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Optimal ordering policies for periodic-review systems with a refined intra-cycle time scale

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Abstract

Chiang [C. Chiang, Optimal ordering policies for periodic-review systems with replenishment cycles, European Journal of Operational Research 170 (2006) 44–56] recently proposed a dynamic programming model for periodic-review systems in which a replenishment cycle consists of a number of small periods (each of identical but arbitrary length) and holding and shortage costs are charged based on the ending inventory of small periods. The current paper presents an alternative (and concise) dynamic programming model. Moreover, we allow the possibility of a positive fixed cost of ordering. The optimal policy is of the familiar (s, S) type because of the convexity of the one-cycle cost function. As in the periodic-review inventory literature, we extend this result to the lost-sales periodic problem with zero lead-time. Computation shows that the long-run average cost is rather insensitive to the choice of the period length. In addition, we show how the proposed model is modified to handle the backorder problem where shortage is charged on a per-unit basis irrespective of its duration. Finally, we also investigate the lost-sales problem with positive lead-time, and provide some computational results.

Keywords: Inventory; Lost sales; Periodic-review systems; Dynamic programming

1. Introduction

Periodic-review inventory systems are commonly found in practice, especially if many different items are purchased from the same supplier and the coordination of ordering and transportation is important. In a recent survey [7, p. 69], material managers indicate the effectiveness of periodic-review systems for reducing inventory levels in a supply chain.

Although most studies on periodic-review inventory models have (implicitly) assumed that the review periods are as small as one day (see, e.g., [5] and references therein), periodic-review systems in practice often have the review periods (i.e., *replenishment cycles* or simply *cycles*) that are a few days or weeks long and regular orders are placed at a review epoch (see, e.g., [2,3] for periodic systems where an emergency order can be placed at a review epoch or virtually at any time between two review epochs). For such periodic-review

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systems, it is appropriate to compute holding and shortage costs based on respectively, the average inventory of a replenishment cycle and the duration of shortage (assuming that demand not immediately filled is backlogged). Due to the difficulties involved in exact analysis, the approximate treatment of such systems is often used in textbooks (e.g., [4, Sec. 5-2] and [6, Sec. 7.9.4]) to obtain easy-to-implement solutions. However, as Chiang [1] recently pointed out, there are many shortcomings with the approximate treatment. Chiang thus proposed a dynamic programming model in which a cycle consists of a number of small periods and holding and shortage costs will be computed based on the ending inventory of small periods (rather than only on the ending inventory of cycles). As periods can be chosen to be any time units (see [1] for a detailed discussion), the period length is tailored to the needs of an application.

In this paper, we present an alternative (and concise) dynamic programming model. Moreover, we allow the possibility of a positive fixed ordering cost that is not considered in [1]. The optimal policy is of the familiar (s, S) type, i.e., if inventory drops to or below s at a review epoch, an order is placed to raise the inventory to a predetermined level S. As in the periodic-review inventory literature (see, e.g., [9]), we extend this result to the lost-sales periodic problem with zero lead-time. Computation shows that the long-run average cost is rather insensitive to the choice of the period length. This indicates that a rough estimate of the period length is acceptable.

In addition, we show how the proposed model is modified to handle the backorder problem where shortage is charged on a per-unit basis irrespective of its duration (as in [4, Sec. 5-2]), which is also not considered in [1]. Finally, we investigate the lost-sales problem with positive lead-time. Although we are unable to derive any properties (as in [1]) that can be used in the dynamic programming computation, we provide some interesting computational results. We advocate that firms use the proposed models for obtaining optimal ordering policies.

2. The backorder model

Suppose that a replenishment cycle, whose length is exogenously determined (as in [1–3]), consists of m periods, each of identical but arbitrary length. Let ξ be a generic demand variable and in particular, let ζ denote the demand of a cycle. Also, let $\varphi^k(\cdot)$ be the probability density function of k-period's demand (the superscript may be omitted for brevity if $k = 1$). Assume first that all demand not immediately satisfied is backlogged. Demand is assumed to be non-negative and independently distributed in disjoint time intervals. In addition, the following notation is used.

λ	mean arrival rate
τ	the (deterministic) supply lead-time, which is an integral multiple of a period
c	the unit procurement cost
K	the fixed cost of ordering
hc	the inventory cost per unit held per cycle
h	the inventory cost per unit held per period (i.e., $h = \text{hc}/m$)
pc	the shortage cost per unit per cycle
p	the shortage cost per unit per period (i.e., $p = \text{pc}/m$)
π	the shortage cost per unit
L	the holding and shortage costs of a replenishment cycle
α	the one-period discount factor (i.e., discounting the cost incurred in one period from now to the present time), $0 < \alpha \leqslant 1$
$\delta(\cdot)$	1 if the argument is positive and 0 otherwise
X	the starting inventory position (i.e., inventory on hand minus backorder plus inventory on order) at a review epoch
$V_n(X)$	the expected discounted cost of ordering, procurement, holding and shortage with n cycles remaining until the end of the planning horizon, given X at a review epoch and an optimal policy is used

For simplicity of formulation, we exclude from $V_n(X)$ the holding and shortage costs during the next τ periods, because these costs are not affected by the decision made at a review epoch. $V_n(X)$ satisfies the functional equation

$$V_n(X) = \min_{X \leqslant R}\{\alpha^\tau[K\delta(R-X) + cR + L(R)] + \alpha^m E_\zeta V_{n-1}(R-\zeta)\} - \alpha^\tau cX, \tag{1}$$

where R (the decision variable) is the inventory position after a possible order is placed at a review epoch. R is assumed to be greater than zero. Notice that the fixed cost of ordering, procurement cost $c(R-X)$ (paid upon delivery), and $L(R)$ are all discounted to the present review epoch. Expression (1) is a familiar dynamic program in the periodic-review inventory literature, except that lead-time may be shorter than the length of a cycle; hence, if L is convex, an (s,S) policy is optimal for the infinite-horizon problem (see, e.g., [5]).

Chiang [1] recently proposed a dynamic programming model in which holding and shortage costs are computed based on the ending inventory of periods, rather than only on the ending inventory of cycles (as in the inventory literature). However, Chiang's model uses two recursions to determine the cost function $V(\cdot)$ described above: an outer one for the number of cycles and an inner one for the number of periods remaining until the end of the planning horizon, and considers only the case of $K = \$0$. In this paper, we use the dynamic program given by (1) that requires only a recursion on the number of cycles. The recursion on the number of periods can be avoided by using the following approach. Let $L_i(R)$, for $i \geqslant \tau$ and $i \leqslant m+\tau-1$, denote the expected holding and shortage costs of the upcoming $(i+1)$th period, given the inventory position of R (after a possible order) at a review epoch. Then

$$L_i(R) = \int_0^R h(R-\xi)\varphi^{i+1}(\xi)\mathrm{d}\xi + \int_R^\infty p(\xi-R)\varphi^{i+1}(\xi)\mathrm{d}\xi. \tag{2}$$

$L_i(R)$ for each i is a convex function. Noticing that holding and shortage costs for each period are discounted to the time of the arrival of a possible order, we can write

$$L(R) = L_\tau(R) + \alpha L_{\tau+1}(R) + \cdots + \alpha^{m-1}L_{\tau+m-1}(R), \tag{3}$$

which is convex in R, and thus an (s,S) policy is optimal for the infinite-horizon model. To compute the two optimal operational parameters s^* and S^* and the long-run average cost $C(s,S)$, we use the following function:

$$G(R) \equiv cR(1-\alpha^m) + L(R) \tag{4}$$

and the discount renewal density $\alpha^m\varphi^m(\cdot)$ in a solution procedure (see, e.g., [9,10]). Note that (4) excludes the multiplier α^τ for simplicity (since it is constant).

To illustrate, consider the base case: a cycle = 10 days, $K = \$20$, $c = \$10$, $\alpha^m = 0.99$, $\tau = 6$ days, $\lambda = 20$ units/cycle with Poisson demand, hc = \$0.1, and pc = \$200. If $m = 10$ (i.e., periods are defined as days), then $\alpha = (0.99)^{0.1}$, $\lambda = 2$ units/period, $h = \$0.01$ and $p = \$20$. By using Zheng and Federgruen's algorithm, we find that $s^* = 38$, $S^* = 88$, and $C(s^*,S^*) = \$18.53$. If $m = 20$ (i.e., the period length is half-a-day or 4 working hours) in the base case, then $\alpha = (0.99)^{0.05}$, $\tau = 12$ periods, $\lambda = 1$ unit/period, $h = \$0.005$, $p = \$10$, and s^* and S^* are found to be 37 and 87, respectively. If the period length is further reduced to 2 hours or even 1 hour (and the relevant data are changed similarly), s^*, S^*, and $C(s^*,S^*)$ are reported in Table 1. Table 1 also records the computational results by varying pc in the base case. We are unable to prove any "properties" that might be conjectured from Table 1 (e.g., the convergence of s with smaller period length). One thing is certain that the shorter the periods, the more *continuously in time* holding and shortage costs are computed [1]; choice of the period length should depend on the particular needs of an application. If the one-day period length is chosen in the first place but a shorter period length should have been used, Table 1 also reports the resulting long-run average cost by using $\hat{s}$ and $\hat{S}$ obtained from the one-day period length. As we can see, $C(s,S)$ is rather insensitive to the choice of the period length, as the percentage error in $C(s,S)$ is quite small. This indicates that a rough estimate of the period length (obtained from the extent of customers' impatience when shortage occurs) is acceptable.

Suppose that the compound Poisson demand distribution is used instead in the above base case (K is scaled to \$80 while other input parameters remain unchanged) to investigate whether or not the bulkiness of demand matters for the choice of the period length. Letting p_i denote the probability of the order size of i units for each customer arrival, we use 4 distributions in the computation. In distribution a, $p_i = 0.2$ for $i = 1,\ldots,5$ and 0 otherwise; in distribution b, $p_1 = p_5 = 1/9$, $p_2 = p_4 = 2/9$, $p_3 = 3/9$, and $p_i = 0$ for $i > 5$; in distribution c, $p_i = 0.2$ for $i = 3,\ldots,7$ and 0 otherwise; and in distribution d, $p_3 = p_7 = 1/9$, $p_4 = p_6 = 2/9$, $p_5 = 3/9$, and $p_i = 0$ for $i < 3$ or $i > 7$. From Tijms [8, pp. 27–29], we can compute $\varphi^k(\cdot)$ (by a recursive routine) that is used

Table 1
Computation of the backorder model

pc	m	s^*	S^*	$C(s^*, S^*)$	$C(\hat{s}, \hat{S})$	Error (%)
\$200	10	38	88	\$18.53	–	–
	20	37	87	18.47	\$18.48	0.05
	40	37	87	18.44	18.46	0.11
	80	37	87	18.42	18.45	0.16
50	10	34	84	17.28	–	–
	20	33	84	17.72	17.74	0.11
	40	33	83	17.69	17.72	0.17
	80	33	83	17.67	17.71	0.23
10	10	27	79	16.70	–	–
	20	27	78	16.65	16.65	0.00
	40	26	78	16.62	16.62	0.00
	80	26	78	16.60	16.61	0.06
5	10	23	76	16.12	–	–
	20	23	76	16.06	16.06	0.00
	40	23	75	16.04	16.04	0.00
	80	22	75	16.02	16.03	0.06

Data: a cycle = 10 days, $K = \$20$, $c = \$10$, $\alpha^m = 0.99$, $\tau = 6$ days, $\lambda = 20$ units/cycle (Poisson demand), hc = \$0.1.

in Zheng and Federgruen's algorithm. We see again from Table 2 that using $\hat{s}$ and $\hat{S}$ obtained from the one-day period length yields almost the same $C(s, S)$ as using s^* and S^* from a shorter period length.

For the special case of $K = \$0$, a (stationary) base-stock policy is optimal and the optimal level R^* is obtained by minimizing $G(R)$ [9]. Denote by Df the first derivative of the function f. It follows that R^* is found by setting $\mathrm{D}G(R) = 0$ and solving the equation

$$\mathrm{D}L_\tau(R) + \alpha \mathrm{D}L_{\tau+1}(R) + \cdots + \alpha^{m-1}\mathrm{D}L_{\tau+m-1}(R) + (1 - \alpha^m)c = 0. \tag{5}$$

Let $\Phi_k(\cdot)$ be the complement of the cumulative distribution function of k-period's demand. Noticing that $\mathrm{D}L_i(R) = h - (h + p)\Phi_{i+1}(R)$, we can simplify (5) to

$$(h + p)[\Phi_{\tau+1}(R) + \cdots + \alpha^{m-1}\Phi_{\tau+m}(R)] = h(1 + \alpha + \cdots + \alpha^{m-1}) + (1 - \alpha^m)c, \tag{6}$$

which is the same as expression (7) of Chiang [1] (since τ is deterministic).

Table 2
Computation of the backorder model

Distribution	m	s^*	S^*	$C(s^*, S^*)$	$C(\hat{s}, \hat{S})$	Error (%)
a	10	117	317	\$62.47	–	
	20	116	315	62.27	\$62.29	0.03
	40	115	315	62.15	62.20	0.08
	80	114	314	62.10	62.16	0.10
b	10	116	316	62.15	–	
	20	115	315	61.97	61.98	0.02
	40	114	314	61.88	61.93	0.08
	80	113	313	61.79	61.86	0.11
c	10	196	435	86.54	–	
	20	193	433	86.22	86.26	0.05
	40	192	431	86.01	86.12	0.13
	80	191	431	85.88	86.02	0.16
d	10	195	434	86.32	–	
	20	192	432	86.01	86.05	0.05
	40	190	431	85.82	85.92	0.12
	80	190	430	85.67	85.80	0.15

Data: a cycle = 10 days, $K = \$80$, $c = \$10$, $\alpha^m = 0.99$, $\tau = 6$ days, $\lambda = 20$ arrivals/cycle (compound Poisson), hc = \$0.1, pc = \$200.

Next, we point out that while Chiang's model [1] does not consider the backorder problem where shortage is charged at π per unit irrespective of its duration [4, p. 238], our model (1) can actually handle it. This is accomplished by modifying $L(R)$ as follows:

$$\begin{aligned}L(R) = {} & h\left\{\int_0^R (R-\xi)\varphi^{\tau+1}(\xi)\mathrm{d}\xi + \alpha\int_0^R (R-\xi)\varphi^{\tau+2}(\xi)\mathrm{d}\xi + \cdots + \alpha^{m-1}\int_0^R (R-\xi)\varphi^{\tau+m}(\xi)\mathrm{d}\xi\right\} \\ & + \pi\left\{\int_R^\infty (\xi-R)\varphi^{\tau+1}(\xi)\mathrm{d}\xi + \alpha\left[\int_R^\infty (\xi-R)\varphi^{\tau+2}(\xi)\mathrm{d}\xi - \int_R^\infty (\xi-R)\varphi^{\tau+1}(\xi)\mathrm{d}\xi\right]\right. \\ & +\alpha^2\left[\int_R^\infty (\xi-R)\varphi^{\tau+3}(\xi)\mathrm{d}\xi - \int_R^\infty (\xi-R)\varphi^{\tau+2}(\xi)\mathrm{d}\xi\right] + \cdots \\ & \left. +\alpha^{m-1}\left[\int_R^\infty (\xi-R)\varphi^{\tau+m}(\xi)\mathrm{d}\xi \quad - \int_R^\infty (\xi-R)\varphi^{\tau+m-1}(\xi)\mathrm{d}\xi\right]\right\}.\end{aligned} \tag{7}$$

It is assumed that if a possible order at the time of its arrival does not clear all the shortages, those shortages not cleared are charged at π per unit again (thus π is really the shortage cost per unit *per cycle*, but its meaning is slightly different from that of pc), as expressed by the first term of the shortage cost expression of (7). The reason for the remaining terms of the shortage cost expression is that when the shortage cost is computed in a period, only the shortages that occur in that period are counted and charged at π per unit and all shortages that occurred in previous periods are excluded (as they were charged already). Let $L_i(R)$ be given as follows:

$$L_i(R) = \int_0^R h(R-\xi)\varphi^{i+1}(\xi)\mathrm{d}\xi + \int_R^\infty (1-\alpha)\pi(\xi-R)\varphi^{i+1}(\xi)\mathrm{d}\xi, \quad \text{for } i \geqslant \tau \text{ and } i \leqslant m+\tau-2, \tag{8}$$

$$L_{\tau+m-1}(R) = \int_0^R h(R-\xi)\varphi^{\tau+m}(\xi)\mathrm{d}\xi + \int_R^\infty \pi(\xi-R)\varphi^{\tau+m}(\xi)\mathrm{d}\xi. \tag{9}$$

We see that $L(R)$ in (7) is in the form of (3) with $L_i(R)$, for $i \geqslant \tau$ and $i \leqslant m+\tau-1$, given by (8) and (9). As $L(R)$ is convex, an (s,S) policy is optimal for the infinite-horizon problem. If an order is always placed at a review epoch (as in [1]), the optimal base-stock level R^* is found by solving the equation:

$$\begin{aligned}& [h+(1-\alpha)\pi]\left[\Phi_{\tau+1}(R) + \cdots + \alpha^{m-2}\Phi_{\tau+m-1}(R)\right] + (h+\pi)\alpha^{m-1}\Phi_{\tau+m}(R) \\ & \quad = h\left(1+\alpha+\cdots+\alpha^{m-1}\right) + (1-\alpha^m)c.\end{aligned} \tag{10}$$

If $\alpha = 1$ (i.e., the undiscounted-cost criterion is used), we can compare R^* to the order-up-to level obtained by using expression (5–9) (whose right-hand side is given by the ratio hc/π) of Hadley and Whitin [4], denoted by $\hat{R}$. Table 3 reports the computational results for 16 problems solved. As we see, Hadley and Whitin's approximate model usually yields the optimal level for problems with small hc/π. This is because for these problems, safety stock is high so that shortage occurs infrequently and thus the expected net inventory is approximately equal to the expected on-hand inventory. For problems with large hc/π, Hadley and Whitin's model may yield a lower level, i.e., $\hat{R} < R^*$, and increase the long-run average cost by more than 2%. The reason for $\hat{R} \leqslant R^*$ is that Hadley and Whitin's model ignores the expected unit years of backorders incurred per cycle [4, p. 239], which should decrease as R increases; hence, incorporating it, as in our model that directly computes the on-hand inventory, will yield an order-up-to level that is at least as large as $\hat{R}$. In Table 4, we use a larger m than in Table 3 and observe a similar result, though the percentage increase in average cost when using Hadley and Whitin's model is smaller (due to the fact that with a larger m, the holding cost is computed more continuously in time). Incidentally, for the problems solved in Tables 3 and 4, the probability that R^* obtained is less than the lead-time demand is approximately 0, and thus an order when arriving will clear any backorders with probability almost 1 (see [2] for a detailed discussion).

Thus far the supply lead-time τ is assumed to be deterministic. If τ is stochastic (integer-valued), an (s,S) policy is still optimal for the infinite-horizon problem, provided that lead-times are generated by an exogenous, sequential supply process that is independent of demand and with the property that orders are received in the same sequence as they are placed, as in the ordinary periodic-review model where holding and shortage

Table 3
Comparison of the proposed model to Hadley and Whitin's model

Distribution	τ	π	R^*	$\hat{R}$	$C(R^*-1,R^*)$	$C(\hat{R}-1,\hat{R})$	Error (%)
Poisson	6	.15	31	29	$1.279	$1.304	1.95
		.2	33	32	1.397	1.399	0.14
		1	39	39	1.934	1.934	–
		10	46	46	2.505	2.505	–
	12	.15	43	41	1.355	1.378	1.70
		.2	45	44	1.488	1.493	0.34
		1	53	53	2.106	2.106	–
		10	60	60	2.765	2.765	–
Compound Poisson[a]	6	.15	93	87	$3.973	$4.040	1.69
		.2	98	95	4.358	4.383	0.57
		1	121	120	6.158	6.159	0.02
		10	143	143	8.108	8.108	–
	12	.15	129	122	4.231	4.319	2.08
		.2	135	131	4.667	4.702	0.75
		1	161	161	6.732	6.732	–
		10	186	186	8.972	8.972	–

Data: a cycle = 10 days, $m=10$, $\lambda=20$ arrivals/cycle, hc = \$0.1.
[a] *Note:* distribution a (i.e., $p_i=0.2$ for $i=1,\dots,5$ and 0 otherwise) is used.

Table 4
Comparison of the proposed model to Hadley and Whitin's model

Distribution	τ	π	R^*	$\hat{R}$	$C(R^*-1,R^*)$	$C(\hat{R}-1,\hat{R})$	Error (%)
Poisson	48	.15	31	29	$1.353	$1.373	1.48
		.2	32	32	1.475	1.475	–
		1	39	39	2.020	2.020	–
		10	46	46	2.591	2.591	–
	96	.15	43	41	1.427	1.445	1.26
		.2	45	44	1.565	1.568	0.19
		1	53	53	2.192	2.192	–
		10	60	60	2.852	2.852	–
Compound Poisson[a]	48	.15	92	87	$4.191	$4.243	1.24
		.2	98	95	4.589	4.607	0.39
		1	121	120	6.415	6.416	0.02
		10	143	143	8.366	8.366	–
	96	.15	128	122	4.444	4.514	1.58
		.2	135	131	4.895	4.921	0.53
		1	161	161	6.988	6.988	–
		10	186	186	9.227	9.227	–

Data: a cycle = 10 days, $m=80$, $\lambda=20$ arrivals/cycle, hc = \$0.1.
[a] *Note:* distribution a (i.e., $p_i=0.2$ for $i=1,\dots,5$ and 0 otherwise) is used.

costs are charged only at the end of cycles (see, e.g., [11, pp. 408–409]). Then $G(R)$ in (4) will be replaced by its expectation, i.e.,

$$G(R) \equiv E_\tau[\alpha^\tau]cR(1-\alpha^m) + E_\tau[\alpha^\tau L(R)] \quad (11)$$

(see also [10]). For the special case of $K = \$0$, minimizing $G(R)$ will yield expression (8) of Chiang [1] for obtaining the optimal level R^*.

3. The lost-sales model

Suppose now that all demand not immediately satisfied is lost. Use the notation in Section 2. X is now defined as the starting on-hand inventory at a review epoch. Assume first $\tau = 0$ (i.e., immediate delivery). Then, $V_n(X)$ is expressed by

$$V_n(X) = \min_{X \leqslant R}\{K\delta(R - X) + cR + L(R) + \alpha^m E_\zeta V_{n-1}(R - \zeta)\} - cX, \tag{12}$$

where R is the on-hand inventory after a possible order is placed (and received) at a review epoch. If $L(R)$ is computed only at the end of cycles, (12) can be viewed as a backorder model in which a credit of $\alpha^m c$ is given to each unit of demand actually backlogged [9]. Hence, an (s, S) policy is still optimal for the infinite-horizon problem. Note that π has a different meaning in the lost-sales model. It should be larger here, for it now includes the sales price [11, p. 386].

Again, Chiang [1] proposed to charge holding and shortage costs based on the ending inventory of periods, but used two subscripts for the cost function $V(\cdot)$ described above and considered only the case of $K = \$0$. In this paper, we use the dynamic program given by (12), but express $L(R)$ to reflect the fact that holding and shortage costs are computed for each period of a replenishment cycle. Notice that any unfulfilled demand in the lost-sales model is charged at π per unit only once, the lost-sales model can be treated as a backorder model in which credits of $\alpha\pi, \alpha^2\pi, \ldots,$ and $\alpha^{m-1}\pi$ are given to each unit of demand actually backlogged, respectively at the end of first period, second period, $\ldots$, and $(m-1)$th period of a cycle. For the mth period, i.e., the last period of a cycle, we still give a credit of $\alpha^m c$ to each unit of demand backlogged, as in the ordinary model (12). Consequently, it follows that

$$\begin{aligned} L(R) = {} & \int_0^R h(R-\xi)\varphi^1(\xi)\mathrm{d}\xi + \int_R^\infty (1-\alpha)\pi(\xi - R)\varphi^1(\xi)\mathrm{d}\xi \\ & + \alpha\left[\int_0^R h(R-\xi)\varphi^2(\xi)\mathrm{d}\xi + \int_R^\infty (1-\alpha)\pi(\xi - R)\varphi^2(\xi)\mathrm{d}\xi\right] + \cdots \\ & + \alpha^{m-2}\left[\int_0^R h(R-\xi)\varphi^{m-1}(\xi)\mathrm{d}\xi + \int_R^\infty (1-\alpha)\pi(\xi - R)\varphi^{m-1}(\xi)\mathrm{d}\xi\right] \\ & + \alpha^{m-1}\left[\int_0^R h(R-\xi)\varphi^{m}(\xi)\mathrm{d}\xi + \int_R^\infty (\pi - \alpha c)(\xi - R)\varphi^{m}(\xi)\mathrm{d}\xi\right], \end{aligned} \tag{13}$$

which is convex in R. Let $L_i(R)$, for $i \geqslant 0$ and $i \leqslant m-2$, be given by (8) (with $\tau = 0$), and $L_{m-1}(R)$ expressed by

$$L_{m-1}(R) = \int_0^R h(R-\xi)\varphi^m(\xi)\mathrm{d}\xi + \int_R^\infty (\pi - \alpha c)(\xi - R)\varphi^m(\xi)\mathrm{d}\xi. \tag{14}$$

We see that $L(R)$ in (13) is written by

$$L(R) = L_0(R) + \alpha L_1(R) + \cdots + \alpha^{m-1}L_{m-1}(R), \tag{15}$$

which is again in the form of (3) with $\tau = 0$. By comparing (14) to (9), we can see that if $(\pi - \alpha c)$ in the lost-sales model is equal to π in the backorder model where shortage is charged irrespective to its duration (with $\tau = 0$), these two models will have the same optimal operational parameters.

Take for example the base case in Section 2 (with $m = 10$) except that $\tau = 0$ and $\pi = \$20$. It is found after solving that $s^* = 21$, $S^* = 71$, and $C(s^*, S^*) = \$16.30$. Again, $C(s, S)$ is not sensitive to the choice of the period length, though computational results are omitted for brevity.

For the special case of $K = \$0$, we differentiate $G(R)$ in (4) with $L(R)$ given by (15) and set the derivative to zero to obtain the following equation:

$$\begin{aligned} & [h + (1-\alpha)\pi]\left[\Phi_1(R) + \cdots + \alpha^{m-2}\Phi_{m-1}(R)\right] + [h + (\pi - \alpha c)]\alpha^{m-1}\Phi_m(R) \\ & \quad = h\left(1 + \alpha + \cdots + \alpha^{m-1}\right) + (1 - \alpha^m)c, \end{aligned} \tag{16}$$

which is the same as expression (16) of Chiang [1]. We have used a different method of deriving (16) for obtaining the optimal base-stock level for the infinite-horizon problem. In the above example, if $K = \$0$ instead (other things being equal), the optimal level R^* is equal to 30.

Suppose now that τ is positive, which is an (deterministic) integral multiple of a period. If holding and shortage costs are computed for each period of a cycle, a concise model with only one subscript for the cost function appears difficult to formulate. Hence, we use expressions (10)–(12) of Chiang [1] and include a positive fixed cost of ordering. Assume that $\tau \leqslant m$ (i.e., at most one order is outstanding at any time, as in [1]) and $\tau \geqslant 2$ (the model for $\tau = 1$ is simpler and thus omitted for brevity, see [1]). Let $V_{n,0}(X,0) \equiv V_n(X)$, and $V_{n,j}(X, Y)$, for $j \neq 0$, denote the expected discounted cost with n cycles and j periods remaining when the starting on-hand and on-order inventory are X and Y, respectively. $V_{n,j}(X, Y)$ is simply $V_{n,j}(X,0)$ for $j = 1,\ldots,m - \tau$. $V_{n,j}(X, Y)$ satisfies the functional equations:

$$V_{n,0}(X,0) = \min_{Z \geqslant 0} \left\{ \alpha^{\tau}[K\delta(Z) + cZ] + L_0(X) + \alpha \int_0^X V_{n-1,m-1}(X-\xi, Z)\varphi(\xi)\mathrm{d}\xi + \alpha V_{n-1,m-1}(0,Z)\int_X^{\infty}\varphi(\xi)\mathrm{d}\xi \right\}, \tag{17}$$

$$V_{n,j}(X,Y) = L_0(X) + \alpha\int_0^X V_{n,j-1}(X-\xi,Y)\varphi(\xi)\mathrm{d}\xi + \alpha V_{n,j-1}(0,Y)\int_X^{\infty}\varphi(\xi)\mathrm{d}\xi,$$

$$j = 1,\ldots,m-1, \text{ and } j \neq m-\tau+1, \tag{18}$$

$$V_{n,m-\tau+1}(X,Y) = L_0(X) + \alpha\int_0^X V_{n,m-\tau}(X-\xi+Y,0)\varphi(\xi)\mathrm{d}\xi + \alpha V_{n,m-\tau}(Y,0)\int_X^{\infty}\varphi(\xi)\mathrm{d}\xi, \tag{19}$$

where $V_{0,0}(X,0) \equiv -cX$, $L_0(X)$ is the one-period holding and shortage costs given by (2) (with p replaced by π), and Z (the decision variable) is the quantity ordered at a review epoch which becomes inventory on order thereafter.

Let $Z_n(X)$ be the (smallest) value of non-negative Z obtained in (17) for a given X. Then the optimal policy at a review epoch with n cycles remaining is to order the amount $Z_n(X)$. We are unable to develop any properties regarding $V_{n,j}(X, Y)$ or $Z_n(X)$ (as in [1]) that can be used in the dynamic programming computation. Nevertheless, we implement the model given by (17)–(19). To be more specific, let Λ be the maximum possible value of X; we stop the dynamic programming computation if there exists n such that $Z_n(X) = Z_{n-1}(X)$ for all $X \leqslant \Lambda$, i.e., the sequence $\{Z_i(X)\}$ seems to have converged. Alternatively, we could adopt a more rigid stopping rule; for instance, run the model until there exists n such that $Z_n(X) = Z_{n-1}(X) = \cdots = Z_{n-4}(X)$ for all $X \leqslant \Lambda$, i.e., $Z_i(X)$ for each X is the same for five consecutive cycles. This is purely computational rather than analytical or methodological, and still has not answered the question: has the sequence $\{Z_i(X)\}$ really converged? Also, a more rigid rule is unnecessary, if one is not interested in obtaining optimal operational parameters for the infinite-horizon problem.

To illustrate, consider the base case in Section 2: a cycle = 10 days, $K = \$20$, $c = \$10$, $\alpha^m = 0.99$, $\tau = 6$ days, $\lambda = 20$ units/cycle (Poisson demand), hc = \$0.1, and $\pi = \$20$. If $m = 10$, it is found after solving that the sequence $\{Z_i(X)\}$ converges (seemingly) to $Z^*(X) = Z_{42}(X)$ (i.e., $n = 42$), and $Z^*(X) = 71$ for $X \leqslant 12$, $Z^*(13) = 70$, $Z^*(X) = 84 - X$ for $14 \leqslant X \leqslant 17$, $Z^*(X) = 85 - X$ for $18 \leqslant X \leqslant 34$, $Z^*(X) = 0$ for $X \geqslant 35$.

Moreover, we vary the value of the three input parameters K, π, and τ in the base case, as reported in Table 5, to investigate how many cycles it takes for $\{Z_i(X)\}$ to converge and their effect on the optimal solutions. There are a total of 27 problems solved. Computational results indicate that it takes an average of about 34 cycles for $\{Z_i(X)\}$ to converge. Also, it is observed that for each problem solved, there is a reorder level denoted by s and for $X \leqslant s$, the order-up-to level is non-increasing in X (this result is similar to Theorem 3.3 of Chiang [1]). Let S be the maximum order-up-to level and X^* the smallest value of X for which the order-up-to level is S. For the example illustrated above, $s = 34$, $S = 85$, and $X^* = 18$. The levels of s, S, and X^* for each problem solved are recorded in Table 5. As we see from Table 5, s is non-decreasing in π or τ (other things being equal) and S is non-decreasing in K. These results are intuitively reasonable, though we cannot provide a formal proof.

Table 5
Computation of the lost-sales model

Input parameters			Operational parameters		
K	τ	π	X^*	s	S
$10	4	$12	10	23	54
		20	12	31	60
		28	10	33	61
	6	12	18	27	59
		20	15	35	64
		28	15	37	66
	8	12	21	31	63
		20	21	39	69
		28	21	42	71
$20	4	$12	12	21	74
		20	11	29	80
		28	11	31	82
	6	12	17	25	79
		20	18	34	85
		28	15	36	86
	8	12	21	29	83
		20	20	38	89
		28	20	40	91
$40	4	$12	9	18	107
		20	8	28	114
		28	8	30	114
	6	12	14	22	109
		20	12	32	118
		28	12	35	119
	8	12	18	26	114
		20	15	37	121
		28	15	39	123

Data: a cycle = 10 days, $c = \$10$, $\alpha^m = 0.99$, $m = 10$, $\lambda = 20$ units/cycle (Poisson demand), hc = \$0.1.

4. Conclusions

This paper presents an alternative dynamic programming model for periodic-review inventory systems with a refined intra-cycle time scale. It also incorporates a positive fixed cost of ordering. The optimal policy is of the familiar (s, S) type because of the convexity of the one-cycle cost function. As in the periodic-review inventory literature, we extend this result to the lost-sales periodic problem with zero lead-time. Computational results show that the long-run average cost is rather insensitive to the choice of the period length. This indicates that a rough estimate of the period length is acceptable.

Moreover, this paper contributes to the periodic-review literature by showing that the backorder problem where shortage is charged per unit irrespective of its duration can be easily handled in the proposed model. It is found that Hadley and Whitin's approximate model usually yields the optimal order-up-to level for problems with a low holding-to-shortage cost ratio. However, if this ratio is high, Hadley and Whitin's model may yield a lower level and increase the long-run average cost by more than 2%.

Finally, this paper investigates the lost-sales problem with positive lead-time. Although we are unable to develop any properties that can be used in the dynamic programming computation, we provide some interesting results. Further research on the lost-sales problem with more than one outstanding order allowed and/or stochastic lead-time is possible, though a dynamic programming model seems complex or difficult to formulate.

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