv} = (\|uv\|/2)(1 + 2r^2)^{1/2},$$

and $0 \leq r \leq 1$.

When not confused, we use m and l instead of m_{uv} and l_{uv} , respectively. In Fig. 3, the shaded region intersected by the three open disks sketches an example of the r -neighborhood region. This region is obviously equivalent to the following point set:

$$NR_r(u, v) = \{P(x) \in \mathcal{R}^2 \mid \|ux\| < \|uv\|, \|vx\| < \|uv\|, \|mx\| < l\}. \tag{1}$$

For any node w located on $NR_r(u, v)$, this region limits the power consumed by path uvw . This property is shown in Lemma 2 and derived in the Appendix.

TABLE 1
The Properties of the Four Main Purely Localizable Structures

	Power stretch factor	Maximum node degree	Planar	Symmetric	Connected
$RNG(V)$	Unbounded	Bounded (with AS) Unbounded (without AS)	Yes	Yes	Yes
$GG(V)$	Bounded	Unbounded	Yes	Yes	Yes
$YG_k(V)$	Partially bounded	Unbounded	No	No	Yes
$LDel^{(k)}(V)$	Partially bounded	Unbounded	No ($k=1$) Yes ($k \geq 2$)	Yes	Yes

Lemma 2. *Given two nodes u and v on \mathbb{R}^2 , for any node w such that $P(w) \in NR_r(u, v)$, $p(uvw) < \|uv\|^\alpha (1 + r^\alpha)$, for all $\alpha \geq 2$.*

This lemma explains why we call such a plane a *neighborhood region*: For any node w located in the region $NR_r(u, v)$, it should be an alternative neighbor for u with respect to v in the sense that the power required for relaying from u to v through w is no greater than $1 + r^\alpha$ times the immediate transmission. Based on this region, the structure is defined below:

Definition 3. *Given a set V of nodes on \mathbb{R}^2 , the r -neighborhood graph of V , denoted as $NG_r(V)$, has an edge uv if and only if $\|uv\| \leq 1$ and $NR_r(u, v)$ contains no node $w \in V$, where $0 \leq r \leq 1$.*

By Definition 3, if edge uv is not in $UDG(V)$ or a node w is inside $NR_r(u, v)$, there is no direct link connecting u and v in $NG_r(V)$, which means that all transmissions between u and v should be relayed through some other node(s) in $NG_r(V)$. Now, we explore the desired properties in our structure. Before this, we shall discuss the following relations:

Lemma 3. *For any set V of nodes on \mathbb{R}^2 ,*

$$RNG(V) \subseteq NG_r(V) \subseteq GG(V),$$

for all $0 \leq r \leq 1$.

Proof. Consider the open disk $D(m, \|uv\|/2)$ defining $GG(V)$. Suppose $uv \in NG_r(V)$, the region $NR_r(u, v)$, has no node inside. Since $D(m, \|uv\|/2)$ is obviously a subregion of $NR_r(u, v)$, for any $0 \leq r \leq 1$, there is also no node in $D(m, \|uv\|/2)$. Therefore, according to the definition of $GG(V)$, we get $uv \in GG(V)$. On the other hand, consider the two open disks $D(u, \|uv\|)$ and $D(v, \|uv\|)$ defining $RNG(V)$. Suppose $uv \in RNG(V)$, no node is inside the intersection of $D(u, \|uv\|)$ and $D(v, \|uv\|)$, which obviously covers the region $NR_r(u, v)$, for any $0 \leq r \leq 1$. Therefore, no node can be inside $NR_r(u, v)$ and we get $uv \in NG_r(V)$. \square

Specifically, as $r = 0$, $NR_0(u, v) \equiv D(m, \|uv\|/2)$, which is the disk defining $GG(V)$. On the contrary, as $r = 1$, $NR_1(u, v) \equiv D(m, \|uv\|)$, which is the disk defining $RNG(V)$. Therefore, $GG(V) \equiv NG_0(V)$ and $RNG(V) \equiv NG_1(V)$. So, we can conclude the following theorem:

Theorem 2. *The r -neighborhood graph is a generalized structure of both the restricted Gabriel graph and the restricted relative neighborhood graph.*

Since a subgraph of a planar graph is always planar, and a supergraph of a connected graph is always connected, with the planarity of $GG(V)$ and connectivity of $RNG(V)$, we can infer the following two theorems:

Theorem 3. *For any set V of nodes on \mathbb{R}^2 , $NR_r(V)$ is planar, for all $0 \leq r \leq 1$.*

Theorem 4. *For any set V of nodes on \mathbb{R}^2 , if the underlying $UDG(V)$ is connected, $NR_r(V)$ is connected, for all $0 \leq r \leq 1$.*

Now, we consider the energy efficiency and node degree of $NR_r(V)$. We will show that the upper bound of $\rho(NG_r(V))$ is increased by r and, contrarily, the upper bound of $d_{\max}(NG_r(V))$ is decreased by r . In other words, the r -neighborhood graph is adjustable to the two objectives through the parameter r . With these results, we can further show that the power stretch factor and maximum node degree are partially bounded in our structure. Before these, a property proposed by Li et al. [15] shall be mentioned. It can be used to simplify our proof.

Lemma 4 [15]. *Given a subgraph $G'(V) \subseteq UDG(V)$ and a constant C , $\rho(G'(V)) \leq C$ if and only if, for any edge uv in $G'(V)$, there is a path $\pi(u, v)$ in $G'(V)$ such that $p_{G'(V)}(u, v) \leq C\|uv\|^\alpha$.*

This lemma indicates that to derive an upper bound for $\rho(NG_r(V))$, it is sufficient to consider only those node pairs having direct links in $UDG(V)$. So, we aim to derive a strictly decreasing function $F(r)$, such that, for any uv in $UDG(V)$, a path $\pi(u, v)$ is in $NR_r(V)$ such that $p(\pi(u, v)) \leq F(r)\|uv\|^\alpha$. To achieve this, we investigate an algorithm called **EXPANSION** with an input of any two nodes (u, v) that outputs subgraph S of $NR_r(V)$ related to (u, v) . Let $P(S)$ be the total transmission power of edges in S , i.e., $P(S) = \sum_{st \in S} p(s, t)$. We can show that there is some path in S connecting (u, v) and $P(S) \leq F(r)\|uv\|^\alpha$.

ALGORITHM EXPANSION

Input: A node pair (u, v) in V .

Output: A subgraph S and a positive value P .

Step 1: $S = \{ \}$, $S' = \{ (u, v) \}$, $Q = \{ u, v \}$, $P = \|uv\|^\alpha$;

Step 2: When some node pair (s, t) is in S such that a node $w \in NR_r(s, t)$

$$S' = S' - (s, t);$$

If $w \notin Q$ then

$$S' = S' \cup (s, w) \cup (w, t);$$

$$Q = Q \cup \{ w \};$$

$$P = P + (\|st\|r)^\alpha;$$

Otherwise,

$$S' = S' \cup (s, w);$$

Step 3: $S = \{ xy \in NG_r(V) \mid (x, y) \in S' \}$;

Step 4: Stop and output E and P .

In this algorithm, S' is a set of node pairs in which an edge st in $NR_r(V)$ can be a part of S only if its two ends (s, t) are in S' as described at Step 3. So, to determine S , we have to discuss the S' first. Initially, S' contains only (u, v) . Then, it will be recursively expanded as follows: For each (s, t) in S' , if a node w is in $NR_r(s, t)$ and not considered before, replace (s, t) with (s, w) and (w, t) ; if a node w is in $NR_r(s, t)$ but considered before, replace (s, t) with (s, w) ; otherwise, keep (s, t) unchanged. We use the set Q to record the considered nodes.

When some (s, t) is in S' such that a node $w \in NR_r(s, t)$, no matter whether w is considered or not, by (1), the replaced node pair(s) must be shorter than $\|st\|$, i.e., $\|sw\| < \|st\|$ and $\|wt\| < \|st\|$. Thus, after finite iterations, each node pair in S' can be replaced by another node pair with the shortest distance. So, the algorithm is terminable. Now, we show that (u, v) is connected by some path in the subgraph S when termination occurs.

Lemma 5. *Given any set V of node on \mathbb{R}^2 , for any two nodes u and v in V , if edge uv is in $UDG(V)$ and $UDG(V)$ is connected, there is some path in S connecting (u, v) .*

Proof. Since Q includes u and v , we can prove this lemma by showing that all nodes in Q are connected in S . For each expansion of S' , we define a dummy graph S'' in which an edge st exists if and only if (s, t) is in S' (note that any edge in S'' is not necessarily in either $UDG(V)$ or $NR_r(V)$). First, we show that, at any iteration, all considered nodes in Q are connected by S'' . Initially, Q is connected by S'' , since $S' = \{(u, v)\}$ and $Q = \{u, v\}$. We assume for induction that all nodes in Q are connected by S'' at the k th iteration. Then, we show that it is true for the next iteration. At the $k + 1$ th iteration, if there is no pair in S' that satisfies the entrance condition of Step 2, the claim is correct, since Q and S'' are unchanged; otherwise, a node pair $(s, t) \in S'$ is expanded. In this case, if the chosen $w \notin Q$, w is connected with all nodes in Q via dummy edges sw and wt ; otherwise, $w \in Q$, which implies that all nodes in Q are still connected by S'' as in the previous iteration. As described above, the distance of any expanded node pair is no longer than the previous one. So, if uv is in $UDG(V)$, all edges in S'' are also in $UDG(V)$. Then, as the algorithm proceeds to Step 3, no nodes can be in the r -neighborhood region of any node pair in S' . With these two facts, all dummy edges in S'' are also in $NR_r(V)$ at termination. So, S is equivalent to the last S'' . Consequently, if $UDG(V)$ is connected, by Theorem 4, all nodes in the last Q are connected to S . \square

Then, we derive a strictly decreasing function $F(r)$ using the value P in this algorithm.

Lemma 6. *Given any set V of n nodes on \mathbb{R}^2 , for any two nodes u and v in V ,*

$$P(S) \leq F(r)\|uv\|^\alpha \text{ and } F(r) = 1 + (n - 2)r^\alpha,$$

for all $0 \leq r \leq 1$ and $\alpha \geq 2$.

Proof. Let $P(S') = \sum_{(s,t) \in S'} p(s, t)$. We show that $P(S') \leq P$ at each iteration of Step 2. Initially, $S' = \{(u, v)\}$. We can get $P(S') = \|uv\|^\alpha = P$. Then, at the first iteration, if no node w is in $NR_r(u, v)$, the claim remains true since neither P nor S is changed; otherwise, a node w is in $NR_r(u, v)$. Since there is no node except u and v in Q so far, (u, v) is replaced by (v, w) and (w, v) . By Lemma 2,

$$P(vw) + P(wu) \leq P(uv)(1 + r^\alpha) = P + (\|uv\|r)^\alpha.$$

Consequently, the new P remains an upper bound of $P(S')$. We assume for induction that $P(S') \leq P$ at the k th iteration. Then, we prove that the claim is true at the next iteration. If the entrance condition of Step 2 is not satisfied or the chosen $w \notin Q$, it can be proved by the same reasons as in the first iteration. Otherwise, (s, t) is replaced by (s, w) only. By (1), $P(st) \leq P(sw)$, which implies that the unchanged P is still an upper bound of $P(S')$. Besides, (1) further implies that the distance between any two nodes in S' is no greater than $\|uv\|$. So, another upper bound P' can be found by replacing $P = P + (\|st\|r)^\alpha$ by $P' = P' + (\|uv\|r)^\alpha$. Moreover, we can observe that the

situation where w is chosen from some $NG_r(s, t)$ that is not in Q never happens more than $n - 2$ times, since, in this case, the size of Q must be increased by 1. Consequently, $P(S') \leq P \leq P' \leq P(uv) + P(uv)r^\alpha(n - 2)$. Finally, we get $F(r) = (1 + r^\alpha)(n - 2)$. \square

With Lemmas 4, 5, and 6, we can conclude the following theorem:

Theorem 5. *For any set V of n nodes on \mathbb{R} , for all $0 \leq r \leq 1$ and $\alpha \geq 2$,*

$$\rho(NG_r(V)) \leq 1 + r^\alpha(n - 2) = F(r).$$

Although this bound is related to the node size n so that $\rho(NR_r(V))$ cannot be bounded, it can still be constant when r is 0 or sufficiently small, i.e., $\rho(NR_r(V))$ is bounded in some range of r . So, we can make the following conclusion:

Corollary 1. *The power stretch factor of the r -neighborhood graph is partially bounded.*

Consider the maximum node degree of the r -neighborhood graph. Since $NR_r(V)$ consists of all edges in $RNG(V)$, the maximum node degree of $NR_r(V)$ is no less than that of $RNG(V)$. In Section 3, we know that $d_{\max}(RNG(V))$ is not always bounded in any case of V . Thus, $d_{\max}(NG_r(V))$ is also unbounded. Fortunately, Theorem 1 indicates that $d_{\max}(RNG(V))$ is bounded in most cases of V , where AS is assumed. Therefore, in the following theorem, we analyze the maximum node degree of the r -neighborhood graph under assumption AS .

Theorem 6. *For any set V of nodes on \mathbb{R}^2 with assumption AS , for all $0 \leq r \leq 1$,*

$$d_{\max}(NG_r(V)) \leq \lfloor \pi / \sin^{-1}(r/2) \rfloor.$$

Proof. To prove this statement, it is sufficient to show that, in $NG_r(V)$, there are no adjacent edges enclosing an angle less than $2\sin^{-1}(r/2)$. Assume for contradiction that two edges uv and uw in $NG_r(V)$ enclose an angle $\theta < 2\sin^{-1}(r/2)$ at node u , where $w, v \in V$. Without loss of generality, we assume that $\|uw\| < \|uv\|$. With assumption AS , all nodes are placed on different positions, i.e., $P(x) \neq P(y)$, for any two nodes $x, y \in V$.

Consider the length of vw : If $\angle uvw$ is obtuse, it is clear that $\|vw\| < \|uv\|$ (note that $\|vw\|$ cannot be equal to $\|uv\|$, since $P(u) \neq P(w)$); see Fig. 4b. Otherwise, if $\angle uvw$ is not obtuse, $\|vw\|$ is less $\|vw'\|$, where $\|uw'\| = \|uv\|$; see Fig. 4a. By the law of cosines, we have

$$\begin{aligned} \|vw'\|^2 &= \|uw'\|^2 + \|uv\|^2 - 2\|uw'\|\|uv\|\cos\theta \\ &= 2\|uv\|^2 - 2\|uv\|^2\cos\theta \\ &< 2\|uv\|^2 - 2\|uv\|^2\cos(2\sin^{-1}(r/2)). \end{aligned} \quad (2)$$

If $\theta' = 2\sin^{-1}(r/2)$, we get $\sin(\theta'/2) = r/2$. Then, one of the corresponding right-angled triangles is as shown in Fig. 4c. In this case, $\cos\theta' = (2 - r^2)/2$. Thus, we can get that $2\sin^{-1}(r/2) = \theta' = \cos^{-1}((2 - r^2)/2)$. Consequently,

$$\begin{aligned} \text{Equation (2)} &= 2\|uv\|^2 - 2\|uv\|^2\cos(\cos^{-1}((2 - r^2)/2)) \\ &= 2\|uv\|^2 - 2\|uv\|^2((2 - r^2)/2) = \|uv\|^2r^2. \end{aligned} \quad (3)$$

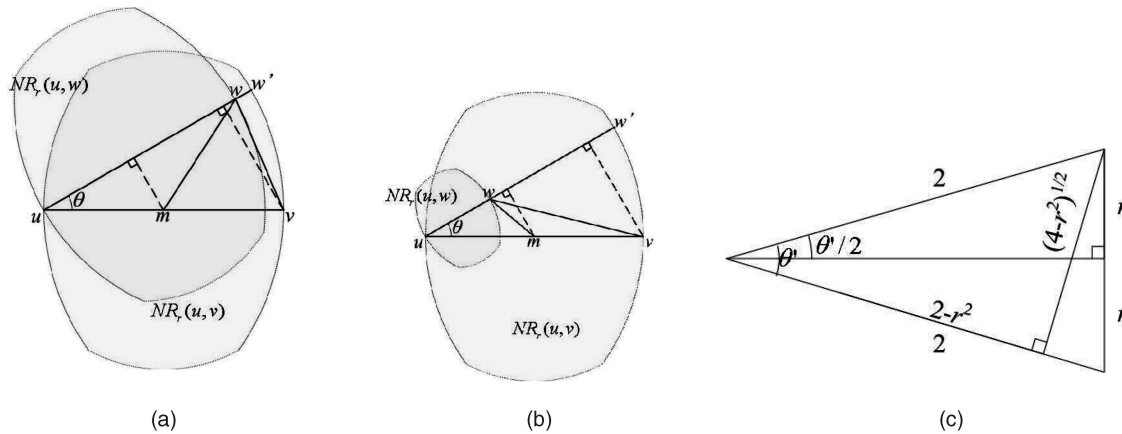


Fig. 4. (a) $\angle uuv$ and $\angle uvm$ are not obtuse. (b) $\angle uuv$ and $\angle uvm$ are obtuse. (c) A right-angled triangle with angle $\theta = 2 \sin^{-1}(r/2)$.

Consequently, we have that, for any case of $\angle uuv$,

$$\|vw\| < \max\{\|uw\|r, \|uv\|\} = \|uv\|. \quad (4)$$

Consider the length of um : If $\angle uvm$ is obtuse, $\|um\| < \|uv\|/2$; see Fig. 4b. Otherwise, $\|mw\|$ is less $\|mw'\|$; see Fig. 4b. By the law of cosine, we have

$$\begin{aligned} \|mw'\|^2 &= \|uw'\|^2 + \|um'\|^2 - 2\|uw'\|\|um'\|\cos\theta \\ &< \|uv\|^2 + \|uv\|^2/4 - \|uv\|^2\cos\theta \\ &< 5\|uv\|^2/4 - \|uv\|^2((2-r^2)/2) = \|uv\|^2((1+2r^2)/4). \end{aligned} \quad (5)$$

Similarly, we have, for any case of $\angle uvm$,

$$\|mw\| < \max\{\|uv\|\sqrt{1+2r^2}/2, \|uv\|/2\} = l. \quad (6)$$

By (4), (6), and the assumption of $\|uw\| < \|uv\|$, w is included in the set of points specified in (1). Therefore, $P(w) \in NR_r(u, v)$. However, it contradicts the assumption that w is in $NR_r(V)$. Thus, we conclude this theorem. \square

However, for those instances of V without AS , Theorem 6 cannot hold anymore. See the instance in Fig. 5, where all nodes except v_i are placed on the outlier of $NR_r(v_i, v_1)$. This will result $n-1$ neighbors adjacent to v_i in $NR_r(V)$.

So, in the next section, we propose an extended version of the r -neighborhood graph. As the readers will see, the extended structure has the partially bounded maximum node degree for all cases of V and inherits almost all of the desired features in $NR_r(V)$.

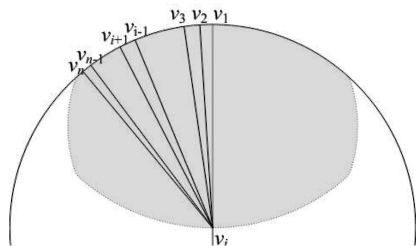


Fig. 5. $d_{\max}(NG_r(V))$ is not bounded if Assumption AS does not hold.

5 THE EXTENDED r -NEIGHBORHOOD GRAPH

In this section, an extended structure of the r -neighborhood graph is given. The main goal is to avoid the unbounded maximum node degree in $NR_r(V)$. In this extension, Assumption AS is not required anymore. Instead, a unique identifier $id(u)$ is available to each node u in V . The structure is defined as follows:

Definition 4. Given a set V of nodes \mathbb{R}^2 , the extended r -neighborhood graph of V , denoted as $NG_r^*(V)$, has an edge uv if and only if $\|uv\| \leq 1$ and there exists no node $w \in V$ satisfying one of the following three conditions:

- D_1 : $P(w) \in NR_r(u, v)$;
- D_2 : $P(w) \in D(m_{uv}, l_{uv}) \cap C(v, \|uv\|)$ and $id(u) > id(w)$;
- D_3 : $P(w) \in D(m_{uv}, l_{uv}) \cap C(u, \|uv\|)$ and $id(v) > id(w)$.

Without D_2 and D_3 , $NG_r^*(V)$ is clearly equivalent to the original r -neighborhood graph. In conditions D_2 and D_3 , the two subregions of $D(m_{uv}, l_{uv})$ intersected by $C(v, \|uv\|)$ and $C(u, \|uv\|)$ are, as depicted in Fig. 6, the solid left arc and the right arc along the outlier of $NR_r(u, v)$, respectively. When a node w is located in these two arcs, the existence of edge wv should be further determined by their identifiers.

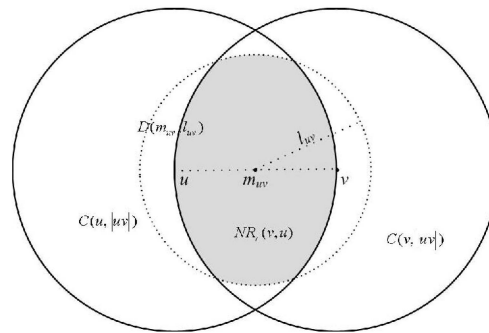


Fig. 6. The r -neighbor region of nodes u and v and the two intersections defined in D_2 and D_3 .

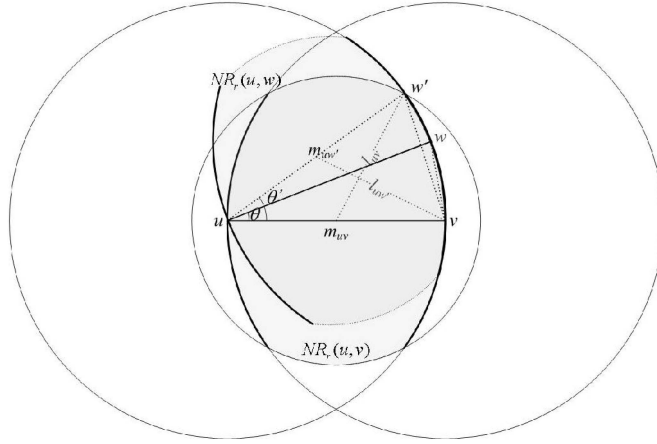


Fig. 7. If $\theta < 2\sin^{-1}(r/2)$ and $\|uw\| = \|uv\|$, either uw or uw cannot be in $NG_r^*(V)$.

Hereafter, we say that a node $w \in V$ blocks an edge uv in $UDG(V)$ if and only if w satisfies one of the three conditions in Definition 4.

In $NG_r^*(V)$, an edge uv of $UDG(V)$ will not only be blocked by some node w in $NR_r(u, v)$, but may also be blocked when either D_2 or D_3 happens. Therefore, $NG_r^*(V)$ constitutes a subgraph of $NR_r(V)$, which means that the maximum node degree of $NG_r^*(V)$ is no worse than its original version. In the following theorem, we show that the upper bound of $d_{\max}(NG_r(V))$ in Theorem 6 remains correct in $d_{\max}(NG_r^*(V))$ and the correctness is, for any case of V , not subject to Assumption AS.

Theorem 7. For any set V of nodes on \mathbb{R}^2 , for all $0 \leq r \leq 1$,

$$d_{\max}(NG_r^*(V)) \leq \left\lceil \frac{\pi}{\sin^{-1}(r/2)} \right\rceil.$$

Proof. Using the same argument as Theorem 6, we assume for contradiction that two edges uv and uw in $NG_r^*(V)$ enclose an angle $\theta' < 2\sin^{-1}(r/2)$ at node u . Without loss of generality, we assume that $\|uw\| \leq \|uv\|$. If $\|uw\| < \|uv\|$, the argument of Theorem 6 has proved the contradiction. Consider $\|uw\| \leq \|uv\|$: Let w' be a point crossed by $C(u, \|uv\|)$ and the outlier of $D(m_{uv}, l_{uv})$, as shown in Fig. 7. The two edges $w'u$ and uw enclose an angle θ . By the law of cosines, we have

$$\begin{aligned} \cos \theta &= \frac{\|uw'\|^2 + (\|uv\|/2)^2 - \|m_{uv}w'\|^2}{\|uw'\|\|uv\|} \\ &= \frac{\|uv\|^2 + (\|uv\|/2)^2 - l_{uv}^2}{\|uv\|\|uv\|} = 1 + r^2/2. \end{aligned}$$

Then, one corresponding right-angle triangulation is as Fig. 4c. In this case, $\sin(\theta/2) = r/2$. Thus, we can get that $\theta < \theta' = 2\sin^{-1}(r/2)$. Since $\|uw\| \leq \|uv\|$, both $P(w)$ and $P(v)$ are on $C(u, \|uv\|)$. The fact that $\theta < \theta'$ further limits $P(w)$ on the arc intersected by $D(m_{uv}, l_{uv})$. Similarly, $P(v)$ is limited on the arc intersected by $D(m_{uv}, l_{uv})$ for the same reason. Therefore, $P(w)$ and $P(v)$ are on the regions defined in D_2 , with respect to edges uv and uw , respectively.

Next, the existence of uv and uw should be determined by their identifiers. If $id(v) > id(w)$, uv is blocked by w . Otherwise, if $id(v) < id(w)$, uw is blocked by v . As a sequel, no matter what the values of $id(v)$ and $id(w)$ are, at least one of the edges enclosing θ' cannot be in $NG_r^*(V)$. Thus, we proved this theorem. \square

From Theorem 7, we can see that $d_{\max}(NG_r^*(V))$ is constant when r is sufficiently large. Therefore, there is some setting of r such that $d_{\max}(NG_r^*(V))$ is bounded by some constant for any set V of n nodes. So, we reach the following conclusion:

Corollary 2. The maximum node degree of the extended r -neighborhood graph is partially bounded.

In the rest, we show that $NG_r^*(V)$ inherits all desired properties achieved by $NR_r(V)$, except the generality for $RNG(V)$. The fact that $NG_r^*(V) \subseteq NG_r(V)$ confirms the planarity of $NG_r^*(V)$, since $NR_r(V)$ is planar for any r . Moreover, when $r = 0$, the two arcs defined in D_2 and D_3 are empty. Thus, whether an edge is in $NG_r^*(V)$ is solely dependent on D_1 , which means that $NG_0^*(V) \equiv NG_0(V) \equiv GG(V)$. Therefore, $NG_r^*(V)$ remains a general structure of $GG(V)$.

However, as shown in Theorem 7, some adjacent edges having the same length in $RNG(V)$ would be avoided in $NG_r^*(V)$. Thus, $RNG(V)$ is not always a subgraph of $NG_r^*(V)$. This means that $NG_1^*(V)$ is not essentially equivalent to $RNG(V)$. Even more, $NG_1^*(V)$ could be a subgraph of $RNG(V)$. Therefore, $NG_r^*(V)$ is no longer a general structure of $RNG(V)$.

About the connectivity, because $RNG(V)$ is not always a subgraph of $NG_r^*(V)$, we cannot ensure the connectivity of $NG_r^*(V)$ directly from that of $RNG(V)$. Therefore, we apply an entirely different logic to prove this property. The idea is based on comparing the lexicographic orders of nodes pairs. This idea has been successfully used to prove the connectivity of $XTC(V)$ [32], another subgraph of $RNG(V)$.

We define a three-field tuple $(\|uv\|, id(u), id(v))$ for each node pair (u, v) . The lexicographic order of (u, v) is smaller than that of another node pair (s, t) if one of the following three cases happens: 1) $\|uv\| < \|st\|$, 2) $\|uv\| = \|st\|$,

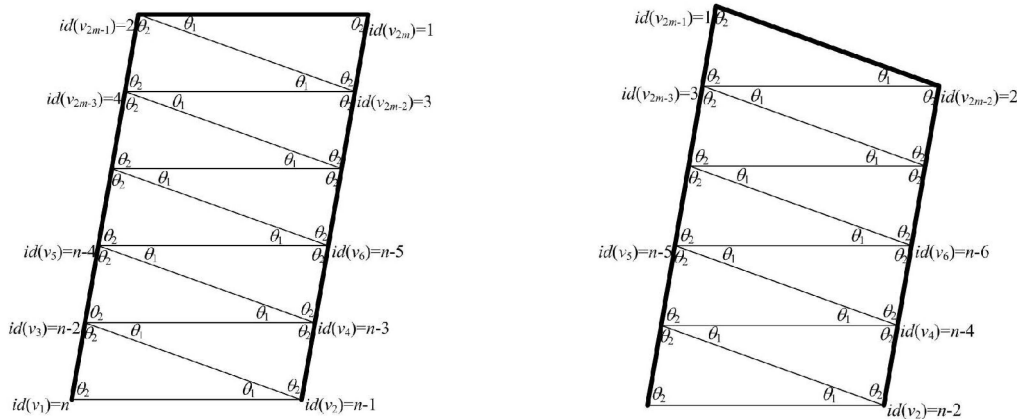


Fig. 8. A worst-case instance V of n nodes in $NG_r^*(V)$: (a) n is even. (b) n is odd.

and $id(u) < id(s)$, or 3) $\|uv\| = \|st\|$, $id(u) = id(s)$ and $id(v) < id(t)$. Now, we prove the connectivity of $NG_r^*(V)$ in Theorem 8.

Theorem 8. For any set V of nodes on \mathbb{R}^2 , if the underlying $UDG(V)$ is connected, $NG_r^*(V)$ is connected, for all $0 \leq r \leq 1$.

Proof. Suppose $UDG(V)$ is connected. Let $U(V)$ be the set of unconnected nodes pairs in $NG_r^*(V)$. We assume for contradiction that some nodes pairs in $NG_r^*(V)$ are not connected, i.e., $U(V)$ is not empty. Let (u, v) be the node pair with smallest lexicographic order in $U(V)$. Assume that edge uv is not in $UDG(V)$, i.e., $\|uv\| > 1$. Since $UDG(V)$ is connected, there must be some path longer than one hop connecting u and v . Let $\pi(u, v)$ be such a path in $UDG(V)$. Since $\|uv\| > 1$, the lengths of each edge on $\pi(u, v)$ is less than $\|uv\|$. When this path is mapped to $NG_r^*(V)$, there are some node pairs on $\pi(u, v)$ unconnected in $NG_r^*(V)$. Thus, some unconnected node pair on $\pi(u, v)$ has length shorter than $\|uv\|$, which, however, contradicts that (u, v) has the smallest lexicographic order in $U(V)$. Therefore, edge uv must be in $UDG(V)$.

Since edge uv is in $UDG(V)$ and not in $NG_r^*(V)$, there must be some node w satisfying one of the three conditions in Definition 4. Besides, either (u, w) or (w, v) is in $U(V)$; otherwise (u, v) can be connected by path uww . We consider the three cases:

1. If D_1 happens, $P(w) \in NR_r(u, v)$. So, we have $\|uw\| < \|uv\|$ and $\|wv\| < \|uv\|$, which means that the lexicographic orders of (u, w) and (w, v) are less than that of (u, v) .
2. If D_2 happens, we have $\|wv\| = \|uv\|$ and $id(u) > id(w)$, which means that the lexicographic order of (w, v) is less than that of (u, v) ;
3. If D_3 happens, we have $\|uw\| = \|uv\|$ and $id(v) > id(w)$, which means that the lexicographic order of (u, w) is less than that of (u, v) .

Therefore, we cannot find any node pair in $U(V)$ having the smallest lexicographic order. In other words, $U(V)$ is empty, which, however, is a contradiction. Thus, we have proven this algorithm. \square

Due to the fact that $NG_r^*(V) \subseteq NG_r(V)$, there may be some paths in $NG_r(V)$ not in $NG_r^*(V)$. Therefore, $\rho(NG_r^*(V))$ is no better or even worse than $\rho(NG_r(V))$. Even so, the upper bound of $\rho_{NG_r^*(V)}(UDG(V))$ can be as good as that proven in Theorem 5. We briefly explain this: All arguments in Theorem 5 are not related to the two additional conditions D_2 and D_3 , except those referred from Lemma 2. Whichever D_1 , D_2 , or D_3 happens, $\|uw\| \leq \|uv\|$, $\|vw\| = \|uv\|$, and $\|mv\| < l$, which means that all inequalities in the proof of Lemma 2 are unchanged. Consequently, Theorem 5 is still correct, even if all conditions of Definition 4 are considered. So, $\rho(NG_r^*(V))$ is also partially bounded.

Below, we show that the bound $1 + r^\alpha(n - 2)$ in Theorem 5 is not only correct, but also *asymptotically tight* to the worst possible value of $\rho_{NG_r^*(V)}(UDG(V))$. In other words, it is very hard to find another upper bound of $\rho_{NG_r^*(V)}(UDG(V))$ better than ours. We apply the same argument as that used to verify the tightness of the length stretch factor [3] and the power stretch factor [15] of $RNG(V)$

Theorem 9. For any $n \geq 2$ and $0 \leq r \leq 1$, there is a set V of n nodes such that

$$\sup_{|V|=n} \rho_{NG_r^*(V)}(UDG(V)) > 1 + r^\alpha(n - 2) - \varepsilon,$$

for any sufficient small $\varepsilon > 0$.

Proof. Let $\theta_1 = 2 \sin^{-1}(r/2) - 2\lambda$ and $\theta_2 = \pi/2 - \sin^{-1}(r/2) + \lambda$, where $\lambda > 0$. We construct a set of n nodes $V = \{v_1, v_2, \dots, v_{2m-1}, v_{2m}, \dots, v_n\}$, where $n \geq 2$ is even and $m = n/2$ as follows:

1. $\|v_1 v_2\| \leq 1$ and $\|v_i v_{i+1}\| = \|v_1 v_2\|$, for $i = 2, 3, \dots, 2m - 1$;
2. $\angle v_i v_{i+1} v_{i+2} = \theta_1$, for $i = 1, 2, \dots, 2m - 2$;
3. $\angle v_{i+2} v_i v_{i+1} = \angle v_i v_{i+2} v_{i+1} = \theta_2$, for $i = 1, 2, \dots, 2m - 2$;
4. $id(v_i) = n - i + 1$, for $i = 1, 2, \dots, n$.

One corresponding $UDG(V)$ is as shown in Fig. 8a. For $i = 1, 2, \dots, 2m - 2$, since $\angle v_i v_{i+1} v_{i+2} = \theta_1 < 2 \sin^{-1}(r/2)$ and $\|v_i v_{i+1}\| = \|v_{i+1} v_{i+2}\|$, by the argument in Theorem 7, we get $P(v_{i+2}) \in D(v_i, v_{i+1}) \cap C(m_{v_i v_{i+1}}, l_{v_i v_{i+1}})$. That is,

$P(v_{i+2})$ is in the regions with respect to edge $v_i v_{i+1}$, defined in D_2 . Moreover, $id(v_i) > id(v_{i+2})$. Thus, edge $v_i v_{i+1}$ is not in $NG_r^*(V)$. Then, the remaining edges are exactly a path (spanning tree) $v_1 v_3 v_5 \dots v_{2m-3} v_{2m-1} v_{2m} v_{2m-2} \dots v_6 v_4 v_2$ of V , connecting all nodes, as the bold links in Fig. 8a. Therefore, we can get that

$$p(\pi_{NG_r^*(V)}^*(v_1, v_2)) = \sum_{i=1}^{2m-2} r^\alpha \|v_i v_{i+2}\|^\alpha + \|v_{2m-1} v_{2m}\|^\alpha.$$

As $\lambda \rightarrow 0$, $\theta_1 \rightarrow 2 \sin^{-1}(r/2)$, which implies that $\|v_i v_{i+2}\| \rightarrow r \|v_1 v_2\| = r \|v_1 v_2\|$ according to (3). Consequently, as $\lambda \rightarrow 0$, we get that

$$\begin{aligned} & \sum_{i=1}^{2m-2} r^\alpha \|v_i v_{i+2}\|^\alpha + \|v_{2m-1} v_{2m}\|^\alpha \\ & \rightarrow \sum_{i=2}^{2h-2} r^\alpha \|v_1 v_2\|^\alpha + \|v_1 v_2\|^\alpha \\ & = \|v_1 v_2\|^\alpha ((n-2)r^\alpha + 1). \end{aligned}$$

On the other hand, since $\|v_1 v_2\| \leq 1$, we get

$$p(\pi_{UDG(V)}^*(u, v)) = \|uv\|^\alpha.$$

Therefore, as $\lambda \rightarrow 0$, $\rho_{NG_r(V)}(UDG(V)) \rightarrow 1 + r^\alpha(n-2)$. That is, $\sup_{|V|=n} \rho_{NG_r(V)}(UDG(V)) > 1 + r^\alpha(n-2) - \varepsilon$, for any sufficient $\varepsilon > 0$. For any odd $n \geq 2$, the result can be obtained by applying the same argument to the instance as shown in Fig. 8b. So, we proved this theorem. \square

Actually, an equivalent structure of $NG_r^*(V)$, without an original version like $NR_r(V)$, was mentioned in our previous paper³ [9]. In that preliminary work, however, only qualitative results were given. To prove the quantitative results, we separate $NR_r(V)$ from $NG_r^*(V)$ in this paper because $NR_r(V)$ has a clearer form in definition that can be used to highlight the main tricky parts in our derivations. Besides, all qualitative results in [9] are reevaluated here using different arguments.

6 PURELY LOCALIZED ALGORITHM

In this section, we propose an efficient purely localized algorithm, named *PLA*, to construct the r -neighborhood graph. This algorithm consists of two main procedures, *GETINF* and *FINDNB*. First, *GETINF* collects a set of nodes' information within one-hop distance, denoted as IN_u . Then, the collected information will be fed into *FINDNB* to determine a set of neighbors in $NR_r(V)$, denoted as NB_u .

ALGORITHM *PLA*

Input: A ratio $0 \leq r \leq 1$.

Output: A set of neighbors adjacent to u .

Step 1: $IN_u := \text{GETINF}(u, r)$;

Step 2: $NB_u := \text{FINDNB}(u, r, IN_u)$;

Step 3: Stop and output NB_u ;

3. The term " r -neighborhood graph" in [9] does not refer to the original version in Definition 3, but to the extended version in Definition 4. In this paper, we reuse the same term to name the original version and rename the previous structure in [9] the extended version.

To collect the one-hop information, the simplest way is to let each node broadcast its information at the maximum transmission range 1 and gather the information from others. However, the severe path loss and the frequent change in topology may cause considerable power in such transmission. Therefore, in *GETINF*, we aim to reduce the transmission range during construction. The main idea is to incrementally raise the transmission power from a small range and then use some rule to stop the increment earlier before the transmission range 1 is reached. The detailed steps are explained as follows: The transmission range is initiated at a small distance d_0 , and then it will be incrementally raised for several rounds. Let d_1 and d_2 be the previous and the current transmission ranges of a round, respectively. In each round, a node broadcasts a request to distance d_2 and waits for the responses from the receiving nodes to gather the nodes' information. To avoid replying to a node for the second time, the request of a node u contains the position $P(u)$ and the previous distance d_1 . As a node v receives this request, it calculates the euclidean distance $\|uv\|$. Then, if $\|uv\| > d_1$, v responds with its information, $P(v)$, to u at distance $\|uv\|$; otherwise, it just neglects the request. In each round, the range is increased by multiplying $\sqrt[3]{2}$, which means the transmission power is multiplied by 2 each time. The process is continued until the following stopping criterion is satisfied. Let v_1 and v_2 be two crossed points intersected by $C(u, \|uv\|)$ and $C(m, l)$; see Fig. 9a. We define $SC(u, v)$ to be the semicircle enclosed by uv_1 and uv_2 with radius ε , where $\varepsilon > 0$ is a small value less than the distance between any pair of nodes in V . Then, given a distance d , a semicircle $\chi(u, d)$ is defined as follows:

$$\chi(u, d) = \bigcup_{\|uv\| \leq d} SC(u, v).$$

We can prove that, if $\chi(u, d)$ is exactly the circle $C(u, \varepsilon)$, like Fig. 9b, then a disk centered at u with d radius can cover all neighbors of u in $NR_r(V)$. In other words, *GETINF* can be halted as $\chi(u, d_2) \equiv C(u, \varepsilon)$. Let $N_u(G(V))$ be the set of neighbors of node u in a graph $G(V)$. This property is proven in Lemma 7.

Lemma 7. *Given a node $u \in V$ and distance $d \in \mathfrak{R}$, if $\chi(u, d) \equiv C(u, \varepsilon)$,*

$$N_u(NG_r(V)) \subseteq \{v \in V | P(v) \in D(u, d)\}.$$

Proof. We assume for contradiction that some node s in $N_u(NG_r(V))$ is not in $\{v \in V | P(v) \in D(u, d)\}$. Since ε is less than the distance between any pair of nodes in V , we get $\|us\| > \varepsilon$. Thus, edge us intersects a point on the circle $C(u, \varepsilon)$. Due to the fact that $\chi(u, d) \equiv C(u, \varepsilon)$, us must intersect at least one semicircle that composes $\chi(u, d)$; see Fig. 9b. Let $SC(u, v)$ be one of the semi-circles intersected by us . Then, us is enclosed by uv_1 and uv_2 in $SC(u, v)$. In other words, $\angle suv \leq \angle v_1 uv$ or $\angle svu \leq \angle vuv_2$. According to the argument in Theorem 7, we can get that $\angle v_1 uv = \angle vuv_2 = 2 \sin^{-1}(r/2)$. Therefore, we have $\angle suv \leq 2 \sin^{-1}(r/2)$. Moreover, since s is not in $\{v \in V | P(v) \in D(u, d)\}$, s must be farther than v from u . So, $P(v) \in NR_r(u, s)$. According to Definition 4, us is not in $NR_r(V)$, which, however, contradicts that s is a neighbor of u in $NR_r(V)$. Thus, we conclude this lemma. \square

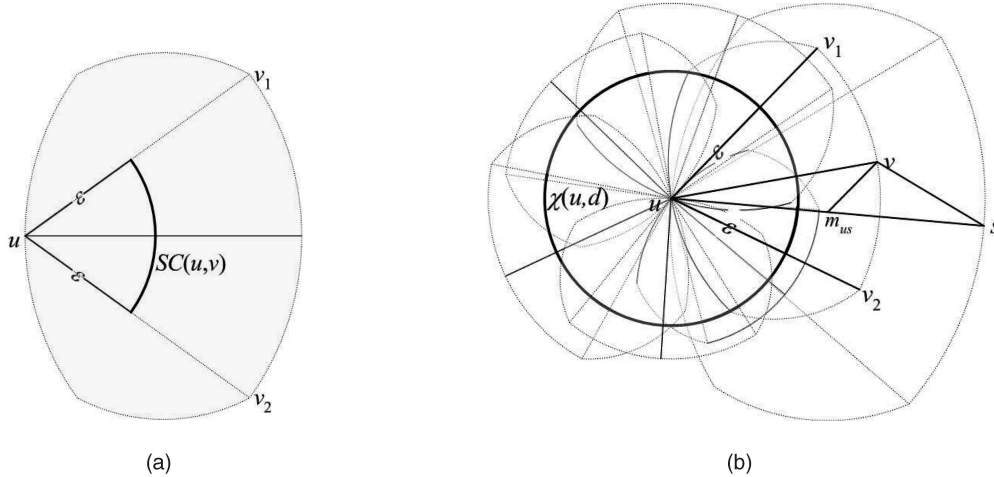


Fig. 9. (a) The semicircle $SC(u, v)$. (b) $\chi(u, d)$ is the union of all $SC(u, v)$, where v is within distance d .

The total transmission power used by GETINF could be as large as $d_0^\alpha(1 + 2^1 + 2^2 + \dots + 2^I)$, where I is the number of rounds. This result could be worse than the maximum transmission power 1 as I is large. Fortunately, when n is large, nodes are closer to and evenly surrounded by each other so that $\chi(u, d)$ has more chance to be quickly shaped as $C(u, \varepsilon)$. So, we can benefit from GETINF in higher probability as the number of nodes increases.

The steps of GETINF are described below. Neglecting the communication overhead at step 2, the execution time of GETINF is dominated by the union operation at step 4. This step can be implemented by some search-and-merge algorithm. Thus, the time complexity of GETINF is $O(n \log n)$.

GETINF(u, r)

- Step 1: $d_1 := 0, d_2 := d_0, IN := \phi, \chi(u, d_2) := \phi$;
 Step 2: Broadcast a request $(P(u), d_1)$ to distance d_2 and gather a set R of responses from nodes within d_1 and d_2 ;
 Step 3: For each $v \in R$ do
 $\chi(u, d_2) := \chi(u, d_2) \cup SC(u, v)$;
 Step 4: $IN := IN \cup R$;
 Step 5: If $d_2 \leq 1$ and $\chi(u, d_2)$ is not the circle $C(u, \varepsilon)$ do
 $d_1 = d_2$;
 $d_2 := d_2 \times 2^{1/\alpha}$;
 Return to step 2;
 Step 6: Stop and output IN ;

Now, we discuss the communication cost of GETINF. As d_0 is multiplied by $\sqrt[\alpha]{2}$ over $\alpha \log_2(1/d_0)$ times, it is larger than 1. Therefore, the number of rounds to increase the transmission range d_2 is dominated by $\alpha \log_2(1/d_0) + 1$. Assume a node's position can be encoded by $\log_2 n$ bits. Each node has to broadcast at most $(\log_2 n)(\alpha \log_2(1/d_0) + 1)$ bits for the request messages. In addition, a node will reply to the same node no more than once. Thus, a node needs at most $(\log_2 n)(n - 1)$ bits to reply to all requests. Combining these results, the communication cost of a node is no more than $(\log_2 n)(\alpha \log_2(1/d_0) + n)$ bits.

Once the information IN_u is collected, node u can start to determine its neighbors in $NR_r(V)$. One instinctive way is

to apply Definition 2 on IN_u directly, as in the following procedure:

- Step 1: $N := IN_u$;
 Step 2: For each node v in N do
 For each node $w \in IN_u$ do
 If $P(w) \in NR_r(u, v)$ do
 $N := N - \{v\}$;
 Step 3: Output N and stop;

In this procedure, the existence of a neighbor v in IN_u is determined by checking whether some node w is located in $NR_r(u, v)$. The correctness is obvious, while, in the worst case, it should take $O(n^2)$ time on each node. This time is usually not tolerable when topology changes frequently. Therefore, we aim to reduce the time complexity in this part. In FINDNB, the main idea is to reverse the original procedure. That is, instead of checking whether some node w can block an edge uv , for each uv , we check whether some edge uv can be blocked by a node w for each w . The procedure is below.

This checking is begun from the farthest to the closet nodes in IN_u . So, we index all elements of IN_u in the nondecreasing order of $\|uw\|$ in Step 2. The set NB contains all candidates that could be a neighbor of u during the process. As a node w is given, we remove from NB all failed candidates that are already blocked by w . After that, w is added into NB to be a new candidate of $N_u(NG_r(V))$. The process continues until all w s in IN_u are considered. Now, we prove the correctness of FINDNB.

FINDNB(u, r, IN_u)

- Step 1: $NB := \phi$;
 Step 2: Index the elements of IN_u in nonincreasing order of $\|uw\|$;
 Step 3: For each node $w \in IN_u$ with *smallest* index do
 For each node $v \in NB$ do
 If $P(w) \in NR_r(u, v)$ do
 $NB := NB - \{v\}$;
 $NB := NB + \{w\}$;
 Step 4: Stop and output NB ;

Theorem 10. For any set V of nodes on \mathbb{R}^2 , $NB_u = N_u(NG_r(V))$, for any $u \in V$.

Proof. We prove this by showing that, for any $v \in V$, $v \in NB_u$ if and only if edge uv is in $NR_r(V)$. Suppose an edge uv is in $NR_r(V)$. By Definition 2, there is no $w \in N_u(UDG(V))$ such that $P(w) \in NR_r(u, v)$. This implies that, once v is added in NB , there is also no $w \in IN_u$ such that v can be removed at Step 3. Since $v \in N_u(NG_r(V)) \subseteq IN_u$ and each node in IN_u can be added to NB , v must be in NB at least one time. So, we can get that v is in the final output of NB_u . Contrarily, we suppose $uv \notin NR_r(V)$. Some node $w \in N_u(UDG(V))$ is located in $NR_r(u, v)$. If $v \notin IN_u$, the result clearly follows by Lemma 7. Otherwise, $v \in IN_u$. In this case, all nodes blocking uv are in IN_u . Besides, every node w blocking uv is always considered after v in GETNB. Therefore, even if v can be added to NB , there must be a node $w \in IN_u$ such that v can be removed from NB at the successive iteration. So we get $v \notin IN_u$. \square

Lemma 7 also implies that, if $uv \in NR_r(V)$, then $v \in N_u$ and $u \in N_v$ and that, if $uv \notin NR_r(V)$, then $v \notin N_u$ and $u \notin N_v$. So, the neighbors (links) determined by GETNB are symmetric.

Corollary 3. Any topology resulted by PLA is symmetric.

Consider the time complexity of FINDNB. Step 2 can be done by some sorting algorithm in $O(n \log n)$. Before a node $w \in IN_u$ is added to NB , any $v \in IN$ blocked by w is removed from NB . Therefore, for any two nodes in NB , neither of them can be blocked by the other. Let s and t be two nodes in NB . The argument of Theorem 7 indicates that, if $\angle sut < 2 \sin^{-1}(r/2)$, then either s blocks t or t block s . Since neither s blocks t nor t blocks s , we get that $\angle sut \geq 2 \sin^{-1}(r/2)$. Therefore, during the process, the size of NB can be never greater than $d_{\max}(NG_r(V))$. Consequently, FINDNB can be done in $O(n \max\{\log n, d_{\max}(NG_r(V))\})$ time. We can observe that this time complexity depends on the parameter r . When r is equal or close to 0 (the worst cases), the time complexity of FINDNB is still $O(n^2)$. However, when r is sufficiently large, such that $d_{\max}(NG_r(V))$ is a constant, FINDNB can be done in $O(n \log n)$.

With a slight modification, PLA can be easily applied on the extended r -neighbors graph and all results can be preserved. We omit the detailed explanation here.

7 CONCLUSION

In this paper, we proposed a purely localized structure to control the topology in wireless networks. We showed that the worst case of the power stretch factor is an increasing function of r and the worst cast of the maximum node degree is contrarily a decreasing function of r . So, the two objectives can be adjusted in our structure. Although the

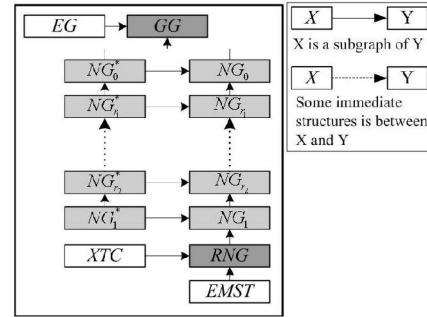


Fig. 10. The relationships of $NG_r(V)$, $NG_r^*(V)$, $GG(V)$, and $RNG(V)$.

power stretch factor is related to n so that our structure is not really a spanner, $\rho(NG_r(V))$ can still be bounded for some range of r . Therefore, the power stretch is partially bounded in our structure. About the maximum node degree, we proposed an upper bound derived for $d_{\max}(NG_r(V))$. However, this result is correct only when no node has two or more neighbors at exactly the same distance. For this reason, an extended structure $NG_r^*(V)$ was given to comprehend this theorem.

Besides, the proposed structure can always result in a connected topology with symmetric edges. Any resulting topology is always planar. The relations between the r -neighborhood graph and existent structures are summarized in Fig. 10. Specially, $NR_r(V)$ is a general structure of both $GG(V)$ and $RNG(V)$.

To construct our structure, we proposed a 1-hop purely localized algorithm, PLA. It can avoid long-distance transmission when collecting information and can be efficiently done in $O(n \log n)$ time when $d_{\max}(NG_r(V))$ is constant.

For further research, a localized topology control approach enables the design of localized routing protocols. For instance, the greedy route discovery in CFG [26] and GPSR [11] are based on GG . We anticipate that the r -neighborhood graph could provide a concrete basis for many interesting extensions due to the sound theoretical results. Moreover, the parameter r can be turned to find the best settings for different scenarios. Another interesting issue for possible further work is to evaluate the stability of the proposed structure when perfect position (range) information is not available or when the accuracy of position information differs from node to node.

APPENDIX

Proof of Lemma 2. Without loss of generality, we assume that $\|uw\| \leq \|vw\|$. Let y be the projection of w on uv so that yw is perpendicular to uv . We can derive that

$$\begin{aligned} \|ym\| &= \frac{\|vw\|^2}{2\|mv\|} - \frac{\|wm\|^2}{2\|mv\|} - \frac{\|mv\|}{2} \quad \text{and} \quad \|yx\| \\ &= \sqrt{\|wv\|^2 - \left(\frac{\|vw\|^2}{2\|mv\|} - \frac{\|wm\|^2}{2\|mv\|} + \frac{\|mv\|}{2} \right)^2}. \end{aligned}$$

Thus,

$$\begin{aligned}
\|uw\|^2 &= \|wy\|^2 + (\|um\| - \|ym\|)^2 \\
&= \|wy\|^2 + (\|mv\| - \|ym\|)^2 \\
&= \|vw\|^2 - \left(\frac{\|wv\|^2}{2\|mv\|} - \frac{\|wm\|^2}{2\|mv\|} + \frac{\|mv\|}{2} \right)^2 \\
&\quad + \left(\frac{3\|mv\|}{2} - \frac{\|wv\|^2}{2\|mv\|} + \frac{\|wm\|^2}{2\|mv\|} \right)^2 \\
&= 2\|mv\|^2 - \|wv\|^2 + 2\|wm\|^2.
\end{aligned}$$

Then, the power consumed by path uvw is as follows:

$$\begin{aligned}
p(uvw) &= \|uw\|^\alpha + \|vw\|^\alpha \\
&= (2\|mv\|^2 - \|wv\|^2 + 2\|wm\|^2)^{\frac{\alpha}{2}} + \|vw\|^\alpha.
\end{aligned}$$

From (1), we get

$$\|wm\| < l = \|uv\| \sqrt{1 + 2r^2} / 2 = \|mv\| \sqrt{1 + 2r^2}$$

and $\|wv\| < \|uv\|$, so

$$\begin{aligned}
&\left(2\|mv\|^2 - \|wv\|^2 + 2\|wm\|^2 \right)^{\frac{\alpha}{2}} + \|vw\|^\alpha \\
&\leq \left((4 + 4r^2)\|mv\|^2 - \|wv\|^2 \right)^{\frac{\alpha}{2}} + \|vw\|^\alpha \\
&\leq \left(\frac{(4 + 4r^2)}{4} \|uv\|^2 - \|uv\|^2 \right)^{\frac{\alpha}{2}} + \|uv\|^\alpha \\
&= \left(r^2 \|uv\|^2 \right)^{\frac{\alpha}{2}} + \|uv\|^\alpha = \|uv\|^\alpha (1 + r^\alpha).
\end{aligned}$$

Thus, we have that $p(uvw) \leq \|uv\|^\alpha (1 + r^\alpha)$. \square

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