
On the solutions of modeling a slug test performed in a two-zone confined aquifer

Yang Shaw-Yang · Yeh Hund-Der

Abstract The slug test is a popular field method for estimating aquifer parameters. As a finite-thickness skin is present at a wellbore, the result of test-data analysis may represent average hydraulic properties of the skin and formation zones. Moench and Hsieh (Int. Assoc. of Hydrogeology, The IAH 17th International Congress on Hydrology of Rocks of Low Permeability, Int. Assoc. of Hydrogeology, Tucson, Arizona, 7–12 January) provided the Laplace-domain solution for a confined aquifer with a finite-thickness skin that satisfies the governing equations, boundary conditions, and continuity requirements at the interface of the skin and formation zones. Two new formulae presented here relate the modified zero-order Bessel functions of the first and second kinds to their first and second derivatives. Their solution is evaluated using the modified Crump approach for the Laplace inversion. The numerical values obtained by the modified Crump approach are compared with others based on finite-element modeling. The results indicate that the values based on finite-element modeling exhibit poor accuracy, attributed to errors in the mass balance equation and interface condition in their mathematical model.

Résumé Le slug-test représente une méthode populaire de terrain pour estimer les paramètres d'un aquifère. Du fait de la présence d'une couche à l'épaisseur limitée sur un site de forages, le résultat de l'analyse des données du test pourrait représenté le terrain mais aussi cette couche. Moench AF et Hsieh PA (Association Internationale d'Hydrogéologie, 7–12 Jan 1985) Analyses de données de slug test dans un puits présentant une couche d'épaisseur finie. 17ème congrès international de l'AIH

sur l'Hydrologie des roches de faible perméabilité, Association Internationale d'Hydrogéologie. Tucson, Arizona, 7–12 Jan. apportent une solution du domaine de Laplace pour un aquifère confiné avec une épaisseur limitée qui satisfait l'équation en question, les conditions aux limites, et les besoins de continuité à l'interface de la couche et de l'encaissant. Deux nouvelles formules présentées dans cet article met en relation les fonctions de Bessel modifiées d'ordre zéro du premier et second type, avec leur dérivées première et deuxième. Leurs solutions sont évaluées en utilisant une approche de Crump modifiée pour l'inversion de Laplace. Les valeurs numériques obtenues avec l'approche de Crump sont comparées avec d'autres approches utilisant les modèles aux éléments finis. Les résultats indiquent que les valeurs basées sur les modèles aux éléments finis montrent une faible précision, du fait d'erreurs dans le bilan hydrologique et les conditions aux limites dans le modèle.

Resumen Los ensayos slug o tipo pulso son métodos populares de campo para estimar parámetros de acuíferos. Debido a que en los sondeos existe una membrana de espesor finito, el resultado del análisis de los datos del ensayo pueden representar las propiedades hidráulicas promedio de las zonas de la formación y dicha membrana. Moench AF y Hsieh PA (Asociación Internacional de Hidrogeólogos, 7–12 Enero 1985) Análisis de datos de prueba slug en un pozo con membrana de espesor finito. 17 Congreso Internacional de la Asociación Internacional de Hidrogeólogos sobre Hidrología de Rocas de Baja Permeabilidad. Asociación Internacional de Hidrogeólogos. Tucson, Arizona. 7–12 Enero. han aportado la solución en el ámbito de Laplace para un acuífero confinado con una membrana de espesor finito que satisface los requerimientos de las ecuaciones dominantes, las condiciones restrictivas, y de continuidad en la interfase de las zonas de formación y membrana. Las dos nuevas fórmulas que se presentan aquí relacionan las funciones modificadas Bessel de orden cero de la primera y segunda clase con su primera y segunda derivadas. Se evalúa su solución usando el enfoque modificado de Crump para la inversión Laplace. Los valores numéricos obtenidos mediante el enfoque modificado de Crump se comparan con otros valores numéricos basados en modelos de elementos finitos. Los resultados indican que los valores basados en el modelo de elementos finitos

Received: 3 July 2006 / Accepted: 8 August 2006
Published online: 18 October 2006

© Springer-Verlag 2006

Y. Shaw-Yang
Department of Civil Engineering,
Vanung University,
Chungli, Taiwan

Y. Hund-Der (✉)
Institute of Environmental Engineering,
National Chiao-Tung University,
Hsinchu, Taiwan
e-mail: hdyeh@mail.nctu.edu.tw
Fax: +886-3-5726050

presentan poca precisión, que se atribuye a errores en la ecuación de balance de masa y a las condiciones de interfase en su modelo matemático.

Keywords Slug test · Analytical solutions · Skin effect · Groundwater hydraulics · Shanks method

Introduction

The slug test is a popular field method to evaluate in-situ aquifer parameters in low permeability aquifers. The test involves a sudden withdrawal from or injection into a test well, resulting in an instantaneous raising or lowering of the water level in the well. An instantaneous head change is thus imposed within a wellbore and the recovery (or falloff) of the water level in the well is measured, often by means of a pressure transducer. The aquifer parameters, e.g., transmissivity and storativity, can then be obtained from the analysis of test data. For engineering applications, the slug test is a simple, quick, and economical approach.

An aquifer is considered a radial two-zone or composite system if the formation properties near the wellbore are apparently changed due to drilling or development of the well. Drilling may create a wellbore skin between the wellbore and formation. The invasion of drilling mud into the aquifer may produce a positive wellbore skin that has lower hydraulic conductivity than that of the undisturbed formation. On the other hand, the extensive well development and/or substantial fracturing of the borehole wall may increase the hydraulic conductivity as a negative wellbore skin.

The objectives of this work are to give a numerical approach for evaluating the Moench and Hsieh (1985b) Laplace-domain solution and compare this with the numerical results obtained from the Yang and Gates (1997) finite-element model. The numerical approach includes the modified Crump approach (IMSL 1997) for performing the numerical inversion for the Moench and Hsieh (1985b) solution (hereinafter referred to as MHS), the series expansion for approximating the Bessel functions in MHS, and the Shanks method (Shanks 1955; Yeh et al. 2003; Yang and Yeh 2005) for accelerating the convergence when evaluating the series of the Bessel functions. It is also proven that MHS satisfies the mathematical model of the slug test in a well with a finite-thickness skin. In addition, two new formulae related the modified zero-order Bessel functions of the first and second kinds to their first and second derivatives, and these formulae may be of use in future studies.

Literature review

Faust and Mercer (1984) provided an infinite-aquifer solution on the response of slug tests in wells containing a finite-thickness skin. They assumed that the skin has a much lower permeability than the adjacent formation

does. Under this condition, the skin effect can lead to very low estimates of hydraulic conductivity if using the type-curve fitting method of Cooper et al. (1967). Moench and Hsieh (1985a) commented on the evaluation of slug tests in a finite-thickness skin by Faust and Mercer (1984). They stated that when the specific storage of the skin is negligibly small, the finite-thickness skin solution becomes equivalent to the infinitesimally thin skin solution published by Ramey et al. (1975). Under a finite-thickness skin condition, the skin properties in the test interval control the early time response, whereas the formation properties relate to the late time response. This causes the well response to depart significantly from the conventional type curves. Moench and Hsieh (1985b) provided a Laplace-domain solution for the mathematical model representing a drill-stem test in a well surrounded by a finite-thickness skin. They found that the standard methods of test-data analysis are adequate for open-well slug tests and may differ markedly in the pressure response for pressurized slug tests. Further, Sageev (1986) investigated the effects of well storage and wellbore skin in a confined aquifer system. He obtained a similar result to that of Moench and Hsieh (1985b). Karasaki et al. (1988) developed the various models of slug-tests and related solutions for linear flow, radial flow with boundaries, two zones, and concentric composite aquifer systems with different flow geometries between the inner and outer regions; they provided type curves for each solution and stated that slug tests suffer the problems of non-uniqueness in matching the test data to type curves. Karasaki (1990) used the time convolution method of Duhamel's theorem to derive the analytical solution of Moench and Hsieh (1985b). The systematized procedure and analysis method were proposed for a drill-stem test.

Recently, Yang and Gates (1997) constructed a finite-element model for a slug test in a confined aquifer system containing a finite-thickness skin. Their analysis shows that the effect of a low-permeability skin on the estimates of hydraulic conductivity could be minimized by the use of late-time test data. Butler and Healey (1998) investigated the estimate of hydraulic conductivities obtained through pumping or slug tests; they simply noted that the hydraulic conductivity estimate from pumping-test data is, on average, larger than that from a series of slug-test data in the same formation. Zlotnik and McGuire (1998a,b) and Hemker (1999) developed the analytical solutions of the transient well inflow in a multi-layered aquifer system. McElwee (2002) examined and used four techniques, including an extended capability nonlinear model, sensitivity analysis, correction for acceleration and velocity effects, and multiple slug tests to improve the analyses of slug tests.

Mathematical model

The governing equations describing the hydraulic head distributions within the skin and formation

zones can respectively be written as (Moench and Hsieh 1985a,b)

$$\frac{\partial^2 h_1}{\partial r^2} + \frac{1}{r} \frac{\partial h_1}{\partial r} = \frac{S_1}{T_1} \frac{\partial h_1}{\partial t}, \quad r_w < r < r_s \quad (1)$$

and

$$\frac{\partial^2 h_2}{\partial r^2} + \frac{1}{r} \frac{\partial h_2}{\partial r} = \frac{\partial h_2}{\partial t}, \quad r > r_s \quad (2)$$

where the subscripts 1 and 2 respectively denote the skin and formation zones, T is the transmissivity, and S is the storativity. The symbol h is the hydraulic head within the skin or formation zone, r is the radial distance from center of well, r_s is the distance from center of well to outer boundary of skin zone, t is the test time, and r_w is the well radius.

The hydraulic heads are initially assumed to be zero within both the skin and formation zones, that is

$$h_1 = h_2 = 0, \quad t = 0 \quad (3)$$

The initial condition for a head in a well, h_w , is

$$h_w = h_0, \quad t = 0 \quad (4)$$

where h_0 is the initial head. The conservation of mass at a well requires that

$$r_c^2 \frac{dh_w}{dt} = 2K_1 b r_w \left(\frac{\partial h_1}{\partial r} \right)_{r=r_w}, \quad r = r_w \quad (5)$$

and the outer boundary condition at infinity is

$$h_2 = 0, \quad r \rightarrow \infty \quad (6)$$

where r_c is the radius of the well casing in the region of changing water level, K is the hydraulic conductivity, and b is the aquifer thickness. The continuity conditions required at the interface between the skin and formation zones are

$$h_1 = h_2, \quad r = r_s \quad (7)$$

and

$$K_1 \frac{\partial h_1}{\partial r} = K_2 \frac{\partial h_2}{\partial r}, \quad r = r_s \quad (8)$$

Applying the dimensionless transforms and the Laplace transforms to Eqs. (1) and (2) and using the initial condition, Eq. (3), gives the following subsidiary equations:

$$\frac{d^2 \bar{h}_1}{dr'^2} + \frac{1}{r'} \frac{d\bar{h}_1}{dr'} = (\beta q)^2 \bar{h}_1, \quad 1 < r' < r'_s \quad (9)$$

and

$$\frac{d^2 \bar{h}_2}{dr'^2} + \frac{1}{r'} \frac{d\bar{h}_2}{dr'} = q^2 \bar{h}_2, \quad r' > r'_s \quad (10)$$

where \bar{h} is the transform dimensionless hydraulic head, p is the Laplace transform variable of dimensionless time, $q = p^{1/2}$, $\beta = (\alpha\delta)^{1/2}$, $\alpha = K_1/K_2$, $\delta = S_{s1}/S_{s2}$, $\gamma = r_c^2/(2r_w^2 S_{s2} b)$, $r' = r/r_w$, and $r'_s = r_s/r_w$. The transform boundary and continuity conditions of Eqs. (5–8) are respectively,

$$\alpha\gamma(p\bar{h}'_w - 1) = \left(\frac{d\bar{h}_1}{dr'} \right)_{r'=1} \quad (11)$$

$$\bar{h}_2 = 0, \quad r' \rightarrow \infty \quad (12)$$

$$\bar{h}_1 = \bar{h}_2, \quad r' = r'_s \quad (13)$$

and

$$\frac{d\bar{h}_1}{dr'} = \alpha \frac{d\bar{h}_2}{dr'}, \quad r' = r'_s \quad (14)$$

The Laplace-domain solutions for dimensionless hydraulic heads within both the skin and formation zones given by Moench and Hsieh (1985b) are respectively

$$\bar{h}_1 = \frac{\alpha\gamma[\Delta_1 K_0(q\beta r') - \Delta_2 I_0(q\beta r')]}{c_1 \Delta_1 - c_2 \Delta_2} \quad (15)$$

and

$$\bar{h}_2 = \frac{\alpha\gamma[\Delta_1 K_0(q\beta r'_s) - \Delta_2 I_0(q\beta r'_s)] K_0(qr')}{[c_1 \Delta_1 - c_2 \Delta_2] K_0(qr'_s)} \quad (16)$$

where the variables Δ_1 , Δ_2 , c_1 , and c_2 are respectively defined as

$$\Delta_1 = \alpha I_0(q\beta r'_s) K_1(qr'_s) + \beta I_1(q\beta r'_s) K_0(qr'_s) \quad (17)$$

$$\Delta_2 = \alpha K_0(q\beta r'_s) K_1(qr'_s) - \beta K_1(q\beta r'_s) K_0(qr'_s) \quad (18)$$

$$c_1 = \alpha\gamma p K_0(q\beta) + \beta q K_1(q\beta) \quad (19)$$

and

$$c_2 = \alpha\gamma p I_0(q\beta) - \beta q I_1(q\beta) \quad (20)$$

When $r' = 1$, the dimensionless head in a well obtained from Eq. (15) is

$$\overline{h}_w = \frac{\alpha\gamma[\Delta_1 K_0(q\beta) - \Delta_2 I_0(q\beta)]}{c_1\Delta_1 - c_2\Delta_2} \quad (21)$$

Moench and Hsieh Solution (1985b)

As shown in this section, the MHS satisfies the governing equations, boundary conditions, and continuity requirements at the interface of the skin and formation zones.

Taking the derivatives of Eq. (15) with respect to r and letting $r'=1$ yields

$$\left. \frac{d\overline{h}_1'}{dr'} \right|_{r'=1} = \frac{-q\beta\alpha\gamma[\Delta_1 K_1(q\beta) + \Delta_2 I_1(q\beta)]}{c_1\Delta_1 - c_2\Delta_2} \quad (22)$$

Substituting Eq. (21) into the left-hand side (LHS) of Eq. (11) and using simple algebraic manipulation, one can obtain Eq. (22). Thus, the boundary condition at a wellbore, Eq. (11), is satisfied.

With $K_0(\infty)=0$ as $r' \rightarrow \infty$, Eq. (16) yields $\overline{h}_2 = 0$. Therefore, the solution satisfies the outer boundary condition, Eq. (12).

Letting $r' = r'_s$, Eqs. (15) and (16) become

$$\overline{h}_1' = \frac{\alpha\gamma[\Delta_1 K_0(q\beta r'_s) - \Delta_2 I_0(q\beta r'_s)]}{c_1\Delta_1 - c_2\Delta_2} = \overline{h}_2' \quad (23)$$

This shows the continuity of dimensionless hydraulic head at the interface of the skin and formation zones.

Taking the derivatives of Eqs. (15) and (16), respectively, with respect to r' and letting $r' = r'_s$ gives

$$\left. \frac{d\overline{h}_1'}{dr'} \right|_{r'=r'_s} = \frac{-\alpha^2\gamma K_1(qr'_s)}{r'_s[c_1\Delta_1 - c_2\Delta_2]} \quad (24)$$

and

$$\left. \frac{d\overline{h}_2'}{dr'} \right|_{r'=r'_s} = \frac{-\alpha\gamma K_1(qr'_s)}{r'_s[c_1\Delta_1 - c_2\Delta_2]} \quad (25)$$

Furthermore, Eq. (24) equals the product of Eq. (25) multiplied by α on both sides. Accordingly, this proves the continuity of flow rate between the skin and formation zones, i.e., Eq. (14).

Following steps similar to those demonstrated above by taking the first and second derivatives with respect to r' for Eqs. (15) and (16) will prove that MHS satisfies the governing equations, Eqs. (9) and (10). McLachlan (1955, pp 192–197) provided the formulae

$$\frac{dI_0(q\beta r')}{dr'} = (q\beta)I_1(q\beta r') \quad (26)$$

$$\frac{dI_1(q\beta r')}{dr'} = \frac{1}{q\beta r'} I_1(q\beta r') + I_0(q\beta r') \quad (27)$$

and

$$\frac{2}{q\beta r'} I_1(q\beta r') = I_0(q\beta r') - I_2(q\beta r') \quad (28)$$

Based on Eqs. (26–28), one can obtain

$$\begin{aligned} \frac{d^2}{dr'^2} [I_0(q\beta r')] + \frac{1}{r'} \frac{d}{dr'} [I_0(q\beta r')] \\ = (q^2\beta^2)I_0(q\beta r') \end{aligned} \quad (29)$$

Likewise,

$$\begin{aligned} \frac{d^2}{dr'^2} [K_0(q\beta r')] + \frac{1}{r'} \frac{d}{dr'} [K_0(q\beta r')] \\ = (q^2\beta^2)K_0(q\beta r') \end{aligned} \quad (30)$$

After taking the first and second derivatives with respect to r' for Eq. (15) and using Eqs. (29) and (30), one can get

$$\frac{d^2\overline{h}_1'}{dr'^2} + \frac{1}{r'} \frac{d\overline{h}_1'}{dr'} = (q\beta)^2\overline{h}_1' \quad (31)$$

Similarly, one can obtain

$$\frac{d^2\overline{h}_2'}{dr'^2} + \frac{1}{r'} \frac{d\overline{h}_2'}{dr'} = (q^2)\overline{h}_2' \quad (32)$$

According to Eqs. (31) and (32), the solutions of Eqs. (15) and (16) respectively satisfy Eqs. (9) and (10).

Numerical evaluation

Bessel functions

Equations (15), (16), and (21) include the Bessel functions $I_0(u)$, $I_1(u)$, $K_0(u)$, and $K_1(u)$. These functions are approximated here by the formulae given in Watson (1958) and Abramowitz and Stegun (1964). The argument u in these formulae may be divided into two ranges; (0, 10) and (10, ∞) for $I_0(u)$ and $I_1(u)$, and (0, 2) and (2, ∞) for $K_0(u)$ and $K_1(u)$ in order to achieve better accuracy. In addition, the formulae are essentially composed of infinite series and may converge slowly, especially when u is small. Therefore, the Shanks method (Shanks 1955; Yeh et al. 2003; Yang and Yeh 2005) is employed to accelerate the convergence when evaluating the series of the Bessel functions. Each function in Eqs. (15), (16), and (21) is calculated to ten decimal places, and thus they bear the same degree of accuracy as those listed in Abramowitz and Stegun (1964).

The Bessel functions of $I_0(u)$, $I_1(u)$, $K_0(u)$, and $K_1(u)$ may be evaluated by the formulae given in Abramowitz and Stegun (1964). For $0 \leq u \leq 10$, the asymptotic expansions for $I_0(u)$ and $I_1(u)$ may be expressed respectively as (Abramowitz and Stegun 1964, p 375)

$$I_0(u) = 1 + \frac{\frac{1}{4}u^2}{(1!)^2} + \frac{(\frac{1}{4}u^2)^2}{(2!)^2} + \frac{(\frac{1}{4}u^2)^3}{(3!)^2} + \dots \tag{33}$$

and

$$I_1(u) = \left(\frac{u}{2}\right) \left[1 + \frac{\frac{1}{4}u^2}{(1!)(2!)} + \frac{(\frac{1}{4}u^2)^2}{(2!)(3!)} + \frac{(\frac{1}{4}u^2)^3}{(3!)(4!)} + \dots \right] \tag{34}$$

The Bessel functions of $I_0(u)$ and $I_1(u)$ for $10 < u < \infty$ are respectively approximated as (Abramowitz and Stegun 1964, p 377)

$$I_0(u) = \frac{e^u}{\sqrt{2\pi u}} \left\{ 1 + \frac{1^2}{(1!)(8u)} + \frac{1^2 \cdot 3^2}{(2!)(8u)^2} + \frac{1^2 \cdot 3^2 \cdot 5^2}{(3!)(8u)^3} + \dots \right\} \tag{35}$$

and

$$I_1(u) = \frac{e^u}{\sqrt{2\pi u}} \left\{ 1 - \frac{4 - 1^2}{(1!)(8u)} + \frac{(4 - 1^2) \cdot (4 - 3^2)}{(2!)(8u)^2} - \frac{(4 - 1^2) \cdot (4 - 3^2) \cdot (4 - 5^2)}{(3!)(8u)^3} + \dots \right\} \tag{36}$$

For $0 \leq u \leq 2$, the asymptotic expansions for $K_0(u)$ and $K_1(u)$ may be written respectively as (Abramowitz and Stegun 1964, p 375)

$$K_0(u) = -\left[\ln\left(\frac{u}{2}\right) + \psi \right] I_0(u) + \frac{\frac{1}{4}u^2}{(1!)^2} + \left(1 + \frac{1}{2}\right) \frac{(\frac{1}{4}u^2)^2}{(2!)^2} + \left(1 + \frac{1}{2} + \frac{1}{3}\right) \frac{(\frac{1}{4}u^2)^3}{(3!)^2} + \dots \tag{37}$$

and

$$K_1(u) = \left[\ln\left(\frac{u}{2}\right) + \psi \right] I_1(u) + \frac{1}{u} - \frac{u}{4} \left\{ \left[\frac{\frac{1}{4}u^2}{(1!)(2!)} + \left(1 + \frac{1}{2}\right) \frac{(\frac{1}{4}u^2)^2}{(2!)(3!)} + \left(1 + \frac{1}{2} + \frac{1}{3}\right) \frac{(\frac{1}{4}u^2)^3}{(3!)(4!)} + \dots \right] + \left[1 + \left(1 + \frac{1}{2}\right) \frac{\frac{1}{4}u^2}{(1!)(2!)} + \left(1 + \frac{1}{2} + \frac{1}{3}\right) \frac{(\frac{1}{4}u^2)^2}{(2!)(3!)} + \dots \right] \right\} \tag{38}$$

where $\psi = 0.577215664901533$ is the Euler's constant. For $2 < u < \infty$, $K_0(u)$ and $K_1(u)$ are respectively taken as (Abramowitz and Stegun 1964, p 378)

$$K_0(u) = \frac{e^u}{\sqrt{2\pi u}} \left\{ 1 - \frac{1^2}{(1!)(8u)} + \frac{1^2 \cdot 3^2}{(2!)(8u)^2} - \frac{1^2 \cdot 3^2 \cdot 5^2}{(3!)(8u)^3} + \dots \right\} \tag{39}$$

and

$$K_1(u) = \frac{e^u}{\sqrt{2\pi u}} \left\{ 1 + \frac{4 - 1^2}{(1!)(8u)} + \frac{(4 - 1^2) \cdot (4 - 3^2)}{(2!)(8u)^2} + \frac{(4 - 1^2) \cdot (4 - 3^2) \cdot (4 - 5^2)}{(3!)(8u)^3} + \dots \right\} \tag{40}$$

Shanks method

The Shanks method, which is also called the ϵ -algorithm (Kahaner 1972), is a non-linear sequence-to-sequence transformation. Shanks (1955) introduced a family of non-linear transforms designated as e_k , e_k^m , \tilde{e}_k , and e_d . Theorems are provided to show the effectiveness of this transform both in accelerating the convergence of slowly convergent sequences and inducing convergence in divergent sequences. The Shanks transform is an iterative algorithm based on the directly summing series. The operator of the i th order transform of S_n is defined as (Shanks 1955)

$$e_m(S_n) = \begin{vmatrix} S_n & S_{n+1} & \dots & S_{n+m} \\ \Delta S_n & \Delta S_{n+1} & \dots & \Delta S_{n+m} \\ \vdots & \vdots & \vdots & \vdots \\ \Delta S_{n+m-1} & \Delta S_{n+m} & \dots & \Delta S_{n+2m-1} \end{vmatrix}^{-1} = \begin{vmatrix} 1 & 1 & \dots & 1 \\ \Delta S_n & \Delta S_{n+1} & \dots & \Delta S_{n+m} \\ \vdots & \vdots & \vdots & \vdots \\ \Delta S_{n+m-1} & \Delta S_{n+m} & \dots & \Delta S_{n+2m-1} \end{vmatrix}^{-1} \tag{41}$$

where $\Delta S_n = S_{n+1} - S_n$ with $n=1, 2, \dots, \infty$ as the number of series and $m=0, 1, 2, \dots, n-2k$ as the number of iterations.

Wynn (1956) derived a mathematical expression from the Shanks transform. This transform for the first and second iterations can then be expressed for computation as follows

$$e_0(S_n) = S_n \tag{42}$$

and

$$e_1(S_n) = [\Delta S_n]^{-1} \tag{43}$$

where $n=1, 2, \dots$. The third and remaining iterations are expressed as

$$e_{i+1}(S_{n-i}) = e_{i-1}(S_{n-i+1}) + [e_i(\Delta S_{n-i})]^{-1} \quad (44)$$

where $i=1, 2, \dots (n-2)$.

The Shanks transform requires use of the even-order terms when approximating the sum of S_n ; the odd-order terms are purely intermediate quantities in the numerical computations. Applying the Shanks transform to evaluate a given series requires setting a convergence criterion, ERR, defined as

$$\left| \frac{e_{2r+2}(S_{n-1}) - e_{2r}(S_n)}{e_{2r+2}(S_{n-1})} \right| \leq ERR, \text{ for } n = 3, 4, \dots \quad (45)$$

Shanks (1955) indicated that the series converges if three successive values of n satisfy the criterion. However, such a suggestion may be too strict. For comparing the implementation of the Shanks transform with the other methods to the same problem, the convergence criterion is to terminate the summing process. When applying the Shanks transform to a given series, it is defined as (Huang et al. 2000)

$$\left| e_{2i}(S_{n-2i}) - e_{2(i-1)}(S_{n-2i+1}) \right| \leq ERR, \quad (46)$$

for $n = 3, 4, \dots$

where $r=1, 2, \dots (n-2)/2$ when n is even and $r= 1, 2, \dots (n-1)/2$ when n is odd.

Numerical inversion

The Laplace transforms are commonly used to solve the differential and integral equations. In many engineering problems, the Laplace-domain solutions for mathematical models are tractable, yet the corresponding solutions in the time domain may be difficult or impossible to solve. Under such circumstances, the methods of numerical Laplace inversion such as the Stehfest method (Stehfest 1970), the Crump method (Crump 1976), or the Talbot method (Talbot 1979) may be used. de Hoog et al. (1982) mentioned that the different numerical methods of inversion give accurate answers for various cases.

The Laplace transform of a real-valued function $f(t)$, $t \geq 0$, is defined as

$$F(p) = \int_0^\infty \exp(-pt)f(t)dt \quad (47)$$

and the inversion of $F(p)$ is given by

$$f(t) = \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \exp(pt)F(p)dp \quad (48)$$

where κ is to the right of any singularities of $F(p)$.

Using the trapezoidal rule with step size π/T_p , Crump (1976) proposed the Laplace inversion formula using an infinite series as follows

$$f(t) = \left(\frac{1}{T_p} \exp(\zeta t) \right) \left\{ \frac{F(\xi)}{2} + \sum_{k=1}^\infty \text{Re} \left[F \left(\xi + \frac{ik\pi}{T_p} \right) \exp \left(\frac{ik\pi t}{T_p} \right) \right] \right\} \quad (49)$$

in which $F(\cdot)$ is constructed to be an infinite set of even and odd period functions, each with a period of $2T_p$, the parameter ξ is an estimate of the maximum of the real parts of singularity functions, and the symbol Re represents the real part of the function. In fact, the Crump method approximates the Laplace inversion by expressing the transform function in a Fourier series.

Crump (1976) used the ϵ -algorithm of Wynn (1956) (also called the Shanks method) to accelerate the convergence of the sum in Eq. (49) and gave the convergence of the sequence of partial sums as

$$S_n = S_{n-1} + \text{Re} \left[F \left(\xi + \frac{in\pi}{T_p} \right) \exp \left(\frac{in\pi t}{T_p} \right) \right], \quad (50)$$

$n = 1, 2, \dots, 2M$

The Crump's accelerated estimate of $f(t)$ is defined as

$$\hat{f}(t) = \left(\frac{1}{T_p} \exp(\zeta t) \right) e_{2M}(S_0) \quad (51)$$

where $S_0 = F(\xi)/2$ and the transform $e_j(S_n)$ is defined as Eq. (44).

de Hoog et al. (1982) applied the Pade approximations to improve the acceleration procedure to the transform sequence in Eq. (51). The Pade approximations represent a function as the quotient of two polynomials

$$f(t) \cong R_N(t) = \frac{a_0 + a_1t + a_2t^2 + \dots + a_nt^n}{1 + b_1t + b_2t^2 + \dots + b_mt^m}, \quad (52)$$

$N = n + m$

where the parameters, a_0, a_1, \dots and b_1, b_2, \dots, b_m , are available for the approximation of $f(t)$, by $R_N(t)$. The Pade approximations are related to Maclaurin expansions in that the coefficients are determined in a similar fashion to make $\hat{f}(t)$ and $R_N(t)$ agree at $t=0$ and the first N derivatives agree at $t=0$ (Gerald and Wheatley 1989). de Hoog et al. (1982) also applied the Shanks method to accelerate the running sums of infinite series.

The dimensionless hydraulic head and water level of a well are determined by inverting Eqs. (15), (16), and (21) with an appropriate numerical algorithm (e.g., Stehfest 1970; Crump 1976; or Talbot 1979). Herein, the modified Crump algorithm (Crump 1976; de Hoog et al. 1982) is adopted to invert the Laplace-domain solutions, Eqs. (15), (16), and (21), to an accuracy of five decimal places. This routine was developed based on an algorithm originally

proposed by Crump (Crump 1976) and modified by de Hoog et al. (de Hoog et al. 1982). A double-precision format is used for all evaluations and the convergence criterion for the Shanks method is set as 10^{-7} . Then, the numerical inversion results are compared with the values obtained from those given by Yang and Gates (1997). The effect of dimensionless hydraulic head distribution is also discussed for a composite aquifer system.

Numerical results and discussion

Based on a composite (two-zone) aquifer model, Yang and Gates (1997) used a finite-element method to simulate the dimensionless head distribution for a slug test. In their simulations, the hydraulic conductivity and storativity are respectively 1.8×10^{-6} cm/sec and 0.01 for a positive skin, 1.8×10^{-5} cm/sec and 0.01 for an undisturbed formation, and 1.8×10^{-4} cm/sec and 0.1 for a negative skin. Initially, MHS is evaluated using a numerical inversion routine INLAP (IMSL 1997). Table 1 gives the values of dimensionless heads in a tested well calculated using the parameters provided by Yang and Gates (1997) for $r_c=5.08$ cm, $r_w=9.15$ cm, and $r_s=30.5$ cm when α (conductivity ratio)=0.1, 1, or 10. Note that the formation

has a negative skin if $\alpha=0.1$ and a positive skin if $\alpha=10$; when the formation has only a single zone, $\alpha=1$. The differences of the tabular values for dimensionless heads between MHS and Yang and Gates simulation results are significant for all cases, especially for a large t and/or a small α . Figure 1 shows the curves of dimensionless head versus time obtained by MHS and those given by Yang and Gates (1997) when $\alpha=0.1, 1, \text{ or } 10$. Obviously, the curves of Yang and Gates (1997) are significantly different from those of MHS. In fact, their mass balance equation—Yang and Gates (1997); Eq. (4)—and interface condition—Yang and Gates (1997), Eq. (5)—are questionable. The LHS term of Eq. (5) is set $r=r_w$ in this paper to represent the mass balance equation at the wellbore. However, Yang and Gates (1997) neglected the well radius of the test well; in other words, they treated the well as a line source. Consequently, the estimated hydraulic heads near the well may have significant errors. In addition, the partial derivative of dimensionless hydraulic head in Eq. (8) was taken with respect to dimensionless time t , instead of r , as used in Yang and Gates (1997). Physically, Eq. (8) should be used to preserve the continuity of the flow rate at the interface of the skin and formation zones.

The dimensionless well water levels estimated from MHS are compared with those given by Yang and Gates

Table 1 Dimensionless water level in a well, estimated by MHS and given in Yang and Gates (1997) for $r_c=5.08$ cm, $r_w=9.15$ cm, and $r_s=30.5$ cm when $\alpha=0.1, 1, \text{ or } 10$

Time (sec)	$\alpha=0.1$		$\alpha=1$		$\alpha=10$	
	MH	YG	MH	YG	MH	YG
2	0.87666	0.8179	0.98649	0.9801	0.99590	0.9978
4	0.82822	0.6905	0.98046	0.9632	0.99415	0.9957
6	0.79230	0.5979	0.97567	0.9484	0.99279	0.9937
8	0.76289	0.5281	0.97153	0.9351	0.99164	0.9916
10	0.73767	0.4738	0.96781	0.9230	0.99061	0.9896
15	0.68617	0.3799	0.95966	0.8966	0.98839	0.9848
20	0.64510	0.3195	0.95257	0.8739	0.98649	0.9802
25	0.61070	0.2772	0.94616	0.8538	0.98480	0.9757
30	0.58102	0.2457	0.94025	0.8356	0.98325	0.9714
35	0.55490	0.2211	0.93472	0.8188	0.98181	0.9673
40	0.53160	0.2014	0.92949	0.8032	0.98046	0.9633
45	0.51057	0.1852	0.92452	0.7887	0.97918	0.9594
50	0.49144	0.1716	0.91976	0.7750	0.97796	0.9556
60	0.45780	0.1500	0.91076	0.7497	0.97567	0.9485
70	0.42908	0.1336	0.90234	0.7268	0.97354	0.9417
80	0.40428	0.1207	0.89438	0.7057	0.97153	0.9352
90	0.38268	0.1103	0.88681	0.6862	0.96962	0.9291
100	0.36377	0.1019	0.87956	0.6680	0.96781	0.9232
140	0.30820	0.0802	0.85314	0.6056	0.96119	0.9017
180	0.27395	0.0694	0.82971	0.5550	0.95531	0.8828
220	0.25222	0.0636	0.80843	0.5126	0.94994	0.8657
260	0.23795	0.0603	0.78881	0.4763	0.94495	0.8501
300	0.22819	0.0583	0.77053	0.4447	0.94025	0.8356
400	0.21380	0.0555	0.72940	0.3811	0.92949	0.8033
500	0.20551	0.0539	0.69325	0.3328	0.91975	0.7750
600	0.19946	0.0526	0.66088	0.2949	0.91075	0.7497
700	0.19445	0.0515	0.63153	0.2644	0.90230	0.7268
800	0.19003	0.0506	0.60468	0.2395	0.89429	0.7057
900	0.18601	0.0497	0.57993	0.2187	0.88664	0.6861
1000	0.18230	0.0489	0.55701	0.2012	0.87928	0.6678

r_c =radius of well casing; r_w =well radius; r_s is the distance from center of well to outer boundary of skin zone. $\alpha=K_1/K_2$, where K_1 is the hydraulic conductivity of the skin, and K_2 is the hydraulic conductivity of the formation. *MH* and *YG* respectively denote the results obtained from MHS and given in Yang and Gates (1997). The hydraulic conductivity and storativity are, respectively, 1.8×10^{-6} cm/sec and 0.01 for a positive skin, 1.8×10^{-5} cm/sec and 0.01 for an undisturbed formation, and 1.8×10^{-4} cm/sec and 0.1 for a negative skin

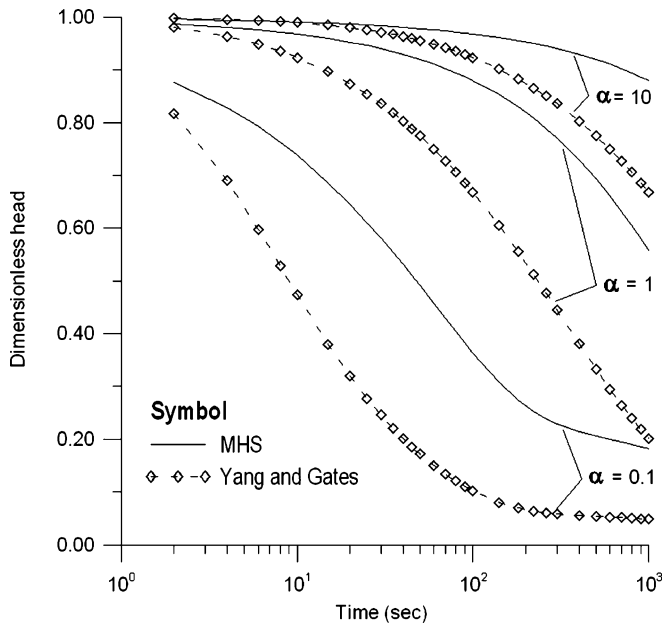


Fig. 1 Curves of dimensionless head versus time obtained by MHS and those given by Yang and Gates (1997) when $\alpha=0.1, 1$ or 10

(1997) based on a finite-element modeling. The result indicates that the Yang and Gates tabular values have very poor accuracy. This problem is mainly attributed to the problem that the interface condition for the flow rate in their mathematical model is incorrect.

Figure 2 displays the curves of dimensionless hydraulic head versus dimensionless distance for $r_c=5.08$ cm, $r_w=9.15$ cm, $r_s=30.5$ cm, and $t=10^3$ sec when $\alpha=0.1, 1$ or 10 . The storativity of the skin and formation zones is assumed as the same to simplify the problems and is set as

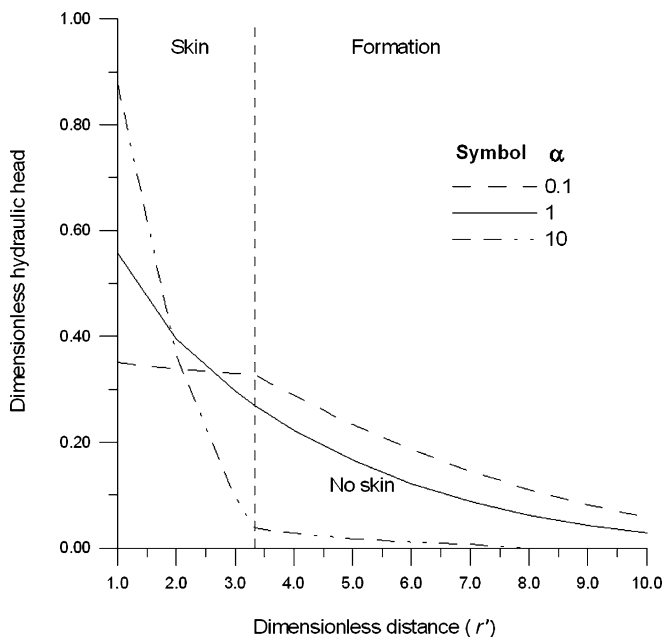


Fig. 2 Curves of dimensionless hydraulic head versus dimensionless distance obtained by MHS for $r_c=5.08$ cm, $r_w=9.15$ cm, $r_s=30.5$ cm, and $t=10^3$ sec when $\alpha=0.1, 1$ or 10

0.01. For the case without a skin zone, the dimensionless hydraulic head gradually decreases with increasing radial distance as shown in Fig. 2. If a finite-thickness skin is present, the relation of dimensionless hydraulic head versus dimensionless distance exhibits two curves with different slope joined at the interface ($r'_s = 3.33$). A negative skin, which has a higher transmissivity than the formation, has a curve with relative mild slope in the skin zone and with steeper slope in the formation zone. In contrast, a positive skin has a very steep slope in the skin zone due to the lower transmissivity and a relative flat slope in the formation zone. In addition, the difference of dimensionless hydraulic head between the skin and formation zones for an aquifer with a positive skin is larger obviously than that with a negative skin. This indicates that the dimensionless hydraulic head distribution depends on the hydraulic properties of both the skin and formation zones. If ignoring the skin zone, the dimensionless hydraulic head in a composite aquifer system may be apparently overestimated or underestimated and the error of the test-data analysis will be inducted.

Conclusions

A detailed derivation has been presented as proof that MHS for a slug test in a well surrounded by a finite-thickness skin satisfies the governing equations, boundary conditions, and continuity equations for hydraulic head and flow. Two new formulae relating the modified zero-order Bessel functions of the first and second kinds to their first and second derivatives are established. An efficient numerical approach is proposed to evaluate MHS. The numerical values of finite-element simulation given in Yang and Gates (1997) are apparently different from those evaluated based on the MHS. The problems in the mass balance equation and interface condition presented by Yang and Gates (1997) make their simulation results differ from the values of MHS. The plots for the spatial and temporal distributions of dimensionless hydraulic head demonstrate the dependence of the head on the hydraulic properties of both the skin and formation zones. In addition, the effect for an aquifer with a positive skin is significantly larger than that with a negative skin.

Notation

- b Thickness of aquifer
- h Hydraulic head in the skin or formation zone
- h' Dimensionless hydraulic head in the skin or formation zone
- h_w Water level in a well
- h'_w Dimensionless water level in a well
- $I_0(\cdot), K_0(\cdot)$ Modified Bessel function of the first and second kinds and order zero, respectively
- $I_1(\cdot), K_1(\cdot)$ Modified Bessel function of the first and second kinds and order one, respectively

K	Hydraulic conductivity
P	Laplace transform variable of dimensionless time
q	$=p^{1/2}$
r	Radial distance from center of well
r_c	Standpipe radius
r_s	Distance from center of well to outer boundary of skin
r_w	Test well radius
r'	$=r/r_w$ Dimensionless radial distance from center of well
r'_s	$=r_s/r_w$ Dimensionless distance from center of well to outer boundary of skin
S_s	Specific storage
t	Test time
t'	$=K_2 t / (r_w)^2 S_{s2}$
α	$=K_2 / K_1$
β	$=(\delta\alpha)^{1/2}$
γ	$=r_c^2 / (2r_w^2 S_{s2} b)$
δ	$=S_{s1} / S_{s2}$

Acknowledgements This study was partly supported by the Taiwan National Science Council under the grant NSC 91-2211-E-009-020. The authors appreciate the comments and suggested revisions of anonymous reviewers that helped to improve the clarity of our presentation.

References

- Abramowitz M, Stegun IA (1964) Handbook of mathematical functions with formulas, graphs and mathematical tables, National Bureau of Standards. Dover, Washington, DC
- Butler Jr JJ, Healey JM (1998) Relationship between pumping-test and slug-test parameters: scale effect or artifact?. *Ground Water* 36(2):305–313
- Cooper Jr HH, Bredehoeft JD, Papadopolis IS (1967) Response of a finite-diameter well to an instantaneous charge of water. *Water Resour Res* 3(1):263–269
- Crump KS (1976) Numerical inversion of Laplace transforms using a Fourier series approximation. *J Assoc Comput Mach* 23(1): 89–96
- de Hoog FR, Knight JH, Stokes AN (1982) An improved method for numerical inversion of Laplace transforms. *Society for Industrial and Applied Mathematics. J Sci Stat Comput* 3(3): 357–366
- Faust CR, Mercer JW (1984) Evaluation of slug tests in wells containing a finite-thickness skin. *Water Resour Res* 20(4):504–506
- Gerald CF, Wheatley PO (1989) *Applied Numerical Analysis*, 5th edn. Addison-Wesley, Boston, MA
- Hemker CJ (1999) Transient well flow in layered aquifer systems: the uniform well-face drawdown solution. *J Hydrol* 225: 19–44
- Huang CP, Yeh HD, Yang SY (2000) Applications of accelerate methods on the evaluation of two types of drawdown solutions. *Proceedings of the 11th hydraulic engineering conference, National Taiwan University, Taipei, Taiwan*, pp D147–D152
- IMSL (1997) *Math/Library*, vols. 1 and 2, Visual Numerics, Houston, TX
- Kahaner DK (1972) Numerical quadrature by the ϵ -algorithm. *Math Comp* 26:689–693
- Karasaki K (1990) A systematized drillstem test. *Water Resour Res* 26(12):2913–2919
- Karasaki K, Long JCS, Witherspoon PA (1988) Analytical models of slug tests. *Water Resour Res* 24(1):115–126
- McElwee CD (2002) Improving the analysis of slug tests. *J Hydro* 269:122–133
- McLachlan NW (1955) *Bessel Functions for Engineers*, 2nd edn. Oxford University Press, London
- Moench AF, Hsieh PA (1985a) Comment on “Evaluation of slug tests in wells containing a finite-thickness skin” by C.R. Faust and J.W. Mercer. *Water Resour Res* 21(9):1459–1461
- Moench AF, Hsieh PA (1985b) Analysis of slug test data in a well with finite thickness skin. *The IAH 17th International congress on Hydrogeology of rocks of low permeability, Int Assoc of Hydrogeol. Tucson, AZ, 7–12 Jan 1985*
- Ramey Jr HJ, Agarwal RG, Martin I (1975) Analysis of “Slug Test” or DST flow period data. *J Can Pet Technol* July/Sept:37–47
- Sageev A (1986) Slug test analysis. *Water Resour Res* 22(8):1323–1333
- Shanks D (1955) Non-linear transformations of divergent and slowly convergent sequences. *J Math Phys* 34:1–42
- Stehfest H (1970) Numerical inversion of Laplace transforms. *Commun ACM* 13(1):47–49
- Talbot A (1979) The accurate numerical inversion of Laplace transforms. *J Inst Math Appl* 23:97–120
- Watson GN (1958) *A treatise on the theory of Bessel functions*, 2nd edn. Cambridge University Press, Cambridge
- Wynn P (1956) On a device for computing the $e_n(S_n)$ transformation. *Math Tables Other Aids Comp* 10:91–96
- Yang YJ, Gates TM (1997) Wellbore skin effect in slug-test data analysis for low-permeability geologic materials. *Ground Water* 35(6):931–937
- Yang SY, Yeh HD (2005) Laplace-domain solutions for radial two-zone flow equations under the conditions of constant-head and partially penetrating well. *J Hydraul Eng ASCE* 131(3):209–216
- Yeh HD, Yang SY, Peng HY (2003) A new closed-form solution for radial two-layer drawdown equation under constant-flux pumping in a finite-radius well. *Adv Water Resour* 26(5):747–757
- Zlotnik VA, McGuire VL (1998a) Multi-level slug test in highly permeable formations: 1. Modification of the Springer-Gelhar (SG) model. *J Hydrol* 204:271–282
- Zlotnik VA, McGuire VL (1998b) Multi-level slug test in highly permeable formations: 2. Hydraulic conductivity identification, method verification, and field applications. *J Hydrol* 204:283–296