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ScienceDirect
Computers & Operations Research 34 (2007) 780–798
computers & operations research
www.elsevier.com/locate/cor

Detecting and adjusting ordinal and cardinal inconsistencies through a graphical and optimal approach in AHP models

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Available online 11 July 2005

Abstract

An AHP model suffering from significant cardinal or/and ordinal inconsistencies in its preference matrix is difficult to rank rationally the alternatives. This study proposes an iterative method to assist a decision maker to detect/adjust inconsistencies and to represent his/her judgments properly. A Gower plot is first used to detect ordinal and cardinal inconsistencies. Two optimization models are then constructed to provide suggeted adjustments upon the request of the decision maker. By examining the Gower plots and numerical suggestions, the decision maker may revise iteratively the preference ratios to improve inconsistencies until all alternatives are ranked.

Keywords: Inconsistency; Gower plot; Preference; Interactive; AHP

1. Introduction

The analytic hierarchy process (AHP) [1] is a popular method for establishing priorities in multicriteria decision problems by evaluating the strength of individual preferences through the pairwise comparison of alternatives at each level of the hierarchy.

Let $A = \{A_i \mid i = 1, \ldots, n\}$ be a set of n alternatives for solving a decision problem. Denote $r_{i,j}$ as $r_{i,j} = w_i/w_j$. The ratio of w_i/w_j measures the relative dominance of A_i over A_j in terms of underlying priority weights $w_1 > 0, \ldots, w_n > 0$, taken to sum up to one by convention. Following Saaty, it is convenient to

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0305-0548/$ - see front matter
doi:10.1016/j.cor.2005.05.010

let $\mathbf{R} = (r_{i,j})$, $i, j \in \{1, \ldots, n\}$, be an $n \times n$ *preference matrix*. $r_{j,i} = 1/r_{i,j}$ is assumed. $\mathbf{R}$ is *ordinally inconsistent* or *intransitive* if for some $i, j, k \in \{1, 2, 3, \ldots, n\}$ there exists $r_{i,j} \geqslant 1$, $r_{j,k} \geqslant 1$ but $r_{i,k} < 1$ (known as *intransitive cycle*). $\mathbf{R}$ is *cardinally inconsistent* if for some $i, j, k \in \{1, 2, 3, \ldots, n\}$ there exists $r_{i,k} \neq r_{i,j} \times r_{j,k}$ [2].

Several methods have been proposed (e.g., [1,3,4]) to rank alternatives in AHP. The ranks they yield do not vary much when the decision makers' preferences are consistent. However, if a preference matrix is ordinally inconsistent or highly cardinally inconsistent, different ranking methods may produce wildly different priorities and rankings. Hence, how to help the decision makers (DMs) to detect and adjust these inconsistencies becomes an important issue.

Genest and Zhang [2] proposed a graphical method to detect the elements in $\mathbf{R}$ that cause major ordinal and cardinal inconsistencies based on a Gower plot [5]. However, they did not propose any systematical way to adjust these inconsistencies.

How to adjust inconsistencies in $\mathbf{R}$ has been addressed by many studies. Most of researchers applied a scaling method for adjusting inconsistencies by using the normalized eigenvector corresponding to the maximum eigenvalue of $\mathbf{R}$ [6–8]. These studies can significantly reduce inconsistencies. However, almost all the elements in the preference matrix will be modified to improve the inconsistencies. Jensen [3] proposed a least squares method to adjust inconsistencies by finding a perfect consistent rank one matrix that minimizes the sum of squared deviations of $r_{i,j}$. Lipovetsky and Conklin [9] considered $\mathbf{R}$ as a contingency table and used χ^2 criteria for localization of outliers among the elicited values. By diminishing the influence of unusual values, the inconsistencies can be reduced. Saaty [10] proposed another method to transform an inconsistent matrix to a near-consistent one based on a principal eigenvector. These methods are mainly based on the eigenvector approach too. Although these methods can largely improve inconsistencies, the adjusted $\mathbf{R}$, however, may be far beyond the real preferences acceptable by the DM. In addition, the DMs should make the final changes, rather than having changes automatically made.

Maas et al. [11] presented an operational method for deriving a linear ranking of alternatives from repeated paired comparisons of those alternatives. In their model, an observed preference between two alternatives that causes intransitivity in the procedure must be reversed if it is of less importance. Larichev and Moshkovich [12] proposed a ZAPROS-LM method for ordering multiattribute alternatives. Their study is based on the assumption of the transitivity of the DM's preferences and the preferential independence of attributes. The DM is asked to change his previous preferences to eliminate intransitivity. Both methods can solve the problems of ordinal inconsistency. However, the problem of cardinal inconsistency has not been addressed.

This study proposes an iterative method to assist a decision maker to detect/adjust inconsistencies and to represent his/her judgments properly. The main features of the proposed method are listed below.

(i) A graphic approach based on Gower plots [2] is applied to represent judgments of the DMs and to detect ordinal and cardinal inconsistencies.
(ii) An optimization model is constructed to help the DMs to adjust ordinal and cardinal inconsistencies simultaneously and efficiently.
(iii) An interactive flow is designed to include the feedbacks from the DMs.
(iv) The DMs may choose to revise their preferences based on the graphical supports and numerical suggestions to improve inconsistencies step by step.
(v) The DMs can control the change process.

The rest of this paper is organized as follows: Section 2 applies three examples to illustrate how to detect inconsistencies based on Gower plots. Section 3 constructs two optimization models to suggest how to efficiently adjust these inconsistencies. Section 4 proposes an interactive flow to help the DMs to detect and adjust inconsistencies and finally rank all alternatives. Section 5 shows the corresponding results and comparisons.

2. Detecting inconsistencies based on Gower plots

Gower plot is a very useful method to graph a skew-symmetric matrix in a two-dimensional (2D) plane. A matrix $\mathbf{M}$ is a skew-symmetric matrix when $\mathbf{M}^{\mathbf{T}} = -\mathbf{M}$, where $\mathbf{M}^{\mathbf{T}}$ denotes the transposition of $\mathbf{M}$. The main idea of Gower plots is to decomposit a skew-symmetric matrix $\mathbf{M}$ into orthonormal eigenvectors by the singular value decomposition technique [13]. Plotting these vectors as points in the plane provides a reasonable representation of $\mathbf{M}$. The mathematical properties of Gower plots are illustrated in the Appendix.

On the basis of Genest and Zhang [2], this section illustrates a way of detecting ordinal and cardinal inconsistencies in a preference matrix $\mathbf{R} = (r_{i,j})$ using Gower [5] plots. Here we first discuss a method of detecting ordinal inconsistencies. Denote $\mathbf{T} = (t_{i,j})$ as a tournament matrix corresponding to $\mathbf{R}$, where

$$t_{i,j} = \begin{cases} 1, & r_{i,j} > 1 \ (A_i \text{ is preferred over } A_j), \\ 0, & r_{i,j} = 1 \ (A_i \text{ and } A_j \text{ are equally preferred}), \\ -1, & r_{i,j} < 1 \ (A_j \text{ is preferred over } A_i). \end{cases}$$

$\mathbf{T}$ is a skew-symmetric matrix and is used to verify the ordinal consistence of $\mathbf{R}$ by Gower plot. Let $\mathbf{T}^*$ be the best approximation of $\mathbf{T}$, based on Appendix. $\mathbf{T}^*$ is expressed as

$$\mathbf{T}^* = \{\lambda_1(u_i v_j - v_i u_j)\} = \{\lambda_1 |P_i||P_j| \sin \theta_{i,j}\},$$

where λ_1 is the largest eigenvalue of $\mathbf{T}$. $P_i = (u_i, v_i)$ stands for the ith point in a 2D plane. $\theta_{i,j}$ denotes the directed angle from points P_i to P_j based on origin. Denote GP($\mathbf{T}$) as a Gower plot of $\mathbf{T}$.

Remark 1. Examining a GP($\mathbf{T}$), $\mathbf{R}$ is close to being ordinally consistent, if (i) the points $P_1, \ldots, P_n$ are equidistant from origin within a 180° arc (half circle); (ii) the angles between two consecutive points are equal to $180/n$ degrees; and (iii) variability $v = \|\mathbf{T}^*\|/\|\mathbf{T}\| = \lambda_1^2 / \sum_{j=1}^{m} \lambda_j^2$, as specified in the appendix, approximates to 1. The points are arranged counterclockwise in the order of preference. GP($\mathbf{T}$) is called an *ordinal Gower plot*.

Remark 1 is derived from and proved by Genest and Zhang [2]. We then have the following definition.

Definition 1 (*Ordinal-inconsistent node*). Plotting the Gower plot of $\mathbf{T}$, GP($\mathbf{T}$), displays the spatial location of alternatives in a 2D plane. Examining the ordinal Gower plot GP($\mathbf{T}$), a node A_i is called an ordinal-inconsistent node if (i) A_i is off the half circle, or (ii) the angles between A_i and its consecutive nodes are not equal to $180/n$ degrees.

Then we discuss a method of detecting cardinal inconsistencies.

Let $\mathbf{S} = (s_{i,j})$, where $s_{i,j} = \ln(r_{i,j})$. $\mathbf{S}$ is then a skew-symmetric matrix. Denote GP($\mathbf{S}$) as a Gower plot of $\mathbf{S}$. Let $\mathbf{S}^*$ be the best approximation of $\mathbf{S}$, based on the Appendix. $\mathbf{S}^*$ is expressed as

$$\mathbf{S}^* = \{\lambda_1(u_i v_j - v_i u_j)\} = \{\lambda_1 |P_i||P_j| \sin\theta_{i,j}\}.$$

Remark 2. Examining a GP($\mathbf{S}$), $\mathbf{R}$ is close to being cardinally consistent if (i) $P_1, \ldots, P_n$ are collinear. This means $s^*_{i,k} + s^*_{k,j} = s^*_{i,j}$, for all $1 \leqslant i, k, j \leqslant n$. (ii) $v = \|\mathbf{S}^*\|/\|\mathbf{S}\| = \lambda_1^2/\sum_{j=1}^m \lambda_j^2$ approximates to 1. The points are arranged counterclockwise in the order of preference. GP($\mathbf{S}$) is called a *cardinal Gower plot* (proved by Genest and Zhang [2]).

Definition 2 (*Cardinal-inconsistent node*). Based on Remark 2, in a cardinal Gower plot GP($\mathbf{S}$), a node A_i is called a cardinal-inconsistent node if A_i is away from the collinear line, i.e. $s^*_{i,k} + s^*_{k,j} \neq s^*_{i,j}$ for some $1 \leqslant k, j \leqslant n$.

Definition 3 (*Degree of cardinal inconsistency*). The degree of cardinal inconsistency for an $\mathbf{R}$ is defined as

$$\sum_{\substack{j=1,\\ j\neq i}}^{n} \sum_{\substack{k>\text{Min}\{i,j\},\\ k\neq i,j}}^{n} |s^*_{i,j} - s^*_{i,k} - s^*_{k,j}|.$$

Three examples are demostrated below.

Example 1. Consider $\mathbf{R}_1$ be a preference matrix with four alternatives A_1, A_2, A_3, and A_4, specified below. Let $\mathbf{T}_1$ be the tournament matrix corresponding to $\mathbf{R}_1$. Let $\mathbf{S}_1 = \ln(\mathbf{R}_1)$.

$$\mathbf{R}_1 = \begin{pmatrix} 1 & 2 & 4 & 5 \\ \frac{1}{2} & 1 & 2 & 5 \\ \frac{1}{4} & \frac{1}{2} & 1 & 3 \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{3} & 1 \end{pmatrix}, \quad \mathbf{T}_1 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{pmatrix},$$

$$\mathbf{S}_1 = \begin{pmatrix} 0 & \ln(2) & \ln(4) & \ln(5) \\ -\ln(2) & 0 & \ln(2) & \ln(5) \\ \ln(4) & -\ln(2) & 0 & \ln(3) \\ -\ln(5) & -\ln(5) & -\ln(3) & 0 \end{pmatrix}.$$

The GP($\mathbf{T}_1$) and GP($\mathbf{S}_1$) are depicted in Fig. 1(a) and (b) with variabilities 97.1% and 99.9%, respectively. The variability stands for the reliability of the graphical representation as defined in the Appendix, and not the degree of consistency.

Examining GP($\mathbf{T}_1$) in Fig. 1(a), on the basis of Remark 1, matrix $\mathbf{R}_1$ is ordinally consistent. Examining GP($\mathbf{S}_1$), on the basis of Remark 2, matrix $\mathbf{R}_1$ is not cardinally consistent because A_4 is away from the collinear line. A_4 is a cardinal-inconsistent node. The consistency ratio (CR)(Saaty [14]) of $\mathbf{R}_1$ is 0.03. Since all points are arranged counterclockwise in the order of preference, the ranking of alternatives is $A_1 \succ A_2 \succ A_3 \succ A_4$. ("$\succ$" means superior to).

Example 2. The second example is selected from Genest and Zhang [2], originally excerpted from Saaty [1]. The six characteristics of the relative desirability for choosing a high school are Learning (A_1),

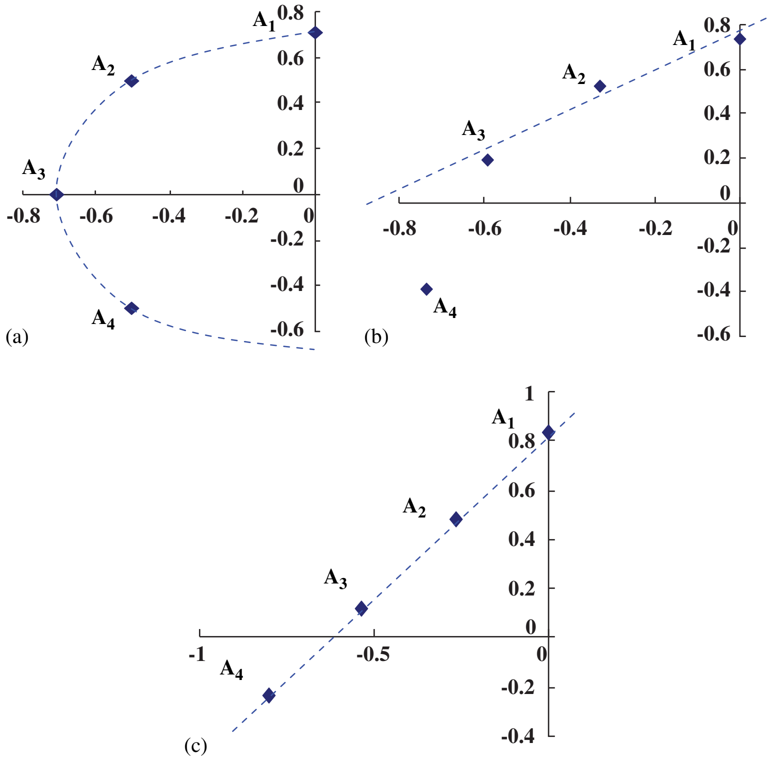

Fig. 1. (a) Ordinal Gower plot of Ex. 1 ($v = 97.1\%$); (b) Cardinal Gower plot of Ex. 1 ($\mathrm{CR} = 0.03,\ v = 99.9\%$); (c) Cardinal plot of Ex. 1 by the proposed approach ($\mathrm{CR} = 0,\ v = 100\%$).

Friends (A_2), School life (A_3), Vocational training (A_4), College preparation (A_5) and Music classes (A_6). The response matrix is given as $\mathbf{R}_2$ with $\mathrm{CR} = 0.223$. $\mathbf{T}_2$ is the tournament matrix corresponding to $\mathbf{R}_2$, and $\mathbf{S}_2 = \ln(\mathbf{R}_2)$. GP($\mathbf{T}_2$) and GP($\mathbf{S}_2$) are shown in Fig. 2(a) and (b), with the variabilities 82.3% and 87.6%, respectively.

$$\mathbf{R}_2 = \begin{pmatrix} 1 & 4 & 3 & 1 & 3 & 4 \\ \frac{1}{4} & 1 & 7 & 3 & \frac{1}{5} & 1 \\ \frac{1}{3} & \frac{1}{7} & 1 & \frac{1}{5} & \frac{1}{5} & \frac{1}{6} \\ 1 & \frac{1}{3} & 5 & 1 & 1 & \frac{1}{3} \\ \frac{1}{3} & 5 & 5 & 1 & 1 & 3 \\ \frac{1}{4} & 1 & 6 & 3 & \frac{1}{3} & 1 \end{pmatrix}, \quad \mathbf{T}_2 = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ -1 & 0 & 1 & 1 & -1 & 0 \\ -1 & -1 & 0 & -1 & -1 & -1 \\ 0 & -1 & 1 & 0 & 0 & -1 \\ -1 & 1 & 1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 1 & -1 & 0 \end{pmatrix}.$$

Examining GP($\mathbf{T}_2$) in Fig. 2(a), $\mathbf{R}_2$ is ordinally inconsistent because Vocational Training (A_4) is off the half circle of the ordinal Gower plot, based on (i) in Remark 1. Here A_4 is an ordinal-inconsistent node. By examining GP($\mathbf{R}_2$) we can see that $\mathbf{R}_2$ is also cardinally inconsistent because Vocational Training (A_4) and School Life (A_3) are far off the collinear line in Fig. 2(b). A_4 and A_3 are cardinal-inconsistent nodes here. The CR of $\mathbf{R}_2$ is 0.223.

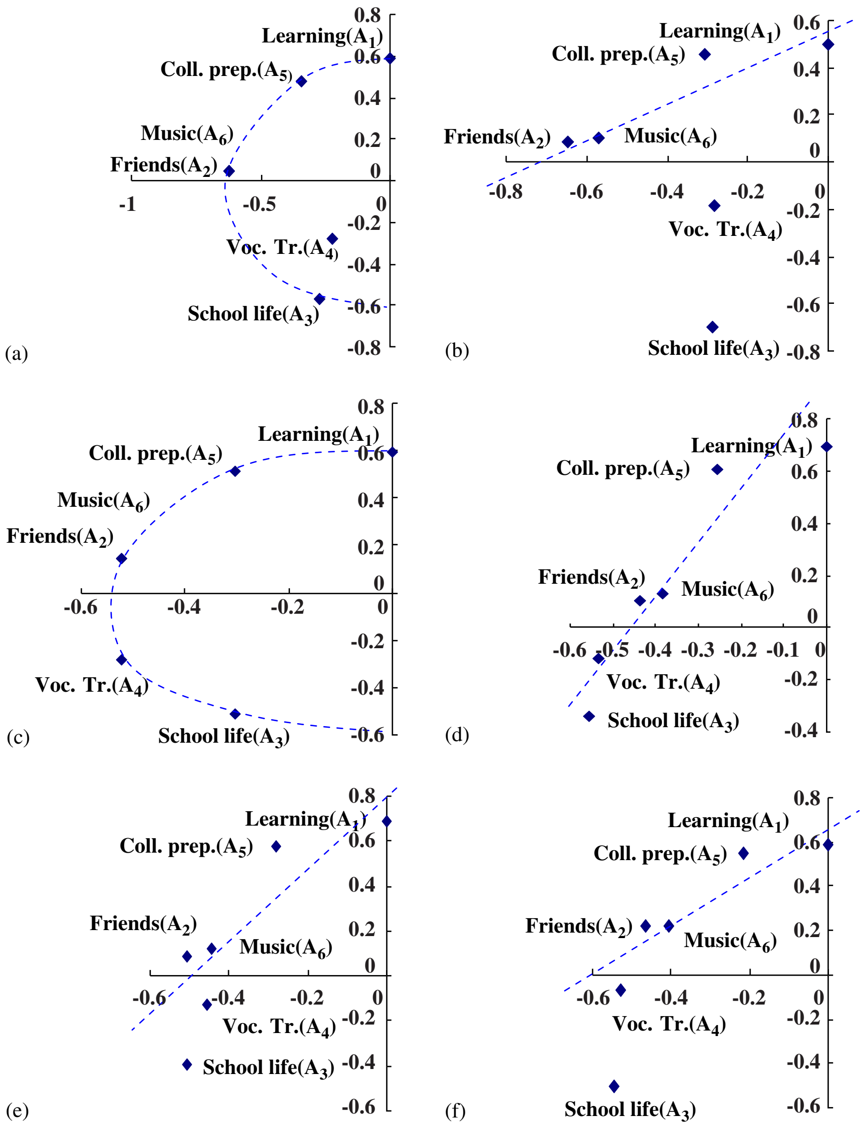

Fig. 2. (a) Ordinal Gower plot of Ex. 2 ($v = 82.3\%$); (b) Cardinal Gower plot of Ex. 2 (CR $= 0.223$, $v = 87.6\%$); (c) Ordinal plot of Ex. 2 by the proposed approach and Genest and Zhang ($v = 95.2\%$); (d) Cardinal plot of Ex. 2 by the proposed approach GO1 (CR $= 0.027$, $v = 99.1\%$); (e) Cardinal plot of Ex. 2 by the proposed approach DM1 (CR $= 0.042$, $v = 97.9\%$); (f) Cardinal plot of Ex. 2 by Genest and Zhang (CR $= 0.078$, $v = 97.8\%$).

Example 3. The third example is a classical AHP problem described by Saaty and Vargas [15], Saaty [16], Lipovetsky and Conklin [9] and Saaty [10]. There are eight criteria for choosing the best home, including Size of house (A_1), Location to bus (A_2), Neighborhood (A_3), Age of house (A_4), Yard space (A_5), Modern facilities (A_6), General condition (A_7) and Financing (A_8). $\mathbf{R}_3$ is a matrix of pairwise comparisons among eight criteria with CR = 0.164. $\mathbf{T}_3$ is the tournament matrix corresponding

$$\mathbf{R}_3=\begin{pmatrix} 1 & 5 & 3 & 7 & 6 & 6 & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{5} & 1 & \frac{1}{3} & 5 & 3 & 3 & \frac{1}{5} & \frac{1}{7} \\ \frac{1}{3} & 3 & 1 & 6 & 3 & 4 & 6 & \frac{1}{5} \\ \frac{1}{7} & \frac{1}{5} & \frac{1}{6} & 1 & \frac{1}{3} & \frac{1}{4} & \frac{1}{7} & \frac{1}{8} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & 3 & 1 & \frac{1}{2} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{4} & 4 & 2 & 1 & \frac{1}{5} & \frac{1}{6} \\ 3 & 5 & \frac{1}{6} & 7 & 5 & 5 & 1 & \frac{1}{2} \\ 4 & 7 & 5 & 8 & 6 & 6 & 2 & 1 \end{pmatrix}, \quad \mathbf{T}_3=\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\ -1 & 0 & -1 & 1 & 1 & 1 & -1 & -1 \\ -1 & 1 & 0 & 1 & 1 & 1 & 1 & -1 \\ -1 & -1 & -1 & 0 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 1 & 0 & -1 & -1 & -1 \\ -1 & -1 & -1 & 1 & 1 & 0 & -1 & -1 \\ 1 & 1 & -1 & 1 & 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

to $\mathbf{R}_3$ and $\mathbf{S}_3=\ln(\mathbf{R}_3)$. Fig. 3(a) and (b) plot the GP($\mathbf{T}_3$) and GP($\mathbf{S}_3$), with 83.99% and 89.01% variabilities.

Examining GP($\mathbf{T}_3$) in Fig. 3(a), based on (ii) in Remark 2, $\mathbf{R}_3$ is not ordinally consistent because there are some angles between two consecutive points not equal to $180/n$. Size (A_1), Neighborhood (A_3) and Condition (A_7) are ordinal-inconsistent nodes because they are located at the same odd position. The CR of $\mathbf{R}_3$ is 0.164. It is clear that if $\mathbf{R}_3$ is ordinally inconsistent then $\mathbf{R}_3$ is also cardinally inconsistent.

3. Proposed model for adjusting inconsistencies

Two models are proposed here to help the DMs to adjust inconsistencies. The first model is adopted when a preference matrix $\mathbf{R}=(r_{i,j})$ has ordinal consistency but also cardinal inconsistency. The second model is applied when $\mathbf{R}=(r_{i,j})$ has both ordinal and cardinal inconsistencies. If $\mathbf{R}$ has ordinal inconsistency then it is also has cardinal inconsistency. Two models for adjusting inconsistencies are discussed here, and an interactive process of ranking all alternatives is illustrated in Section 4.

Given a preference matrix $\mathbf{R}=(r_{i,j})$, let $\mathbf{R}'=(r'_{i,j})$ be the suggested preference matrix of $\mathbf{R}$. Suppose $\mathbf{R}$ is cardinally inconsistent, the model for adjusting $\mathbf{R}$ is formulated below.

Model 1 (A model of adjusting cardinal inconsistence)

$$\text{Min} \quad \text{Obj1}=\sum_{\substack{j=1,\\ j\neq i}}^{n}\ \sum_{\substack{k>\text{Min}\{i,j\},\\ k\neq i,j}}^{n} |x_{i,j}-x_{i,k}-a_{k,j}|$$

$$\text{s.t.} \quad \underline{x_{i,j}}\leqslant x_{i,j}\leqslant \overline{x_{i,j}}, \quad \forall 1\leqslant j\leqslant n, \quad j\neq i, \tag{3.1}$$

where A_i is a cardinal-inconsistent node.

$a_{k,j}=\ln(r_{k,j})$ is a constant where $r_{k,j}$ is obtained directly from the given preference matrix $\mathbf{R}$. $x_{i,j}$ and $x_{i,k}$ are variables which represent the adjusted $\ln(r'_{i,j})$ and $\ln(r'_{i,k})$, respectively. The objective of Model 1 is to minimize the cardinal inconsistency by referring to Definition 3. In expression (3.1), $\underline{x_{i,j}}$ and $\overline{x_{i,j}}$ are, respectively, the lower and upper bounds of $x_{i,j}$, which can be specified by the DM. For example, if the DM specifies $r_{1,4}$ within a tolerable range $3\leqslant r_{1,4}\leqslant 5$ (i.e. $\underline{x_{1,4}}=\ln(3)$, $\overline{x_{1,4}}=\ln(5)$), the proposed model will find the most consistent solutions during the specified ranges. The DM can choose to accept this suggestion to achieve a higher consistent decision, or adjust the range of $r_{i,j}$ to obtain other suggestions, or do nothing to retain his original judgment. If the DM does not wish to adjust some $r_{i,j}$, he can set $r_{i,j}$ to be a specific value (for example $r_{1,4}=4$ by setting $\overline{x_{1,4}}=\underline{x_{1,4}}=\ln(4)$). If the DM does not

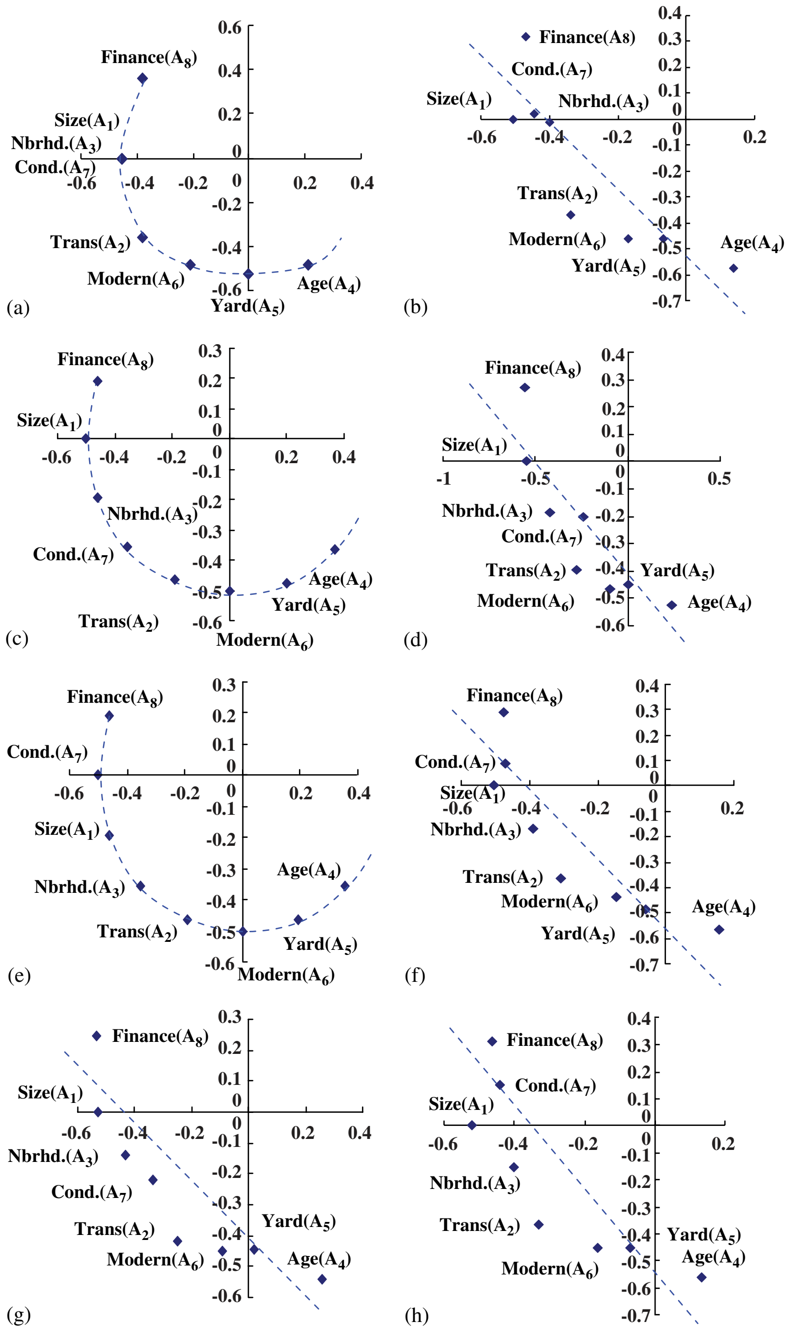

Fig. 3. (a) Ordinal Gower plot of Ex. 3 ($v = 83.99\%$); (b) Cardinal Gower plot of Ex. 3 ($\text{CR} = 0.164,\ v = 89.01\%$); (c) Ordinal Gower plot of Ex. 3 by the proposed approach GO1 ($v = 90.26\%$); (d) Cardinal Gower plot of Ex. 3 by the proposed approach GO1 ($\text{CR} = 0.072,\ v = 97\%$); (e) Ordinal plot of Ex. 2 by the proposed approach GO2 and Saaty ($v = 90.7\%$); (f) Cardinal Gower plot of Ex. 3 by the proposed approach GO2 ($\text{CR} = 0.074,\ v = 97.14\%$); (g) Cardinal Gower plot of Ex. 3 by the proposed approach DM1 ($\text{CR} = 0.099,\ v = 94.2\%$); (h) Cardinal Gower plot of Ex. 3 by Saaty (2003) ($\text{CR} = 0.085,\ v = 97.24\%$).

set any bounds, the default bounds $\underline{x_{i,j}} = \ln(\frac{1}{9})$ and $\overline{x_{i,j}} = \ln(9)$ are used, following the Saaty response scale $[\frac{1}{9} \leqslant r_{i,j} \leqslant 9]$.

The range specified by the DM implies his tolerance about changes in judgment. The wider the range, the higher the possibility for large changes.

Model 1 is a linear program which leads to an optimal solution. Take Example 1 for instance; the alternative A_4 is a cardinal inconsistency node because A_4 is away from the collinear line in Fig. 1(b). All links connecting with A_4 are considered to be adjusted. The lower $(\underline{x_{i,j}})$ and upper $(\overline{x_{i,j}})$ bounds can be set by the DM. Suppose the DM does not set any bounds, the default bounds $\underline{x_{i,j}} = \ln(\frac{1}{9})$ and $\overline{x_{i,j}} = \ln(9)$ are used. The model of adjusting cardinal inconsistencies in Example 1 is formulated as follows:

Model 1 (Adjusting cardinal inconsistency for Example 1)

$$\begin{aligned} \text{Min} \quad \text{Obj1} &= |x_{1,4} - a_{1,2} - x_{2,4})| + |x_{1,4} - a_{1,3} - x_{3,4})| + |x_{2,4} - a_{2,3} - x_{3,4})| \\ &= |x_{1,4} - \ln(2) - x_{2,4}| + |x_{1,4} - \ln(4) - x_{3,4}| + |x_{2,4} - \ln(2) - x_{3,4}| \\ \text{s.t.} \quad & \ln(\tfrac{1}{9}) \leqslant x_{i,j} \leqslant \ln(9). \end{aligned}$$

The optimal solution of this program is $x_{1,4}{=}2.079$ $(r_{1,4}{=}8)$, $x_{2,4}{=}1.386$ $(r_{2,4}{=}4)$, $x_{3,4}{=}0.693$ $(r_{3,4}{=}2)$, with $\text{Obj1} = 0$. The cardinal Gower plot after adjustment is shown in Fig. 1(c), with $v = 100\%$ and $\text{CR} = 0$. That implies if the DM accepts this suggestion, he can make a higher consistent decision. If he feels that the change is too large, he can narrow down the ranges of $r_{i,j}$ and obtain another suggestions or stop adjusting the process.

The adjustment of cardinal inconsistency is to change the magnitudes of some preferences, which can be achieved conveniently by Model 1. However, the adjustment of ordinal inconsistency, which is to change the directions of given preferences, is more difficult to treat. Consider the following remark:

Remark 3. Given a preference matrix $\mathbf{R} = (r_{i,j})$, let $\mathbf{R}' = (r'_{i,j})$ be the suggested preference matrix of $\mathbf{R}$. $r_{i,j}$ is considered to be "reversed" into $r'_{i,j}$ if

(i) $r_{i,j} \geqslant 1$ but $r'_{ij} < 1$,
(ii) $r_{i,j} < 1$ but $r'_{i,j} > 1$.

For example, if $r_{i,j}{=}2$ is adjusted to $r'_{i,j}{=}\frac{1}{3}$, then $r_{i,j}$ is said to be "reversed" into $r'_{i,j}$. Note that the range of $r_{i,j}$ and $r'_{i,j}$ here is based on Saaty response scale $[\frac{1}{9} \leqslant r_{i,j} \leqslant 9]$, i.e. $r_{i,j}, r'_{i,j} \in \{\frac{1}{9}, \frac{1}{8}, \ldots, \frac{1}{2}, 1, 2, \ldots, 9\}$.

The following proposition is proposed based on Remark 3.

Proposition 1. *For a preference matrix* $\mathbf{R} = (r_{i,j})$ *and its suggested preference matrix* $\mathbf{R}' = (r'_{i,j})$, $r_{i,j}$ *is "reversed" into* $r'_{i,j}$ if (i) $r_{i,j} \neq 1$, *but* $\ln(r'_{i,j})\ln(r_{i,j}) < 0$, (ii) $r_{i,j} = 1$, *but* $\ln(r'_{i,j}) < 0$.

Proof. From Remark 3 we know that (i) if $r_{i,j} > 1$ but $r'_{i,j} < 1$, then $\ln(r_{i,j}) > 0$ and $\ln(r'_{i,j}) < 0$. (ii) if $r_{i,j} < 1$ but $r'_{i,j} > 1$, then $\ln(r_{i,j}) < 0$ and $\ln(r'_{i,j}) > 0$. Both (i) and (ii) result in $\ln(r'_{i,j})\ln(r_{i,j}) < 0$. (iii) if $r_{i,j} = 1$ but $r'_{i,j} < 1$, then $\ln(r'_{i,j}) < 0$. □

Based on Proposition 1, every preference relation $r_{i,j}$ can be classified into one of the following six cases:

(1) if $(r_{i,j} \geqslant 2)$ and ($r_{i,j}$ no reverse) then $\ln(r'_{i,j})\ln(r_{i,j}) \geqslant 0$ and $\ln(r'_{i,j}) \geqslant \ln(2)$;

Table 1
Mapping table for 0/1 variables

Case	$u_{i,j}$	$p_{i,j}$	$q_{i,j}$	Note	Constraints
(1)	0	1	0	$r_{i,j} \geqslant 2$ and no reverse	$\ln(r'_{i,j})\ln(r_{i,j}) \geqslant 0$ and $x_{i,j} \geqslant \ln(2)$
(2)	0	0	0	$r_{i,j} = 1$ and no reverse	$\ln(r'_{i,j}) \geqslant 0$
(3)	0	0	1	$r_{i,j} \leqslant \frac{1}{2}$ and no reverse	$\ln(r'_{i,j})\ln(r_{i,j}) \geqslant 0$ and $\ln(r'_{i,j}) \leqslant \ln(\frac{1}{2})$
(4)	1	1	0	$r_{i,j} \geqslant 2$ and "reverse"	$\ln(r'_{i,j})\ln(r_{i,j}) < 0$ and $\ln(r'_{i,j}) \leqslant \ln(\frac{1}{2})$
(5)	1	0	0	$r_{i,j} = 1$ and "reverse"	$\ln(r'_{i,j}) \leqslant \ln(\frac{1}{2})$
(6)	1	0	1	$r_{i,j} \leqslant \frac{1}{2}$ and "reverse"	$\ln(r'_{i,j})\ln(r_{i,j}) < 0$ and $\ln(r'_{i,j}) \geqslant \ln(2)$

(2) if ($r_{i,j} = 1$) and ($r_{i,j}$ no reverse) then $\ln(r'_{i,j}) \geqslant 0$;
(3) if ($r_{i,j} \leqslant \frac{1}{2}$) and ($r_{i,j}$ no reverse) then $\ln(r'_{i,j})\ln(r_{i,j}) \geqslant 0$ and $\ln(r'_{i,j}) \leqslant \ln(\frac{1}{2})$;
(4) if ($r_{i,j} \geqslant 2$) and ($r_{i,j}$ reverse) then $\ln(r'_{i,j})\ln(r_{i,j}) < 0$ and $\ln(r'_{i,j}) \leqslant \ln(\frac{1}{2})$;
(5) if ($r_{i,j} = 1$) and ($r_{i,j}$ reverse) then $\ln(r'_{i,j}) \leqslant \ln(\frac{1}{2})$;

(6) if ($r_{i,j} \leqslant \frac{1}{2}$) and ($r_{i,j}$ reverse) then $\ln(r'_{i,j})\ln(r_{i,j}) < 0$ and $\ln(r'_{i,j}) \geqslant \ln(2)$. (3.2)

Based on Saaty's $[\frac{1}{9}, 9]$ ratio scale, $r_{i,j} > 1$ indicates $r_{i,j} \geqslant 2$ and $r_{i,j} < 1$ implies $r_{i,j} \leqslant \frac{1}{2}$. Denote $u_{i,j}$, $p_{i,j}$, and $q_{i,j}$ as three binary variables to represent the six cases, as listed in Table 1. $u_{i,j}$ is used to indicate the reverse condition. If $r_{i,j}$ is reversed, $u_{i,j} = 1$; otherwise, $u_{i,j} = 0$. $p_{i,j}$ and $q_{i,j}$ are used to distinguish the value of $r_{i,j}$. $p_{i,j} = 1$ and $q_{i,j} = 0$ indicate $r_{i,j} \geqslant 2$. $p_{i,j} = 0$ and $q_{i,j} = 0$ indicate $r_{i,j} = 1$. $p_{i,j} = 0$ and $q_{i,j} = 1$ indicate $r_{i,j} \leqslant \frac{1}{2}$. Consider the following proposition.

Proposition 2. The "If ... then" conditions in (3.2) can be formulated as the following inequalities:

$$\textit{Case}\ (1): M(u_{i,j} - p_{i,j} + q_{i,j} + 1) + \ln(r'_{i,j})\ln(r_{i,j}) \geqslant 0,$$
$$M(u_{i,j} - p_{i,j} + q_{i,j} + 1) + \ln(r'_{i,j}) \geqslant \ln(2), \tag{3.3}$$

$$\textit{Case}\ (2): M(u_{i,j} + p_{i,j} + q_{i,j}) + \ln(r'_{i,j}) \geqslant 0, \tag{3.4}$$

$$\textit{Case}\ (3): M(u_{i,j} + p_{i,j} - q_{i,j} + 1) + \ln(r'_{i,j})\ln(r_{i,j}) \geqslant 0,$$
$$- M(u_{i,j} + p_{i,j} - q_{i,j} + 1) + \ln(r'_{i,j}) \leqslant \ln(\tfrac{1}{2}), \tag{3.5}$$

$$\textit{Case}\ (4): - M(-u_{i,j} - p_{i,j} + q_{i,j} + 2) + \ln(r'_{i,j})\ln(r_{i,j}) < 0,$$
$$- M(-u_{i,j} - p_{i,j} + q_{i,j} + 2) + \ln(r'_{i,j}) \leqslant \ln(\tfrac{1}{2}), \tag{3.6}$$

$$\textit{Case}\ (5): -M(-u_{i,j} + p_{i,j} + q_{i,j} + 1) + \ln(r'_{i,j}) \leqslant \ln(\tfrac{1}{2}), \tag{3.7}$$

$$Case\ (6): -M(-u_{i,j}+p_{i,j}-q_{i,j}+2)+\ln(r'_{i,j})\ln(r_{i,j})<0,$$

$$M(-u_{i,j}+p_{i,j}-q_{i,j}+2)+\ln(r'_{i,j})\geqslant \ln(2), \tag{3.8}$$

$$M(-p_{i,j}+q_{i,j}+1)+r_{i,j}\geqslant 2, \tag{3.9}$$

$$-M(p_{i,j}-q_{i,j}+1)+r_{i,j}\leqslant \tfrac{1}{2}, \tag{3.10}$$

$$M(p_{i,j}+q_{i,j})+r_{i,j}\geqslant 1, \tag{3.11}$$

$$-M(p_{i,j}+q_{i,j})+r_{i,j}\leqslant 1, \tag{3.12}$$

$$p_{i,j}+q_{i,j}\leqslant 1, \tag{3.13}$$

$$u_{i,j}, p_{i,j}, q_{i,j} \in \{0,1\},\ \ M \text{ is a big value.} \tag{3.14}$$

Proof. When $r_{i,j}\geqslant 2$ and $r_{i,j}$ *is not reversed* ($u_{i,j}=0$, $p_{i,j}=1$, $q_{i,j}=0$), Case (1) is activated, which enforces $\ln(r'_{i,j})\ln(r_{i,j})\geqslant 0$ and $\ln(r'_{i,j})\geqslant \ln(2)$. Case (2)–Case (6) can be activated correspondingly. Expressions (3.9)–(3.12) are used to define the relations among $p_{i,j}$, $q_{i,j}$ and $r_{i,j}$. For example, in (3.9), $p_{i,j}=1$ and $q_{i,j}=0$ enforce $r_{i,j}\geqslant 2$. (3.11) and (3.12) enforce $r_{i,j}=1$ when $p_{i,j}=0$ and $q_{i,j}=0$. (3.13) implies that $p_{i,j}=1$ and $q_{i,j}=1$ are not allowed. □

Remark 4. If A_m is an ordinal-inconsistent node, then there exists at least one intransitive cycle connecting to A_m.

Remark 5. An intransitive cycle can be eliminated if one of the links in the cycle can be reversed. The ordinal inconsistency can therefore be improved.

From the above discussion, if there is a cardinal-inconsistent node A_i and an ordinal-inconsistent node A_m in **R**, the model of adjusting cardinal and ordinal Inconsistencies is formulated below.

Model 2 (A model of adjusting cardinal and ordinal inconsistencies)

$$\text{Min}\quad \{\text{Obj1}, \text{Obj2}\}$$

$$\text{Obj1}=\sum_{\substack{j=1,\\ j\neq i}}^{n}\ \sum_{\substack{k>\text{Min}\{i,j\},\\ k\neq i,j}}^{n} |x_{i,j}-x_{i,k}-a_{k,j}|$$

$$\text{Obj2}=\sum_{\substack{j=1,\\ j\neq m}}^{n} u_{m,j}r_{m,j},$$

$$\text{s.t.}\quad \sum_{\substack{j=1,\\ j\neq m}}^{n} u_{m,j}\leqslant 1, \tag{3.15}$$

(3.1), (3.3)–(3.14).

Obj2 is the weighted sum of reverse preference. Since the model is designed to reverse a DM's preferences as less as possible, Obj2 is minimized. The preference ratio $r_{i,j}$ is considered as a weight in Obj2 to be minimized because it is easier for a DM to reverse his preference with a smaller $r_{i,j}$. For example, it is easier for a DM to change a preference ratio of 3 to $\frac{1}{2}$ than to change a preference ratio of 9 to $\frac{1}{2}$.

If $r_{i,j} < 1$, $r_{j,i}$ should substitute for $r_{i,j}$ in Obj2. (3.15) sets that at most one preference relation $r_{i,j}$ can be reversed at one time.

Model 2 is a multi-objective linear optimization problem which can be solved by many techniques to obtain a global optimum. One of the commonly used methods is formulated below.

$$\begin{array}{ll} \text{Min} & \alpha \\ \text{s.t.} & \text{Obj1} \leqslant \alpha; \quad \text{Obj2} \leqslant \alpha; \end{array}$$

All other constraints are in Model 2.

Take Example 2 for instance, A_4 (an ordinal-inconsistent node) is the one causing the ordinal inconsistency by examining GP($\mathbf{T}_2$) in Fig. 2(a). Examining GP($\mathbf{S}_2$) in Fig. 2(b); A_3 and A_4 (cardinal-inconsistent nodes) cause major cardinal inconsistencies because they are farthest away from the collinear line. Example 2 can be formulated as follows:

Model 2 (Adjusting cardinal and ordinal inconsistency for Example 2)

$$\begin{array}{lll} \text{Min} & \alpha & \\ \text{s.t.} & \text{Obj1} \leqslant & \alpha; \text{Obj2} \leqslant \alpha; \\ & \text{Obj1}= & |x_{1,4} - a_{1,2} - x_{2,4}| + |x_{1,4} - a_{1,3} - x_{3,4}| + |x_{1,4} - a_{1,5} - x_{5,4}| \\ & & +|x_{1,4} - a_{1,6} - x_{6,4}| + |x_{2,4} - a_{2,3} - x_{3,4}| + |x_{2,4} - a_{2,5} - x_{5,4}| \\ & & +|x_{2,4} - a_{2,6} - x_{6,4}| + |x_{3,4} - a_{3,5} - x_{5,4}| + |x_{3,4} - a_{3,6} - x_{6,4}| \\ & & +|x_{5,4} - a_{5,6} - x_{6,4}| + |x_{1,3} - a_{1,2} - x_{2,3}| + |x_{1,3} - x_{1,4} - x_{4,3}| \\ & & +|x_{1,3} - a_{1,5} - x_{5,3}| + |x_{1,3} - a_{1,6} - x_{6,3}| + |x_{2,3} - x_{2,4} - x_{4,3}| \\ & & +|x_{2,3} - a_{2,5} - x_{5,3}| + |x_{2,3} - a_{2,6} - x_{6,3}| + |x_{5,3} - a_{5,6} - x_{6,3}|, \\ & \text{Obj2}= & u_{1,4} \times 1 + u_{2,4} \times 3 + u_{3,4} \times 5 + u_{5,4} \times 1 + u_{6,4} \times 3, \\ & & u_{1,4} + u_{2,4} + u_{3,4} + u_{5,4} + u_{6,4} \leqslant 1, \end{array}$$

(3.3)–(3.13),

$$u_{i,j},\, p_{i,j}, q_{i,j} \in \{0, 1\},\ M = 1000, \quad \forall (i,j) \in \{(1,4), (2,4), (3,4), (5,4), (6,4)\},$$

$$\ln(\tfrac{1}{9}) \leqslant x_{i,j} \leqslant \ln(9),$$
$$\forall (i,j) \in \{(1,4), (2,4), (3,4), (5,4), (6,4), (1,3), (2,3), (5,3), (6,3)\}.$$

Applying Model 2 to Example 2 yields $u_{1,4} = u_{2,4} = u_{3,4} = u_{5,4} = u_{6,4} = 0$, $x_{1,4} = 2.079$ ($r_{1,4} = 8$), $x_{2,4}=0.693$ ($r_{2,4}=2$), $x_{3,4}=-0.693$ ($r_{3,4}=\frac{1}{2}$), $x_{5,4}=1.792$ ($r_{5,4}=6$), $x_{6,4}=0.693$ ($r_{6,4}=2$), $x_{1,3}= 2.197$ ($r_{1,3}=9$), $x_{2,3}=1.099$ ($r_{2,3}=3$), $x_{5,3}=2.197$ ($r_{5,3}=9$), $x_{6,3}=1.099$ ($r_{6,3}=3$), Obj1 = 5.808 and Obj2 = 0. The ordinal and cardinal Gower plots after adjustment are shown in Fig. 2(c) and (d), with variabilities 95.2% and 99.1%, respectively. Model 2 is solved by optimization software [17].

4. The iterative process for ranking alternatives

This section proposes an interactive flow to help the DMs to detect and adjust inconsistencies and finally ranking all alternatives. Based on Gower plots, the DM can visualize the consistencies of his preferences. If there are inconsistencies, the DM may decide to adjust these inconsistencies by himself, or may ask for suggested adjustments and adjust inconsistencies step by step, or make no change. It depends on the DM to control the adjusting process.

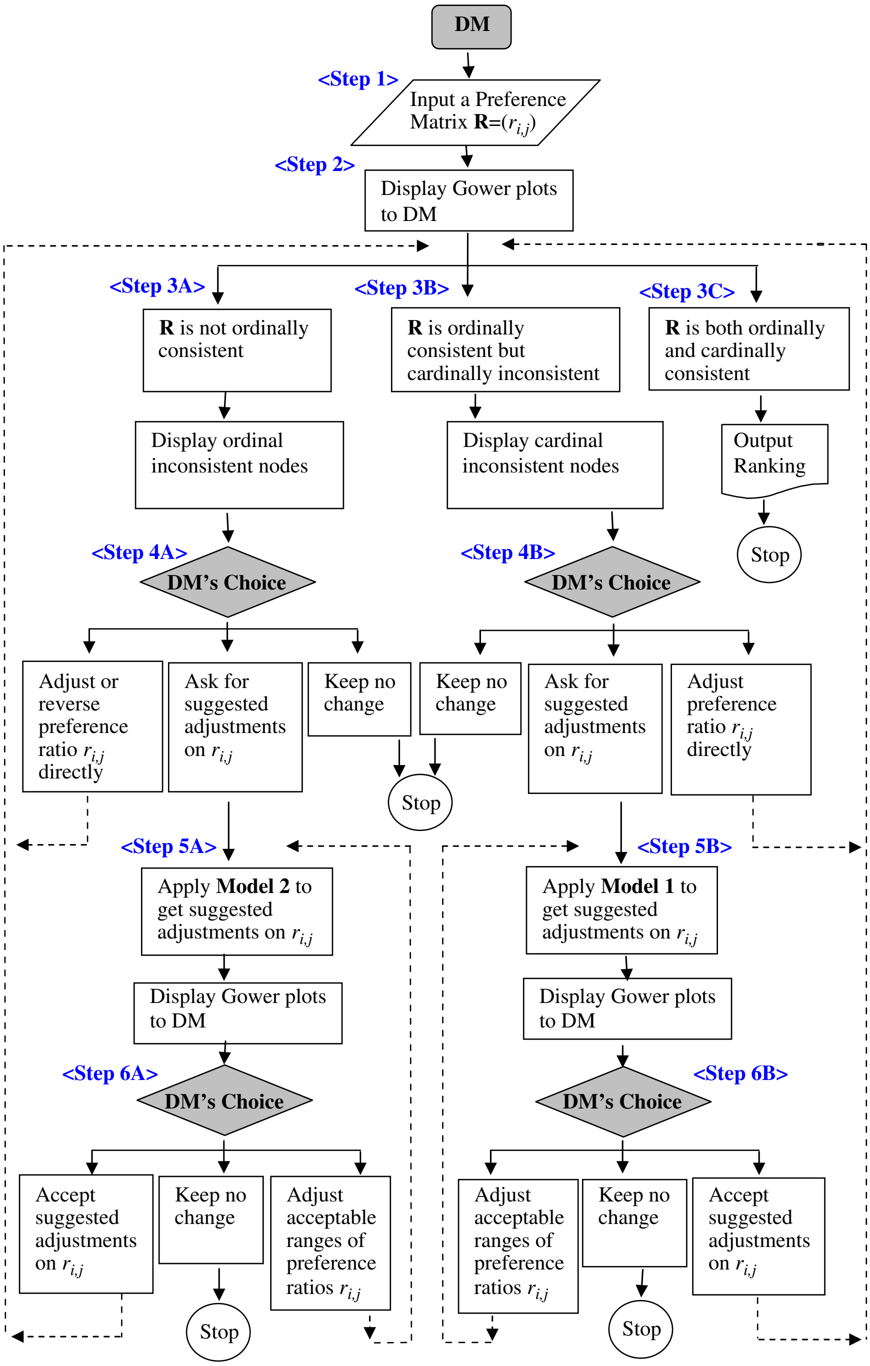

Fig. 4. Flowchart for detecting and adjusting inconsistencies.

The interactive flow is depicted in Fig. 4 and is illustrated as follows:

⟨Step 1⟩ A DM inputs a preference matrix $\mathbf{R} = (r_{i,j})$.

⟨Step 2⟩ An ordinal Gower plot GP($\mathbf{T}$) and a cardinal Gower plot GP($\mathbf{S}$) are displayed to the DM, where $\mathbf{T}$ is a tournament matrix corresponding to $\mathbf{R}$ and $\mathbf{S} = \ln(\mathbf{R})$.

⟨Step 3⟩ $\mathbf{R}$ can be grouped into the following three types.

3A: If $\mathbf{R}$ is not ordinally consistent, the ordinal-inconsistent nodes are displayed to the DM.
3B: If $\mathbf{R}$ is ordinally consistent but cardinally inconsistent, the cardinal-inconsistent nodes are depicted to the DM.
3C: If $\mathbf{R}$ is both ordinally and cardinally consistent, the alternatives are ranked directly. Alternatives are arranged counterclockwise in the order of preference in Gower plots. Stop.

⟨Step 4⟩ The DM decides how to treat inconsistencies.

4A: Following Step 3A, the DM may take one of the following three actions:

(1) Adjust or reverse the targeted preference ratio $r_{i,j}$ by himself directly. Go to Step 3.
(2) Ask for suggeted adjustments on $r_{i,j}$. The DM may wish to find the optimal adjustments to reduce inconsistencies.
(3) Make no change. The DM does not wish to adjust inconsistencies. Stop.

4B: Following Step 3B, the DM may take one of the following three actions :

(1) Adjust the targeted $r_{i,j}$ by himself directly. Go to Step 3.
(2) Ask for suggeted adjustments on $r_{i,j}$.
(3) Make no change. Stop.

⟨Step 5⟩ Apply proposed models to obtain suggested adjustments on $r_{i,j}$.

5A: Following Step 4A, apply Model 2 to obtain the suggested adjustments on $r_{i,j}$. The corresponding Gower plots are displayed to the DM.
5B: Following Step 4B, apply Model 1 to obtain the suggested adjustments on $r_{i,j}$. The corresponding Gower plots are displayed to the DM.

⟨Step 6⟩ The DM decides to accept suggested adjustments, or obtain other suggestions on $r_{i,j}$, or make no change.

6A: Following Step 5A, the DM may take one of the following three actions:

(1) Accept the suggested adjustments. Go to Step 3.
(2) Make no change. Stop.
(3) The DM dose not accept the suggested adjustments on $r_{i,j}$ and ask for other suggestions by adjusting the acceptable ranges of targeted preference ratios $r_{i,j}$. Go to Step 5A.

6B: Following Step 5B, the DM may take one of the following three actions:

(1) Accept suggested adjustments on $r_{i,j}$. Go to Step 3.
(2) Make no change. Stop.

(3) The DM does not accept the suggested adjustments on $r_{i,j}$ and ask for other suggestions by adjusting the acceptable ranges of targeted preference ratios. Go to Step 5B.

Based on the support of Gower plots and the proposed models, the DM may iteratively adjust preference ratios to reduce inconsistencies until consistencies are achieved or the DM may not wish to adjust inconsistencies anymore. The final ranks for all alternatives can then be decided.

5. Results and comparisons

This section takes Examples 2 and 3 as instances to illustrate the interactive approach for detecting and adjusting inconsistencies based on the DM's responses. The comparisons of proposed approach with that of Genest's and Saaty's are also described.

The interactive flow of Example 2 is illustrated as follows:

Step 1: A DM inputs a preference matrix $\mathbf{R}_2$.

Step 2: The corresponding ordinal and cardinal Gower plots are depicted in Fig. 2(a) and (b), respectively.

Step 3: $\mathbf{R}_2$ is not ordinally consistent, which belongs to Step 3A. A_4 is the ordinal-inconsistent node because A_4 is off the half circle in Fig. 2(a). $\mathbf{R}_2$ is also cardinally inconsistent with CR = 0.223.

Step 4A: Suppose the DM decides to ask for suggested adjustments on $r_{i,j}$.

Step 5A: Applying Model 2 to $\mathbf{R}_2$ yields $u_{1,4}=u_{2,4}=u_{3,4}=u_{5,4}=u_{6,4}=0$, with Obj1 = 5.808, Obj2 = 0. As described in Section 3.2, the suggested optimal adjustments are $x_{1,4} = 2.079$ ($r_{1,4} = 8$), $x_{2,4} = 0.693$ ($r_{2,4} = 2$), $x_{3,4} = -0.693$ ($r_{3,4} = \frac{1}{2}$), $x_{5,4} = 1.792$ ($r_{5,4} = 6$), $x_{6,4} = 0.693$ ($r_{6,4} = 2$), $x_{1,3} = 2.197$ ($r_{1,3} = 9$), $x_{2,3} = 1.099$ ($r_{2,3} = 3$), $x_{5,3} = 2.197$ ($r_{5,3} = 9$), $x_{6,3} = 1.099$ ($r_{6,3} = 3$). The CR can be significantly improved from 0.223 to 0.027. The ordinal and cardinal Gower plots after suggested adjustments are shown in Figs. 2(c) and (d).

Step 6A: If the DM accepts the suggested adjustments, the process goes to Step 3C since $\mathbf{R}_2$ is both ordinally and cardinally consistent. The ranking of the relative desirability of sex characteristics for choosing a high school is Learning (A_1) $\succ$ Colleage preparation (A_5) $\succ$ Music (A_6) $\succ$ Friends (A_2) $\succ$ Vocational training (A_4) $\succ$ School life (A_3). The results are listed in Table 2 at the column labeled GO1.

If the DM does not accept these adjustments, he can ask for other suggestions by respecifying acceptable ranges for targeted preference ratios. Suppose he sets some acceptable ranges as $1 \leqslant r_{1,4} \leqslant 5$, $1 \leqslant r_{5,4} \leqslant 3$, $3 \leqslant r_{1,3} \leqslant 6$, $5 \leqslant r_{5,3} \leqslant 7$, applying Model 2 (Step 5A) based on the new ranges ($\ln(1) \leqslant x_{1,4} \leqslant \ln(5)$, $\ln(1) \leqslant x_{5,4} \leqslant \ln(3)$, $\ln(3) \leqslant x_{1,3} \leqslant \ln(6)$, $\ln(5) \leqslant x_{5,4} \leqslant \ln(7)$) yields $u_{1,4} = u_{2,4} = u_{3,4} = u_{5,4} = u_{6,4} = 0$, with Obj1 = 9.116, Obj2 = 0. The suggested adjustments are $x_{1,4} = 1.609$ ($r_{1,4} = 5$), $x_{2,4} = 0.693$ ($r_{2,4} = 2$), $x_{3,4} = -0.693$ ($r_{3,4} = \frac{1}{2}$), $x_{5,4} = 1.099$ ($r_{5,4} = 3$), $x_{6,4} = 0.693$ ($r_{6,4} = 2$), $x_{1,3} = 1.792$ ($r_{1,3} = 6$), $x_{2,3} = 1.099$ ($r_{2,3} = 3$), $x_{5,3} = 1.792$ ($r_{5,3} = 6$), $x_{6,3} = 1.099$ ($r_{6,3} = 3$). The results are listed in Table 2 in the column labeled as DM1, where CR = 0.042. The corresponding cardinal Gower plot is shown in Fig. 2(e). The DM can adjust his acceptable ranges again to improve inconsistencies step by step.

Genest and Zhang [2] suggested that $r_{1,4}$, $r_{5,4}$ should be revised first because Learning (A_1) and College preparation (A_5) are farthest away from Vocational training (A_4) on the cardinal Gower plot. They set both $r_{1,4}$, $r_{5,4}$ from 1 to 5 based on the work of Jensen [18]. Then, they suggested revising $r_{1,3}$, $r_{5,3}$ from 3 and 5 upward to 9. CR can be improved from 0.223 to 0.078. The results are listed in Table 2. Fig. 2(f) depicts the cardinal Gower plot after adjustment with 97.8%.

Table 2
Results and comparisons of Example 2

Method	Original	Genest	GO1	DM1
Ordinal-inconsistent node(s)			A_4	A_4
Cardinal-inconsistent node(s)		A_4 then A_3	A_4 and A_3	A_4 and A_3
Boundary set by DM				$1 \leqslant r_{1,4} \leqslant 5$; $1 \leqslant r_{5,4} \leqslant 3$; $3 \leqslant r_{1,3} \leqslant 6$; $5 \leqslant r_{5,3} \leqslant 7$;
Original preferences/suggested adjustments	$r_{1,4}=1$; $r_{2,4}=3$; $r_{3,4}=\frac{1}{5}$; $r_{5,4}=1$; $r_{6,4}=3$; $r_{1,3}=3$; $r_{2,3}=7$; $r_{5,3}=5$; $r_{6,3}=6$;	$r_{1,4}=5$; $r_{5,4}=5$; $r_{1,3}=9$; $r_{5,3}=9$;	$r_{1,4}=8$; $r_{2,4}=2$; $r_{3,4}=\frac{1}{2}$; $r_{5,4}=6$; $r_{6,4}=2$; $r_{1,3}=9$; $r_{2,3}=3$; $r_{5,3}=9$; $r_{6,3}=3$;	$r_{1,4}=5$; $r_{2,4}=2$; $r_{3,4}=\frac{1}{2}$; $r_{5,4}=3$; $r_{6,4}=2$; $r_{1,3}=6$; $r_{2,3}=3$; $r_{5,3}=6$; $r_{6,3}=3$;
Cardinal consistency (measured by Obj 1)	23.739	15.46	5.808	9.116
Ordinal consistency (measured by Obj 2)		0	0	0
CR	0.223	0.078	0.027	0.042

The cardinal consistency can be measured by both Obj1 and CR. The smaller Obj1 implies that the cardinal Gower plot is closer to being collinear which yields a higher cardinal consistency. Following Saaty, the smaller CR indicates a higher consistency. The proposed approach can improved both CR and Obj1 more significantly than Genest and Zhang's. Besides, the proposed approach is more flexible because the DM can set his acceptable ranges for adjusted preferences and see the corresponding optimal adjustments and Gower plots immediately and iteratively.

The interactive flow of Example 3 is illustrated as follows:

Step 1: A DM inputs a preference matrix $\mathbf{R}_3$.

Step 2: The corresponding ordinal and cardinal Gower plots are depicted in Fig. 3(a) and (b), respectively.

Step 3: $\mathbf{R}_3$ is not ordinal consistent (Step 3A) because there are some angles between two consecutive points not equal to $180/n$. Size (A_1), Neighborhood (A_3) and Condition (A_7) can be arbitrarily chosen as a revise node because they are located at the same odd position. In order to compare the results with that of Saaty's [10], Condition (A_7) is selected as an ordinal-inconsistent node first.

Step 4A: Suppose the DM decides to ask for suggested adjustments on $r_{i,j}$.

Step 5A: Applying Model 2 to Example 3 yields $u_{1,7}=1$, $u_{2,7}=u_{3,7}=u_{4,7}=u_{5,7}=u_{6,7}=u_{8,7}=0$, $x_{1,7}=0.916$ ($r_{1,7}=2$), $x_{2,7}=-0.693$ ($r_{2,7}=\frac{1}{2}$), $x_{3,7}=0.693$ ($r_{3,7}=2$), $x_{4,7}=-1.974$ ($r_{4,7}=(\frac{1}{7})$, $x_{5,7}=-0.875$ ($r_{5,7}=\frac{1}{2}$), $x_{6,7}=-0.693$ ($r_{6,7}=\frac{1}{2}$), $x_{8,7}=1.253$ ($r_{8,7}=4$), Obj1 = 10.333 and Obj2 = 3. In order to achieve ordinal consistency, $r_{1,7}$ is suggested to be reversed from $\frac{1}{3}$ to 2. The results are listed in Table 3 in the column labeled as GO1. The ordinal and cardinal Gower plots are graphed in Figs. 3(c) and 3(d) with 90.26% and 97% variabilities respectively. In Fig. 3(c), the ranks of alternatives are $A_8 \succ A_1 \succ A_3 \succ A_7 \succ A_*$, where A_* denotes the remaining alternatives.

Table 3
Results and comparisons of Example 3

Method	Original	Saaty	GO1	GO2	DM1
Ordinal-inconsistent node(s)			A_7 $u_{1,7}=1$	A_7 $u_{3,7}=1$	A_7 $u_{1,7}=1$
Cardinal-inconsistent node(s)			A_7	A_7	A_7
Boundary/constraint set by DM				$u_{1,7}=0$	$\frac{1}{5}\leqslant r_{2,7}\leqslant\frac{1}{4}$; $4\leqslant r_{3,7}\leqslant 6$;
Suggested adjustment/ boundary set by DM	$r_{1,7}=\frac{1}{3}$; $r_{2,7}=\frac{1}{5}$; $r_{3,7}=6$; $r_{4,7}=\frac{1}{7}$; $r_{5,7}=\frac{1}{5}$; $r_{6,7}=\frac{1}{5}$; $r_{8,7}=2$;	$r_{3,7}=\frac{1}{2}$	$r_{1,7}=2$; $r_{2,7}=\frac{1}{2}$; $r_{3,7}=2$; $r_{4,7}=\frac{1}{7}$; $r_{5,7}=\frac{1}{2}$; $r_{6,7}=\frac{1}{2}$; $r_{8,7}=4$;	$r_{1,7}=\frac{1}{2}$; $r_{2,7}=\frac{1}{4}$; $r_{3,7}=\frac{1}{2}$; $r_{4,7}=\frac{1}{9}$; $r_{5,7}=\frac{1}{9}$; $r_{6,7}=\frac{1}{4}$; $r_{8,7}=2$;	$r_{1,7}=2$; $r_{2,7}=\frac{1}{4}$; $r_{3,7}=4$; $r_{4,7}=\frac{1}{7}$; $r_{5,7}=\frac{1}{2}$; $r_{6,7}=\frac{1}{2}$; $r_{8,7}=3$;
Cardinal consistency (measured by Obj 1)	28.341	16.479	10.333	11.663	18.342
Ordinal consistency (measured by Obj 2)		6	3	6	3
CR	0.164	0.085	0.072	0.074	0.099

Step 6A: Suppose the DM does not wish to reverse $r_{1,7}$, he can set $u_{1,7}=0$ to obtain other suggestions. Applying Model 2 (Step 5A) again yields $u_{3,7}=1$, $u_{1,7}=u_{2,7}=u_{4,7}=u_{5,7}=u_{6,7}=u_{8,7}=0$, $x_{1,7}=-0.693$ ($r_{1,7}=\frac{1}{2}$), $x_{2,7}=-1.253$ ($r_{2,7}=\frac{1}{4}$), $x_{3,7}=-0.693$ ($r_{3,7}=\frac{1}{2}$), $x_{4,7}=-2.197$ ($r_{4,7}=\frac{1}{9}$), $x_{5,7}=-2.197$ ($r_{5,7}=\frac{1}{9}$), $x_{6,7}=-1.504$ ($r_{6,7}=\frac{1}{4}$), $x_{8,7}=0.693$ ($r_{8,7}=2$), Obj1 = 11.663 and Obj2 = 6. The results are listed in Table 3 in the column labeled as GO2. The ordinal and cardinal Gower plots are depicted in Fig. 3(e) and (f) with variabilities 90.7% and 97.14%, respectively. The ranks of alternatives are $A_8 \succ A_7 \succ A_1 \succ A_3 \succ A_*$, where the ranks of A_7 and A_3 are reversed. The top four rankings of alternatives are changed from $A_8 \succ A_1 \succ A_3 \succ A_7$ to$A_8 \succ A_7 \succ A_1 \succ A_3$ based on the DM's preferences.

If the DM accepts to reverse $r_{1,7}$ as GO1 in Table 3 but wishes to set some acceptable ranges as $\frac{1}{5}\leqslant r_{2,7}\leqslant\frac{1}{4}, 4\leqslant r_{3,7}\leqslant 6$, applying Model 2 (Step 5A) yields $u_{1,7}=1$, $u_{2,7}=u_{3,7}=u_{4,7}=u_{5,7}=u_{6,7}=u_{8,7}=0$, $x_{1,7}=0.693$ ($r_{1,7}=2$), $x_{2,7}=-1.386$ ($r_{2,7}=\frac{1}{4}$), $x_{3,7}=1.386$ ($r_{3,7}=4$), $x_{4,7}=-1.946$ ($r_{4,7}=\frac{1}{7}$), $x_{5,7}=-0.847$ ($r_{5,7}=\frac{1}{2}$), $x_{6,7}=-0.693$ ($r_{6,7}=\frac{1}{2}$), $x_{8,7}=1.099$ ($r_{8,7}=3$), Obj1 = 18.342 and Obj2 = 3. The results are listed in Table 3 in the column labeled as DM1. The cardinal Gower plot is shown in Fig. 3(g) with 94.2% variability. The ranks of alternatives are $A_8 \succ A_1 \succ A_3 \succ A_7 \succ A_*$.

Saaty [10] suggested to reverse $r_{3,7}$ from 6 to $\frac{1}{2}$ based on the principal eigenvector approach. Fig. 3(e) and (h) show the ordinal and cardinal Gower plots after adjustments, with 90.7% and 97.24% variabilities.

The CR can be effectively improved from 0.164 to 0.085. The ranks of alternatives are $A_8 \succ A_7 \succ A_1 \succ A_3 \succ A_*$.

Comparing Saaty's approach with GO1 in Table 3, the proposed approach can improve cardinal inconsistency better than Saaty's, measured by both Obj1 and CR. Saaty suggested to reverse $r_{3,7}$ from 6 to $\frac{1}{2}$, but the proposed approach suggested to reverse $r_{1,7}$ from $\frac{1}{3}$ to 2. The change for achieving ordinal consistency is also smaller by the proposed approach, which can be measured by Obj2. In addition, the proposed approach is more flexible and interactive because the DM can choose optimal reverse nodes and set his acceptable ranges for adjusted preferences. The ranks of alternatives may be changed based on the DM's preferences.

Some people may question that many pairwise comparisons $r_{i,j}$ are affected in the above example by the proposed approach. However, it totally depends on the DM's choice to make a large change on only one $r_{i,j}$ or small changes on a few $r_{i,j}$s. If the DM wishes to change only one $r_{i,j}$ at a time (practically useful for sensitivity analysis), he could set all other $r_{i,j}$ as a specific number with $\overline{x_{i,j}} = \underline{x_{i,j}} = \ln(r_{i,j})$. Take Example 3 for instance, if the DM wishes to change $r_{3,7}$ only, he could set bounds of $r_{1,7}$, $r_{2,7}$, $r_{4,7}$, $r_{5,7}$, $r_{6,7}$ and $r_{8,7}$ as $\ln(\frac{1}{3})$, $\ln(\frac{1}{5})$, $\ln(\frac{1}{7})$, $\ln(\frac{1}{5})$, $\ln(\frac{1}{5})$ and $\ln(2)$, respectively, where $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$, $\frac{1}{5}$, $\frac{1}{5}$ and 2 are original pairwise comparisons in matrix $\mathbf{R}_3$. Applying Model 2 to Example 3 again yields $u_{3,7} = 1$, $u_{1,7} = u_{2,7} = u_{4,7} = u_{5,7} = u_{6,7} = u_{8,7} = 0$, $r_{3,7} = \frac{1}{2}$, Obj1 = 16.479 and Obj2 = 6. The results are exactly the same as those of Saaty's. In this case, Saaty's solution is one of the options that the DM can choose by the proposed approach.

6. Conclusions

This study proposes an approach to represent the judgments of the decision makers (DMs) and provide an interactive method to assist the DMs to detect and adjust inconsistencies. A graphic approach based on Gower plots is applied to represent the judgments of DMs and to detect ordinal and cardinal inconsistencies. Two global optimization models are constructed to assist the DMs to adjust these ordinal and cardinal inconsistencies simultaneously and efficiently. The DMs can choose to revise their preferences based on the graphical supports and numerical suggestions to improve inconsistencies step by step. The proposed approach is flexible and provides options to assist the DMs to make more consistent decisions. The DMs can completely maintain control of change process.

Appendix

The mathematical properties of Gower plots are briefly illustrated here. The singular values of a matrix $\mathbf{M}$ of rank n are the positive square roots of the eigenvalues of the symmetric matrix $\mathbf{M^T M}$, where $\mathbf{M^T}$ stands for transposition of $\mathbf{M}$. If $\mathbf{M}$ is skew-symmetric, i.e. $\mathbf{M^T} = -\mathbf{M}$, the singular values of the matrix $\mathbf{M}$ are equal to the norm of its purely imaginary eigenvalues. Let $\lambda_1 \geqslant \cdots \geqslant \lambda_m \geqslant 0$ (and $\lambda_{m+1} = 0$ if n is an odd number) represent these singular values, with m indicating the integer part of $n/2$. Using singular value decomposition (Horn and Johnson [13]), a skew-symmetric matrix $\mathbf{M}$ can be decomposed to

$$\mathbf{M} = \sum_{j=1}^{m} \lambda_j (\mathbf{U}_{2j-1}\mathbf{U}_{2j}^T - \mathbf{U}_{2j}\mathbf{U}_{2j-1}^T),$$

where $\mathbf{U}_{2j-1}$ and $\mathbf{U}_{2j}$ are orthonormal eigenvectors of $\mathbf{M}^{\mathbf{T}}\mathbf{M}$ corresponding to λ_j^2.

$$\mathbf{M}^* = \lambda_1(\mathbf{U}\mathbf{V}^{\mathbf{T}} - \mathbf{V}\mathbf{U}^{\mathbf{T}}),$$

with $\mathbf{U} = \mathbf{U}_1$ and $\mathbf{V} = \mathbf{U}_2$, provides the best approximation of a skew-symmetric matrix $\mathbf{M}$ of rank two, because the first term of $\mathbf{M}$ gives the best least-squares fit of rank two to $\mathbf{M}$ (Eckart and Young [19]). Denote $\mathbf{GP}(\mathbf{M})$ as a Gower plot of a skew-symmetric matrix $\mathbf{M}$. Let $\mathbf{U} = (u_1, \ldots, u_n)^{\mathrm{T}}$ and $\mathbf{V} = (v_1, \ldots, v_n)^{\mathrm{T}}$. $\mathbf{M}^*$ can be expressed as $\mathbf{M}^* = \{\lambda_1(u_i v_j - v_i u_j)\} = \{\lambda_1 |P_i||P_j| \sin \theta_{i,j}\}$, in which $\theta_{i,j}$ denotes the directed angle from points P_i to P_j based on the origin. Plotting the vectors as n points $P_j = (u_j, v_j)$ in the plane provides a reasonable 2D representation of M. Such a graphical display is unique up to a rotation if $\lambda_1 > \lambda_2$ [5]. The measure of the reliability of the graphical representation of M is provided by the variability $v = \|\mathbf{M}^*\|/\|\mathbf{M}\| = \lambda_1^2/\sum_{j=1}^{m} \lambda_j^2$.

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